A THEOREM OF ROLEWICZ'S TYPE IN SOLID FUNCTION SPACES

CONSTANTIN BUŞE AND SEVER S. DRAGOMIR

Abstract. Let $\mathbf{R}_+$ be the set of all non-negative real numbers, $\mathbf{I} \in \{\mathbf{R}, \mathbf{R}_+\}$ and $\mathcal{U}_{\mathbf{I}} = \{U(t,s) : t \geq s \in I\}$ be a strongly measurable and exponentially bounded evolution family of bounded linear operators acting on a Banach space X. Let $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ be a strictly increasing function and E be a normed function space over $\mathbf{I}$ satisfying some properties, see Section 2. We prove that if

$$\phi \circ (\chi_{[s,\infty)}(\cdot)||U(\cdot,s)x||)$$

defines an element of the space E for every $s \in \mathbf{I}$ and all $x \in X$ and if there exists $M > 0$ such that

$$\sup_{s\in\mathbf{I}} |\phi \circ (\chi_{[s,\infty)}(\cdot)||U(\cdot,s)x||)|_E = M < \infty, \quad \forall x \in X, ||x|| \leq 1$$

then $\mathcal{U}_{\mathbf{I}}$ is uniformly exponentially stable. In particular if $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ is a nondecreasing function such that $\psi(t) > 0$ for all $t > 0$ and if there exists $K > 0$ such that

$$\sup_{s\in\mathbf{I}} \int_s^\infty \psi(||U(t,s)x||)dt = K < \infty, \quad \forall x \in X, ||x|| \leq 1$$

then $\mathcal{U}_{\mathbf{I}}$ is uniformly exponentially stable. For $\mathbf{I} = \mathbf{R}_+$, ψ continuous and $\mathcal{U}_{\mathbf{R}_+}$ strongly continuous this last result is due to S. Rolewicz. Some related results for periodic evolution families are also proved.

1. Introduction

Let $\mathbf{T} = \{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup on a Banach space X, and $\omega_0(\mathbf{T}) := \lim_{t\to\infty} \frac{\ln[||T(t)||]}{t}$ be its growth bound. It is a well known theorem of Datko [9], that if the function $t \mapsto ||T(t)x||$ belongs to $L^2(\mathbf{R}_+)$ for all $x \in X$ then $\omega_0(\mathbf{T})$ is negative, i.e. $\mathbf{T}$ is uniformly exponentially stable. This result was generalized by Pazy [15] who showed that the exponent $p = 2$ may be replaced by $1 \leq p < \infty$, and by Datko [10], who showed the following result:

Let $\mathcal{U}_{\mathbf{R}_+} = \{U(t,s) : t \geq s \geq 0\}$ be a strongly continuous and exponentially bounded evolution family of bounded linear operators acting on X, see definitions below. In what follows we consider that $U(t,s) = 0$ if $t < s$. Let us consider the function

$$t \mapsto U_s^x(t) := \chi_{[s,\infty)}(t)||U(t,s)x|| : \mathbf{I} \to \mathbf{R}_+, \quad s \in \mathbf{I}, \quad x \in X.$$

Date: April 18, 2000.

1991 *Mathematics Subject Classification.* 47A30, 93D05, 35B35, 35B40, 46A30.

Key words and phrases. solid space, evolution family of bounded linear operators, operator semigroup, q-periodic evolution family, exponentially stable, Rolewicz's theorem.

If there exists $1 \leq p < \infty$ such that U_s^x belongs to $L^p(\mathbf{R}_+)$ for all $s \geq 0$ and every $x \in X$ and if, in addition,

$$\sup_{s\geq 0} ||U_s^x||_p = M(x) < \infty \quad \forall x \in X,$$

then the family $\mathcal{U}_{\mathbf{R}_+}$ is uniformly exponentially stable, that is, there exist the constants $N > 0$ and $\nu > 0$ such that

$$||U(t,s)|| \leq Ne^{-\nu(t-s)}, \quad \forall t \geq s \geq 0.$$

The lastly result was generalized by S. Rolewicz [17]. More exactly, S. Rolewicz has proved that if $\psi : \mathbf{R}_+ \to \mathbf{R}_+$ is a continuous and nondecreasing function such that $\psi(t) > 0$ for all $t > 0$, $\psi \circ U_s^x$ belongs to $L^1(\mathbf{R}_+)$ for all $s \geq 0$ and if, in addition,

$$\sup_{s\geq 0} ||\psi \circ U_s^x|| < \infty, \quad \forall x \in X, \quad ||x|| \leq 1$$

then $\mathcal{U}_{\mathbf{R}_+}$ is uniformly exponentially stable, see also [18].

A shorter proof of Rolewicz's theorem was given by Q. Zheng [23] (cf. Neerven [14, page 111]) who also removed the continuity assumption about ψ. Other proofs of (the semigroup case) Rolewicz's theorem was offered by W. Littman [12], and van Neerven [14, Theorem 3.2.2]. Some related results have been obtained by K.M. Przyłuski [16], G. Weiss [20] and J. Zabczyk [22].

The paper is organized as follows. Section 2 contains the necessary definitions for the paper to be selfcontained. In this section we also state the main result. In Section 3 we prove this result and consider some natural consequences. Section 4 is devoted to some dual results connected with a classical result of Barbashin while the last section deals with certain integral characterization of nonuniform exponential stability.

2. Definitions and Notations

Let X be a real or complex Banach space. We shall denote by $\mathcal{L}(X)$ the Banach space of all bounded linear operators acting on X. We also denote by $||\cdot||$ the norms of vectors and operators in X and $\mathcal{L}(X)$, respectively.

A family $\mathcal{U}_{\mathbf{I}} := \{U(t,s) : t \geq s \in \mathbf{I}\}$ is said to be an *evolution family of bounded linear operators on* X, iff:

- (e_1) $U(t,s)U(s,r) = U(t,r)$ and $U(t,t) = Id$ for all $t \geq r \geq s \in \mathbf{I}$; Id is the identity operator in $\mathcal{L}(X)$.

The evolution family $\mathcal{U}_{\mathbf{I}}$ is said to be:

- (e_2) *strongly continuous* if for every $x \in X$ the function

$$(t,s) \mapsto U(t,s)x : \{(t,s) : t \geq s \in \mathbf{I}\} \to X$$

  is continuous;
- (e_3) *strongly measurable* if for every $x \in X$ and any $s \in \mathbf{I}$ the function

$$t \mapsto ||U(t,s)x|| : [s,\infty) \to \mathbf{R}_+$$

  is measurable;

- (e_4) *exponentially bounded* if there are $M_1 \geq 1$ and $\omega_1 > 0$ such that

$$||U(t,s)|| \leq M_1 e^{\omega_1(t-s)} \text{ for all } t \geq s \in \mathbf{I};$$

- (e_5) *q-periodic* (with fixed $q > 0$) if

$$U(t+q, s+q) = U(t,s) \text{ for all } t \geq s \in \mathbf{I}.$$

It is easy to see that a q-periodic and strongly continuous evolution family on X is an exponentially bounded evolution family on X (see e.g. [4, Lemma 4.1]).

Let $(\mathbf{I}, \mathcal{L}, m)$ be the Lebesgue measure space, and $\mathcal{M}(\mathbf{I})$ be the linear space of all measurable functions $f : \mathbf{I} \to \mathbf{R}$, identifiyng the functions which are equal a.e. on $\mathbf{I}$. We consider a function $\rho : \mathcal{M}(\mathbf{I}) \to [0, \infty]$ with the following properties:

- $(\mathbf{n_1})$ $\rho(f) = 0$ if and only if $f = 0$;
- $(\mathbf{n_2})$ $\rho(af) = |a|\rho(f)$ for any scalar $a \in \mathbf{R}$ and any $f \in \mathcal{M}(\mathbf{I})$, with $\rho(f) < \infty$;
- $(\mathbf{n_3})$ $\rho(f+g) \leq \rho(f) + \rho(g)$ for all $f, g \in \mathcal{M}(\mathbf{I})$.

Let $F = F_\rho$ be the set of all $f \in \mathcal{M}(\mathbf{I})$ such that $|f|_F := \rho(f) < \infty$. It is clear that $(F, |\cdot|)$ is a normed linear space. The normed linear subspace E of F is said to be a *solid space over* $\mathbf{I}$, (see also [19], [21] for similar notions), if the following two conditions hold:

- $(\mathbf{n_4})$ if $f \in E, g \in E$ and $|f| \leq |g|$ a.e., then $|f|_E \leq |g|_E$;
- $(\mathbf{n_5})$ $\chi_{[0,t]} \in E$ for all $t > 0$.

A solid space E over $\mathbf{I}$ has the *ideal property* if for all $f \in \mathcal{M}(\mathbf{I})$ and any $g \in E$, from $|f| \leq |g|$ a.e. it follows that $f \in E$. It is clear that F_ρ has the ideal property.

Let E be a solid space over $\mathbf{I}$. We say that E satisfies the *hypothesis* (H) if the following condition holds:

- $(\mathbf{n_6})$ if the sequence $(A_n)_{n=0}^{\infty}$ is such that $A_n \in \mathcal{L}$, $m(A_n) < \infty$ and $\chi_{A_n} \in E$ then $|\chi_{A_n}|_E \to \infty$ as $n \to \infty$.

Let E be a solid space. For all $t > 0$, we define

$$\Psi_E(t) := |\chi_{[0,t]}|_E \text{ and } \Psi_E(\infty) = \lim_{t\to\infty} \Psi_E(t).$$

It is clear that if E is a solid space which satisfies the hypothesis (H), then $\Psi_E(\infty) = \infty$, but the converse statement is not true, see e.g. [5, Example 1.1]. However if E is rearrangement invariant (see e.g. [14, page 222] or [11] for this class of spaces) and $\Psi_E(\infty) = \infty$ then E satisfies the hypothesis (H). In this paper we shall prove the following:

Theorem 2.1 *Let $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ be a strictly increasing function, $\mathcal{U}_\mathbf{I} = \{U(t,s) : t \geq s \in \mathbf{I}\}$ be a strongly measurable and exponentially bounded evolution family of bounded linear operators acting on a Banach space X and E be a solid space over $\mathbf{I}$. We suppose that E has the ideal property, $\Psi_E(\infty) = \infty$ and*

$$|\chi_{[0,t]}|_E \leq |\chi_{[\tau,t+\tau]}|_E \quad \forall t \geq 0, \forall \tau \in \mathbf{I}. \tag{1}$$

χ_A is the characteristic function of the set A. If for all $x \in X$ and every $s \in \mathbf{I}$, $\phi \circ U_s^x$ defines an element of the space E and, in addition, there exists $M > 0$ such that

$$\sup_{s\in\mathbf{I}} |\phi \circ U_s^x|_E = M < \infty, \quad \forall x \in X, ||x|| \leq 1 \tag{2}$$

then $\mathcal{U}_{\mathbf{I}}$ is uniformly exponentially stable, i.e., there exist $N > 0$ and $\nu > 0$ such that

$$||U(t,s)|| \leq Ne^{-\nu(t-s)}, \quad t \geq s \in \mathbf{I}. \tag{3}$$

For $E := L^p(\mathbf{R}_+, \mathbf{C})$ the condition (1) is verified with equality. The condition (1) is essential in the proof of Theorem 2.1, see [5, Example 3.2], but it may be except in the autonomous case [14, Theorem 3.1.5] and it may also be except in the periodic case [4, Theorem 4.5]. In the paper [1] the authors replaced the continuity assumptions of solutions, by measurability.

3. Proof and Consequences of Theorem 2.1

Proof of Theorem 2.1. We shall prove the Theorem in two steps.
Step 1. Here we shall state that $\mathcal{U}_{\mathbf{I}}$ is uniformly bounded. Upon replacing ϕ by some multiple of itself we may assume that $\phi(1) = 1$. Also we may assume that $\phi(0) = 0$. Let N be a positive integer number such that $|\chi_{[0,N]}|_E > M$, $t_0 \in \mathbf{I}$, $t \geq t_0 + N$ and $x \in X$, $||x|| \leq 1$. For $t - N \leq \tau \leq t$ we have

$$\begin{aligned} e^{-\omega_1 N}\chi_{[t-N,t]}(u)||U(t,t_0)x|| &\leq e^{-\omega_1(t-\tau)}\chi_{[t-N,t]}(u)||U(t,\tau)||||U(\tau,t_0)x|| \\ &\leq M_1||U(u,t_0)x||, \quad \forall u \geq t_0, \end{aligned}$$

therefore in view of $(\mathbf{n_4})$ it follows that:

$$|\phi \circ (\frac{1}{M_1 e^{\omega_1 N}}||U(t,t_0)x||\chi_{[t-N,t]}(\cdot))|_E \leq |\phi \circ U^x_{t_0}|_E. \tag{4}$$

However,

$$\begin{aligned} |\phi(\tfrac{1}{M_1 e^{\omega_1 N}}||U(t,t_0)x||)\chi_{[0,N]}(\cdot)|_E &\leq |\phi(\tfrac{1}{M_1 e^{\omega_1 N}}||U(t,t_0)x||)\chi_{[t-N,t]}(\cdot)|_E \\ &= |\phi \circ (\tfrac{1}{M_1 e^{\omega_1 N}}||U(t,t_0)x||\chi_{[t-N,t]}(\cdot))|_E. \end{aligned}$$

Now from (2) and (4) we have

$$\phi(\frac{1}{M_1 e^{\omega_1 N}}||U(t,t_0)x||)|\chi_{[0,N]}(\cdot)|_E \leq M,$$

therefore using the fact that $\phi(1) = 1$ it follows that

$$||U(t,t_0)x|| \leq M_1 e^{\omega_1 N} \text{ for all } x \in X \text{ with } ||x|| \leq 1.$$

Now it is not hard to see that there exists a constant $K_1 > 0$ such that

$$\sup_{t \geq s \in \mathbf{I}} ||U(t,s)|| = K_1 < \infty.$$

Step 2. We consider the function $t \mapsto \Phi(t) : \mathbf{R}_+ \to \mathbf{R}_+$ defined by

$$\Phi(t) = \begin{cases} \int_0^t \phi(s)ds, & \text{if } t < 1 \\ \phi(t), & \text{if } t \geq 1. \end{cases}$$

It is clear that Φ is strictly increasing, $\Phi(1) = 1$ and $\Phi \leq \phi$. Moreover the inequality (2) from Theorem 2.1 remains valid when we replace ϕ by Φ. Let $s \in \mathbf{I}$, $x \in X$, $||x|| \leq 1$ and $t > s$. For all $u \geq s$ we have

$$\begin{aligned} \chi_{[s,t]}(u)||U(t,s)x|| \leq \quad & K_1\chi_{[s,t]}(u)||U(u,s)x|| \\ &\leq K_1||U(u,s)x||. \end{aligned}$$

As before, it follows that

$$\Phi(\frac{1}{K_1}||U(t,s)x||) \leq \frac{M}{|\chi_{[0,t-s]}|_E} \quad x \in X, ||x|| \leq 1. \tag{5}$$

From (5) for $t-s$ sufficiently large it results

$$||U(t,s)|| \leq K_1 \Phi^{-1}\big(\frac{M}{|\chi_{[0,t-s]}|_E}\big).$$

The proof of Theorem 2.1 is finished if we use the following lemma.

Lemma 3.1 *Let $\mathcal{U}_{\mathbf{I}} = \{U(t,s) : t \geq s \in \mathbf{I}\}$ be an exponentially bounded linear operator on a Banach space X. If there exists a function $g : \mathbf{R}_+ \to \mathbf{R}_+$ such that*

$$\inf_{t>0} g(t) < 1 \text{ and } ||U(t,s)|| \leq g(t-s) \text{ for all } t \geq s \in \mathbf{I},$$

then $\mathcal{U}_{\mathbf{I}}$ is uniformly exponentially stable, i.e., (3) holds.

For the proof of Lemma 3.1 we refer to [6, Lemma 4].

Corollary 3.2 *Let $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ be a non-decreasing function such that $\phi(t) > 0$ for all $t > 0$ and $\mathcal{U}_{\mathbf{I}}$ a strongly measurable and exponentially bounded evolution family of bounded linear operators acting on X. If there exists a $K > 0$ such that*

$$\sup_{s \in \mathbf{I}} \int_s^\infty \phi(||U(t,s)x||)dt = K < \infty \quad \forall x \in X, ||x|| \leq 1,$$

then $\mathcal{U}_{\mathbf{I}}$ is uniformly exponentially stable.

Proof. It follows by Theorem 2.1 putting $E = L^1(\mathbf{I}, \mathbf{R}_+)$ and using the fact that ϕ can be replaced by a function ψ which is strictly increasing on $\mathbf{R}_+$ and $\psi \leq \phi$. Such a function can be defined in the following manner:

Let $\phi(1) = 1$ and $a = \int_0^1 \phi(t)dt$. The function

$$t \mapsto \psi(t) := \begin{cases} \int_0^t \phi(s)ds, & \text{if } t \leq 1 \\ \frac{at}{at+1-a}, & \text{if } t > 1 \end{cases}$$

has the desired properties.

Theorem 3.3 *Let $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ be a non-decreasing function such that $\phi(t) > 0$ for all $t > 0$, $\mathcal{U}_{\mathbf{I}}$ be a strongly continuous and q-periodic evolution family of bounded linear operators on X, and E be a solid space over $\mathbf{R}_+$ which has the ideal property and satisfies the hypothesis (H). If $\phi \circ U_0^x$ defines an element of the space E for all $x \in X$, then $\mathcal{U}_{\mathbf{I}}$ is uniformly exponentially stable.*

Proof. Is sufficient to consider the case when $\mathbf{I} = \mathbf{R}_+$ because if the restriction $\mathcal{U}_{\mathbf{I}}^0$ of $\mathcal{U}_{\mathbf{I}}$ to the set $\{(t,s) : t \geq s \geq 0\}$ is uniformly exponentially stable then $\mathcal{U}_{\mathbf{I}}$ is uniformly exponentially stable, too. We shall modify the first step of the Theorem 2.1. The argument is standard, see [15, Theorem 4.4.1], [7, Theorem 2.1], [14, Theorem 2.2] or [5, Theorem 3.1]. In fact we can prove that if $\phi \circ U_0^x$ defines an element of the space E for some $x \in X$, $||x|| \leq 1$ then

$$\lim_{t\to\infty} ||U(t,t_0)x|| = 0.$$

Indeed, if not, then

$$\limsup_{t\to\infty} ||U(t,0)x|| > 0$$

and there exists a $\delta > 0$ and a sequence $(t_n)_{n=0}^{\infty}$ with $t_0 > 0$ and $t_{n+1} - t_n > \frac{1}{\omega_1}$ such that $||U(t_n,0)x|| > \delta$ for all positive integers n. Let

$$J_n = [t_n - \frac{1}{\omega_1}, t_n], A_n = \cup_{k=0}^n J_k \text{ and } t \in J_n.$$

We have

$$\phi(\delta) \leq \phi(||U(t_n,0)x||) \leq \phi(M_1 e||U(t,0)x||).$$

Therefore, as ϕ can be considered strictly increasing, it follows that:

$$\delta \leq M_1 e||U(t,0)x|| \quad \forall t \in A_n, \forall n \in \mathbf{N}.$$

Now in view of hypothesis (H) it results:

$$\infty = \lim_{n\to\infty} \phi(\frac{\delta}{M_1 e})|\chi_{A_n}(\cdot)|_E \leq |\phi \circ U_0^x|_E,$$

which is a contradiction. Using the linearity of $U(t,0)$ and the boundedness uniform principle it follows that there exists a constant $K_2 > 0$ such that

$$\sup_{t\geq 0} ||U(t,0)|| = K_2 < \infty.$$

Moreover in view of $(\mathbf{e_4})$ and $(\mathbf{e_5})$ it easily follows, see e.g. [5, Proof of Theorem 3.1] that

$$\sup_{t\geq s\geq 0} ||U(t,s)|| \leq K_2 M_1 e^{\omega_1 q} < \infty.$$

From here the proof can be continued as in the proof of Theorem 2.1.

4. The dual results

A reformulation of an old result of E. A. Barbashin [2, Theorem 5.1] says:

Let $\mathcal{U}_{\mathbf{R}_+}$ be an exponentially bounded evolution family of bounded linear operators on X. We suppose that the function

$$s \mapsto ||U(t,s)|| : [0,t] \to \mathbf{R}_+$$

is measurable for all $t > 0$. If

$$\sup_{t\geq 0} \int_0^t ||U(t,s)|| ds < \infty$$

then $\mathcal{U}_{\mathbf{R}_+}$ is uniformly exponentially stable.

See also [13] and [3] for similar facts.

The following theorem is a generalization of the above result in the case $\mathbf{I} = \mathbf{R}$.

Theorem 4.1 *Let $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ be a non-decreasing function such that $\phi(t) > 0$ for all $t > 0$ and $\mathcal{U}_{\mathbf{R}} = \{U(t,s) : t \geq s\}$ an exponentially bounded evolution family of linear operators on X. We assume that the function*

$$s \mapsto ||U(t,s)|| : (-\infty, t] \to \mathbf{R}_+$$

is measurable for all $t \in \mathbf{R}$. If

$$\sup_{t\in\mathbf{R}} \int_{-\infty}^{t} \phi(||U(t,s)||)ds < \infty$$

then $\mathcal{U}_{\mathbf{R}}$ is uniformly exponentially stable.

Proof. Let X^* be the dual space of X and $U(t,s)^*$ the adjunct operator of $U(t,s)$ for $t \geq s$. Let $t \in \mathbf{R}$, $u = -t$ and

$$V(s,u) := U(-u,-s)^* \in \mathcal{L}(X^*).$$

We have

$$\begin{aligned} \int_{-\infty}^{t} \phi(||U(t,s)||)ds &= \int_{-\infty}^{t} \phi(||U(t,s)^*||)ds \\ &= \int_{-t}^{\infty} \phi(||U(t,-s)^*||)ds \\ &= \int_{u}^{\infty} \phi(||U(-u,-s)^*||)ds \\ &= \int_{u}^{\infty} \phi(||V(s,u)||)ds. \end{aligned}$$

It is clear that the family $\mathcal{V}_{\mathbf{R}} := \{V(s,u) : s \geq u \in \mathbf{R}\}$ is an exponentially bounded evolution family of bounded linear operators on X^* and, in addition, the function

$$s \mapsto ||V(s,u)|| : [u, \infty) \to \mathbf{R}_+$$

is measurable for all $u \in \mathbf{R}$.

From the uniform variant of Corollary 3.2 it follows that $\mathcal{V}_{\mathbf{R}}$ is uniformly exponentially stable. Hence $\mathcal{U}_{\mathbf{R}}$ is uniformly exponentially stable, too.

Theorem 4.2 *Let $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ be a non-decreasing function such that $\phi(t) > 0$ for all $t > 0$ and $\mathcal{U}_{\mathbf{R}}$ be a q-periodic evolution family of bounded linear operators on X. We assume that the function*

$$t \mapsto ||U(0,-t)|| : [0, \infty) \to \mathbf{R}_+$$

is measurable. If

$$\int_{0}^{\infty} \phi(||U(0,-t)||)dt < \infty$$

then $\mathcal{U}_{\mathbf{R}}$ is uniformly exponentially stable.

Proof. As in the proof of Theorem 4.1 it results that

$$\int_{0}^{\infty} \phi(||V(t,0)||)dt < \infty$$

and apply Theorem 3.3 for $E = L^1(\mathbf{R}_+)$.

Corollary 4.3 *Let ϕ and $\mathcal{U}_{\mathbf{R}}$ as in Theorem 4.2. We assume that the function*

$$s \mapsto ||U(t,s)|| : [0,t] \to \mathbf{R}_+$$

is measurable on $[0,t]$ for all $t > 0$. If

$$\sup_{t\geq 0} \int_0^t \phi(||U(t,s)||)ds = N_0 < \infty \tag{6}$$

then $\mathcal{U}_{\mathbf{R}}$ is uniformly exponentially stable.

Proof. From (6) for $t = nq$, $n \in \mathbf{N}$ it follows that

$$\begin{aligned} N_0 &\geq \sup_{n\in\mathbf{N}} \int_0^{nq} \phi(||U(nq,s)||)ds = \sup_{n\in\mathbf{N}} \int_0^{nq} \phi(||U(0,s-nq)||)ds \\ &= \sup_{n\in\mathbf{N}} \int_0^{nq} \phi(||U(0,-t)||)dt = \int_0^{\infty} \phi(||U(0,-t)||)dt. \end{aligned}$$

Now we can apply Theorem 4.2.

5. Nonuniform exponential stability

An evolution family $\mathcal{U}_{\mathbf{I}} = \{U(t,s) : t \geq s \in \mathbf{I}\}$ of bounded linear operators on X is said to be *exponentially stable* if there exists a constant $\nu > 0$ and a function $N : \mathbf{I} \to (0,\infty)$ such that

$$||U(t,s)|| \leq N(s)e^{-\nu(t-s)} \quad \forall t \geq s \in \mathbf{I}.$$

It is easy to see that the function $N(\cdot)$ can be chosen to be non-decreasing on $\mathbf{I}$. In the case $\mathbf{I} = \mathbf{R}_+$ we have the following Datko's theorem version for non-uniform exponential stability.

Theorem 5.1 *A strongly continuous and exponentially bounded evolution family $\mathcal{U}_{\mathbf{R}_+} = \{U(t,s) : t \geq s \geq 0\}$ is exponentially stable if and only if there exists an $\alpha > 0$ such that*

$$\int_s^{\infty} e^{\alpha t}||U(t,s)x||dt < \infty \quad \forall x \in X, \forall s \geq 0.$$

For the proof of Theorem 5.1 and its other variants we refer to [8, Theorem 2.1], [7, Theorem 2.2] or [5, Theorem 3.2]. The extension of Theorem 5.1 for the case $\mathbf{I} = \mathbf{R}$ can be easily obtained. Moreover we have:

Theorem 5.2 *Let $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ be a non-decreasing function such that $\phi(t) > 0$ for all $t > 0$ and $\mathcal{U}_{\mathbf{I}}$ be a strongly measurable and exponentially bounded evolution* *family of bounded linear operators on X. If there exists an $\alpha > 0$ such that*

$$\int_s^{\infty} \phi(e^{\alpha t}||U(t,s)x||)dt < \infty \quad \forall s \in \mathbf{I}, \forall x \in X$$

then $\mathcal{U}_{\mathbf{I}}$ is exponentially stable.

The proof of Theorem 5.2 follows as in [7, Theorem 2.2]. The Barbashin's theorem version for exponential stability is:

Theorem 5.3 *Let $\phi : \mathbf{R}_+ \to \mathbf{R}_+$ be a non-decreasing function such that $\phi(t) > 0$ for all $t > 0$ and $\mathcal{U}_{\mathbf{R}}$ be an exponentially bounded evolution family of bounded linear operators on X. We assume that the function*

$$s \mapsto ||U(t,s)|| : (-\infty, t] \to \mathbf{R}_+$$

is measurable for all $t \in \mathbf{R}$. If there exists an $\alpha > 0$ such that

$$\int_{-\infty}^{t} \phi(e^{-\alpha s}||U(t,s)||)ds < \infty, \quad \forall t \in \mathbf{R}$$

then $\mathcal{U}_{\mathbf{R}}$ is exponentially stable.

Proof. As in the Proof of Theorem 4.1, it follows that the family $\mathcal{V}_{\mathbf{R}} := \{V(s,u) : s \geq u \in \mathbf{R}\}$, where $V(s,u) := U(-u,-s)^*$, is exponentially stable, that is, there exist $\nu > 0$ and a function $N : \mathbf{R} \to (0, \infty)$ such that

$$||V(s,u)|| \leq N(u)e^{-\nu(s-u)} \quad \forall s \geq u \in \mathbf{R}.$$

Let $\alpha := -u \geq \beta := -s$. Then

$$||U(\alpha,\beta)|| \leq N(-\alpha)e^{-\nu(\alpha-\beta)} \leq N(-\beta)e^{-\nu(\alpha-\beta)},$$

that is, $\mathcal{U}_{\mathbf{R}}$ is exponentially stable.

Acknowledgement 1. *The authors would like to thank the anonymous referees for their comments and suggestions on preliminary versions of this paper, which have led to a substantial improvement of its readability.*

References

[1] Bainov D. D. and Konstantinov S. I., Exponential stability of the solutions of linear homogeneous differential equations with impulse effect in Banach spaces, *Dokl. Bulg. Acad. Sc.* **40**(1987), 29-30.

[2] Barbashin E. A., *Introduction in the Theory of Stability*, Izd. Nauka, Moskow, (1967), (Russian).

[3] Buşe C. and Giurgiulescu M., A new proof for a Barbashin's theorem in the periodic case, preprint in *Evolution Equations and Semigroups*, (http:// ma1serv.mathematik.uni-karlsruhe.de/evolve-l/index.html).

[4] Buşe C. and Pogan A., Individual Exponential Stability for Evolution Families of Linear and Bounded Operators, to appear in *New Zealand Journal of Mathematics*, as preprint in *RGMIA Research Report Collection,* **2**, (6) (1999), (http://rgmia.vu.edu.au/v2n6.html).

[5] Buşe C., Asymptotic stability of evolutors and normed function spaces, *Rend. Sem. Mat. Univ. Pol. Torino*, **55**, 2(1997), 109-122.

[6] Buşe C., On the Perron-Bellman theorem for evolutionary processes with exponential growth in Banach spaces, *New Zealand Journal of Mathematics*, **27** (1998), 183-190.

[7] Buşe C., Nonuniform exponential stability and Orlicz functions, *Comm. Math. Prace Matematyczne*, Poznań, Poland **36**, (1996), 39-47.
[8] Buşe C., On nonuniform exponential stability of evolutionary processes, *Rend. Sem. Univ. Pol. Torino*, **52**, 4(1994), 395-406.
[9] Datko R., Extending a theorem of A.M. Liapunov to Hilbert space, *J. Math. Anal. Appl.*, **32**, (1970), 610-616.
[10] Datko R., Uniform asymptotic stability of evolutionary processes in a Banach space, *SIAM J. Math. Analysis,* **3**, (1973), 428-445.
[11] Krein S. G., Petunin Yu. I., Semeonov E. M., *Interpolation of Linear Operators,* Transl. Math. Monogr. **54**, Am. Math. Soc. Providence (1982).
[12] Littman W., A generalization of a theorem of Datko and Pazy, in Lect. Notes in Control and Inform. Sci. 130, Springer-Verlag (1989), 318-323.
[13] Megan M., *Proprietes qualitatives des systems lineaires controles dans les espaces de dimension infinie*, Monographies Mathematiques, Timisoara, (1988).
[14] J.M.A.M. van Neerven, *The Asymptotic Behaviour of Semigroups of Linear Operators,* Birkhäuser Verlag Basel Boston Berlin, (1996).
[15] Pazy A., *Semigroups of linear operators and applications to partial differential equations,* Springer-Verlag, Berlin, Heidelberg, New-York, Tokyo, (1983).
[16] Przyłuski K. M., On a discrete time version of a problem of A. J. Pritchard and J. Zabczyk, Proc. Roy. Soc. Edinburgh, Sect A, **101** (1985), 159-161.
[17] Rolewicz S., On uniform N-equistability, *J. Math. Anal. Appl.* **115** (1986), 434-441.
[18] Rolewicz S., *Functional Analysis and Control Theory*, D. Reidel and PWN-Polish Scientific Publishers, Dordrecht-Warszawa, 1987.
[19] Rolewicz S., *Metric Linear Spaces*, 2-nd ed., D. Reidel and PWN-Polish Scientific Publishers, Dordrecht-Warszawa, 1985.
[20] Weiss G., Weakly l^p-stable linear operators are power stable, *Int. J. Systems Sci.* **20** (1989), 2383-2328.
[21] Zaanen A. C., *Integration*, 2nd ed., North Holland (1967)
[22] Zabczyk A., Remarks on the control of discrete-time distributed parameter systems, *SIAM J. Control*, **12** (1974), 731-735.
[23] Zheng Q., The exponential stability and the perturbation problem of linear evolution systems in Banach spaces, *J. Sichuan Univ.*, **25** (1988), 401-411 (in Chinese).

[C. Buşe] Department of Mathematics, West University of Timişoara, Bd. V. Pârvan 4, 1900-Timişoara, ROMÂNIA.

[S. S. Dragomir] School of Communications and Informatics, Victoria University of Technology, P. O. Box 14428, Melbourne City MC, Victoria 8001, Australia.