ON THE LUPAŞ-BEESACK-PEČARIĆ INEQUALITY FOR ISOTONIC LINEAR FUNCTIONALS

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Abstract. Some inequalities related to the Lupaş-Beesack-Pečarić result for $m-\Psi-$convex and $M-\Psi-$convex functions and applications are given.

1. Introduction

Let L be a linear class of real-valued functions $g:E\to\mathbb{R}$ having the properties

(L1) $f,g\in L$ imply $(\alpha f+\beta g)\in L$ for all $\alpha,\beta\in\mathbb{R}$;
(L2) $\mathbf{1}\in L$, i.e., if $f_0(t)=1$, $t\in E$ then $f_0\in L$.

An *isotonic linear functional* $A:L\to\mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f+\beta g)=\alpha A(f)+\beta A(g)$ for all $f,g\in L$ and $\alpha,\beta\in\mathbb{R}$.
(A2) If $f\in L$ and $f\geq 0$, then $A(f)\geq 0$.
The mapping A is said to be *normalised if*
(A3) $A(\mathbf{1})=1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Thus, they provide, for example, Jessen's inequality, which is a functional form of Jensen's inequality (see [2] and [3]).

We recall Jessen's inequality (see also [9]).

Theorem 1. *Let $\phi:I\subseteq\mathbb{R}\to\mathbb{R}$ (I is an interval), be a convex function and $f:E\to I$ such that $\phi\circ f$, $f\in L$. If $A:L\to\mathbb{R}$ is an isotonic linear and normalised functional, then*

$$\phi(A(f))\leq A(\phi\circ f). \tag{1.1}$$

A counterpart of this result was proved by Beesack and Pečarić in [2] for compact intervals $I=[\alpha,\beta]$.

Theorem 2. *Let $\phi:[\alpha,\beta]\subset\mathbb{R}\to\mathbb{R}$ be a convex function and $f:E\to[\alpha,\beta]$ such that $\phi\circ f$, $f\in L$. If $A:L\to\mathbb{R}$ is an isotonic linear and normalised functional, then*

$$A(\phi\circ f)\leq\frac{\beta-A(f)}{\beta-\alpha}\phi(\alpha)+\frac{A(f)-\alpha}{\beta-\alpha}\phi(\beta). \tag{1.2}$$

Remark 1. *Note that (1.2) is a generalisation of the inequality*

$$A(\phi)\leq\frac{b-A(e_1)}{b-a}\phi(a)+\frac{A(e_1)-a}{b-a}\phi(b) \tag{1.3}$$

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due to Lupaş [1] *(see for example* [2, Theorem A]*), which assumed* $E = [a, b]$*,* L *satisfies (L1), (L2),* $A : L \to \mathbb{R}$ *satisfies (A1), (A2),* $A(\mathbf{1}) = 1$*,* ϕ *is convex on* E *and* $\phi \in L$*,* $e_1 \in L$*, where* $e_1(x) = x$*,* $x \in [a, b]$*.*

The following inequality is well known in the literature as the Hermite-Hadamard inequality

$$\varphi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \varphi(t)\,dt \leq \frac{\varphi(a)+\varphi(b)}{2}, \tag{1.4}$$

provided that $\varphi : [a, b] \to \mathbb{R}$ is a convex function.

Using Theorem 1 and Theorem 2, we may state the following generalisation of the Hermite-Hadamard inequality for isotonic linear functionals ([4] and [5]).

Theorem 3. *Let* $\phi : [a, b] \subset \mathbb{R} \to \mathbb{R}$ *be a convex function and* $e : E \to [a, b]$ *with* $e,\ \phi \circ e \in L$*. If* $A \to \mathbb{R}$ *is an isotonic linear and normalised functional, with* $A(e) = \frac{a+b}{2}$*, then*

$$\varphi\left(\frac{a+b}{2}\right) \leq A(\phi \circ e) \leq \frac{\varphi(a)+\varphi(b)}{2}. \tag{1.5}$$

For other results concerning convex functions and isotonic linear functionals, see [4] – [9] and the recent monograph [12].

2. The Concepts of $m - \Psi-$Convex and $M - \Psi-$Convex Functions

Assume that the mapping $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ (I is an interval) is convex on I and $m \in \mathbb{R}$. We shall say that the mapping $\phi : I \to \mathbb{R}$ is $m-\Psi-$ *lower convex* if $\phi - m\Psi$ is a convex mapping on I (see [11]). We may introduce the class of functions

$$\mathcal{L}(I, m, \Psi) := \{\phi : I \to \mathbb{R} | \phi - m\Psi \text{ is convex on } I\}. \tag{2.1}$$

Similarly, for $M \in \mathbb{R}$ and Ψ as above, we can introduce the class of $M - \Psi-$*upper convex* functions by

$$\mathcal{U}(I, M, \Psi) := \{\phi : I \to \mathbb{R} | M\Psi - \phi \text{ is convex on } I\}. \tag{2.2}$$

The intersection of these two classes will be called the class of $(m, M) - \Psi-$*convex functions* and will be denoted by (see [11])

$$\mathcal{B}(I, m, M, \Psi) := \mathcal{L}(I, m, \Psi) \cap \mathcal{U}(I, M, \Psi). \tag{2.3}$$

Remark 2. *If* $\Psi \in \mathcal{B}(I, m, M, \Psi)$*, then* $\phi - m\Psi$ *and* $M\Psi - \phi$ *are convex and then* $(\phi - m\Psi) + (M\Psi - \phi)$ *is also convex which shows that* $(M - m)\Psi$ *is convex, implying that* $M \geq m$ *(as* Ψ *is assumed not to be the trivial convex function* $\Psi(t) = 0$*,* $t \in I$*).*

The above concepts may be introduced in the general case of a convex subset in a real linear space, but we do not consider this extension here.

In [10], S.S. Dragomir and N.M. Ionescu introduced the concept of $g-$*convex dominated mappings*, for a mapping $f : I \to \mathbb{R}$. We recall this, by saying, for a given convex function $g : I \to \mathbb{R}$, the function $f : I \to \mathbb{R}$ is $g-$*convex dominated* iff $g + f$ and $g - f$ are convex mappings on I. In [10], the authors pointed out a number of inequalities for convex dominated functions related to Jensen's, Fuchs', Pečarić's, Barlow-Marshall-Proschan and Vasić-Mijalković results, etc.

We observe that the concept of $g-$convex dominated functions can be obtained as a particular case from $(m, M) - \Psi-$convex functions by choosing $m = -1$, $M = 1$ and $\Psi = g$.

The following lemma holds (see also [11]).

Lemma 1. *Let $\Psi, \phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable functions on $\mathring{I}$ and Ψ is a convex function on $\mathring{I}$.*

(i) *For $m \in \mathbb{R}$, the function $\phi \in \mathcal{L}\left(\mathring{I}, m, \Psi\right)$ iff*

$$m\left[\Psi(x) - \Psi(y) - \Psi'(y)(x-y)\right] \leq \phi(x) - \phi(y) - \phi'(y)(x-y) \tag{2.4}$$

for all $x, y \in \mathring{I}$.

(ii) *For $M \in \mathbb{R}$, the function $\phi \in \mathcal{U}\left(\mathring{I}, M, \Psi\right)$ iff*

$$\phi(x) - \phi(y) - \phi'(y)(x-y) \leq M\left[\Psi(x) - \Psi(y) - \Psi'(y)(x-y)\right] \tag{2.5}$$

for all $x, y \in \mathring{I}$.

(iii) *For $M, m \in \mathbb{R}$ with $M \geq m$, the function $\phi \in \mathcal{B}\left(\mathring{I}, m, M, \Psi\right)$ iff both (2.4) and (2.5) hold.*

Proof. Follows by the fact that a differentiable mapping $h : I \to \mathbb{R}$ is convex on $\mathring{I}$ iff $h(x) - h(y) \geq h'(y)(x-y)$ for all $x, y \in \mathring{I}$. ∎

Another elementary fact for twice differentiable functions also holds (see also [11]).

Lemma 2. *Let $\Psi, \phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable on $\mathring{I}$ and Ψ is convex on $\mathring{I}$.*

(i) *For $m \in \mathbb{R}$, the function $\phi \in \mathcal{L}\left(\mathring{I}, m, \Psi\right)$ iff*

$$m\Psi''(t) \leq \phi''(t) \quad \text{for all } t \in \mathring{I}. \tag{2.6}$$

(ii) *For $M \in \mathbb{R}$, the function $\phi \in \mathcal{U}\left(\mathring{I}, M, \Psi\right)$ iff*

$$\phi''(t) \leq M\Psi''(t) \quad \text{for all } t \in \mathring{I}. \tag{2.7}$$

(iii) *For $M, m \in \mathbb{R}$ with $M \geq m$, the function $\phi \in \mathcal{B}\left(\mathring{I}, m, M, \Psi\right)$ iff both (2.6) and (2.7) hold.*

Proof. Follows by the fact that a twice differentiable function $h : I \to \mathbb{R}$ is convex on $\mathring{I}$ iff $h''(t) \geq 0$ for all $t \in \mathring{I}$. ∎

We consider the $p-$logarithmic mean of two positive numbers given by

$$L_p(a,b) := \begin{cases} a & \text{if } \quad b = a, \\ \left[\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } \quad a \neq b \end{cases}$$

and $p \in \mathbb{R} \setminus \{-1, 0\}$.

The following proposition holds (see also [11]).

Proposition 1. *Let $\phi : (0, \infty) \to \mathbb{R}$ be a differentiable mapping.*

(i) *For $m \in \mathbb{R}$, the function $\phi \in \mathcal{L}\left((0,\infty), m, (\cdot)^p\right)$ with $p \in (-\infty, 0) \cup (1, \infty)$ iff*

$$mp(x-y)\left[L_{p-1}^{p-1}(x,y) - y^{p-1}\right] \leq \phi(x) - \phi(y) - \phi'(y)(x-y) \tag{2.8}$$

for all $x, y \in (0, \infty)$.

(ii) *For $M \in \mathbb{R}$, the function $\phi \in \mathcal{U}((0,\infty), M, (\cdot)^p)$ with $p \in (-\infty, 0) \cup (1, \infty)$ iff*

$$\phi(x) - \phi(y) - \phi'(y)(x-y) \leq Mp(x-y)\left[L_{p-1}^{p-1}(x,y) - y^{p-1}\right] \tag{2.9}$$

for all $x, y \in (0,\infty)$.

(iii) *For $M, m \in \mathbb{R}$ with $M \geq m$, the function $\phi \in \mathcal{B}((0,\infty), M, (\cdot)^p)$ with $p \in (-\infty, 0) \cup (1, \infty)$ iff both (2.8) and (2.9) hold.*

The proof follows by Lemma 1 applied for the convex mapping $\Psi(t) = t^p$, $p \in (-\infty, 0) \cup (1, \infty)$ and via some elementary computation. We omit the details.

The following corollary is useful in practice.

Corollary 1. *Let $\phi : (0,\infty) \to \mathbb{R}$ be a differentiable function.*

(i) *For $m \in \mathbb{R}$, the function ϕ is $m-$quadratic-lower convex (i.e., for $p = 2$) iff*

$$m(x-y)^2 \leq \phi(x) - \phi(y) - \phi'(y)(x-y) \tag{2.10}$$

for all $x, y \in (0,\infty)$.

(ii) *For $M \in \mathbb{R}$, the function ϕ is $M-$quadratic-upper convex iff*

$$\phi(x) - \phi(y) - \phi'(y)(x-y) \leq M(x-y)^2 \tag{2.11}$$

for all $x, y \in (0,\infty)$.

(iii) *For $m, M \in \mathbb{R}$ with $M \geq m$, the function ϕ is $(m, M)-$quadratic convex if both (2.10) and (2.11) hold.*

The following proposition holds (see also [11]).

Proposition 2. *Let $\phi : (0,\infty) \to \mathbb{R}$ be a twice differentiable function.*

(i) *For $m \in \mathbb{R}$, the function $\phi \in \mathcal{L}((0,\infty), m, (\cdot)^p)$ with $p \in (-\infty, 0) \cup (1, \infty)$ iff*

$$p(p-1)mt^{p-2} \leq \phi''(t) \quad \text{for all } t \in (0,\infty). \tag{2.12}$$

(ii) *For $M \in \mathbb{R}$, the function $\phi \in \mathcal{U}((0,\infty), M, (\cdot)^p)$ with $p \in (-\infty, 0) \cup (1, \infty)$ iff*

$$\phi''(t) \leq p(p-1)Mt^{p-2} \quad \text{for all } t \in (0,\infty). \tag{2.13}$$

(iii) *For $m, M \in \mathbb{R}$ with $M \geq m$, the function $\phi \in \mathcal{B}((0,\infty), m, M, (\cdot)^p)$ with $p \in (-\infty, 0) \cup (1, \infty)$ iff both (2.12) and (2.13) hold.*

As can be easily seen, the above proposition offers the practical criterion of deciding when a twice differentiable mapping is $(\cdot)^p -$lower or $(\cdot)^p -$upper convex and which weights the constant m and M are.

The following corollary is useful in practice.

Corollary 2. *Assume that the mapping $\phi : (a,b) \subseteq \mathbb{R} \to \mathbb{R}$ is twice differentiable.*

(i) *If $\inf_{t\in(a,b)} \phi''(t) = k > -\infty$, then ϕ is $\frac{k}{2}-$quadratic lower convex on (a,b);*

(ii) *If $\sup_{t\in(a,b)} \phi''(t) = K < \infty$, then ϕ is $\frac{K}{2}-$quadratic upper convex on (a,b).*

3. Lupaş-Beesack-Pečarić Inequality for $m-\Psi-$Convex and $M-\Psi-$Convex Functions

In [11], S.S. Dragomir proved the following inequality of Jessen's type for $m-\Psi-$convex and $M-\Psi-$convex functions.

Theorem 4. *Let $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $f : E \to I$ such that $\Psi \circ f$, $f \in L$ and $A : L \to \mathbb{R}$ be an isotonic linear and normalised functional.*

(i) *If $\phi \in \mathcal{L}(I, m, \Psi)$ and $\phi \circ f \in L$, then we have the inequality*

$$m\left[A(\Psi \circ f) - \Psi(A(f))\right] \leq A(\phi \circ f) - \phi(A(f)). \tag{3.1}$$

(ii) *If $\phi \in \mathcal{U}(I, M, \Psi)$ and $\phi \circ f \in L$, then we have the inequality*

$$A(\phi \circ f) - \phi(A(f)) \leq M\left[A(\Psi \circ f) - \Psi(A(f))\right]. \tag{3.2}$$

(iii) *If $\phi \in \mathcal{B}(I, m, M, \Psi)$ and $\phi \circ f \in L$, then both (3.1) and (3.2) hold.*

The following corollary is useful in practice.

Corollary 3. *Let $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$, $f : E \to I$ such that $\Psi \circ f$, $f \in L$ and $A : L \to \mathbb{R}$ be an isotonic linear and normalised functional.*

(i) *If $\phi : I \to \mathbb{R}$ is twice differentiable and $\phi''(t) \geq m\Psi''(t)$, $t \in \mathring{I}$ (where m is a given real number), then (3.1) holds, provided that $\phi \circ f \in L$.*

(ii) *If $\phi : I \to \mathbb{R}$ is twice differentiable and $\phi''(t) \leq M\Psi''(t)$, $t \in \mathring{I}$ (where M is a given real number), then (3.2) holds, provided that $\phi \circ f \in L$.*

(iii) *If $\phi : I \to \mathbb{R}$ is twice differentiable and $m\Psi''(t) \leq \phi''(t) \leq M\Psi''(t)$, $t \in \mathring{I}$, then both (3.1) and (3.2) hold, provided $\phi \circ f \in L$.*

We now prove the following new result.

Theorem 5. *Let $\Psi : [\alpha, \beta] \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $f : I \to [\alpha, \beta]$ such that $\Psi \circ f$, $f \in L$ and $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional.*

(i) *If $\phi \in \mathcal{L}(I, m, \Psi)$ and $\phi \circ f \in L$, then we have the inequality*

$$\begin{aligned} & m\left[\frac{\beta - A(f)}{\beta - \alpha}\Psi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha}\Psi(\beta) - A(\Psi \circ f)\right] \\ \leq\ & \frac{\beta - A(f)}{\beta - \alpha}\phi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha}\phi(\beta) - A(\phi \circ f). \end{aligned} \tag{3.3}$$

(ii) *If $\phi \in \mathcal{U}(I, M, \Psi)$ and $\phi \circ f \in L$, then*

$$\begin{aligned} & \frac{\beta - A(f)}{\beta - \alpha}\phi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha}\phi(\beta) - A(\phi \circ f) \\ \leq\ & M\left[\frac{\beta - A(f)}{\beta - \alpha}\Psi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha}\Psi(\beta) - A(\Psi \circ f)\right]. \end{aligned} \tag{3.4}$$

(iii) *If $\phi \in \mathcal{B}(I, m, M, \Psi)$ and $\phi \circ f \in L$, then both (3.3) and (3.4) hold.*

Proof. The proof is as follows.

(i) As $\phi \in \mathcal{L}(I, m, \Psi)$ and $\phi \circ f \in L$, it follows that $\phi - m\Psi$ is convex and $(\phi - m\Psi) \circ f \in L$.
Applying Lupaş-Beesack-Pečarić's inequality for the convex function $\phi - m\Psi$, we get

$$A((\phi - m\Psi) \circ f) \leq \frac{\beta - A(f)}{\beta - \alpha}(\phi - m\Psi)(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha}(\phi - m\Psi)(\beta). \tag{3.5}$$

However,

$$A((\phi - m\Psi)\circ f) = A(\phi\circ f) - mA(\Psi\circ f)$$

and then, after some simple computation, (3.5) is equivalent to (3.3).

(ii) Goes likewise and we omit the details.

(iii) Follows by (i) and (ii).

∎

The following corollary is useful in practice.

Corollary 4. *Let $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable convex function on $\mathring{I}$, $f : E \to I$ such that $\Psi \circ f$, $f \in L$ and $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional.*

(i) *If $\phi : I \to \mathbb{R}$ is twice differentiable, $\phi \circ f \in L$ and $\phi''(t) \geq m\Psi''(t)$, $t \in \mathring{I}$ (where m is a given real number), then (3.3) holds.*

(ii) *If $\phi : I \to \mathbb{R}$ is twice differentiable, $\phi \circ f \in L$ and $\phi''(t) \leq M\Psi''(t)$, $t \in \mathring{I}$ (where m is a given real number), then (3.4) holds.*

(iii) *If $m\Psi''(t) \leq \phi''(t) \leq M\Psi''(t)$, $t \in \mathring{I}$, then both (3.3) and (3.4) hold.*

Some particular important cases of the above corollary are embodied in the following propositions.

Proposition 3. *Assume that the function $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ is twice differentiable on $\mathring{I}$.*

(i) *If $\inf_{t\in\mathring{I}} \phi''(t) = k > -\infty$, then we have the inequality:*

$$\frac{k}{2}\left[(\alpha+\beta)A(f) - \alpha\beta - A\left(f^2\right)\right] \leq \frac{\beta - A(f)}{\beta-\alpha}\phi(\alpha) + \frac{A(f)-\alpha}{\beta-\alpha}\phi(\beta) - A(\phi\circ f), \tag{3.6}$$

provided that $\phi \circ f, f^2, f \in L$.

(ii) *If $\sup_{t\in\mathring{I}} \phi''(t) = K < \infty$, then we have the inequality*

$$\frac{\beta - A(f)}{\beta-\alpha}\phi(\alpha) + \frac{A(f)-\alpha}{\beta-\alpha}\phi(\beta) - A(\phi\circ f) \leq \frac{K}{2}\left[(\alpha+\beta)A(f) - \alpha\beta - A\left(f^2\right)\right]. \tag{3.7}$$

provided that $\phi \circ f, f^2, f \in L$.

(iii) *If $-\infty < k \leq \phi''(t) \leq K < \infty$, $t \in \mathring{I}$, then both (3.6) and (3.7) hold, provided that $\phi \circ f, f^2, f \in L$.*

Proof. The proof is as follows.

(i) Consider the auxiliary mapping $h(t) := \phi(t) - \frac{1}{2}kt^2$. Then $h''(t) = \phi''(t) - k \geq 0$ i.e., h is convex, or, equivalently, $\phi \in \mathcal{L}\left(I, \frac{1}{2}k, (\cdot)^2\right)$. Applying Corollary 4, we may state

$$\frac{k}{2}\left[\frac{\beta - A(f)}{\beta-\alpha}\alpha^2 + \frac{A(f)-\alpha}{\beta-\alpha}\beta^2 - A\left(f^2\right)\right] \leq \frac{\beta - A(f)}{\beta-\alpha}\phi(\alpha) + \frac{A(f)-\alpha}{\beta-\alpha}\phi(\beta) - A(\phi\circ f),$$

which is clearly equivalent to (3.6)

(ii) Goes likewise and we omit the details.

(iii) Follows by (i) and (ii).

∎

Another result is the following one.

Proposition 4. *Assume that the mapping $\phi : [\alpha, \beta] \subset (0, \infty) \to \mathbb{R}$ is twice differentiable on (α, β), Let $p \in (-\infty, 0) \cup (1, \infty)$ and define $g_p : [\alpha, \beta] \to \mathbb{R}$, $g_p(t) = \phi''(t)\, t^{2-p}$.*

(i) *If $\inf_{t \in \mathring{I}} g_p(t) = \gamma > -\infty$, then we have the inequality*

$$\begin{aligned} &\frac{\gamma}{p(p-1)} \left[pL_{p-1}^{p-1}(\alpha, \beta) A(f) - \alpha\beta (p-1) L_{p-2}^{p-2}(\alpha, \beta) - A(f^p) \right] \\ \leq\ & \frac{\beta - A(f)}{\beta - \alpha} \phi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha} \phi(\beta) - A(\phi \circ f), \end{aligned} \tag{3.8}$$

provided that $\phi \circ f, f^p, f \in L$.

(ii) *If $\sup_{t \in \mathring{I}} g_p(t) = \Gamma < \infty$, then we have the inequality*

$$\begin{aligned} &\frac{\beta - A(f)}{\beta - \alpha} \phi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha} \phi(\beta) - A(\phi \circ f) \\ \leq\ & \frac{\Gamma}{p(p-1)} \left[pL_{p-1}^{p-1}(\alpha, \beta) A(f) - \alpha\beta (p-1) L_{p-2}^{p-2}(\alpha, \beta) - A(f^p) \right]. \end{aligned} \tag{3.9}$$

(iii) *If $-\infty < \gamma \leq g_p(t) \leq \Gamma < \infty$, $t \in \mathring{I}$, then we have both (3.8) and (3.9).*

Proof. The proof is as follows.

(i) Consider the auxiliary mapping $h_p(t) = \phi(t) - \frac{\gamma}{p(p-1)} t^p$. Then

$$\begin{aligned} h_p''(t) &= \phi''(t) - \gamma t^{p-2} = t^{p-2} \left(t^{2-p} \phi''(t) - \gamma \right) \\ &= t^{p-2} (g_p(t) - \gamma) \geq 0. \end{aligned}$$

That is, h_p is convex, or, equivalently, $\phi \in \mathcal{L}\left(I, \frac{\gamma}{p(p-1)}, (\cdot)^p \right)$. Applying Corollary 4, we may state

$$\begin{aligned} &\frac{\gamma}{p(p-1)} \left[\frac{\beta - A(f)}{\beta - \alpha} \alpha^p + \frac{A(f) - \alpha}{\beta - \alpha} \beta^p - A(f^p) \right] \\ \leq\ & \frac{\beta - A(f)}{\beta - \alpha} \phi(\alpha) + \frac{A(f) - \alpha}{\beta - \alpha} \phi(\beta) - A(\phi \circ f), \end{aligned}$$

which is equivalent to (3.8).

(ii) Goes likewise.

(iii) Follows by (i) and (ii).

∎

The following proposition also holds.

Proposition 5. *Assume that the mapping $\phi : [\alpha, \beta] \subset (0, \infty) \to \mathbb{R}$ is twice differentiable on (α, β). Define $l(t) = t^2 \phi''(t)$, $t \in [\alpha, \beta]$.*

(*i*) *If* $\inf_{t\in(\alpha,\beta)} l(t) = s > -\infty$, *then we have the inequality*

$$s\left[A(\ln f) + \ln\left[I\left(\frac{1}{\alpha},\frac{1}{\beta}\right)\right] + 1 - \frac{A(f)}{L(\alpha,\beta)}\right] \le \frac{\beta - A(f)}{\beta-\alpha}\phi(\alpha) + \frac{A(f)-\alpha}{\beta-\alpha}\phi(\beta) - A(\phi\circ f), \tag{3.10}$$

provided that $\phi\circ f, \ln f$ *and* $f \in L$, *and* $I(\cdot,\cdot)$ *denotes the identric mean, i.e., we recall it*

$$I(u,v) := \begin{cases} u & \text{if } v = u, \\ \frac{1}{e}\left(\frac{u^u}{v^v}\right)^{\frac{1}{u-v}}, & v \ne u. \end{cases}$$

(*ii*) *If* $\sup_{t\in(\alpha,\beta)} l(t) = S < \infty$, *then we have the inequality*

$$\frac{\beta - A(f)}{\beta-\alpha}\phi(\alpha) + \frac{A(f)-\alpha}{\beta-\alpha}\phi(\beta) - A(\phi\circ f) \le S\left[A(\ln f) + \ln\left[I\left(\frac{1}{\alpha},\frac{1}{\beta}\right)\right] + 1 - \frac{A(f)}{L(\alpha,\beta)}\right]. \tag{3.11}$$

(*iii*) *If* $-\infty < s \le l(t) \le S < \infty$ *for* $t \in (\alpha,\beta)$, *then both (3.10) and (3.11) hold.*

Proof. The proof is as follows.

(*i*) Define the auxiliary function $h(t) = \phi(t) + s\ln t$. Then

$$h''(t) = \phi''(t) - \frac{s}{t^2} = \frac{1}{t^2}\left(\phi''(t)t^2 - s\right) \ge 0,$$

showing that h is convex, or, equivalently, $\phi \in \mathcal{L}(I, s, -\ln(\cdot))$. Applying Corollary 4, we may state that:

$$s\left[\frac{\beta - A(f)}{\beta-\alpha}\cdot[-\ln(\alpha)] + \frac{A(f)-\alpha}{\beta-\alpha}\cdot[-\ln(\beta)] + A(\ln f)\right] \le \frac{\beta - A(f)}{\beta-\alpha}\phi(\alpha) + \frac{A(f)-\alpha}{\beta-\alpha}\phi(\beta) - A(\phi\circ f),$$

which is equivalent to (3.10).

(*ii*) Goes likewise.

(*iii*) Follows by (*i*) and (*ii*).

∎

Finally, the following result also holds.

Proposition 6. *Assume that the mapping* $\phi : [\alpha,\beta] \subset (0,\infty) \to \mathbb{R}$ *is twice differentiable on* (α,β). *Define* $\tilde{I}(t) = t\phi''(t)$, $t \in I$.

(*i*) *If* $\inf_{t\in(\alpha,\beta)} \tilde{I}(t) = \delta > -\infty$, *then we have the inequality*

$$\delta\left[A(f)\ln I(\alpha,\beta) - \frac{G^2(\alpha,\beta)}{L(\alpha,\beta)} + A(f) - A(f\ln f)\right] \le \frac{\beta - A(f)}{\beta-\alpha}\phi(\alpha) + \frac{A(f)-\alpha}{\beta-\alpha}\phi(\beta) - A(\phi\circ f), \tag{3.12}$$

provided that $\phi \circ f, f \ln f, f \in L$ *and* $G(\alpha, \beta) = \sqrt{ab}$ *is the geometric mean and* $L(\alpha, \beta)$ *is the logarithmic mean, i.e., we recall it*

$$L(\alpha,\beta) := \begin{cases} \alpha & \text{if} \quad \beta = \alpha, \\ \frac{\beta-\alpha}{\ln\beta - \ln\alpha} & \text{if} \quad \beta \neq \alpha. \end{cases}$$

(ii) *If* $\sup_{t\in(\alpha,\beta)} \tilde{I}(t) = \Delta < \infty$, *then we have the inequality*

$$\begin{aligned} &\frac{\beta - A(f)}{\beta - \alpha}\phi(\alpha) + \frac{A(f)-\alpha}{\beta-\alpha}\phi(\beta) - A(\phi\circ f) \\ \leq \quad & \Delta\left[A(f)\ln I(\alpha,\beta) - \frac{G^2(\alpha,\beta)}{L(\alpha,\beta)} + A(f) - A(f\ln f)\right] \end{aligned} \tag{3.13}$$

(iii) *If* $-\infty < \delta \leq \tilde{I}(t) \leq \Delta < \infty$ *for* $t \in (\alpha, \beta)$, *then both (3.12) and (3.13) hold.*

Proof. The proof is as follows.

(i) Define the auxiliary mapping $h(t) = \phi(t) - \delta t \ln t$, $t \in (\alpha, \beta)$. Then

$$h''(t) = \phi''(t) - \frac{\delta}{t} = \frac{1}{t^2}\left[\phi''(t)t - \delta\right] = \frac{1}{t}\left[\tilde{I}(t) - \delta\right] \geq 0$$

which shows that h is convex or, equivalently, $\phi \in \mathcal{L}(I, \delta, (\cdot)\ln(\cdot))$. Applying Corollary 4, we can write

$$\begin{aligned} &\delta\left[\frac{\beta-A(f)}{\beta-\alpha}\cdot[\alpha\ln\alpha] + \frac{A(f)-\alpha}{\beta-\alpha}\cdot[\beta\ln\beta] - A(f\ln f)\right] \\ \leq \quad & \frac{\beta-A(f)}{\beta-\alpha}\phi(\alpha) + \frac{A(f)-\alpha}{\beta-\alpha}\phi(\beta) - A(\phi\circ f), \end{aligned}$$

which is clearly equivalent to (3.12).

(ii) Goes similarly.

(iii) Follows by (i) and (ii).

∎

4. Applications for Hermite-Hadamard Inequalities

a) Assume that $\phi : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is a twice differentiable function satisfying the condition $-\infty < k \leq \phi''(t) \leq K < \infty$ for $t \in (a, b)$. If in Propostion 3 we choose $A(f) := \frac{1}{b-a}\int_a^b f(t)\,dt$, $f = e$, i.e., $e(x) = x$, $x \in [a, b]$ and take into account that

$$A\left(f^2\right) = \frac{b^2 + ab + a^2}{3},$$

then we may state the inequality (see also [12, p. 40])

$$\frac{k(b-a)^2}{12} \leq \frac{\phi(b)+\phi(a)}{2} - \frac{1}{b-a}\int_a^b \phi(x)\,dx \leq \frac{K(b-a)^2}{12}. \tag{4.1}$$

b) Now, if we assume that $\phi : [a, b] \subset (0, \infty) \to \mathbb{R}$ is twice differentiable on (a, b) and $-\infty < \gamma \leq t^{2-p}\phi''(t) \leq \Gamma < \infty$, $t \in (a, b)$, $p \in (-\infty, 0) \cup (1, \infty)$, then, applying

Proposition 4 for integrals, we may state the inequality

$$\begin{aligned}(4.2)\quad & \frac{\gamma}{p(p-1)}\left[pL_{p-1}^{p-1}(a,b)A(a,b)-(p-1)G^{2}(a,b)L_{p-2}^{p-2}(a,b)-L_{p}^{p}(a,b)\right] \\ \leq\ & \frac{\phi(b)+\phi(a)}{2}-\frac{1}{b-a}\int_{a}^{b}\phi(x)\,dx \\ \leq\ & \frac{\Gamma}{p(p-1)}\left[pL_{p-1}^{p-1}(a,b)A(a,b)-(p-1)G^{2}(a,b)L_{p-2}^{p-2}(a,b)-L_{p}^{p}(a,b)\right].\end{aligned}$$

c) Suppose that the twice differentiable function $\phi:[a,b]\subset(0,\infty)\to\mathbb{R}$ satisfies the condition $-\infty<s\leq t^{2}\phi''(t)\leq S<\infty$. Then by Proposition 5 applied for the integral functional, we may state the following inequality

$$\begin{aligned}(4.3)\quad s\ln\left[\frac{I(a,b)\,I\left(\frac{1}{a},\frac{1}{b}\right)}{\exp\left(\frac{A(a,b)-L(a,b)}{L(a,b)}\right)}\right] & \leq \frac{\phi(b)+\phi(a)}{2}-\frac{1}{b-a}\int_{a}^{b}\phi(x)\,dx \\ & \leq S\ln\left[\frac{I(a,b)\,I\left(\frac{1}{a},\frac{1}{b}\right)}{\exp\left(\frac{A(a,b)-L(a,b)}{L(a,b)}\right)}\right]\end{aligned}$$

or, equivalently,

$$\begin{aligned}(4.4)\quad \left[\frac{I(a,b)\,I\left(\frac{1}{a},\frac{1}{b}\right)}{\exp\left(\frac{A(a,b)-L(a,b)}{L(a,b)}\right)}\right]^{s} & \leq \frac{\exp\left[\frac{\phi(b)+\phi(a)}{2}\right]}{\exp\left[\frac{1}{b-a}\int_{a}^{b}\phi(x)\,dx\right]} \\ & \leq \left[\frac{I(a,b)\,I\left(\frac{1}{a},\frac{1}{b}\right)}{\exp\left(\frac{A(a,b)-L(a,b)}{L(a,b)}\right)}\right]^{S}.\end{aligned}$$

d) Finally, if we assume that the twice differentiable function $\phi:[a,b]\subset(0,\infty)\to\mathbb{R}$ satisfies the condition $-\infty<\delta\leq t\phi''(t)\leq 1<\infty$, then by Proposition 6 applied for the integral functional, we may state the following inequality:

$$\begin{aligned}(4.5)\quad & \delta A(a,b)\ln\left[\left(\frac{I(a,b)}{\sqrt{I(a^{2},b^{2})}}\right)\cdot\exp\left(\frac{L(a,b)A(a,b)-G^{2}(a,b)}{L(a,b)A(a,b)}\right)\right] \\ \leq\ & \frac{\phi(b)+\phi(a)}{2}-\frac{1}{b-a}\int_{a}^{b}\phi(x)\,dx \\ \leq\ & \Delta A(a,b)\ln\left[\left(\frac{I(a,b)}{\sqrt{I(a^{2},b^{2})}}\right)\cdot\exp\left(\frac{L(a,b)A(a,b)-G^{2}(a,b)}{L(a,b)A(a,b)}\right)\right],\end{aligned}$$

or, equivalently,

$$(4.6)\quad \left[\left(\frac{I(a,b)}{\sqrt{I(a^{2},b^{2})}}\right)\cdot\exp\left(\frac{L(a,b)A(a,b)-G^{2}(a,b)}{L(a,b)A(a,b)}\right)\right]^{\delta A(a,b)}$$

$$
\begin{aligned}
&\leq \frac{\exp\left[\frac{\phi(b)+\phi(a)}{2}\right]}{\exp\left[\frac{1}{b-a}\int_a^b \phi(x)\,dx\right]} \\
&\leq \left[\left(\frac{I(a,b)}{\sqrt{I(a^2,b^2)}}\right)\cdot \exp\left(\frac{L(a,b)\,A(a,b)-G^2(a,b)}{L(a,b)\,A(a,b)}\right)\right]^{\Delta A(a,b)}.
\end{aligned}
$$

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