ON SOME GRÜSS TYPE INEQUALITY IN 2-INNER PRODUCT SPACES AND APPLICATIONS

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Abstract. In this paper, we shall give a generalization of the Grüss type inequality and obtain some applications of the Grüss type inequality in terms of 2-inner product spaces.

1. Introduction

Let X be a linear space of dimension greater than 1 and $(\cdot,\cdot|\cdot)$ be a real-valued function on $X \times X \times X$ satisfying the following conditions:

($2I_1$) $(x, x|z) \geq 0$,
$(x, x|z) = 0$ if and only if x and z are linearly dependent,
($2I_2$) $(x, x|z) = (z, z|x)$,
($2I_3$) $(x, y|z) = (y, x|z)$,
($2I_4$) $(\alpha x, y|z) = \alpha(x, y|z)$ for any real number α,
($2I_5$) $(x + x', y|z) = (x, y|z) + (x', y|z)$.

$(.,.|.)$ is called a *2-inner product* and $(X, (\cdot,\cdot|\cdot))$ is called a *2-inner product space* (or a *2-pre-Hilbert space*) ([3]).

Some basic properties of the 2-inner product $(\cdot,\cdot|\cdot)$ are as follows ([3], [4]):

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(1) For all $x, y, z \in X$,

$$|(x, y|z)| \leq \sqrt{(x, x|z)}\sqrt{(y, y|z)}.$$

(2) For all $x, y \in X, (x, y|y) = 0$.

(3) If $(X, (\cdot, \cdot))$ is an inner product space, then the 2-inner product $(\cdot, \cdot|\cdot)$ is defined on X by

$$(x, y|z) = \begin{vmatrix} (x|y) & (x|z) \\ (y|z) & (z|z) \end{vmatrix} = (x|y)\|z\|^2 - (x|z)(y|z)$$

for all $x, y, z \in X$.

Under the same assumptions over X, the real-valued function $\|\cdot, \cdot\|$ on $X \times X$ satisfying the following conditions:

($2N_1$) $\|x, y\| = 0$ if and only if x and y are linearly dependent,

($2N_2$) $\|x, y\| = \|y, x\|$,

($2N_3$) $\|\alpha x, y\| = |\alpha|\|x, y\|$ for all real number α,

($2N_4$) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

$\|\cdot, \cdot\|$ is called a *2-norm* on X and $(X, \|\cdot, \cdot\|)$ is called a *linear 2-normed space* ([7]).

Note that it is easy to show that the 2-norm $\|\cdot, \cdot\|$ is non-negative and, for all $x, y \in X$ and real numbers α, $\|x, y + \alpha x\| = \|x, y\|$.

For any non-zero $x_1, x_2, ..., x_n$ in X, let $V(x_1, x_2, ..., x_n)$ denote the subspace of X generated by $x_1, x_2, ..., x_n$. Whenever the notation $V(x_1, x_2, ..., x_n)$ is used, by it will understood $x_1, x_2, ..., x_n$ to be linearly independent.

Note that, on any 2-inner product space $(X, (\cdot, \cdot|\cdot))$, $\|x, y\| = \sqrt{(x, x|y)}$ defines a 2-norm for which we have

$$(x, y|z) = \frac{1}{4}(\|x + y, z\|^2 - \|x - y, z\|^2), \tag{1.1}$$

$$\|x + y, z\|^2 + \|x - y, z\|^2 = 2(\|x, z\|^2 + \|y, z\|^2) \tag{1.2}$$

for all $x, y, z \in X$. On the other hand, if $(X, \|\cdot, \cdot\|)$ is a linear 2-normed space in which the condition (1.2) is satisfied for all $x, y, z \in X$, then we can define a 2-inner product $(\cdot, \cdot|\cdot)$ on X by the condition (1.1).

For a 2-inner product space $(X, (\cdot, \cdot|\cdot))$, Cauchy-Schwarz's inequality

$$|(x, y|z)| \le (x, x|z)^{1/2}(y, y|z)^{1/2} = \|x, z\|\|y, z\|, \tag{1.3}$$

a 2-dimensional analogue of Cauchy-Schwarz's inequality, holds.

For further details on 2-inner product spaces and linear 2-normed spaces, refer to the papers ([2]-[5], [9], [10]).

Y. J. Cho et al. ([1]), S. S. Dragomir et al. ([6]) studied the inequalities of 2-inner product spaces and obtained some related results.

In this paper, we shall give a generalization of the Grüss type inequality and obtain some applications of the Grüss type inequality in terms of 2-inner product spaces.

2. The Main Results

In 1935, G. Grüss proved the integral inequality

$$\left|\frac{1}{b-a}\int_b^a f(x)g(x)dx - \frac{1}{b-a}\int_b^a f(x)dx \cdot \frac{1}{b-a}\int_b^a g(x)dx\right| \\ \le \frac{1}{4}(M-m)(N-n)$$

if f and g are two integrable functions on [a,b] satisfying the condition:

$$m \le f(x) \le M, \quad n \le g(x) \le N$$

for all $x \in [a, b]$([8]).

In this section, we shall give a generalization of the Grüss type inequality in terms of 2-inner product spaces.

Theorem 2.1. *Let $(X, (\cdot, \cdot|\cdot))$ be a 2-inner product space and $x, y, z, e \in X$ with $\|e, z\| = 1$ and $z \notin V(x, e, y)$. If m, n, M, N are real numbers such that*

$$(Me - x, x - me|z) \ge 0, \quad (Ne - y, y - ne|z) \ge 0, \tag{2.1}$$

then we have the inequality

$$|(x,y|z)-(x,e|z)(e,y|z)| \le \frac{1}{4}|M-m||N-n|. \tag{2.2}$$

Proof. Note that

$$(x,y|z)-(x,e|z)(e,y|z) = (x-(x,e|z)e, y-(e,y|z)e|z).$$

By the Cauchy-Schwarz's inequality (1.3),

$$\begin{aligned} &|(x-(x,e|z)e, y-(e,y|z)e|z)|^2 \\ &\le \|x-(x,e|z)e,z\|^2\|y-(e,y|z)e,z\|^2 \\ &= (\|x,z\|^2-|(x,e|z)|^2)(\|y,z\|^2-|(e,y|z)|^2). \end{aligned} \tag{2.3}$$

On the other hand, we have

$$(M-(x,e|z))((x,e|z)-m)-(Me-x,x-me|z) = \|x,z\|^2-|(x,e|z)|^2 \tag{2.4}$$

and

$$(N-(e,y|z))((e,y|z)-n)-(Ne-y,y-ne|z) = \|y,z\|^2-|(e,y|z)|^2. \tag{2.5}$$

Since $(Me-x,x-me|z) \ge 0, (Ne-y,y-ne|z) \ge 0$, we have

$$(M-(x,e|z))((x,e|z)-m) \ge \|x,z\|^2-|(x,e|z)|^2 \tag{2.6}$$

and

$$(N-(e,y|z))((e,y|z)-n) \ge \|y,z\|^2-|(e,y|z)|^2. \tag{2.7}$$

Also, by the inequality $4ab \le (a+b)^2$ for $a,b \in R$, we have

$$(M-(x,e|z))((x,e|z)-m) \le \frac{1}{4}(M-m)^2 \tag{2.8}$$

and, similarly,

$$(N - (e, y|z))((e, y|z) - n) \le \frac{1}{4}(N - n)^2. \tag{2.9}$$

Thus, using (2.3)∼(2.9), we have the inequality

$$|(x, y|z) - (x, e|z)(e, y|z)|^2 \le \frac{1}{16}|M - m|^2|N - n|^2$$

and so we have the desired inequality (2.2). This completes the proof.

The mapping $(\cdot, \cdot|\cdot)_{\overline{p}} : R^n \to R$ given by

$$(\overline{x}, \overline{y}|\overline{z})_{\overline{p}} = \frac{1}{2}\sum_{i,j=1}^{n} p_i p_j (x_i z_j - x_j z_i)(y_i z_j - y_j z_i),$$

where $\overline{x} = (x_1, x_2, \dots, x_n)$, $\overline{y} = (y_1, y_2, \dots, y_n)$, $\overline{z} = (z_1, z_2, \dots, z_n) \in R^n$ and $\overline{p} = (p_1, p_2, \dots, p_n) > \overline{0}$, that is, $p_i > 0$ for all $i = 1, 2, \dots, n$, is obviously a 2-inner product on R^n generating the 2-norm on R^n

$$\|\overline{x}, \overline{y}\|_{\overline{p}} = \left[\frac{1}{2}\sum_{i,j=1}^{n} p_i p_j (x_i z_j - x_j z_i)^2\right]^{1/2}.$$

Propsition 2.2. *Let $(R^n, (\cdot, \cdot|\cdot)_{\overline{p}})$ be a 2-inner product space and $\overline{x}, \overline{y}, \overline{z}, \overline{e} \in R^n$ such that $\|\overline{e}, \overline{z}\| = 1$ and $\overline{z} \notin V(\overline{x}, \overline{y}, \overline{e})$. If m, n, M, N are real numbers such that*

$$(M\overline{e} - \overline{x}, \overline{x} - m\overline{e}|\overline{z})_{\overline{p}} \ge 0, \quad (N\overline{e} - \overline{y}, \overline{y} - n\overline{e}|\overline{z})_{\overline{p}} \ge 0,$$

then we have the inequality

$$\begin{aligned} &\Bigg|\frac{1}{2}\sum_{i,j=1}^{n} p_i p_j (x_i z_j - x_j z_i)(y_i z_j - y_j z_i) \\ &\quad - \left(\frac{1}{2}\sum_{i,j=1}^{n} p_i p_j (x_i z_j - x_j z_i)(e_i z_j - e_j z_i)\right) \\ &\quad\quad \times \left(\frac{1}{2}\sum_{i,j=1}^{n} p_i p_j (e_i z_j - e_j z_i)(y_i z_j - y_j z_i)\right)\Bigg| \\ &\le \frac{1}{4}|M - m||N - n|. \end{aligned}$$

Next, let $(\cdot,\cdot|\cdot)$ be a 2-inner product and $\{(\cdot,\cdot|\cdot)_i\}_{i\in N}$ be a sequence of 2-inner products satisfying the following condition:

$$\|x,z\|^2 > \sum_{i=1}^{\infty} \|x,z\|_i^2$$

for all x, z being linearly independent. Let $p \in N$. Define a mapping

$$(x,y|z)_p = (x,y|z) - \sum_{i=1}^{p}(x,y|z)_i,$$

for $x, y, z \in X$ and $z \notin V(x,y)$. Then the mapping $(\cdot,\cdot|\cdot)_p$ satisfies the properties:

(1) $(x,x|z)_p \geq 0$,
(2) $(\alpha x + \beta x', y|z)_p = \alpha(x,y|z)_p + \beta(x' + y|z)_p$,
(3) $(x,y|z)_p + (y,x|z)_p$,
(4) $(x,x|z)_p = (z,z|x)_p$

for every $x, x', y, z \in X$ and $\alpha, \beta \in R$.

By Theorem 2.1, we have the following:

Proposition 2.3. *If there exist real numbers m, n, M, N are real numbers such that*

$$(Me - x, x - me|z)_p \geq 0, \quad (Ne - y, y - ne|z)_p \geq 0,$$

then we have

$$|(x,y|z)_p - (x,e|z)_p(e,y|z)_p| \leq \frac{1}{4}|M - m||N - n|.$$

3. Applications for Isotonic functionals

Let E be a nonempty set, $F(E,R)$ be the real algebra of all real-valued functions defined on E and L be a subalgebra of $F(E,R)$. A functional A

is said to be *isotonic* if $f \geq g$, that is, $f(t) \geq g(t)$ for every $t \in E$, implies $A(f) \geq A(g)$ for all $f, g \in L$. A functional A is said to be *normalized* on L if $\mathbf{1} \in L$, that is, $\mathbf{1}(t) = 1$ for all $t \in E$ implies $A(\mathbf{1}) = 1$.

For some inequalities involving linear isotonic functionals is given in [8].

Suppose that $fgh^2, fh^2, gh^2 \in L$ for all $f, g \in L$. For a isotonic linear functional $A : L \to R$, we define a functional $(\cdot, \cdot|\cdot)_A : L \times L \times L \to R$ by

$$(f, g|h)_A = A(fgh^2)$$

for every $f, g, h \in L$. Then we have the following properties:

(1) $(f, f|h)_A = A(f^2h^2) \geq 0$,
(2) $(\alpha f + \beta f', g|h)_A = \alpha(f, g|h)_A + \beta(f', g|h)_A$,
(3) $(f, g|h)_A = (g, f|h)_A$,
(4) $(f, f|h)_A = (h, h|f)_A$.

for every $f, f', g, h \in L$ and $\alpha, \beta \in R$.

Theorem 3.1. *Let L be as above, $fgh^2, fh^2, gh^2, f, g, e, h \in L$ with $\|e, h\| = 1$ and $h \notin V(f, g, e)$. If m, n, M, N are real numbers such that*

$$m \leq f \leq M, \qquad n \leq g \leq N \tag{3.1}$$

and $A : L \to R$ is an isotonic linear functional, then we have the following inequality

$$|A(fgh^2) - A(fh^2)A(gh^2)| \leq \frac{1}{4}(M - m)(N - n).$$

Proof. Choose $e=\mathbf{1}$. Then since $\|e, h\| = 1$, $(e, e|h)_A = 1$, $A(e^2h^2) = A(h^2) = 1$ and we have

$$(Me - f, f - me|h)_A = A((M - f)(f - m)h^2) \geq 0$$

and

$$(Ne - g, g - ne|h)_A = A((N - g)(g - n)h^2) \geq 0.$$

Applying Theorem 2.1 for $(\cdot,\cdot|\cdot)_A$, we have

$$|(f,g|h)_A - (f,e|h)(e,g|h)_A| \le \frac{1}{4}(M-m)(N-n).$$

This completes the proof.

Corollary 3.2. *Let $fg, f, g, e, h \in L$. Suppose $\mathbf{1} \in L$ and $A : L \to R$ is a normalized isotonic linear functional. If m, n, M, N satisfy (3.1), then we have the following inequality*

$$|A(fg) - A(f)A(g)| \le \frac{1}{4}(M-m)(N-n).$$

Let $L^2_{[a,b]}$ be a real Hilbert space of square integrable mapping on $[a,b]$, that is, $\int\limits_a^b |f^2|dm < \infty$ if $f \in L^2_{[a,b]}$. Define a mapping $(\cdot,\cdot|\cdot) : L^2_{[a,b]} \times L^2_{[a,b]} \times L^2_{[a,b]} \to R$ by

$$\begin{aligned}&(f,g|l)\\ &= \frac{1}{2}\int_a^b\int_a^b (f(x)l(y) - f(y)l(x))(g(x)l(y) - g(y)l(x))dm(x)dm(y).\end{aligned}$$

Then $(\cdot,\cdot|\cdot)$ is a 2-inner product on $L^2_{[a,b]}$ generating the 2-norm

$$\|f,l\| = \left(\frac{1}{2}\int_a^b\int_a^b (f(x)l(y) - f(y)l(x))^2 dm(x)dm(y)\right)^{1/2}.$$

Proposition 3.3. *Let $(L^2_{[a,b]}, (\cdot,\cdot|\cdot)$ be a 2-inner product space and $f, g, e, h \in L^2_{[a,b]}$ with $\|e,h\| = 1$ and $h \notin V(f,g,e)$. There exist real numbers m, n, M, N such that*

$$m \le f \le M, \quad n \le g \le N.$$

Then we have the inequality

$$\begin{aligned}
&\Big|\int_a^b\int_a^b (f(x)h(y)-f(y)h(x))(g(x)h(y)-g(y)h(x))dm(x)dm(y)\\
&\qquad-\left[\int_a^b\int_a^b (f(x)h(y)-f(y)h(x))(h(y)-h(x))dm(x)dm(y)\right]\\
&\qquad\quad\times\left[\int_a^b\int_a^b (h(y)-h(x))(g(x)h(y)-g(y)h(x))dm(x)dm(y)\right]\Big|\\
&\le\frac{1}{4}|M-m||N-n|.
\end{aligned}$$

Proof. By Theorem 3.1, applied to

$$A(f,g|h)=A(fgh^2)=\int_a^b\int_a^b \det(f,h)\det(g,h)dm(x)dm(y),$$

where

$$\det(f,h)=\begin{vmatrix} f(x) & f(y)\\ h(x) & h(y)\end{vmatrix},$$

the result follows.

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