A GENERALISATION OF THE TRAPEZOIDAL RULE FOR THE RIEMANN-STIELTJES INTEGRAL AND APPLICATIONS

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Abstract. A generalisation of the trapezoid rule for the Riemann-Stieltjes integral and applications for special means are given.

1. Introduction

The following inequality is well known in the literature as the *"trapezoid inequality"*:

$$\left|\frac{f(a)+f(b)}{2}\cdot(b-a)-\int_a^b f(t)\,dt\right| \leq \frac{1}{12}(b-a)^3\|f''\|_\infty, \tag{1.1}$$

where the mapping $f:[a,b]\to\mathbb{R}$ is assumed to be twice differentiable on (a,b), with its second derivative $f'':(a,b)\to\mathbb{R}$ bounded on (a,b), that is, $\|f''\|_\infty := \sup_{t\in(a,b)}|f''(t)| < \infty$. The constant $\frac{1}{12}$ is sharp in (1.1) in the sense that it cannot be replaced by a smaller constant.

If $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ is a division of the interval $[a,b]$ and $h_i = x_{i+1} - x_i$, $\nu(h) := \max\{h_i | i = 0, ..., n-1\}$, then the following formula, which is called the *"trapezoid quadrature formula"*

$$T(f, I_n) = \sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}\cdot h_i \tag{1.2}$$

approximates the integral $\int_a^b f(t)\,dt$ with an error of approximation $R_T(f, I_n)$ which satisfies the estimate

$$|R_T(f,I_n)| \leq \frac{1}{12}\|f''\|_\infty\sum_{i=0}^{n-1}h_i^3 \leq \frac{b-a}{12}\|f''\|_\infty[\nu(h)]^2. \tag{1.3}$$

In (1.3), the constant $\frac{1}{12}$ is sharp as well.

If the second derivative does not exist or f'' is unbounded on (a,b), then we cannot apply (1.3) to obtain a bound for the approximation error. It is important, therefore, that we consider the problem of estimating $R_T(f, I_n)$ in terms of lower derivatives.

Define the following functional associated to the trapezoid inequality

$$\Psi(f;a,b) := \frac{f(a)+f(b)}{2}\cdot(b-a)-\int_a^b f(t)\,dt \tag{1.4}$$

where $f:[a,b]\to\mathbb{R}$ is an integrable mapping on $[a,b]$.

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The following result is known [3]:

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous mapping on $[a,b]$. Then*

$$|\Psi(f;a,b)| \leq \begin{cases} \frac{(b-a)^2}{4} \|f'\|_\infty & \text{if } f' \in L_\infty[a,b]; \\ \frac{(b-a)^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}}} \|f'\|_p & \text{if } f' \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{b-a}{2} \|f'\|_1, \end{cases} \tag{1.5}$$

where $\|\cdot\|_p$ are the usual $p-$norms, i.e.,

$$\|f'\|_\infty := ess \sup_{t\in[a,b]} |f'(t)|,$$

$$\|f'\|_p := \left(\int_a^b |f'(t)|^p dt\right)^{\frac{1}{p}},\ p>1$$

and

$$\|f'\|_1 := \int_a^b |f'(t)|\, dt,$$

respectively.

The following corollary for composite formulae holds [3].

Corollary 1. *Let f be as in Theorem 1. Then we have the quadrature formula*

$$\int_a^b f(x)\,dx = T(f,I_n) + R_T(f,I_n), \tag{1.6}$$

where $T(f,I_n)$ is the trapezoid rule and the remainder $R_T(f,I_n)$ satisfies the estimation

$$|R_T(f,I_n)| \leq \begin{cases} \frac{1}{4}\|f'\|_\infty \sum_{i=0}^{n-1} h_i^2 & \text{if } f' \in L_\infty[a,b]; \\ \frac{1}{2(q+1)^{\frac{1}{q}}} \|f'\|_p \left(\sum_{i=0}^{n-1} h_i^{q+1}\right)^{\frac{1}{q}} & \text{if } f' \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}\|f'\|_1 \nu(h). \end{cases} \tag{1.7}$$

A more general result concerning a trapezoid inequality for functions of bounded variation has been proved in the paper [4].

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and denote $\bigvee_a^b(f)$ as its total variation on $[a,b]$. Then we have the inequality*

$$|\Psi(f;a,b)| \leq \frac{1}{2}(b-a)\bigvee_a^b(f). \tag{1.8}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

The following corollary which provides an upper bound for the approximation error in the trapezoid quadrature formula, for f of bounded variation, holds [4].

Corollary 2. *Assume that $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$. Then we have the quadrature formula* (1.6) *where the reminder satisfies the estimate*

$$|R_T(f,I_n)|\leq\frac{1}{2}\nu(h)\bigvee_a^b(f). \tag{1.9}$$

The constant $\frac{1}{2}$ is sharp.

For other recent results on the trapezoid inequality see [5]-[10], or the book [11] where further references are given.

The following theorem generalizing the classical trapezoid inequality for mappings of bounded variation holds [12].

Theorem 3. *Let $f:[a,b]\to\mathbb{K}\,(\mathbb{K}=\mathbb{R},\mathbb{C})$ be a $p-H-$Hölder type mapping, that is, it satisfies the condition*

$$|f(x)-f(y)|\leq H|x-y|^p \text{ for all } x,y\in[a,b], \tag{1.10}$$

where $H>0$ and $p\in(0,1]$ are given, and $u:[a,b]\to\mathbb{K}$ is a mapping of bounded variation on $[a,b]$. Then we have the inequality:

$$|\Psi(f,u;a,b)|\leq\frac{1}{2^p}H(b-a)^p\bigvee_a^b(u), \tag{1.11}$$

where $\Psi(f,u;a,b)$ is the generalized trapezoid functional

$$\Psi(f,u;a,b):=\frac{f(a)+f(b)}{2}[u(b)-u(a)]-\int_a^b f(t)\,du(t). \tag{1.12}$$

The constant $C=1$ on the right hand side of (1.11) *cannot be replaced by a smaller constant.*

The following corollaries are natural consequences of (1.11):

Corollary 3. *Let f be as above and $u:[a,b]\to\mathbb{R}$ be a monotonic mapping on $[a,b]$. Then we have*

$$|\Psi(f,u;a,b)|\leq\frac{1}{2^p}H(b-a)^p|u(b)-u(a)|. \tag{1.13}$$

Corollary 4. *Let f be as above and $u:[a,b]\to\mathbb{K}$ be a Lipschitzian mapping with the constant $L>0$. Then*

$$|\Psi(f,u;a,b)|\leq\frac{1}{2^p}HL(b-a)^{p+1}. \tag{1.14}$$

Corollary 5. *Let f be as above and $G:[a,b]\to\mathbb{R}$ be the cumulative distribution function of a certain random variable X. Then*

$$\left|\frac{f(a)+f(b)}{2}-\int_a^b f(t)\,dG(t)\right|\leq\frac{1}{2^p}H(b-a)^p. \tag{1.15}$$

Remark 1. *If we assume that $g:[a,b]((a,b))\to\mathbb{K}$ is continuous, then $u(x)=\int_a^x g(t)\,dt$ is differentiable, $u(b)=\int_a^b g(t)\,dt$, $u(a)=0$, and $\bigvee_a^b(u)=\int_a^b|g(t)|\,dt$. Consequently, by (1.11), we obtain*

$$\left|\frac{f(a)+f(b)}{2}\cdot\int_a^b g(t)\,dt-\int_a^b f(t)\,g(t)\,dt\right| \leq \frac{1}{2^p}H(b-a)^p\int_a^b|g(t)|\,dt. \tag{1.16}$$

The following theorem which complements, in a sense, the previous result also holds [13].

Theorem 4. *Let $f:[a,b]\to\mathbb{K}$ be a mapping of bounded variation on $[a,b]$ and $u:[a,b]\to\mathbb{K}$ be a $p-H-$Hölder type mapping, that is, it satisfies the condition:*

$$|u(x)-u(y)|\leq H|x-y|^p \text{ for all } x,y\in[a,b], \tag{1.17}$$

where $H>0$ and $p\in(0,1]$ are given. Then we have the inequality:

$$|\Psi(f,u;a,b)|\leq\frac{1}{2^p}H(b-a)^p\bigvee_a^b(f). \tag{1.18}$$

The constant $C=1$ on the right hand side of (1.18) cannot be replaced by a smaller constant.

The following corollary is a natural consequence of the above result.

Corollary 6. *Let $f:[a,b]\to\mathbb{K}$ be as in Theorem 4 and u be an $L-$Lipschitzian mapping on $[a,b]$, that is,*

$$|u(t)-u(s)|\leq L|t-s| \text{ for all } t,s\in[a,b], \tag{1.19}$$

where $L>0$ is fixed. Then we have the inequality

$$|\Psi(f,u;a,b)|\leq\frac{L}{2}(b-a)\bigvee_a^b(f). \tag{1.20}$$

Remark 2. *If $f:[a,b]\to\mathbb{R}$ is monotonic and u is of $p-H-$Hölder type, then the inequality (1.18) becomes:*

$$|\Psi(f,u;a,b)|\leq\frac{1}{2^p}H(b-a)|f(b)-f(a)|. \tag{1.21}$$

In addition, if u is $L-$Lipschitzian, then the inequality (1.20) can be replaced by

$$|\Psi(f,u;a,b)|\leq\frac{L}{2}(b-a)|f(b)-f(a)|. \tag{1.22}$$

Remark 3. *If f is Lipschitzian with a constant $K>0$, then it is obvious that f is of bounded variation on $[a,b]$ and $\bigvee_a^b(f)\leq K(b-a)$. Consequently, the inequality (1.18) becomes*

$$|\Psi(f,u;a,b)|\leq\frac{1}{2^p}HK(b-a)^{p+1}, \tag{1.23}$$

and the inequality (1.20) becomes

$$|\Psi(f,u;a,b)|\leq\frac{LK}{2}(b-a)^2. \tag{1.24}$$

We now point out some results in estimating the integral of a product.

Corollary 7. *Let $f:[a,b]\to\mathbb{R}$ be a mapping of bounded variation on $[a,b]$ and g be continuous on $[a,b]$. Put $\|g\|_\infty := \sup_{t\in[a,b]}|g(t)|$. Then we have the inequality:*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(s)\,ds-\int_a^b f(t)\,g(t)\,dt\right|\le\frac{\|g\|_\infty}{2}(b-a)\bigvee_a^b(f). \tag{1.25}$$

Remark 4. *Now, if in the above corollary we assume that f is monotonic, then* (1.25) *becomes*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(s)\,ds-\int_a^b f(t)\,g(t)\,dt\right| \le \frac{\|g\|_\infty|f(b)-f(a)|(b-a)}{2}, \tag{1.26}$$

and if in Corollary 7 we assume that f is $K-$Lipschitzian, then the inequality (1.25) *becomes*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(s)\,ds-\int_a^b f(t)\,g(t)\,dt\right|\le\frac{\|g\|_\infty K(b-a)^2}{2}. \tag{1.27}$$

The following corollary is also a natural consequence of Theorem 4.

Corollary 8. *Let f and g be as in Corollary 7. Put*

$$\|g\|_p := \left(\int_a^b |g(s)|^p\,ds\right)^{\frac{1}{p}};\ p>1.$$

Then we have the inequality

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(s)\,ds-\int_a^b f(t)\,g(t)\,dt\right| \le \frac{1}{2^{\frac{p-1}{p}}}\|g\|_p(b-a)^{\frac{p-1}{p}}\bigvee_a^b(f). \tag{1.28}$$

2. The Results

The following theorem holds.

Theorem 5. *Let $u:[a,b]\to\mathbb{R}$ be of $H-r$-Hölder type, i.e., we recall this*

$$|u(x)-u(y)|\le H|x-y|^r,\ \text{for any } x,y\in[a,b] \text{ and some } H>0, \tag{2.1}$$

where $r\in(0,1]$ is given, and $f:[a,b]\to\mathbb{R}$ is of bounded variation.
Then we have the inequality:

$$\left|\int_a^b f(t)du(t)-[(u(b)-u(x))f(b)+(u(x)-u(a))f(a)]\right| \le H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(f)\le H(b-a)^r\bigvee_a^b(f) \tag{2.2}$$

for any $x\in[a,b]$.
The constant $\frac{1}{2}$ is sharp in the sense that we cannot put a smaller constant instead.

Proof. Using the integration by parts formula, we may state:

$$\int_a^b (u(t)-u(x))df(t) = [u(b)-u(x)]f(b)-[u(a)-u(x)]f(a)-\int_a^b f(t)du(t). \tag{2.3}$$

It is well known that if $m:[a,b]\to\mathbb{R}$ is continuous and $n:[a,b]\to\mathbb{R}$ is of bounded variation, the Riemann-Stieltjes integral $\int_a^b m(t)dn(t)$ exists, and

$$\left|\int_a^b m(t)dn(t)\right| \le \sup_{t\in[a,b]}|m(t)|\cdot\bigvee_a^b(n).$$

Thus,

$$\begin{aligned}&\left|\int_a^b (u(t)-u(x))df(t)\right|\\ \le\ & \sup_{t\in[a,b]}|u(t)-u(x)|\bigvee_a^b(f)\le\sup_{t\in[a,b]}\{H|t-x|^r\}\bigvee_a^b(f)\\ =\ & H\max\{|b-x|^r,|x-a|^r\}\bigvee_a^b(f)=H[\max(b-x,x-a)]^r\bigvee_a^b(f)\\ =\ & H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(f).\end{aligned}$$

Finally, as

$$\left|x-\frac{a+b}{2}\right|\le\frac{1}{2}(b-a)\text{ for any }x\in[a,b]$$

we get the last inequality in (2.2).

To prove the sharpness of the constant $\frac{1}{2}$, we assume that (2.2) holds with the constant $c>0$, i.e.,

$$\begin{aligned}&\left|\int_a^b f(t)du(t)-[(u(b)-u(x))f(b)+(u(x)-u(a))f(a)]\right|\\ \le\ & H\left[c(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r\bigvee_a^b(f).\end{aligned} \tag{2.4}$$

Choose $u(t)=t$ which is of $(1-1)$-Hölder type and $f:[a,b]\to\mathbb{R}$, $f(t)=0$ if $t\in\{a,b\}$ and $f(t)=1$ if $t\in(a,b)$, which is of bounded variation, in (2.4).

We get:

$$|b-a|\le 2\left[c(b-a)+\left|x-\frac{a+b}{2}\right|\right],\text{ for any }x\in[a,b].$$

For $x=\frac{a+b}{2}$, we get:

$$|b-a|\le 2c(b-a),\text{ i.e. }c\ge\frac{1}{2}.$$

∎

Remark 5. *If u is Lipschitz continuous function, i.e.*

$$|u(x)-u(y)| \leq L|x-y| \text{ for any } x, y \in [a,b], (\text{ and some } L>0),$$

the inequality (2.2) becomes:

$$\left|\int_a^b f(t)du(t) - [(u(b)-u(x))f(b) + (u(x)-u(a))f(a)]\right| \tag{2.5}$$
$$\leq \quad L \cdot \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \cdot \bigvee_a^b(f) \leq L(b-a)\bigvee_a^b(f).$$

Corollary 9. *If f is of bounded variation on $[a,b]$ and u is absolutely continuous with $u' \in L_\infty[a,b]$ then instead of L in (2.5) we can put*

$$||u'||_\infty = ess \sup_{t\in[a,b]} |u'(t)|.$$

Corollary 10. *If $g : [a,b] \to \mathbb{R}$ is Riemann integrable on $[a,b]$ and if we choose $u(t) = \int_a^t g(s)ds$, then*

$$\left|\int_a^b f(t)g(t)dt - f(b)\int_x^b g(s)ds - f(a)\int_a^x g(s)ds\right| \tag{2.6}$$
$$\leq \quad \|g\|_\infty \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b(f) \leq \|g\|_\infty (b-a)\bigvee_a^b(f).$$

Remark 6. *If in (2.6) we choose $x = \frac{a+b}{2}$, we get the best inequality in the class, i.e.,*

$$\left|\int_a^b f(t)g(t)dt - f(b)\int_{\frac{a+b}{2}}^b g(s)ds - f(a)\int_a^{\frac{a+b}{2}} g(s)ds\right| \tag{2.7}$$
$$\leq \quad \frac{1}{2}\|g\|_\infty (b-a)\bigvee_a^b(f).$$

3. Approximating Riemann-Stieltjes Integral

Let $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ a division of $[a,b]$. Denote $h_i := x_{i+1} - x_i$, and $\nu(I_n) = \sup_{i=\overline{0,n-1}} h_i$ then construct the sums

$$S(f,u,I_n,\boldsymbol{\xi}) = \sum_{i=0}^{n-1}[u(x_{i+1}) - u(\xi_i)]f(x_{i+1}) + \sum_{i=0}^{n-1}[u(\xi_i) - u(x_i)]f(x_i), \tag{3.1}$$

where $\xi_i \in [x_i, x_{i+1}], i = \overline{0, n-1}$ and $\boldsymbol{\xi} = (\xi_0, \xi_1, \cdots \xi_{n-1})$.

We can state the following theorem concerning the approximation of Riemann-Stieltjes integral:

Theorem 6. *Let f, u be as in Theorem 5 and $I_n, \boldsymbol{\xi}$ as defined above. Then:*

$$\int_a^b f(t)du(t) = S(f,u,I_n,\boldsymbol{\xi}) + R(f,u,I_n,\boldsymbol{\xi}) \tag{3.2}$$

when $S(f,u,I_n,\boldsymbol{\xi})$ is defined by (3.1) and the remainder $R(f,u,I_n,\boldsymbol{\xi})$ satisfies the estimate:

$$\begin{aligned}(3.3)\qquad |R(f,u,I_n,\boldsymbol{\xi})| &\le H\cdot\left[\frac{1}{2}\nu(I_n)+\sup_{i=\overline{0,n-1}}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^r\bigvee_a^b(f)\\ &\le H\cdot\nu^r(I_n)\bigvee_a^b(f).\end{aligned}$$

Proof. We apply (2.2) on $[x_i,x_{i+1}]$ to get:

$$\begin{aligned}&\left|\int_{x_i}^{x_{i+1}}f(t)du(t)-[u(x_{i+1})-u(\xi_i)]f(x_{i+1})-[u(\xi_i)-u(x_i)]f(x_i)\right|\\ &\le H\cdot\left[\frac{1}{2}h_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^r\bigvee_{x_i}^{x_{i+1}}(f)\le H\cdot h_i^r\bigvee_{x_i}^{x_{i+1}}(f).\end{aligned}$$

Summing on i from 0 to $n-1$, and using the generalised triangle inequality we get:

$$\begin{aligned}&\left|\int_a^b f(t)du(t)-S(f,u,I_n,\boldsymbol{\xi})\right|\\ &\le H\cdot\sum_{i=0}^{n-1}\left[\frac{1}{2}h_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^r\cdot\bigvee_{x_i}^{x_{i+1}}(f)\\ &\le H\sup_{i=\overline{0,n-1}}\left[\frac{1}{2}h_i+\left|\boldsymbol{\xi}-\frac{x_i+x_{i+1}}{2}\right|\right]^r\bigvee_a^b(f)\\ &\le H\left[\frac{1}{2}\nu(I_n)+\sup_{i=\overline{0,n-1}}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^r\bigvee_a^b(f)\\ &\le H\nu^r(I_n)\bigvee_a^b(f),\end{aligned}$$

and the theorem is proved. ∎

Remark 7. *It is obvious that if $\nu(I_n)\to 0$ then (3.2) provides an approximation for the Riemann-Stieltjes integral $\int_a^b f(t)du(t)$.*

Corollary 11. *If we consider the sum*

$$\begin{aligned}&S_M(f,u,I_n)\\ &=\sum_{i=0}^{n-1}\left[u(x_{i+1})-u\left(\frac{x_i+x_{i+1}}{2}\right)\right]f(x_{i+1})+\sum_{i=0}^{n-1}\left[u\left(\frac{x_i+x_{i+1}}{2}\right)-u(x_i)\right]f(x_i)\end{aligned}$$

then:

$$(3.4)\qquad \int_a^b f(t)du(t)=S_M(f,u,I_n)+R_M(f,u,I_n)$$

and the remainder $R_M(f,u,I_n)$ satisfies the estimate

$$(3.5)\qquad |R_M(f,u,I_n)|\le\frac{1}{2^r}H\nu^r(I_n)\bigvee_a^b(f).$$

The following corollary in approximating the integral $\int_a^b f(t)g(t)dt$ holds.

Corollary 12. *If f, g are as in Corollary 10, then*

$$\int_a^b f(t)g(t)dt = P(f,g,I_n,\boldsymbol{\xi}) + R_P(f,g,I_n,\boldsymbol{\xi})$$

where

$$P(f,g,I_n,\boldsymbol{\xi}) = \sum_{i=0}^{n-1} f(x_{i+1}) \int_{\xi_i}^{x_{i+1}} g(s)ds + \sum_{i=0}^{n-1} f(x_i) \int_{x_i}^{\xi_i} g(s)ds.$$

and the remainder $R_P(f,g,I_n,\boldsymbol{\xi})$ satisfies the estimate:

$$\begin{aligned} |R_P(f,g,I_n,\boldsymbol{\xi})| &\leq \|g\|_\infty \left[\frac{1}{2}\nu(I_n) + \sup_{i=\overline{0,n-1}} \left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right] \bigvee_a^b (f) \\ &\leq \|g\|_\infty \nu(I_n) \bigvee_a^b (f). \end{aligned}$$

Remark 8. *If in the above corollary we choose $\xi_i = \frac{x_i+x_{i+1}}{2}$ $\left(i = \overline{0,n-1}\right)$ then we get the best formula in the class, i.e.,*

$$P_M(f,g,I_n,\boldsymbol{\xi}) = \sum_{i=0}^{n-1} f(x_{i+1}) \int_{\frac{x_i+x_{i+1}}{2}}^{x_{i+1}} g(s)ds + \sum_{i=0}^{n-1} f(x_i) \int_{x_i}^{\frac{x_i+x_{i+1}}{2}} g(s)ds$$

and

$$R_{P_M}(f,g,I_n,\boldsymbol{\xi}) \leq \frac{1}{2} \|g\|_\infty \nu(I_n) \bigvee_a^b (f).$$

4. Application for Special Means

Consider the means:

1. Arithmetic mean

$$A(a,b) := \frac{a+b}{2}; a,b \geq 0;$$

2. Geometric mean

$$G(a,b) := \sqrt{ab}; a,b \geq 0;$$

3. Harmonic mean

$$H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}; a,b > 0;$$

4. Logarithmic mean

$$L(a,b) := \begin{cases} \dfrac{b-a}{\ln b - \ln a}; & a,b > 0, a = b \\ a, & a = b. \end{cases}$$

5. Identric mean

$$I(a,b) := \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}; & a,b>0, a=b \\ a, & a=b. \end{cases}$$

6. p- Logarithmic mean

$$L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}; & a,b>0, a=b \\ a, & a=b. \end{cases} \quad , \quad p \in \mathbb{R}\setminus\{-1,0\}.$$

It is well known that $L_p(a,b)$ is monotically increasing as a function of $p \mapsto L_p(a,b)$ denoting that $L_{-1} = L$ and $L_0 = I$.

In Section 2 we proved the following inequality:

$$\begin{aligned} &\left|\int_a^b f(t)g(t)dt - f(b)\int_x^b g(s)ds - f(a)\int_a^x g(s)ds\right| \\ \leq\ & \|g\|_\infty \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right] \bigvee_a^b (f) \leq \|g\|_\infty (b-a) \bigvee_a^b (f). \end{aligned}$$

We can use this inequality in the sequel for different selections of f and g.

1. If we choose: $f(x) = x^p$ and $g(x) = x^q, x \in [a,b],\ a,b>0$ we get the inequalities:

$$\begin{aligned} &\left|(b-a)L_{p+q}^{p+q}(a,b) - b^p(b-x)L_q^q(x,b) - a^p(x-a)L_q^q(a,x)\right| \\ \leq\ & b^q p(b-a)^2 L_{p-1}^{p-1}(a,b) \end{aligned}$$

for any $q>0$ and

$$\begin{aligned} &\left|(b-a)L_{p+q}^{p+q}(a,b) - b^p(b-x)L_q^q(x,b) - a^p(x-a)L_q^q(a,x)\right| \\ \leq\ & a^q p(b-a)^2 L_{p-1}^{p-1}(a,b) \end{aligned}$$

for any $q<0$, $q \neq -1$. Particularly, for $x = A(a,b)$ we obtain:

$$\begin{aligned} &\left|2L_{p+q}^{p+q}(a,b) - b^p L_q^q(A(a,b),b) - a^p L_q^q(a,A(a,b))\right| \\ \leq\ & b^q p(b-a)L_{p-1}^{p-1}(a,b) \end{aligned}$$

for any $q>0$, respectively,

$$\begin{aligned} &\left|2L_{p+q}^{p+q}(a,b) - b^p L_q^q(A(a,b),b) - a^p L_q^q(a,A(a,b))\right| \\ \leq\ & a^q p(b-a)L_{p-1}^{p-1}(a,b) \end{aligned}$$

for any $q<0$, $q\neq -1$.

2. If we choose: $f(x) = x^p$ and $g(x) = \frac{1}{x}, x \in [a,b], a,b>0$ we get the inequality:

$$\begin{aligned} &\left|(b-a)L_{p-1}^{p-1}(a,b) - b^p(b-x)L_{-1}^{-1}(x,b) - a^p(x-a)L_{-1}^{-1}(a,x)\right| \\ \leq\ & \frac{p}{a}(b-a)^2 L_{p-1}^{p-1}(a,b). \end{aligned}$$

Particularly, for $x = A(a, b)$ we obtain:

$$\left|2L_{p-1}^{p-1}(a,b) - b^p L_{-1}^{-1}(A(a,b),b) - a^p L_{-1}^{-1}(a,A(a,b))\right| \leq \frac{p}{a}(b-a)L_{p-1}^{p-1}(a,b).$$

3. If we choose: $f(x) = x^p$ and $g(x) = \ln x, x \in [a, b], a, b > 0$ we get the inequality:

$$\begin{aligned} &\Big|\frac{b-a}{p+1}[(p\ln b + \ln b - 1)L_p^p(a,b) + a^{p+1}L_{-1}^{-1}(a,b)] \\ &-b^p(b-x)\ln(L_0(x,b)) - a^p(x-a)\ln(L_0(a,x))\Big| \\ \leq\ & p(b-a)^2(\ln b)L_{p-1}^{p-1}(a,b). \end{aligned}$$

Particularly, for $x = A(a, b)$ we obtain:

$$\begin{aligned} &\Bigg|\frac{2}{p+1}[(p\ln b + \ln b - 1)L_p^p(a,b) + a^{p+1}L_{-1}^{-1}(a,b)] \\ &-b^p\ln(L_0(A(a,b),b)) - a^p\ln(L_0(a,A(a,b)))\Bigg| \\ \leq\ & p(b-a)\ln bL_{p-1}^{p-1}(a,b). \end{aligned}$$

4. If we choose: $f(x) = \frac{1}{x}$ and $g(x) = x^q, x \in [a, b], a, b > 0$ we get the inequalities:

$$\left|G^2(a,b)(b-a)L_{q-1}^{q-1}(a,b) - a(b-x)L_q^q(x,b) - b(x-a)L_q^q(a,x)\right| \leq (b-a)^2 b^q$$

for any $q > 0$ and

$$\left|G^2(a,b)(b-a)L_{q-1}^{q-1}(a,b) - a(b-x)L_q^q(x,b) - b(x-a)L_q^q(a,x)\right| \leq (b-a)^2 a^q$$

for any $q < 0, q \neq -1$.
Particularly, for $x = A(a, b)$ we obtain:

$$\left|2G^2(a,b)L_{q-1}^{q-1}(a,b) - aL_q^q(A(a,b),b) - bL_q^q(a,A(a,b))\right| \leq (b-a)b^q$$

for any $q > 0$, respectively:

$$\left|2G^2(a,b)L_{q-1}^{q-1}(a,b) - aL_q^q(A(a,b),b) - bL_q^q(a,A(a,b))\right| \leq (b-a)a^q$$

for any $q < 0, q \neq -1$.

5. If we choose: $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x}$ we get the inequality:

$$\left|b - a - a(b-x)L_{-1}^{-1}(x,b) - b(x-a)L_{-1}^{-1}(a,x)\right| \leq \frac{(b-a)^2}{a}.$$

Particularly, for $x = A(a, b)$ we obtain:

$$\left|2 - aL_{-1}^{-1}(A(a,b),b) - bL_{-1}^{-1}(a,A(a,b))\right| \leq \frac{b-a}{a}.$$

6. If we choose: $f(x) = \frac{1}{x}$ and $g(x) = \ln x$ we get the inequality:

$$\begin{aligned} & \Big| G^2(a,b) \cdot \frac{b-a}{2} \cdot \ln(G^2(a,b)) \cdot L_{-1}^{-1}(a,b) - a(b-x)\ln(L_0(x,b)) \\ & -b(x-a)\ln\left(L_0(a,x)\right) \Big| \\ \le \; & (b-a)^2 \ln b. \end{aligned}$$

Particularly, for $x = A(a,b)$ we obtain:

$$\left| G^2(a,b)\ln(G^2(a,b))L_{-1}^{-1}(a,b) - a\ln(L_0(a,A(a,b))\right| \le (b-a)\ln b$$

7. If we choose: $f(x) = \ln x$ and $g(x) = x^q$ we get the inequalities:

$$\begin{aligned} & \Big| \frac{b-a}{q+1}[(q\ln b + \ln b - 1)L_q^q(a,b) + a^{q+1}L_{-1}^{-1}(a,b)] \\ & -(\ln b)(b-x)L_q^q(x,b) - (\ln a)(a-x)L_q^q(a,x) \Big| \\ \le \; & (b-a)^2 b^q L_{-1}^{-1}(a,b) \text{ for any } q > 0, \end{aligned}$$

and

$$\begin{aligned} & \Big| \frac{b-a}{q+1}[(q\ln b + \ln b - 1)L_q^q(a,b) + a^{q+1}L_{-1}^{-1}(a,b)] \\ & -(\ln b)(b-x)L_q^q(x,b) - (\ln a)(a-x)L_q^q(a,x) \Big| \\ \le \; & (b-a)^2 a^q L_{-1}^{-1}(a,b) \text{ for any } q < 0,\ q \ne -1. \end{aligned}$$

Particularly, for $x = A(a,b)$ we obtain:

$$\begin{aligned} & \Big| \frac{2}{q+1}[(q\ln b + \ln b - 1)L_q^q(a,b) + a^{q+1}L_{-1}^{-1}(a,b)] \\ & -\ln b L_q^q(A(a,b),b) - \ln a L_q^q(a,A(a,b)) \Big| \\ \le \; & (b-a)b^q L_{-1}^{-1}(a,b) \text{ for any } q > 0, \end{aligned}$$

respectively:

$$\begin{aligned} & \Big| \frac{2}{q+1}[(q\ln b + \ln b - 1)L_q^q(a,b) + a^{q+1}L_{-1}^{-1}(a,b)] \\ & -\ln b L_q^q(A(a,b),b) - \ln a L_q^q(a,A(a,b)) \Big| \\ \le \; & (b-a)a^q L_{-1}^{-1}(a,b) \text{ for any } q < 0, q \ne -1. \end{aligned}$$

8. If we choose: $f(x) = \ln x$ and $g(x) = \frac{1}{x}$ we get the inequality:

$$\begin{aligned} & \left| \frac{b-a}{2}\ln G^2(a,b)L_{-1}^{-1}(a,b) - (b-x)\ln b L_{-1}^{-1}(x,b) - (a-x)\ln a L_{-1}^{-1}(a,x) \right| \\ \le \; & \frac{(b-a)^2}{a} L_{-1}^{-1}(a,b). \end{aligned}$$

Particularly, for $x = A(a,b)$ we obtain:

$$\begin{aligned} & \left|\ln G^2(a,b) L_{-1}^{-1}(a,b) - \ln b L_{-1}^{-1}(A(a,b),b) - \ln a L_{-1}^{-1}(a,A(a,b))\right| \\ \leq \; & \frac{b-a}{a} L_{-1}^{-1}(a,b). \end{aligned}$$

9. If we choose: $f(x) = \ln x$ and $g(x) = \ln x$ we get the inequality:

$$\begin{aligned} & \left| \frac{b-a}{G^2(a,b)} [b(\ln a^a b^b - 2) \ln(L_0(a,b)) + b \ln a^a b^b - \ln^2 b^b] \right. \\ & \left. -(b-x) \ln b \ln(L_0(x,b)) - (x-a) \ln a \ln(L_0(a,x)) \right| \\ \leq \; & (b-a)^2 \ln b L_{-1}^{-1}(a,b). \end{aligned}$$

Particularly, for $x = A(a,b)$ we obtain:

$$\begin{aligned} & \left| \frac{2}{G^2(a,b)} [b(\ln a^a b^b - 2) - \ln(L_0(a,b)) + b \ln a^a b^b - (\ln b^b)^2] \right. \\ & \left. - \ln a \ln(L_0(a, A(a,b)) \right| \\ \leq \; & (b-a) \ln b L_{-1}^{-1}(a,b). \end{aligned}$$

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