LOGARITHMICALLY COMPLETELY MONOTONIC RATIOS OF MEAN VALUES AND AN APPLICATION

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Abstract. In the article, some strictly Logarithmically completely monotonic ratios of mean values are presented.

A function f is said to be completely monotonic on an interval I, if f has derivatives of all orders on I and satisfies

$$(-1)^n f^{(n)}(x) \geq 0 \tag{1}$$

for $x \in I$ and $n \geq 0$. If inequality (1) is strict, then f is said to be strictly completely monotonic on I.

Completely monotonic functions have remarkable applications in different mathematical branches. For instance, they play a role in potential theory [3], probability theory [4, 7, 10], physics [5], numerical and asymptotic analysis [8, 18], and combinatorics [1]. A detailed collection of the most important properties of completely monotonic functions can be found in [17, Chapter IV], and in an abstract in [2].

A positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^n [\ln f(x)]^{(n)} \geq 0 \tag{2}$$

for $x \in I$ and $n \in \mathbb{N}$. If inequality (2) is strict, then f is said to be strictly logarithmically completely monotonic.

The terminology "(strictly) logarithmically completely monotonic function" was named first by F. Qi, B.-N. Guo and Ch.-P. Chen in [11, 12, 13]. It was also showed in these papers that a (strictly) logarithmically completely monotonic function is also (strictly) completely monotonic.

The generalized logarithmic mean or Stolarsky mean $L_r(a,b)$ of two positive numbers a and b was introduced in [9, 15, 16] and [6, p. 6] for $a = b$ by $L_r(a,b) = a$ and for $a \neq b$ by

$$L_r(a,b) = \left(\frac{b^{r+1} - a^{r+1}}{(r+1)(b-a)}\right)^{1/r}, \quad r \neq -1,\ 0; \tag{3}$$

$$L_{-1}(a,b) = \frac{b-a}{\ln b - \ln a}; \tag{4}$$

$$L_0(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}. \tag{5}$$

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Here $L_{-1}(a,b) \triangleq L(a,b)$ and $L_0(a,b) \triangleq I(a,b)$ are the logarithmic and identric means, respectively. When $a \neq b$, $L_r(a,b)$ is a strictly increasing function of r. Further,

$$L_1(a,b) \triangleq A(a,b), \quad L_{-2}(a,b) \triangleq G(a,b), \tag{6}$$

where A and G are the arithmetic and geometric means, respectively.

For $a \neq b$, the following well known inequalities hold

$$H(a,b) < G(a,b) < L(a,b) < I(a,b) < A(a,b), \tag{7}$$

where H is the harmonic mean.

In this paper, the (logarithmically) complete monotonicity of some ratios of mean values are obtained. Our main results are as follows.

Theorem 1. *The ratios*

$$\frac{A(x,x+1)}{I(x,x+1)}, \quad \frac{A(x,x+1)}{G(x,x+1)} = \frac{G(x,x+1)}{H(x,x+1)}, \tag{8}$$

$$\frac{A(x,x+1)}{H(x,x+1)}, \quad \frac{I(x,x+1)}{G(x,x+1)}, \quad \frac{I(x,x+1)}{H(x,x+1)} \tag{9}$$

of mean values A, G, H and I are strictly logarithmically completely monotonic in $(0,\infty)$ *and the ratio*

$$\frac{A(x,x+1)}{L(x,x+1)} \tag{10}$$

is strictly completely monotonic in $(0,\infty)$.

Proof. Define for $x > 0$

$$\phi_{A/I}(x) = \ln \frac{A(x,x+1)}{I(x,x+1)} = \ln \frac{x+1/2}{(x+1)(1+1/x)^x}. \tag{11}$$

Differentiating directly, using the following representations for $x > 0$, $s \geq 0$ and $n \in \mathbb{N}$

$$\ln x = \int_0^\infty \frac{e^{-t} - e^{-xt}}{t} \, \mathrm{d}t, \tag{12}$$

$$\frac{1}{(x+s)^n} = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-(x+s)t} \, \mathrm{d}t \tag{13}$$

and the power series expansion of $te^{t/2} - e^t + 1$ at 0, we conclude that

$$\begin{aligned}(-1)^n \phi_{A/I}^{(n)}(x) &= -\int_0^\infty (te^{t/2} - e^t + 1) t^{n-2} e^{-(x+1)t} \, \mathrm{d}t \\ &= \sum_{k=3}^\infty \left(\frac{1}{k} - \frac{1}{2^{k-1}}\right) \frac{1}{(k-1)!} \int_0^\infty t^{n+k-2} e^{-(x+1)t} \, \mathrm{d}t \\ &> 0.\end{aligned} \tag{14}$$

This means that the ratio $\frac{A(x,x+1)}{I(x,x+1)}$ is strictly logarithmically monotonic in $(0,\infty)$.

Define for $x > 0$

$$\phi_{A/G}(x) = \ln \frac{A(x,x+1)}{G(x,x+1)} = \ln\left(x + \frac{1}{2}\right) - \frac{1}{2} \ln x - \frac{1}{2} \ln(x+1), \tag{15}$$

then, by argument as above, we have for any nonnegative integer n

$$\begin{aligned}(-1)^n\phi_{A/G}^{(n)}(x)&=\frac12\int_0^\infty(e^t+1-2e^{t/2})t^{n-1}e^{-(x+1)t}\,\mathrm{d}t\\&=\frac12\sum_{k=2}^\infty\left(1-\frac{1}{2^{k-1}}\right)\frac{1}{k!}\int_0^\infty t^{n+k-1}e^{-(x+1)t}\,\mathrm{d}t\\&>0.\end{aligned}\tag{16}$$

This reveals that the ratio $\frac{A(x,x+1)}{G(x,x+1)}$ is strictly logarithmically completely monotonic in $(0,\infty)$.

Define for $x>0$

$$\phi_{A/H}(x)=\ln\frac{A(x,x+1)}{H(x,x+1)}=2\ln\left(x+\frac12\right)-\ln x-\ln(x+1),\tag{17}$$

then we have for nonnegative integer n

$$\begin{aligned}(-1)^n\phi_{A/H}^{(n)}(x)&=\int_0^\infty(e^t-2e^{t/2}+1)t^{n-1}e^{-(x+1)t}\,\mathrm{d}t\\&=\sum_{k=2}^\infty\left(1-\frac{1}{2^{k-1}}\right)\frac{1}{k!}\int_0^\infty t^{n+k-1}e^{-(x+1)t}\,\mathrm{d}t\\&>0.\end{aligned}\tag{18}$$

Therefore, it follows that the ratio $\frac{A(x,x+1)}{H(x,x+1)}$ is strictly logarithmically completely monotonic in $(0,\infty)$.

Define for $x>0$

$$\phi_{I/G}(x)=\ln\frac{I(x,x+1)}{G(x,x+1)}=x\ln\left(1+\frac1x\right)+\frac12\ln(x+1)-\frac12\ln x-1.\tag{19}$$

By standard argument above, differentiation for nonnegative integer n yields

$$\begin{aligned}(-1)^n\phi_{I/G}^{(n)}(x)&=\frac12\int_0^\infty(2e^t-te^t-t-2)t^{n-2}e^{-(x+1)t}\,\mathrm{d}t\\&=\frac12\sum_{k=3}^\infty\frac{k-2}{k!}\int_0^\infty t^{n+k-2}e^{-(x+1)t}\,\mathrm{d}t\\&>0.\end{aligned}\tag{20}$$

This shows that the ratio $\frac{I(x,x+1)}{G(x,x+1)}$ is also strictly logarithmically completely monotonic in $(0,\infty)$.

Define for $x>0$

$$\phi_{I/H}(x)=\ln\frac{I(x,x+1)}{H(x,x+1)}=x\ln\left(1+\frac1x\right)+\ln\left(x+\frac12\right)-\ln x-1.\tag{21}$$

By the same procedure as above, we obtain for $n\in\mathbb{N}$

$$\begin{aligned}(-1)^n\phi_{I/H}^{(n)}(x)&=\int_0^\infty(e^t+te^{t/2}-te^t-t-1)t^{n-2}e^{-(x+1)t}\,\mathrm{d}t\\&=\sum_{k=3}^\infty\left(1-\frac1k-\frac{1}{2^{k-1}}\right)\frac{1}{(k-1)!}\int_0^\infty t^{n+k-2}e^{-(x+1)t}\,\mathrm{d}t\\&>0.\end{aligned}\tag{22}$$

Thus, it is proved that $\frac{I(x,x+1)}{H(x,x+1)}$ is also strictly logarithmically completely monotonic in $(0,\infty)$.

Define for $x>0$

$$\phi_{A/L}(x)=\frac{A(x,x+1)}{L(x,x+1)}=\left(x+\frac{1}{2}\right)\ln\left(1+\frac{1}{x}\right). \tag{23}$$

Straightforward differentiating, using formulas (12) and (13) and expanding the function $2e^t-te^t-t-2$ at 0 yields

$$\begin{aligned}(-1)^n\phi_{A/L}^{(n)}(x)&=-\frac{1}{2}\int_0^\infty(2e^t-te^t-t-2)t^{n-2}e^{-(x+1)t}\,\mathrm{d}t\\&=\frac{1}{2}\sum_{k=3}^\infty\frac{k-2}{k!}\int_0^\infty t^{n+k-2}e^{-(x+1)t}\,\mathrm{d}t\\&>0.\end{aligned} \tag{24}$$

This tell us that the ratio $\frac{A(x,x+1)}{L(x,x+1)}$ is strictly completely monotonic in $(0,\infty)$.

The proof is complete. □

In the final, as an applications of Theorem 1, we give the following remark.

Remark 1. As stated above, a strictly logarithmically completely monotonic function is also strictly completely monotonic. As a result, we deduce from Theorem 1 that

$$e\left(1-\frac{1}{2x+1}\right)<e\sqrt{\frac{x}{x+1}}<\left(1+\frac{1}{x}\right)^x<e\left(1-\frac{1}{2x+2}\right) \tag{25}$$

for $x>0$. Inequality (25) can be found in [14, 19].

By using the right-hand side of (25), Yang in [19] obtained a strengthened Hardy's inequality:

$$\sum_{n=1}^\infty\lambda_{n+1}\left(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n}\right)^{1/\Lambda_n}<e\sum_{n=1}^\infty\left[1-\frac{\lambda_n}{2(\Lambda_n+\lambda_n)}\right]\lambda_na_n, \tag{26}$$

where $0<\lambda_{n+1}\le\lambda_n$, $\Lambda_n=\sum_{m=1}^n\lambda_m$ and $a_n\ge0$ for $n\in\mathbb{N}$, $0<\sum_{n=1}^\infty\lambda_na_n<\infty$.

In particular, if setting $\lambda_n\equiv1$, then (26) becomes the following strengthened Carleman's inequality [19]:

$$\sum_{n=1}^\infty(a_1a_2\cdots a_n)^{1/n}<e\sum_{n=1}^\infty\left[1-\frac{1}{2(n+1)}\right]a_n. \tag{27}$$

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