ON SOME INEQUALITIES OF SYMMETRIC MEANS AND MIXED MEANS

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ABSTRACT. We improve some inequalities involving the symmetric means. We also prove some mixed-mean inequalities for certain families of means.

1. Introduction

Let $M_{n,r}(\mathbf{x})$ be the generalized weighted power means: $M_{n,r}(\mathbf{x}) = (\sum_{i=1}^n q_i x_i^r)^{\frac{1}{r}}$, where $q_i > 0, 1 \leq i \leq n$ with $\sum_{i=1}^n q_i = 1$ and $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. Here $M_{n,0}(\mathbf{x})$ denotes the limit of $M_{n,r}(\mathbf{x})$ as $r \to 0^+$. Let $r \in \{0, 1, \cdots, n\}$, the r-th symmetric function $E_{n,r}$ of $\mathbf{x}$ and its mean $P_{n,r}$ are defined by

$$E_{n,r}(\mathbf{x}) = \sum_{1 \leq i_1 < \cdots < i_r \leq n} \prod_{j=1}^r x_{i_j}, 1 \leq r \leq n; E_{n,0} = 1; P^r_{n,r}(\mathbf{x}) = \frac{E_{n,r}(\mathbf{x})}{\binom{n}{r}}.$$

Unless specified, we always assume $x_i > 0, 1 \leq i \leq n$ and we define $\sigma_n := \sum_{i=1}^n q_i(x_i - A_n)^2$.

To any given $\mathbf{x}, t \geq 0$ we associate $\mathbf{x}' = (1 - x_1, 1 - x_2, \cdots, 1 - x_n), \mathbf{x}_t = (x_1 + t, \cdots, x_n + t)$. When there is no risk of confusion, we shall write $M_{n,r}$ for $M_{n,r}(\mathbf{x})$, $M_{n,r,t}$ for $M_{n,r}(\mathbf{x}_t)$ and $M'_{n,r}$ for $M_{n,r}(\mathbf{x}')$ if $x_i < 1, 1 \leq i \leq n$. The meanings of $E_{n,r}, E'_{n,r}, E_{n,r,t}, P_{n,r}, P'_{n,r}, P_{n,r,t}$ are similar. We also define $A_n = M_{n,1}, G_n = M_{n,0}, H_n = M_{n,-1}$ and similarly for $A'_n, G'_n, H'_n, A_{n,t}, G_{n,t}, H_{n,t}$.

The following counterpart of the arithmetic mean-geometric mean inequality, due to Ky Fan, was first published in the monograph *Inequalities* by Beckenbach and Bellman [5]:

Theorem 1.1. *For $x_i \in (0, 1/2], q_i = 1/n$, $1 \leq i \leq n$,*

$$\frac{A'_n}{G'_n} \leq \frac{A_n}{G_n} \tag{1.1}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

We refer the reader to the survey article[2] and the references therein for an account of Ky Fan's inequality. See also [11]-[16] for recent developments in this subject. Among numerous sharpenings of Ky Fan's inequality in the literature, we note the following inequalities connecting the three classical means(with $q_i = 1/n$ here):

$$(\frac{H_n}{H'_n})^{n-1}\frac{A_n}{A'_n} \leq (\frac{G_n}{G'_n})^n \leq (\frac{A_n}{A'_n})^{n-1}\frac{H_n}{H'_n}. \tag{1.2}$$

The right-hand side inequality of (1.2) is due to P.F.Wang and W.L.Wang[24] and the left-hand side inequality was proved recently by H. Alzer, S. Ruscheweyh and L. Salinas[3]. The result of P.F.Wang and W.L.Wang is more general, they have shown

Theorem 1.2. *For $1 \leq r \leq n-1, x_i \in (0, 1/2], 1 \leq i \leq n$,*

$$\ln P_{n,r} - \ln P_{n,r+1} \geq \ln P'_{n,r} - \ln P'_{n,r+1}. \tag{1.3}$$

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By letting $q_i = w_i/W_n, W_n = \sum_{i=1}^{n} w_i, w_i > 0$(note for different n's, q_i's take different values), we have the following Popoviciu-type inequality due to C.L.Wang[23] which generalizes (1.1):

Theorem 1.3. *For $x_i \in (0, 1/2], 1 \leq i \leq n$,*

$$W_{n+1}\big(\ln(\frac{A_{n+1}}{A'_{n+1}}) - (\frac{G_{n+1}}{G'_{n+1}})\big) \geq W_n\big(\ln(\frac{A_n}{A'_n}) - (\frac{G_n}{G'_n})\big).$$

We also note the following result of P.S. Bullen and M. Marcus[8]:

Theorem 1.4. *For $1 \leq r \leq k \leq n$,*

$$(k+1)\big(\ln(P_{n+1,r}) - \ln(P_{n+1,k+1})\big) \geq k\big(\ln(P_{n,r}) - \ln(P_{n,k})\big)$$

with equality holding if and only if $x_1 = \cdots = x_{n+1}$.

One way of obtaining refinements of known inequalities of means is to study the behavior of means under equal increments of their variables. This was initiated by L. Hoehn and I. Niven[19] and was further developed in [1],[6], [13]and [14]. In particular, one can check that Theorems 1.1-1.3 still hold with $M'_{n,r}, P'_{n,r}$'s replaced by the corresponding $M_{n,r,t}, P_{n,r,t}$'s.

It is then natural to ask whether one has for $1 \leq r \leq k \leq n, t \geq 0$,

$$(k+1)\big(\ln(\frac{P_{n+1,r}}{P_{n+1,r,t}}) - \ln(\frac{P_{n+1,k+1}}{P_{n+1,k+1,t}})\big) \geq k\big(\ln(\frac{P_{n,r}}{P_{n,r,t}}) - \ln(\frac{P_{n,k}}{P_{n,k,t}})\big). \tag{1.4}$$

Our first goal in the paper is to provide a refinement of Theorem 1.2 and also prove (1.4) for $r = 1$. We then move on to study other families of means, such as those considered by E. Beckenbach[4] and M.Dresher[10]. Our motivation comes from the existence of mixed-mean inequalities among the family of generalized power means, described as follows:

For fixed $\mathbf{x} = (x_1, \cdots, x_n), \mathbf{w} = (w_1, \cdots, w_n), w_i > 0$, we define $\mathbf{x}_i = (x_1, \cdots, x_i), \mathbf{w}_i = (w_1, \cdots, w_i), W_i = \sum_{j=1}^{i} w_j$, $M_{i,r} = M_{i,r}(\mathbf{x}_i) = M_{i,r}(\mathbf{w}_i/W_i, \mathbf{x}_i), \mathbf{M}_{i,r} = (M_{1,r}, \cdots, M_{i,r})$. Then we have([22])

Theorem 1.5. *If $1 > s$ and for $2 \leq k \leq n-1, n \geq 2$, $W_n w_k - W_k w_n > 0$. Then*

$$W_{n-1}M_{n-1,s}(\mathbf{M}_{n-1,1}) + w_n M_{n,s} \leq W_n M_{n,s}(\mathbf{M}_{n,1})$$

with equality holding if and only if $x_1 = \cdots = x_n$ and the inequality reverses when $s > 1$.

It follows from this the following mixed-mean inequality(see also [7])

Theorem 1.6. *If $r > s$ and for $2 \leq k \leq n-1, n \geq 2$, $W_n w_k - W_k w_n > 0$. Then*

$$M_{n,s}(\mathbf{M}_{n,r}) \geq M_{n,r}(\mathbf{M}_{n,s}),$$

with equality holding if and only if $x_1 = \cdots = x_n$.

One expects similar mixed-mean inequalities among other families of means under certain conditions(on the weights) and in the last part of the paper, we will establish some results of this type.

2. Results on Symmetric Means

For most parts of the section, we need the following key lemma due to C.Wu, W.Wang and L. Fu[25](see also p. 317, [2]), we include its proof for completeness.

Lemma 2.1. *Let $2 \leq r \leq n, \mathbf{x} = (x_1, \cdots, x_n), x_1 \leq x_2 \leq \cdots \leq x_n$. There exists $\mathbf{y} = (y_1, \cdots, y_r)$ with $x_1 \leq y_1 \leq \cdots \leq y_r \leq x_n$ such that $P_{n,i}(\mathbf{x}) = P_{r,i}(\mathbf{y}), 0 \leq i \leq r$. Moreover, if $x_1, \cdots, x_n$ are not all equal, then $y_1, \cdots, y_r$ are also not all equal.*

Proof. Let $f(t)=\prod_{i=1}^n(t-x_i)$, where we assume that the x_i's are not all equal. Then we conclude that the zeros of f^{n-r}, which we denote by $y_1,\cdots y_r$ with $x_1\leq y_1, y_r\leq x_n$, are also not all equal. We define

$$f^{(n-r)}(t)=\frac{n!}{r!}\sum_{i=0}^{r}(-1)^i g_i(\mathbf{x})t^{r-i}.$$

Then we have for $0\leq i\leq r$,

$$g_i(\mathbf{x})=\frac{r!}{n!}E_{n,i}(\mathbf{x})\prod_{j=0}^{n-r-1}(n-i-j). \tag{2.1}$$

Since

$$f^{(n-r)}(t)=\frac{n!}{r!}\sum_{i=0}^{r}(-1)^i E_{r,i}(\mathbf{y})t^{r-i},$$

we conclude that $g_i(\mathbf{x})=E_{r,i}(\mathbf{y})$ for $0\leq i\leq r$ and the lemma follows from this and (2.1). □

We remark here it follows from the proof of the lemma that for any $t\geq 0$, $P_{n,i}(\mathbf{x}_t)=P_{r,i}(\mathbf{y}_t)$. For an application of the lemma, we note the following result(see [18], Theorems 51 and 52, be aware of the changes in notation):

Theorem 2.1.

$$P_{n,n}\leq P_{n,n-1}\leq\cdots\leq P_{n,2}\leq P_{n,1}, \tag{2.2}$$

and for $0<r<n$,

$$P^{r-1}_{n,r-1}P^{r+1}_{n,r+1}\leq P^{2r}_{n,r}. \tag{2.3}$$

In [18], it shows (2.3) implies (2.2). We now use Lemma 2.1 to show the two are equivalent.

Theorem 2.2. *Inequalities* (2.2), (2.3) *and* $P_{n,1}\geq P_{n,n}$ *are equivalent.*

Proof. Plainly (2.2) implies $P_{n,1}\geq P_{n,n}$ and via a change of variables $x_i\rightarrow 1/x_{n-i+1}$, $P_{n,1}\geq P_{n,n}$ is equivalent to $P_{n,n-1}\geq P_{n,n}$ and then Lemma 2.1 gives (2.2). To show (2.3) implies (2.2), we let $f(t)=\ln(P_{n,r}(\mathbf{x}_t)/P_{n,r+1}(\mathbf{x}_t))$ and note that $f'(t)\leq 0$ implies (2.2) since $\lim_{t\rightarrow\infty}f(t)=0$. As $\mathbf{x}$ is arbitrary, it suffices to show $f'(0)\leq 0$, which is equivalent to (2.3). Now we show (2.3) follows from (2.2). For a given $\mathbf{x}=(x_1,\cdots,x_n)$, we define $\mathbf{x}^{-1}=(1/x_1,\cdots,1/x_n)$. Note $P^j_{n,j}(\mathbf{x}^{-1})=P^{n-j}_{n,n-j}(\mathbf{x})/P^n_{n,n}(\mathbf{x})$. Hence (2.2) implies $P_{n,1}(\mathbf{x}^{-1})\geq P_{n,2}(\mathbf{x}^{-1})$ or $P^{2(n-1)}_{n,n-1}\geq P^{n-2}_{n,n-2}P^n_{n,n}$. This combined with Lemma 2.1 gives (2.3). □

We now look at the following inequalities in the unweighted case($q_i=1/n, n\geq 2$):

$$(n-1)(M^2_{n,2}-A^2_n)\quad\geq A^2_n-G^2_n\quad\geq\frac{1}{n-1}(M^2_{n,2}-A^2_n), \tag{2.4}$$

$$(n-1)(M^2_{n,2}-A^2_n)\quad\geq A^2_n-A_nH_n\quad\geq\frac{1}{n-1}(M^2_{n,2}-A^2_n). \tag{2.5}$$

Inequality (2.4) is due to Diananda[9]. Theorem 5.1 of [13] implies $f'(0)\leq 0, g'(0)\leq 0$ with $f(t)=(1-1/n)\ln A_{n,t}+(1/n)\ln H_{n,t}-\ln G_{n,t}$ and $g(t)=\ln G_{n,t}-(1/n)\ln A_{n,t}-(1-1/n)\ln H_{n,t}$. Inequality (2.5) follows from this and a change of variables $x_i\rightarrow 1/x_{n-i+1}$.

We note the two left-hand side inequalities of (2.4), (2.5) give refinements of (2.3). Since $M^2_{n,2}-A^2_n=(n-1)(P^2_{n,1}-P^2_{n,2})$, the left-hand side inequality of (2.4) is equivalent to $(n-1)^2P^2_{n,2}\leq n(n-2)P^2_{n,1}+P^2_{n,n}$. By a change of variables $x_i\rightarrow 1/x_{n-i+1}$, this is

$$(n-1)^2P^{n-2}_{n,n-2}P^n_{n,n}\leq n(n-2)P^{2n-2}_{n,n-1}+P^{2n-2}_{n,n}.$$

It follows then from Lemma 2.1 that for $2 \leq r \leq n$,

$$(r-1)^2 P_{n,r-2}^{r-2} P_{n,r}^{r} \leq r(r-2) P_{n,r-1}^{2r-2} + P_{n,r}^{2r-2} \leq (r-1)^2 P_{n,r-1}^{2r-2}.$$

Similarly, the left-hand side inequality of (2.5) gives for $2 \leq r \leq n$,

$$(r-1)^2 P_{n,r-2}^{r-2} P_{n,r}^{r} \leq r(r-2) P_{n,r-1}^{2r-2} + P_{n,r-1}^{r-1} P_{n,r}^{r}/P_{n,1} \leq (r-1)^2 P_{n,r-1}^{2r-2}.$$

The two right-hand inequalities of (2.4) and (2.5) are relatively easy. For example, the right-hand side inequality of (2.4) is equivalent to $P_{n,2} \geq P_{n,n}$. We now give a uniform treatment of the two right-hand side inequalities.

Theorem 2.3. *For $t \geq 0, 1 < r \leq n$, $f(t;\alpha) = P_{n,1,t}^{\alpha} - P_{n,r,t}^{\alpha}$ is a decreasing function of t for $\alpha \leq r/(r-1)$ and $P_{n,1,t}^{\alpha} - P_{n,r,t}^{\alpha}$ is an increasing function for $\alpha \geq r$. In particular, for $n \geq 3, q_i = 1/n$, one has*

$$(1-1/n)\frac{G_n^{n/(n-2)} A_n^{(n-3)/(n-2)}}{H_n^{1/(n-2)}} + 1/n M_{n,2}^2 \leq A_n^2. \tag{2.6}$$

Proof. The first assertion of the theorem follows from Proposition 3.1 in [16] and Lemma 2.1. Apply this to $r = n-1$ so that $f'(0;(n-1)/(n-2)) \leq 0$, (2.6) follows from this by a change of variables $x_i \to 1/x_{n-i+1}$. □

Note when $n \geq 3$ and by the well-known Sierpiński's inequality: $A_n H_n^{n-1} \leq G_n^n$,

$$\max\{G_n^2, A_n H_n\} \leq \frac{G_n^{n/(n-2)} A_n^{(n-3)/(n-2)}}{H_n^{1/(n-2)}}.$$

Hence (2.6) gives a refinement of the right-hand side inequalities of (2.4) and (2.5).

Now we give a generalization of (1.3).

Theorem 2.4. *For $2 \leq r \leq n, x_1 \leq x_2 \cdots \leq x_n$,*

$$\frac{\sigma_n}{2(n-1)x_1^2} \geq \ln P_{n,r-1} - \ln P_{n,r} \geq \frac{\sigma_n}{2(n-1)x_n^2}, \tag{2.7}$$

$$\frac{r\sigma_n}{2(n-1)x_1^{2-r/(r-1)}} \geq P_{n,1}^{r/(r-1)} - P_{n,r}^{r/(r-1)} \geq \frac{r\sigma_n}{2(n-1)x_n^{2-r/(r-1)}}. \tag{2.8}$$

Proof. We note first in our case $\sigma_n = (n-1)(P_{n,1}^2 - P_{n,2}^2)$. By Lemma 2.1, there exists $\mathbf{y} = (y_1, \cdots, y_r)$ with $x_1 \leq y_1 \leq \cdots \leq y_r \leq x_n$ such that $P_{n,i}(\mathbf{x}) = P_{r,i}(\mathbf{y}), 0 \leq i \leq r$. Further note that $P_{r,r}(\mathbf{y}) = G_r, P_{r,r-1}(\mathbf{y}) = G_r^{r/(r-1)}/H_r^{1/(r-1)}$. By a result of the author[15], we have

$$(P_{r,1}^2(\mathbf{y}) - P_{r,2}^2(\mathbf{y}))/2y_1^2 \geq \ln P_{r,r-1}(\mathbf{y}) - \ln P_{r,r}(\mathbf{y}) \geq (P_{r,1}^2(\mathbf{y}) - P_{r,2}^2(\mathbf{y}))/2y_r^2.$$

Inequality (2.7) then follows from this and Lemma 2.1. Similarly, (2.8) follows from Theorem 3.1 in [16] and Lemma 2.1. □

Theorem 2.5. *Inequality (1.4) holds for $r = 1 \leq k \leq n$ with equality holding if and only if $x_1 = \cdots = x_n$ when $k \neq n$ and $x_{n+1} = A_n$ when $k = n$.*

Proof. We use the idea in [8] and we may assume $k < n$. In our case, (1.4) is equivalent to

$$\Big(\frac{P_{n+1,k+1}^{k+1}/P_{n,k}^{k}}{P_{n+1,k+1,t}^{k+1}/P_{n,k,t}^{k}}\Big)\Big(\frac{P_{n,1}}{P_{n,1,t}}\Big)^k \leq \Big(\frac{P_{n+1,1}}{P_{n+1,1,t}}\Big)^{k+1}. \tag{2.9}$$

Using the relation

$$P_{n+1,k+1}^{k+1} = \frac{n-k}{n+1} P_{n,k+1}^{k+1} + \frac{k+1}{n+1} x_{n+1} P_{n,k}^{k},$$

we can express the first factor on the left-hand side of (2.9) as

$$\frac{(n-k)P_{n,k+1}^{k+1}/P_{n,k}^{k}+(k+1)x_{n+1}}{(n-k)P_{n,k+1,t}^{k+1}/P_{n,k,t}^{k}+(k+1)(x_{n+1}+t)}.$$

Similarly, by Theorem 2.3 with $\alpha=0$,

$$\begin{aligned}(\frac{P_{n+1,1}}{P_{n+1,1,t}})^{k+1} &= (\frac{P_{n+1,1}(x_{n+1},P_{n,1},\cdots,P_{n,1})}{P_{n+1,1}(x_{n+1}+t,P_{n,1}+t,\cdots,P_{n,1}+t)})^{k+1} \\ &\geq (\frac{P_{n+1,k+1}(x_{n+1},P_{n,1},\cdots,P_{n,1})}{P_{n+1,k+1,t}(x_{n+1}+t,P_{n,1}+t,\cdots,P_{n,1}+t)})^{k+1} \\ &= \frac{(n-k)P_{n,1}^{k+1}+(k+1)x_{n+1}P_{n,1}^{k}}{(n-k)P_{n,1,t}^{k+1}+(k+1)(x_{n+1}+t)P_{n,1,t}^{k}}.\end{aligned}$$

Thus our conclusion will follow provides that

$$\frac{(n-k)P_{n,k+1}^{k+1}/P_{n,k}^{k}+(k+1)x_{n+1}}{(n-k)P_{n,k+1,t}^{k+1}/P_{n,k,t}^{k}+(k+1)(x_{n+1}+t)} \leq \frac{(n-k)P_{n,1}+(k+1)x_{n+1}}{(n-k)P_{n,1,t}+(k+1)(x_{n+1}+t)}.$$

If $x_1=\cdots=x_n$ then equality holds above, otherwise little calculation shows the above is equivalent to

$$\frac{P_{n,1}-P_{n,k+1}^{k+1}/P_{n,k}^{k}}{P_{n,1,t}-P_{n,k+1,t}^{k+1}/P_{n,k,t}^{k}}-1 \geq \frac{-t(n+1)}{(n-k)P_{n,1,t}+(k+1)(x_{n+1}+t)}.$$

Since only the denominator on the right-hand side above depends on x_{n+1}, it suffices to show the left-hand side above ≥ 0(the case $x_{n+1}\rightarrow\infty$) and this last inequality follows by using the method in the proof of Theorem 2.4 combined with the case $s=-1$ in Theorem 3.1 of [13] and Lemma 2.1. □

Corollary 2.1. *For $r=1\leq k\leq n$,*

$$(k+1)(\frac{P_{n+1,k}^{k}}{P_{n+1,k+1}^{k+1}}-\frac{1}{A_{n+1}}) \geq k(\frac{P_{n,k-1}^{k-1}}{P_{n,k}^{k}}-\frac{1}{A_n}).$$

Proof. It follows from Theorem 2.5 that the function $f(t)=(k+1)\big(\ln(A_{n+1,t})-\ln(P_{n+1,k+1,t})\big)-k\big(\ln(A_{n,t})-\ln(P_{n,k,t})\big)$ is a decreasing function of t and the conclusion follows from $f'(0)\leq 0$. □

3. The Derived Means and Mixed-Mean Inequalities

Let $I=[m,M]$ with $0<m<M$ and fix $n,q_i,1\leq i\leq n$, we may think $M_{n,r}$'s as a family of mappings, parameterized by r, from $(0,\infty)^n$ to $(0,\infty)$, such that $M_{n,r}(\mathbf{x})\in I$ for $\mathbf{x}\in I^n$, $M_{n,r}(k\mathbf{x})=kM_{n,r}(\mathbf{x})$ for $k>0$ and $M_{n,r}\geq M_{n,s}$ for $r>s$. We may thus regard any family of mappings satisfying the above properties to be certain family of "generalized" means. One example is given by the family of $R_r=M_{n,r}^{r}/M_{n,r-1}^{r-1}$'s, parameterized by r. Note for $r>s$, $R_r\geq R_s$(see [4]). Furthermore, we may think of R_r's as the family of means derived from that of $M_{n,r}$'s since $R_r=1/f'(0)$ with $f(t)=\ln M_{n,r,t}$. More generally, we can think of R_r's as a subfamily of the family of $R_{r,s}$'s, parameterized by both r and s. Here

$$R_{r,s}=(\frac{M_{n,r}^{r}}{M_{n,s}^{s}})^{\frac{1}{r-s}}, r\neq s; R_{r,r}=\exp(\frac{\sum_{i=1}^{n}q_ia_i^r\ln a_i}{\sum_{i=1}^{n}q_ia_i^r}).$$

We note for fixed r(resp. s), $R_{r,s}$ is an increasing function of s(resp. r)(Theorem 3.2, [21]). Moreover, we have the Beckenbach-Dresher inequality(see [4, 10, 17]):

Theorem 3.1. *Let $r \geq 1 \geq s \geq 0, r \neq s$ and $x_i, y_i > 0$ for $i = 1, 2, \cdots, n$, then*

$$(\frac{\sum_{i=1}^{n}(x_i+y_i)^r}{\sum_{i=1}^{n}(x_i+y_i)^s})^{1/(r-s)} \leq (\frac{\sum_{i=1}^{n}x_i^r}{\sum_{i=1}^{n}x_i^s})^{1/(r-s)} + (\frac{\sum_{i=1}^{n}y_i^r}{\sum_{i=1}^{n}y_i^s})^{1/(r-s)}, \tag{3.1}$$

and the inequality is reversed for $1 \geq r \geq 0 \geq s$.

Similarly, we can consider the family of $P_{n,r,r-1} = P^r_{n,r}/P^{r-1}_{n,r-1}$'s, parameterized by r with $1 \leq r \leq n$. Theorem 2.1 implies $P_{n,r,r-1} \leq P_{n,s,s-1}$ for $r > s$. Observe $P_{n,r,r-1} = 1/f'(0)$ with $f(t) = \ln P_{n,r,t}$, so we may think of $P_{n,r,r-1}$'s as the family of means derived from that of $P_{n,r}$'s. More generally, we may also think of $P_{n,r,r-1}$'s as a subfamily of $P_{n,r,s}$'s, parameterized by both r and s with $0 \leq s < r \leq n$. Here

$$P_{n,r,s} = (\frac{P^r_{n,r}}{P^s_{n,s}})^{1/(r-s)}.$$

Analogue to Theorem 3.1, we have the following result of M. Marcus and L. Lopes[20](see also pp. 33-35 in [5]):

Theorem 3.2. *Let $0 \leq s < r \leq n$ and $x_i, y_i > 0$ for $i = 1, 2, \cdots, n$, then*

$$P_{n,r,s}(\mathbf{x}+\mathbf{y}) \geq P_{n,r,s}(\mathbf{x}) + P_{n,r,s}(\mathbf{y}),$$

with equality holding if and only if $r = 1$ or there exists a constant λ such that $\mathbf{x} = \lambda\mathbf{y}$.

We want to establish certain mixed-mean inequalities among each families we considered above. From now on the notations we use will be consistent with those defined in the paragraph containing Theorem 1.6. First, we state a Lemma of C. Tarnavas and D. Tarnavas[22].

Lemma 3.1. *Let $f : R \rightarrow R$ be a convex function and suppose for $n \geq 2, 1 \leq k \leq n-1$, $W_nw_k - W_kw_n > 0$. Then*

$$\frac{1}{W_{n-1}}\sum_{k=1}^{n-1}w_kf(W_{n-1}A_k) \geq \frac{1}{W_n}\sum_{k=1}^{n}w_kf(W_nA_k - w_nx_k).$$

The equality holds if $n = 2$ or $x_1 = \cdots = x_n$ when $f(x)$ is strictly convex. When $f(x)$ is concave, then the above inequality is reversed.

We now relate Lemma 3.1 to Schur convexity. We first recall a few notations, which the reader can find in [5],$\S29-\S31$. An $n \times n$ matrix $S = [s_{ij}]$ is a doubly stochastic matrix if $s_{ij} \geq 0$ for $1 \leq i, j \leq n$, and $\sum_{j=1}^{n} s_{ij} = \sum_{i=1}^{n} s_{ij} = 1, 1 \leq i, j \leq n$. Let $I^n = I \times I \times \cdots \times I$ (n copies), where I is an interval of the real line. A function $f : I^n \rightarrow R$ is Schur convex if for every doubly stochastic matrix S, $f(S\mathbf{x}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in I^n$ and f is Schur concave if the inequality is reversed. If f also has continuous partial derivatives on I^n, then f is Schur convex if and only if

$$(x_i - x_j)(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}) \geq 0. \tag{3.2}$$

Consider the case $w_i = 1, 1 \leq i \leq n$ and let $\mathbf{u}$ be an $n(n-1)$-tuple $((n-1)A_1, \cdots, (n-1)A_1, (n-1)A_2, \cdots, (n-1)A_2, \cdots, (n-1)A_{n-1}, \cdots, (n-1)A_{n-1})$ with each of the term $(n-1)A_i, 1 \leq i \leq n-1$ repeating n times and let $\mathbf{v}$ be an $n(n-1)$-tuple $(nA_1 - x_1, \cdots, nA_1 - x_1, nA_2 - x_2, \cdots, nA_2 - x_2, \cdots, nA_n - x_n, \cdots, nA_n - x_n)$ with each of the term $nA_i - x_i, 1 \leq i \leq n$ repeating $n-1$ times. On writing $x_i = iA_i - (i-1)A_{i-1}$, it is easy to see that there exists a doubly stochastic matrix S such that $\mathbf{v} = S\mathbf{u}$. Hence for a Schur convex function f, we have $f(\mathbf{v}) \leq f(\mathbf{u})$. Now consider $f(\mathbf{x}) = (\sum_{i=1}^{n(n-1)} x_i^r / \sum_{i=1}^{n(n-1)} x_i^s)^{1/(r-s)}, r > s$. It is easy to check via (3.2) that f is Schur concave for $s \leq 0 \leq r \leq 1$ and Schur convex for $0 \leq s \leq 1 \leq r$. Hence we obtain

Lemma 3.2. *Let $w_i = 1, 1 \leq i \leq n$. For $0 \leq s \leq 1 \leq r$,*

$$\left(\frac{M^r_{n,r}(n\mathbf{A}_n - \mathbf{x}_n)}{M^s_{n,s}(n\mathbf{A}_n - \mathbf{x}_n)}\right)^{1/(r-s)} \leq \left(\frac{M^r_{n-1,r}((n-1)\mathbf{A}_{n-1})}{M^s_{n-1,s}((n-1)\mathbf{A}_{n-1})}\right)^{1/(r-s)},$$

and the above inequality reverses when $s \leq 0 \leq r \leq 1$.

Define $P_{1,2} = P_{1,1}$, we have

Lemma 3.3. *For $n \geq 2$,*

$$\begin{aligned} P_{n-1,2}((n-1)\mathbf{A}_{n-1}) &\leq P_{n,2}(n\mathbf{A}_n - \mathbf{x}_n), \\ \frac{P^2_{n-1,2}((n-1)\mathbf{A}_{n-1})}{P_{n-1,1}((n-1)\mathbf{A}_{n-1})} &\leq \frac{P^2_{n,2}(n\mathbf{A}_n - \mathbf{x}_n)}{P_{n,1}(n\mathbf{A}_n - \mathbf{x}_n)}, \end{aligned}$$

with equality holding in both cases if and only if $n = 2$ or $x_1 = \cdots = x_n$.

Proof. We may assume $n \geq 3$ here. Write $a_i = (n-1)A_i, 1 \leq i \leq n-1; b_j = nA_j - x_j, 1 \leq j \leq n$. Note $n\sum_{i=1}^{n-1} a_i = (n-1)\sum_{i=1}^n b_i$, hence it is enough to prove the first assertion of the lemma.

Lemma 3.1 with $f(x) = x^2$ implies $(n-1)\sum_{i=1}^n b_i^2 \leq n\sum_{i=1}^{n-1} a_i^2$. On expanding $(n\sum_{i=1}^{n-1} a_i)^2 = ((n-1)\sum_{i=1}^n b_i)^2$, we obtain

$$\begin{aligned} n^2\sum_{i=1}^{n-1} a_i^2 + 2n^2 \sum_{1\leq i\neq j\leq n-1} a_ia_j &= (n-1)^2\sum_{i=1}^{n} b_i^2 + 2(n-1)^2 \sum_{1\leq i\neq j\leq n} b_ib_j \\ &\leq n(n-1)\sum_{i=1}^{n-1} a_i^2 + 2(n-1)^2 \sum_{1\leq i\neq j\leq n} b_ib_j. \end{aligned}$$

Hence

$$n\sum_{i=1}^{n-1} a_i^2 + 2n^2 \sum_{1\leq i\neq j\leq n-1} a_ia_j \leq 2(n-1)^2 \sum_{1\leq i\neq j\leq n} b_ib_j. \tag{3.3}$$

Using $M_{n,2} \geq A_n = P_{n,1} \geq P_{n,2}$, we obtain

$$\frac{1}{n-1}\sum_{i=1}^{n-1} a_i^2 \geq \frac{1}{\binom{n-1}{2}} \sum_{1\leq i\neq j\leq n-1} a_ia_j.$$

So by (3.3),

$$\frac{1}{\binom{n-1}{2}} \sum_{1\leq i\neq j\leq n-1} a_ia_j \leq \frac{1}{\binom{n}{2}} \sum_{1\leq i\neq j\leq n} b_ib_j,$$

which is just what we want. □

We now extend the result of Theorems 1.5, 1.6 to the symmetric means case:

Theorem 3.3. *Define $\mathbf{P}_{n,2} = (P_{1,2}, \cdots, P_{n,2}), \mathbf{P}^2_{n,2}/\mathbf{P}_{n,1} = (P^2_{1,2}/P_{1,1}, \cdots, P^2_{n,2}/P_{n,1})$, then*

$$\begin{aligned} P_{n,2} + (n-1)P_{n-1,2}(\mathbf{P}_{n-1,1}) &\leq nP_{n,2}(\mathbf{P}_{n,1}), \\ \frac{P^2_{n,2}}{P_{n,1}} + (n-1)\frac{P^2_{n-1,2}(\mathbf{A}_{n-1})}{P_{n-1,1}(\mathbf{A}_{n-1})} &\leq n\frac{P^2_{n,2}(\mathbf{P}_{n,1})}{P_{n,1}(\mathbf{P}_{n,1})} \end{aligned} \tag{3.4}$$

with equality holding if and only if $x_1 = \cdots = x_n$. It follows that

$$P_{n,1}(\mathbf{P}_{n,2}) \leq P_{n,2}(\mathbf{P}_{n,1});\ P_{n,1}(\mathbf{P}^2_{n,2}/\mathbf{P}_{n,1}) \leq P^2_{n,2}(\mathbf{P}_{n,1})/P_{n,1}(\mathbf{P}_{n,1})$$

with equality holding if and only if $x_1 = \cdots = x_n$.

Proof. Since the proofs are similar, we will only prove (3.4) here. We use the idea in [22]. By Lemma 3.3

$$\begin{aligned} P_{n,2}+(n-1)P_{n-1,2}(\mathbf{P}_{n-1,1}) &\leq P_{n,2}+P_{n,2}(n\mathbf{A}_n-\mathbf{x}_n) \\ &\leq P_{n,2}(n\mathbf{A}_n-\mathbf{x}_n+\mathbf{x}_n)=nP_{n,2}(\mathbf{P}_{n,1}), \end{aligned}$$

where the last inequality follows from Theorem 3.2 for the case $r=2, s=0$. $\square$

Now for $r>s$, define $(\mathbf{M}^r_{i,r}/\mathbf{M}^s_{i,s})^{1/(r-s)}=((M^r_{1,r}/M^s_{1,s})^{1/(r-s)},\cdots,(M^r_{i,r}/M^s_{i,s})^{1/(r-s)})$ and by repeating the proof of Theorem 3.3 using the Lemma 3.2 and (3.1), we obtain

Theorem 3.4. *Let $w_i=1, x_i>0$, then for $r>s, 0\leq s\leq 1\leq r$,*

$$\begin{aligned} (\mathbf{M}^r_{n,r}/\mathbf{M}^s_{n,s})^{1/(r-s)}+(n-1)(M^r_{n-1,r}/M^s_{n-1,s}(\mathbf{A}_n))^{1/(r-s)} &\geq n(M^r_{n,r}/M^s_{n,s}(\mathbf{A}_n))^{1/(r-s)}, \\ A_n((\mathbf{M}^r_{i,r}/\mathbf{M}^s_{i,s})^{1/(r-s)}) &\geq (M^r_{n,r}/M^s_{n,s}(\mathbf{A}_n))^{1/(r-s)}, \end{aligned}$$

and the above inequality reverses for $s\leq 0\leq r\leq 1$.

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