A Generalization of Multiplication Table

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Abstract

In this note, we generalize the concept of multiplication table by connecting with lattice points. Then we introduce and proof a generalization of Erdös multiplication table theorem.

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Consider the following $n \times n$ Multiplication Table (we call after this $\mathrm{MT}_{n\times n}$):

1	2	3	$\cdots$	n
2	4	6	$\cdots$	$2n$
3	6	9	$\cdots$	$3n$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
n	$2n$	$3n$	$\cdots$	n^2

One of the wonderful results about $\mathrm{MT}_{n\times n}$ is the following theorem [2]:

Erdös Multiplication Table Theorem. *Suppose* $M(n) = \#\{ij | 1 \leq i, j \leq n\}$, *then*

$$\lim_{n\to\infty} \frac{M(n)}{n^2} = 0.$$

In fact $M(n)$ is the number of distinct numbers that you can find in $\mathrm{MT}_{n\times n}$. Asymptotic behavior of $M(n)$ is an open problem! The following table include some computational results about $M(n)$ by Maple software.

n	$M(n)$	$\frac{M(n)}{n^2} \approx$
10	42	0.4200000000
50	800	0.3200000000
100	2906	0.2906000000
200	11131	0.2782750000
1000	248083	0.2480830000
2000	959759	0.2399397500
2500	1483965	0.2374344000
3000	2121063	0.2356736667
4000	3723723	0.2327326875

It is shown that [1] there is some constant $c > 0$ such that

$$M(n) = O\big(\frac{n^2}{\log^c n}\big).$$

Now, consider lattice points on a quarter of plan;

$$L_2(n) := \big\{(a,b) \in \mathbb{N}^2 : 1 \leq a, b \leq n\big\}.$$

Clearly, $\mathrm{MT}_{n\times n}$ is generated by multiplying point's entries in $L_2(n)$. This idea is generalizable! Consider the following lattice in $\mathbb{R}^k$:

$$L_k(n) := \big\{(a_1, a_2, \cdots, a_k) \in \mathbb{N}^k : 1 \leq a_1, a_2, \cdots, a_k \leq n\big\}.$$

Generalized Multiplication Table. A $k-$dimensional multiplication table, denoted by $\mathrm{MT}^k_{n\times n}$, is a $k-$dimensional array of n^k numbers in $\mathbb{R}^k$ in which every number generated by multiplying entries of corresponding lattice point in $L_k(n)$.

Theorem 1 *Suppose*

$$M_k(n) = \#\big\{a_1 a_2 \cdots a_k : a_1, a_2, \cdots, a_k \in \mathbb{N}, 1 \leq a_1, a_2, \cdots, a_k \leq n\big\}.$$

Then we have

$$\lim_{n\to\infty} \frac{M_k(n)}{n^k} = 0,$$

and more precisely, there is some constant $c > 0$ such that

$$M_k(n) = O\big(\frac{n^k}{\log^c n}\big).$$

Proof: According to the definition of $M_k(n)$, we have

$$M_{k+1}(n) < nM_k(n).$$

Considering this fact with Erdös's result and with Linnik-Vinogradov's result yield the results of theorem, respectively. $\square$

We end this short note with the following table inclosing the values of $M_k(n)$ for some k and n. For generating this table we used the following kind of program in Maple (for example for computing $M_3(100)$ here):

```
with(stats):
n:=10:
M[3](n):=describe[count](convert(seq(seq(seq(i*j*k,i=1..n),j=1..n),k=1..n),'list'));
```

n	$M_2(n)$	$M_3(n)$	$M_4(n)$	$M_5(n)$
10	42	120	275	546
20	152	732	2670	8052
30	308	1909	8679	31856
40	517	3919	21346	OCCOC*
50	800	7431	49076	OCCOC*

*Out of our computer's computational capacity!

References

[1] http://www.research.att.com/cgi-bin/access.cgi/as/njas/sequences/eismum.cgi

[2] C. Pomerance, Paul Erdös, *Notices of Amer. Math. Soc.*, vol. 45, no. 1, 1998, 19-23.