$2k$-INNER PRODUCTS AND $2k$-RIEMANNIAN METRICS

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Abstract. The notion of $2k$-inner product is introduced as a generalization of usual inner product and Q-inner product([4]-[8]). As a consequence, is defined the notion of $2k$-normed space and some properties, e.g. uniformly convexity, Gâteaux differentiability and Riesz propriety of the dual, are given. Also, the notion of $2k$-Riemannian metric is introduced.

1. Introduction

In the last decade, the second author gave (see [4] – [9]) an extension of the usual notion of inner product, namely the quaternionic inner product, or, for short, the Q-inner product. Some of the properties of an inner product and of the associated norm, such as:

(i) uniform convexity,
(ii) Gâteaux differentiability,
(iii) equivalence of Birkhoff orthogonality with the inner product orthogonality,
(iv) the Riesz form of linear continuous functionals

were reobtained in this new framework.

The present paper is devoted to a generalization of both the classical inner product and the Q-inner product. In the first section we introduce the concept of $2k$-inner products and prove the properties (i)-(ii) above. Also, it is proved that a $2k$-inner product space is a smooth space of (BD)-type in the sense of Dragomir, and two remarkable identities, equivalent with the parallelogram identity, are given. The following two sections deal with the properties (iii) and (iv) and some results related to projections are obtained. The paper concludes with a generalization of Riemannian metrics, namely $2k$-Riemannian metrics.

2. Main Properties of $2k$-Inner Products

Let X be a real linear space and $k \neq 0$ a natural number. As usual, we shall denote $X^{2k} = \underbrace{X \times \ldots \times X}_{2k\ times}$. We introduce the following new concept:

Definition 1. *A mapping $(\cdot, \ldots, \cdot) : X^{2k} \to \mathbb{R}$ is said to be a $2k$-inner product if:*

(i) $(\alpha_1 x_1 + \alpha_2 x_2, x_3, \ldots, x_{2k+1}) = \alpha_1 (x_1, x_3, \ldots, x_{2k+1}) + \alpha_2 (x_2, x_3, \ldots, x_{2k+1})$, $\alpha_1, \alpha_2 \in \mathbb{R}$;
(ii) $\left(x_{\sigma(1)}, \ldots, x_{\sigma(2k)}\right) = (x_1, \ldots, x_{2k})$, $\quad \sigma \in S_{2k}$, *where S_{2n} denotes the set of all permutations of the indices $\{1, \ldots, 2k\}$;*
(iii) $(x, \ldots, x) > 0$ *if $x \neq 0$;*

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(iv) *Cauchy-Buniakowski-Schwarz's inequality (CBS for short)*

$$|(x_1,\ldots,x_{2k})|^{2k} \le \prod_{i=1}^{2k} (x_i,\ldots,x_i)$$

with equality if and only if $x_1,\ldots,x_{2k}$ *are linear dependent.*

The pair $(X,(\cdot,\ldots,\cdot))$ is called $2k$*-inner product space*. Let us remark that our notion is different from the n-inner product of Misiak ([10]).

For $k=1$ we have the usual notion of inner product and for $k=2$ we obtain the notion of Q*-inner product* from [4]-[8]. Also, it follows that

$$(0,x_2,\ldots,x_{2k}) = 0 \text{ and } \quad (\alpha x_1,\ldots,\alpha x_{2k}) = \alpha^{2k}(x_1,\ldots,x_{2k}).$$

Example 1. 1.

I) $X=\mathbb{R}^n,\ (x_1,\ldots,x_{2k}) = \sum_{i=1}^{n}\left(\prod_{j=1}^{2k} x_j^i\right)$ *if* $x_j = \left(x_j^1,\ldots,x_j^n\right)$

II) *Let* $(\Omega,\mathcal{A},\mu)$ *be a measure space consisting of a set* Ω*, a* σ*-algebra* $\mathcal{A}$ *of subsets of* Ω*, and a countably additive and positive measure* μ *on* $\mathcal{A}$ *with* $\mu(\Omega)<\infty$*. Then on* $X = L^{2k}(\Omega,\mathcal{A},\mu)$ *we have the* $2k$*-inner product*

$$(x_1,\ldots,x_{2k}) = \int_\Omega \prod_{i=1}^{2k} x_i(t)\, d\mu(t).$$

A remarkable class of $2k$-inner products is provided by:

Proposition 1. *An usual inner product* $(\cdot,\cdot)$ *on* X *gives rise to a* $2k$*-inner product on* X *for every* k*.*

Proof. By induction after k. Let us suppose that the given inner product yields the $2k$-inner product $(\cdot,\ldots,\cdot)_{2k}$. Then:

$$\begin{aligned} & (x_1,\ldots,x_{2k+2})_{2k+2} \\ := {} & \frac{1}{2k+1}[(x_1,x_2)(x_3,\ldots,x_{2k+2})_{2k} + (x_1,x_3)(x_2,x_4,\ldots,x_{2k+2})_{2k} + \ldots \\ & + (x_1,x_{2k+2})(x_3,\ldots,x_{2k+1})_{2k}] \end{aligned}$$

is a $(2k+2)$-inner product. ∎

In the following we call *simple* the above type of $2k$-inner products.

Example 2. 1.

(i) *For* $k=2$ *(*[6, p. 76], [8, p. 20]*) we have the following 4-inner product:*

$$(x_1,x_2,x_3,x_4)_4 = \frac{1}{3}[(x_1,x_2)(x_3,x_4) + (x_1,x_3)(x_2,x_4) + (x_1,x_4)(x_2,x_3)]$$

(ii) *For* $k=3$ *we have the 6-inner product*

$$\begin{aligned} & (x_1,\ldots,x_6)_6 \\ = {} & \frac{1}{15}\{(x_1,x_2)[(x_3,x_4)(x_5,x_6) + (x_3,x_5)(x_4,x_6) + (x_3,x_6)(x_4,x_5)] \\ & + (x_1,x_3)[(x_2,x_4)(x_5,x_6) + (x_2,x_5)(x_4,x_6) + (x_2,x_6)(x_4,x_5)] \\ & + (x_1,x_4)[(x_2,x_3)(x_5,x_6) + (x_2,x_5)(x_3,x_6) + (x_2,x_6)(x_3,x_5)] \\ & + (x_1,x_5)[(x_2,x_3)(x_4,x_6) + (x_2,x_4)(x_3,x_6) + (x_2,x_6)(x_3,x_4)] \\ & + (x_1,x_6)[(x_2,x_3)(x_4,x_5) + (x_2,x_4)(x_3,x_5) + (x_2,x_5)(x_3,x_4)]\}. \end{aligned}$$

(iii) *In the general case we have* $(2k-1)!! = 1 \cdot 3 \cdot \ldots \cdot (2k-1)$ *terms. So, for* $k = 4$ *we have* $7!! = 3 \cdot 5 \cdot 7 = 105$ *terms.*

The previous proposition leads to the definition of orthogonal basis. Let us suppose that X has dimension n and let $B = \{e_i\}_{1 \le i \le n}$ be a basis for X. For $k = 1$ as usual B is said to be *orthogonal* if $(e_i, e_j) = \delta_{ij}$ and for $k > 1$ we define recurrently using the relation from the proof of Proposition 1. For example, B is orthogonal for a Q-inner product if:

$$(e_{i_1}, e_{i_2}, e_{i_3}, e_{i_4}) = \frac{1}{3}\left(\delta_{i_1 i_2}\delta_{i_3 i_4} + \delta_{i_1 i_3}\delta_{i_2 i_4} + \delta_{i_1 i_4}\delta_{i_2 i_3}\right).$$

Then, for $i \neq j$, we have $(e_i, e_i, e_j, e_j) = \frac{1}{3}$ and $(e_i, e_i, e_i, e_j) = 0$.

A first property is:

Proposition 2. *If* $(\cdot, \ldots, \cdot)$ *is a* $2k$*-inner product then* $\|\cdot\|_{2k} : X \to \mathbb{R}_+, \|x\|_{2k} = (x, \ldots, x)^{\frac{1}{2k}}$ *is a norm on* X *for which the following generalization of parallelogram identity holds:*

$$\|x+y\|_{2k}^{2k} + \|x-y\|_{2k}^{2k} = 2\sum_{i=0}^{k}\binom{2k}{2(k-i)}\left(\underbrace{x, \ldots, x}_{2i\ times}, \underbrace{y, \ldots, y}_{2(k-i)\ times}\right).$$

Proof. By definition of the $2k-$norm, we get

$$\|x+y\|_{2k}^{2k} = \sum_{i=0}^{2k}\binom{2k}{i}\left(\underbrace{x, \ldots, x}_{i\ times}, \underbrace{y, \ldots, y}_{2k-i\ times}\right).$$

However,

$$\left(\underbrace{x, \ldots, x}_{i\ times}, \underbrace{y, \ldots, y}_{2k-i\ times}\right) \le \|x\|_{2k}^{i}\|y\|_{2k}^{2k-i}$$

and then

$$\|x+y\|_{2k}^{2k} \le \sum_{i=0}^{2k}\binom{2k}{i}\|x\|_{2k}^{i}\|y\|_{2k}^{2k-i} = \left(\|x\|_{2k} + \|y\|_{2k}\right)^{2k}$$

which gives the triangle inequality. The relations:

$$\|x\|_{2k} \ge 0, \|x\|_{2k} = 0 \Leftrightarrow x = 0$$

and $\|\lambda x\|_{2k} = |\lambda| \, \|x\|_{2k}$, λ a real number, immediately follow. The parallelogram identity is obvious. ∎

Remark 1. 1.

(i) *For Example 1 part I, we have*

$$\|x\|_{2k} = \left(\sum_{i=1}^{n}\left(x^i\right)^{2k}\right)^{\frac{1}{2k}}$$

if $x = \left(x^i\right)_{1 \le i \le n}$.

(ii) *CBS has the form*

$$|\left(x_1, \ldots, x_{2k}\right)| \le \prod_{i=1}^{2k}\|x_i\|_{2k}.$$

(iii) *If $(\cdot,\dots,\cdot)_{2k}$ is a simple $2k$-inner product with the inner product $(\cdot,\cdot)$ as generator then $\|\cdot\|_{2k}$ is exactly the norm $\|\cdot\|$ of $(\cdot,\cdot)$. Also, we have*
$$(x,\dots,x,y)_{2k}=\|x\|_{2k}^{2(k-1)}(x,y),$$
a relation important for orthogonality theory, see Remark 1 part (ii) of Section 3.

The previous result leads to:

Definition 2. *A real normed space is said to be a $2k$-normed space if its norm is defined by a $2k$-inner product.*

An important property of $2k-$normed spaces is provided by:

Theorem 1. *A $2k$-normed space is uniformly convex.*

Proof. Let $0<\varepsilon<2$ and $x,y\in X$ with $\|x\|_{2k}\le 1$, $\|y\|_{2k}\le 1$ and $\|x-y\|_{2k}\ge\varepsilon$. Applying the parallelogram identity and the CBS inequality, we have that
$$\begin{aligned}\|x+y\|_{2k}^{2k} &\le 2\sum_{i=0}^{k}\binom{2k}{2(k-i)}\|x\|_{2k}^{2i}\|y\|_{2k}^{2(k-i)}-\|x-y\|_{2k}^{2k}\\ &\le 2^{2k}-\varepsilon^{2k}=2^{2k}\left[1-\left(\frac{\varepsilon}{2}\right)^{2k}\right]\end{aligned}$$
and then
$$\left\|\frac{x+y}{2}\right\|\le 1-\left[1-\left(1-\left(\frac{\epsilon}{2}\right)^{2k}\right)^{\frac{1}{2k}}\right].$$
Putting
$$\delta(\varepsilon)=1-\left(1-\left(\frac{\varepsilon}{2}\right)^{2k}\right)^{\frac{1}{2k}}$$
we have $\delta(\varepsilon)>0$, which gives the desired result. ∎

Another remarkable result of this section is:

Theorem 2. *The norm of a $2k$-normed space is Gâteaux differentiable with:*
$$\tau(x,y):=(\|\cdot\|_{2k}')(x)(y)=\frac{(x,\dots,x,y)}{\|x\|_{2k}^{2k-1}},\quad x\neq 0.$$

Proof. Let $x,y\in X$, $x\neq 0$ and $t\neq 0$ a real number. Since
$$\frac{1}{t}\left(\|x+ty\|_{2k}^{2k}-\|x\|_{2k}^{2k}\right)=\frac{1}{t}\sum_{i=0}^{2k-1}\binom{2k}{i}\Big(\underbrace{x,\dots,x}_{i\ times},\underbrace{ty,\dots,ty}_{2k-i\ times}\Big),$$
we have
$$\lim_{t\to 0}\frac{1}{t}\left(\|x+y\|_{2k}^{2k}-\|x\|_{2k}^{2k}\right)=2k(x,\dots,x,y).$$
Also, from:
$$\frac{1}{t}\left(\|x+ty\|_{2k}-\|x\|_{2k}\right)=\frac{1}{t}\cdot\frac{\|x+ty\|_{2k}^{2k}-\|x\|_{2k}^{2k}}{\left(\|x+ty\|_{2k}^{k}+\|x\|_{2k}^{k}\right)\sum\limits_{i=1}^{k}\|x+ty\|_{2k}^{k-i}\|x\|_{2k}^{i-1}}$$

we get:

$$\lim_{t\to 0}\frac{1}{t}\left(\|x+ty\|_{2k}-\|x\|_{2k}\right)=\frac{2k\,(x,\dots,x,y)}{2\|x\|_{2k}^{k}k\|x\|_{2k}^{k-1}},$$

which is the required relation. ∎

Let us recall, following [9], the following notions:

Definition 3. 1.

(i) *On a normed linear space* $(X,\|\cdot\|)$ *the semi-inner-product* $(\cdot,\cdot)_T : X\times X\to\mathbb{R}$,

$$(x,y)_T := \lim_{t\downarrow 0}\frac{1}{2t}\left(\|y+tx\|^2-\|y\|^2\right)$$

is called semi-inner-product in the Tapia sense.

(ii) *A smooth normed space is called of (D)-type if there exists:*

$$(x,y)_T^{'} := \lim_{t\to 0}\frac{1}{t}\left[(x,y+tx)_T-(x,y)_T\right]$$

and a space of (D)-type is called of (BD)-type if there exists a real number k *so that* $(x,y)_T^{'}\le k^2\|y\|^2$. *The least number* k *is called the boundedness modulus.*

The following result is known.

Proposition 3. *(*[9, p. 1]*) A normed linear space is smooth if and only if* $(\cdot,\cdot)_T$ *is linear in the first variable.*

A straightforward computation for the $2k-$normed spaces gives:

Proposition 4. *A* $2k$*-normed space is smooth since*

$$(x,y)_T=\frac{(y,\dots,y,x)}{\|y\|_{2k}^{2(k-1)}}.$$

Also, a $2k$*-normed space is of (BD)-type with boundedness modulus* 1 *because* $(x,y)_T^{'}=\|y\|_{2k}^2$.

We finish this section with two identities in a $2k$-inner space. A simple calculation gives the equivalences:

$$a^2+c^2=2b^2 \Longleftrightarrow \frac{1}{b+c}+\frac{1}{a+b}=\frac{2}{a+c}$$

$$a^2+c^2=2b^2 \Longleftrightarrow \frac{a}{b+c}+\frac{c}{a+b}=\frac{2b}{a+c}.$$

Using the above parallelogram identity let

$$a = \|x+y\|_{2k}^k,\quad c=\|x-y\|_{2k}^k \text{ and}$$

$$b = \left(\sum_{i=0}^{k}\binom{2k}{2(k-i)}\Big(\underbrace{x,\dots,x}_{2i\ times},\underbrace{y,\dots,y}_{2(k-i)\ times}\Big)\right)^{\frac{1}{2}}$$

to obtain:

$$\frac{1}{\|x-y\|_{2k}^{k}+\left(\sum_{i=0}^{k}\binom{2k}{2(k-i)}\left(\underbrace{x,\dots,x}_{2i\ times},\underbrace{y,\dots,y}_{2(k-i)\ times}\right)\right)^{\frac{1}{2}}}$$
$$+\frac{1}{\|x+y\|_{2k}^{k}+\left(\sum_{i=0}^{k}\binom{2k}{2(k-i)}\left(\underbrace{x,\dots,x}_{2i\ times},\underbrace{y,\dots,y}_{2(k-i)\ times}\right)\right)^{\frac{1}{2}}}$$
$$=\frac{2}{\|x+y\|_{2k}^{k}+\|x-y\|_{2k}^{k}}$$

and

$$\frac{\|x+y\|_{2k}^{k}}{\|x-y\|_{2k}^{k}+\left(\sum_{i=0}^{k}\binom{2k}{2(k-i)}\left(\underbrace{x,\dots,x}_{2i\ times},\underbrace{y,\dots,y}_{2(k-i)\ times}\right)\right)^{\frac{1}{2}}}$$
$$+\frac{\|x-y\|_{2k}^{k}}{\|x+y\|_{2k}^{k}+\left(\sum_{i=0}^{k}\binom{2k}{2(k-i)}\left(\underbrace{x,\dots,x}_{2i\ times},\underbrace{y,\dots,y}_{2(k-i)\ times}\right)\right)^{\frac{1}{2}}}$$
$$=\frac{2\left(\sum_{i=0}^{k}\binom{2k}{2(k-i)}\left(\underbrace{x,\dots,x}_{2i\ times},\underbrace{y,\dots,y}_{2(k-i)\ times}\right)\right)^{\frac{1}{2}}}{\|x+y\|_{2k}^{k}+\|x-y\|_{2k}^{k}}.$$

3. $2k$-Orthogonality

We shall begin with:

Definition 4. *If $x,y\in(X,(\cdot,\dots,\cdot))$ then x is said to be $2k$-orthogonal to y if $(x,\dots,x,y)=0$ and we denote this fact by $x\perp_{2k}y$.*

Remark 2. 1.
- (i) *Obviously, $x\perp_{2k}x\Rightarrow x=0$.*
- (ii) *From Remark 1 part (iii), it follows that for a simple $2k$-inner product generated by $(\cdot,\cdot)$ we have $x\perp_{2k}y\Leftrightarrow x\perp_2 y$.*

Let us recall that on a normed space $(X,\|\cdot\|)$, x is called *Birkhoff orthogonal to* y if $\|x+\lambda y\|\geq\|x\|$ for all real λ and denote this fact by $x\perp_B y$. The following characterization of Birkhoff orthogonality is due by R. C. James:

Proposition 5. *([11, p. 92]) $x\perp_B y\Leftrightarrow\tau_-(x,y)\leq 0\leq\tau_+(x,y)$ where:*

$$\tau_-(x,y):=\lim_{t\downarrow 0}\frac{1}{t}\left(\|x+ty\|-\|x\|\right),\quad \tau_+(x,y):=\lim_{t\uparrow 0}\frac{1}{t}\left(\|x+ty\|-\|x\|\right).$$

The following lemma is useful:

Lemma 1. *If* $(X, (\cdot, \ldots, \cdot))$ *is a* $2k$*-inner product space then the* $2k$*-orthogonality is equivalent with Birkhoff orthogonality.*

Proof. If $x \perp_B y$ then applying Proposition 5 it results that

$$0 \leq \tau_- (x, y) \leq 0 \leq \tau_+ (x, y)$$

which implies

$$\tau (x, y) = \tau_- (x, y) = \tau_+ (x, y) = 0$$

and then $x \perp_{2k} y$. Conversely, if $x \perp_{2k} y$ and $x \neq 0$ then

$$\tau_- (x, y) = \tau_+ (x, y) = \frac{(x, \ldots, x, y)}{\|x\|_{2k}^{2k-1}} = 0$$

and applying Proposition 5 we have the conclusion. ∎

This result has an important consequence. Thus, applying Ex. 24 from [3, V. 66] it results that $x \perp_{2k} y$ is equivalent with $y \perp_{2k} x$ if and only if $\|\cdot\|_{2k}$ is generated by an usual inner product. For example, this is the case of simple $2k$-inner products, see Remark 1 part (iii) or Remark 1 part (ii).

Definition 5. *Given a subset* $Y \subset (X, (\cdot, \ldots, \cdot))$, *the set* $Y^{\perp_{2k}} = \{z \in X; z \perp_{2k} y$ *for all* $y \in Y\}$ *is called the* $2k$*-orthogonal complement of* Y.

Remark that $Y \cap Y^{\perp_{2k}} = \{0\}$ and if $\lambda \in \mathbb{R}$ and $z \in Y^{\perp_{2k}}$ then $\lambda z \in Y^{\perp_{2k}}$ showing that $Y^{\perp_{2k}}$ is a linear subspace. However, from Proposition 4 X is smooth and applying Ex. 26 from [3, V. 66] it results that $Y^{\perp_{2k}}$ is a linear subspace.

The following orthogonal decomposition theorem holds.

Proposition 6. *Let* Y *be a closed linear subspace in a complete* $2k$*-inner product space* $(X, (\cdot, \ldots, \cdot))$. *Then, for* $x \in X$ *there exists a unique* $y \in Y$ *and* $z \in Y^{\perp_{2k}}$ *such that* $x = y + z$.

Proof. Existence. From uniform convexity it follows that X is reflexive ([11, p. 368]), and thus there exists a projection of x on Y, i.e., an element $y \in Y$ such that

$$\|x - y\|_{2k} \leq \|x - y'\|_{2k}$$

for all $y' \in Y$. Denoting $z = x - y$ we have the required relation.

Now, we prove that $z \in Y^{\perp_{2k}}$. For $y' \in Y$ we have

$$\|z + \lambda y'\|_{2k} = \|x - (y - \lambda y')\|_{2k} \geq \|x - y\|_{2k} = \|z\|_{2k}$$

for all real λ and then $z \perp_B y'$. Applying Lemma 1 we obtain $z \in Y^{\perp_{2k}}$.

Unicity. The above y is in $P_Y (x)$, where $P_Y (x)$ denotes the set of best approximation elements in Y referring to x. Since X is uniformly convex it results that X is strictly convex and then $P_Y (x)$ contains a unique element ([11, p. 110]). ∎

In the following we obtain some results in the spirit of [10], which appear as a counterpart of the above results.

Let $a \in X \backslash \{0\}$ and denote by $X (a)$ the linear subspace generated by a. Let us consider the mapping

$$pr_a : X \to X, pr_a (x) := \frac{(a, \ldots, a, x)}{||a||_{2k}^{2k}} a.$$

It follows that:

Proposition 7. 1.

(i) *pr_a is independent of the choice of a in $X(a)$ i.e. for $\lambda \in \mathbb{R}$ we have $pr_{\lambda a} = pr_a$.*

(ii) *pr_a is a projection onto $X(a)$.*

(iii) *For arbitrary $x \in X$, a is $2k$-orthogonal to $x - pr_a x$ and*

$$\|pr_a(x)\|_{2k} \leq \|x\|_{2k}.$$

Proof. The proof is as follows.

(i) We observe that

$$pr_{\lambda a}(x) = \frac{(\lambda a, \ldots, \lambda a, x)}{\|\lambda a\|_{2k}^{2k}} \lambda a = \frac{\lambda^{2k}(a, \ldots, a, x)}{\lambda^{2k}\|a\|_{2k}^{2k}} a = pr_a(x).$$

(ii) We note that pr_a is onto because $pr_a(a) = a$. Obviously, pr_a is linear and:

$$pr_a(pr_a(x)) = \frac{(a, \ldots, a, pr_a(x))}{\|a\|_{2k}^{2k}} a = \frac{(a, \ldots, a)(a, \ldots, a, x)}{\|a\|_{2k}^{4k}} a = pr_a(x).$$

(iii) We remark that

$$\begin{aligned}(a, \ldots, a, x - pr_a(x)) &= (a, \ldots, a, x) - (a, \ldots, a, pr_a(x)) \\ &= (a, \ldots, a, x) - \frac{(a, \ldots, a)(a, \ldots, a, x)}{\|a\|_{2k}^{2k}} = 0\end{aligned}$$

and

$$\|pr_a(x)\|_{2k} = \frac{|(a, \ldots, a, x)|\|a\|_{2k}}{\|a\|_{2k}^{2k}} = \frac{|(a, \ldots, a, x)|}{\|a\|_{2k}^{2k-1}} \leq \frac{\|a\|_{2k}^{2k-1}\|x\|_{2k}}{\|a\|_{2k}^{2k-1}} = \|x\|_{2k},$$

and the proposition is proved.

∎

4. The Riesz Property

Let us denote by X^* the usual dual of X, that is, the space of linear continuous functionals $f : X \to \mathbb{R}$. Fix an element $y \in X$ and consider the functional $f : X \to \mathbb{R}$, $f(x) := (x, y, \ldots, y)$. It follows that $f \in X^*$ with

$$|f(x)| \leq \|x\|_{2k}\|y\|_{2k}^{2k-1} \text{ for all } x \in X,$$

hence

$$\|f\| \leq \|y\|_{2k}^{2k-1}.$$

Also,

$$\|f\|\|y\|_{2k} \geq f(y) = \|y\|_{2k}^{2k},$$

so that

$$\|f\| = \|y\|_{2k}^{2k-1}.$$

Conversely, we shall show that any $f \in X^*$ has the above form if X is complete, obtaining the following generalization of the Riesz representation theorem:

Theorem 3. *If $(X, (\cdot, \ldots, \cdot))$ is a complete $2k$-inner product space and $f \in X^*$ then there exists an element $y \in X$ such that $f(x) = (x, y, \ldots, y)$ for all $x \in X$ and $\|f\| = \|y\|_{2k}^{2k-1}$.*

Proof. If $f = 0$ then $y = 0$. If $f \neq 0$ let $x_0 \in X$ with $f(x_0) \neq 0$. Applying the Proposition 6 for x_0 and $Y = Ker(f)$ which is a closed linear subspace of X, there is a unique $y_0 \in Ker(f)$ and a unique $z_0 \in \; Ker(f)^{\perp_{2k}}$ such that $x_0 = y_0 + z_0$. It results that $z_0 \notin Ker(f)$.

Let $\lambda \in \mathbb{R}$ with

$$\lambda^{2k-1} = \frac{f(x_0)}{\|z_0\|_{2k}^{2k}}$$

and $y = \lambda z_0$. Because $f(x) z_0 - f(z_0) x \in Ker(f)$ for all $x \in X$ we have

$$z_0 \perp_{2k} (f(x) z_0 - f(z_0) x),$$

that is,

$$(f(x) z_0 - f(z_0) x, z_0, \ldots, z_0) = 0$$

which implies

$$\begin{aligned} f(x) &= \frac{f(z_0)}{\|z_0\|_{2k}^{2k}} (x, z_0, \ldots, z_0) = \lambda^{2k-1} (x, z_0, \ldots, z_0) \\ &= (x, \lambda z_0, \ldots, \lambda z_0) = (x, y, \ldots, y) \end{aligned}$$

for all $x \in X$. ∎

Finally, we shall prove the theorem of unicity for the representation element.

Theorem 4. *Let $(X, (\cdot, \ldots, \cdot))$ be a complete $2k$-inner product space and $f \in X^* \setminus \{0\}$. Then there exists an unique $u \in X$ with $\|u\|_{2k} = 1$ such that $f(x) = \|f\| (x, u, \ldots, u)$ for all $x \in X$.*

Proof. Existence. As above, there exists a $z_0 \in \; Ker(f)^{\perp_{2k}} \setminus \{0\}$ such that

$$f(x) = \frac{f(z_0)}{\|z_0\|_{2k}} \left(x, \frac{z_0}{\|z_0\|_{2k}}, \ldots, \frac{z_0}{\|z_0\|_{2k}} \right)$$

for all $x \in X$ and

$$\|f\| = \frac{f(z_0)}{\|z_0\|_{2k}}.$$

With

$$\lambda = \left(\frac{f(z_0)}{|f(z_0)|} \right)^{1/2k-1}$$

we get

$$\begin{aligned} f(x) &= \|f\| \frac{f(z_0)}{|f(z_0)|} \left(x, \frac{z_0}{\|z_0\|_{2k}}, \ldots, \frac{z_0}{\|z_0\|_{2k}} \right) \\ &= \|f\| \lambda^{2k-1} \left(x, \frac{z_0}{\|z_0\|_{2k}}, \ldots, \frac{z_0}{\|z_0\|_{2k}} \right) = \|f\| (x, u, \ldots, u), \end{aligned}$$

where $u = \frac{\lambda z_0}{\|z_0\|_{2k}}$. Obviously $\|u\|_{2k} = 1$.

Unicity. We have $f(u) = \|f\|$. Since $(X, (,))$ is strictly convex and u satisfy the last relations, by the Krein theorem ([11, p. 110]), it follows that u is unique. ∎

5. $2k$-Riemannian manifolds

Let M be a smooth, n-dimensional manifold, $C^{\infty}(M)$ the ring of smooth real functions on M and $\mathcal{X}(M)$ the Lie algebra of vector fields on M.

Definition 6. *We say that M is endowed with a $2k$-Riemannian metric if every tangent space T_xM is endowed with a $2k$-inner product $g_x : (T_xM)^{2k} \to \mathbb{R}$. The pair $\left(M, g = (g_x)_{x\in M}\right)$ is said to be a $2k$-Riemannian manifold.*

Obviously, for $k = 1$ we obtain the usual notion of Riemannian metric. For $k = 2$ we prefer to say 4-Riemannian manifold because there already exists the notion of the *quaternionic manifold* ([1]).

The above definition get a symmetric $(0, 2k)$-tensor field $g : (\mathcal{X}(M))^{2k} \to C^{\infty}(M)$:

$$g(X_1, \dots, X_{2k})(x) = g_x(X_1(x), \dots, X_{2k}(x)),$$
$$X_1, ..., X_{2k} \in \mathcal{X}(M), x \in M.$$

In a local chart $\left(x^i\right)_{1\leq i\leq n}$ this tensor field has the components

$$g_{i_1\dots i_{2k}} := g\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_{2k}}}\right).$$

Unfortunately, the main tool in the geometry of usual Riemannian metrics, namely *the Levi-Civita connection*, does not seem to admit a "mot-à-mot" extension for the general case $k \geq 2$.

Definition 7. *A symmetric linear connection $\nabla = \left(\Gamma^i_{jm} = \Gamma^i_{mj}\right)$ we called the Levi-Civita connection of g if the covariant derivative of g with respect to ∇ vanish:*

$$g_{i_1\dots i_{2k}|a} = 0,\ 1 \leq i_1, \dots, i_{2k},\ a \leq n,$$

where:

$$g_{i_1\dots i_{2k}|a} := \frac{\partial g_{i_1\dots i_{2k}}}{\partial x^a} - \Gamma^j_{ai_1} g_{ji_2\dots i_{2k}} - \dots - \Gamma^j_{ai_{2k}} g_{i_1\dots i_{2k-1}j}.$$

Denoting

$$g_{i_1\dots i_{2k}\ ,a} = \frac{\partial g_{i_1\dots i_{2k}}}{\partial x^a},$$

let us try the usual Christoffel process:

$$\begin{aligned}
g_{i_1\dots i_{2k}\ ,i_{2k+1}} &= \Gamma^j_{i_{2k+1}i_1} g_{ji_2\dots i_{2k}} + \dots + \Gamma^j_{i_{2k+1}i_{2k}} g_{i_1\dots i_{2k-1}j} \\
g_{i_2\dots i_{2k+1},i_1} &= \Gamma^j_{i_1i_2} g_{ji_3\dots i_{2k+1}} + \dots + \Gamma^j_{i_1i_{2k+1}} g_{i_2\dots i_{2k}j} \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
g_{i_{2k+1}\dots i_{2k-1}\ ,i_{2k}} &= \Gamma^j_{i_{2k}i_{2k+1}} g_{ji_1\dots i_{2k-1}} + \dots + \Gamma^j_{i_{2k}i_{2k-1}} g_{i_{2k+1}\dots i_{2k-2}j}.
\end{aligned}$$

At this stage, we do not know of any method to find Γ.

Let us treat in detail the case $k = 2$:

$$\begin{aligned}
g_{i_1i_2i_3i_4,i_5} &= \Gamma^j_{i_5i_1} g_{ji_2i_3i_4} + \Gamma^j_{i_5i_2} g_{i_1ji_3i_4} + \Gamma^j_{i_5i_3} g_{i_1i_2ji_4} + \Gamma^j_{i_5i_4} g_{i_1i_2i_3j} \\
g_{i_2i_3i_4i_5,i_1} &= \Gamma^j_{i_1i_2} g_{ji_3i_4i_5} + \Gamma^j_{i_1i_3} g_{i_2ji_4i_5} + \Gamma^j_{i_1i_4} g_{i_2i_3ji_5} + \Gamma^j_{i_1i_5} g_{i_2i_3i_4j} \\
g_{i_3i_4i_5i_1,i_2} &= \Gamma^j_{i_2i_3} g_{ji_4i_5i_1} + \Gamma^j_{i_2i_4} g_{i_3ji_5i_1} + \Gamma^j_{i_2i_5} g_{i_3i_4ji_1} + \Gamma^j_{i_2i_1} g_{i_3i_4i_5j} \\
g_{i_4i_5i_1i_2,i_3} &= \Gamma^j_{i_3i_4} g_{ji_5i_1i_2} + \Gamma^j_{i_3i_5} g_{i_4ji_1i_2} + \Gamma^j_{i_3i_1} g_{i_4i_5ji_2} + \Gamma^j_{i_3i_2} g_{i_4i_5i_1j} \\
g_{i_5i_1i_2i_3,i_4} &= \Gamma^j_{i_4i_5} g_{ji_1i_2i_3} + \Gamma^j_{i_4i_1} g_{i_5ji_2i_3} + \Gamma^j_{i_4i_2} g_{i_5i_1ji_3} + \Gamma^j_{i_4i_3} g_{i_5i_1i_2j}.
\end{aligned}$$

Subtracting the last two relations from the sum of first three we obtain:

$$\begin{aligned} & \Gamma^j_{i_1i_2} g_{ji_3i_4i_5} + \Gamma^j_{i_5i_1} g_{ji_2i_3i_4} + \Gamma^j_{i_5i_2} g_{ji_1i_3i_4} - \Gamma^j_{i_3i_4} g_{ji_1i_2i_5} \\ = & \frac{1}{2}\left(g_{i_1i_2i_3i_4,i_5} + g_{i_2i_3i_4i_5,i_1} + g_{i_3i_4i_5i_1,i_2} - g_{i_4i_5i_1i_2,i_3} - g_{i_5i_1i_2i_3,i_4}\right). \end{aligned}$$

Let us recall that another method to find the Christoffel coefficients is to compute *the Euler-Lagrange equations*:

$$E_a(L) := \frac{\partial L}{\partial x^a} - \frac{d}{dt}\left(\frac{\partial L}{\partial y^a}\right)$$

for *the kinetic energy* $L : TM = (x^a, y^a) \to \mathbb{R}$:

$$L(x,y) := \frac{1}{2}\|y\|^2 = \frac{1}{2} g_{ij}(x)\, y^i y^j$$

because:

$$g^{ia} E_a(L) = \ddot{x}^i + \Gamma^i_{jm}\dot{x}^j \dot{x}^m,$$

where $\left(g^{ab}\right)$ is the inverse of (g_{ab}).

In the general case, for:

$$L = \frac{1}{2}\|y\|^2_{2k} = \frac{1}{2}(y, \dots, y)^{\frac{1}{k}} = \frac{1}{2}\left(g_{i_1\dots i_{2k}}(x)\, y^{i_1}\dots y^{i_{2k}}\right)^{\frac{1}{k}}$$

we have:

$$\frac{\partial L}{\partial x^a} = \frac{1}{2k}\left(g_{\dots} y^{\cdots}\right)^{\frac{1}{k}-1} g_{i_1\dots i_{2k},a} y^{i_1}\dots y^{i_{2k}}$$

$$\frac{\partial L}{\partial y^a} = \left(g_{\dots} y^{\cdots}\right)^{\frac{1}{k}-1} g_{ai_2\dots i_{2k}} y^{i_2}\dots y^{i_{2k}}$$

and then:

$$E_a(L) = \frac{1}{2k}\left(g_{\dots} y^{\cdots}\right)^{\frac{1}{k}-1} g_{i_1\dots i_{2k},a} y^{i_1}\dots y^{i_{2k}} - \frac{d}{dt}\left[\left(g_{\dots} y^{\cdots}\right)^{\frac{1}{k}-1} g_{ai_2\dots i_{2k}} y^{i_2}\dots y^{i_{2k}}\right].$$

Therefore, in the expression of $E_a(L)$ the second derivative does not appear separately.

However, if the $2k$-Riemannian metric g is generated, via Proposition 1, by a classical Riemannian metric g^{cl}, then the Levi-Civita connection of g^{cl} is Levi-Civita for g. For example, if $k = 4$:

$$\begin{aligned} & g_{i_1i_2i_3i_4|a} \\ = & \frac{1}{3}\left(g^{cl}_{i_1i_2|a} g^{cl}_{i_3i_4} + g^{cl}_{i_1i_2} g^{cl}_{i_3i_4|a} + g^{cl}_{i_1i_3|a} g^{cl}_{i_2i_4} + g^{cl}_{i_1i_3} g^{cl}_{i_2i_4|a} + g^{cl}_{i_1i_4|a} g^{cl}_{i_2i_3} + g^{cl}_{i_1i_4} g^{cl}_{i_2i_3|a}\right) \end{aligned}$$

and then, if $g^{cl}_{ij|a} = 0$ it follows that $g_{i_1i_2i_3i_4|a} = 0$.

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