INEQUALITIES FOR A WEIGHTED MULTIPLE INTEGRAL

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Abstract. In the article, using Taylor's formula for functions of several variables, the author establishes some inequalities for the weighted multiple integral of a function defined on an m-dimensional rectangle, if its partial derivatives of $(n+1)$-th order remain between bounds. From which Iyengar's inequality is generalized and related results in references could be deduced.

1. Main Results

For given points $a=(a_1,\cdots,a_m)$, $b=(b_1,\cdots,b_m)\in\mathbb{R}^m$ and $a_i<b_i$, $i=1,2,\cdots,m$, denote the m-rectangles by

$$Q_m=\prod_{i=1}^m[a_i,b_i],\quad Q_m(t)=\prod_{i=1}^m\big[a_i,c_i(t)\big], \tag{1}$$

where $c_i(t)=(1-t)a_i+tb_i$, $i=1,2,\cdots,m$, $t\in[0,1]$.

Let $\boldsymbol{\nu}=(\nu_1,\cdots,\nu_m)$ be a multi-index, that is, $\nu_i=\text{integer}\geqslant 0$, with $|\boldsymbol{\nu}|=\sum\limits_{i=1}^m\nu_i$. Let f be a function of several variables defined on Q_m, and its partial derivatives of $(n+1)$-th order remain between the upper and lower bounds $M_{n+1}(\boldsymbol{\nu})$ and $N_{n+1}(\boldsymbol{\nu})$ as follows

$$N_{n+1}(\boldsymbol{\nu})\leqslant D^{\boldsymbol{\nu}}f(x)\leqslant M_{n+1}(\boldsymbol{\nu}),\quad x\in Q_m, \tag{2}$$

where we define

$$D^{\boldsymbol{\nu}}f(x)=\partial^{n+1}f(x)\bigg/\prod_{i=1}^m\partial x_i^{\nu_i}. \tag{3}$$

Let $w(x)\geqslant 0$ be an integrable function of several variables defined on the m-rectangle Q_m, which is not identically zero for $x\in Q_m$. Define

$$h_{s,\boldsymbol{\nu}}(t)=\int\limits_{Q_m(t)}w(x)\prod_{i=1}^m(x_i-s_i)^{\nu_i}\,dx, \tag{4}$$

where $s=(s_1,s_2,\cdots,s_m)\in\mathbb{R}^m$, $t\in[0,1]$.

In this article, using Taylor's formula for functions with several variables, we obtain some inequalities for a weighted multiple integral $\int_{Q_m}w(x)f(x)\,dx$ with weight $w(x)\geqslant 0$ on the m-rectangle Q_m in terms of the values of the partial

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derivatives of the function f at points a and b and the bounds $M_{n+1}(\boldsymbol{\nu})$ and $N_{n+1}(\boldsymbol{\nu})$ of $D^{\boldsymbol{\nu}} f(x)$, that is

Main Theorem. *Let $f \in C^{n+1}(Q_m)$ and $N_{n+1}(\boldsymbol{\nu}) \leqslant D^{\boldsymbol{\nu}} f(x) \leqslant M_{n+1}(\boldsymbol{\nu})$ hold for any $x \in Q_m$ and $|\boldsymbol{\nu}| = n+1$, where $M_{n+1}(\boldsymbol{\nu})$ and $N_{n+1}(\boldsymbol{\nu})$ are constants depending on n and $\boldsymbol{\nu}$. Let $w(x)$ be an integrable function of several variables over Q_m, which is not identically zero. Then, for any $t \in (0,1)$,*

(i) *if n is an even, we have*

(5)
$$\begin{aligned}
&\sum_{|\boldsymbol{\nu}|=n+1} \frac{M_{n+1}(\boldsymbol{\nu}) h_{b,\boldsymbol{\nu}}(1) - N_{n+1}(\boldsymbol{\nu}) h_{b,\boldsymbol{\nu}}(t)}{\prod_{i=1}^{m} (\nu_i!)} + \sum_{|\boldsymbol{\nu}|=n+1} \frac{N_{n+1}(\boldsymbol{\nu})}{\prod_{i=1}^{m} (\nu_i!)} h_{a,\boldsymbol{\nu}}(t) \\
&\leqslant \int_{Q_m} w(x) f(x)\,dx - \sum_{k=0}^{n} \sum_{|\boldsymbol{\nu}|=k} \frac{D^{\boldsymbol{\nu}} f(b)}{\prod_{i=1}^{m} (\nu_i!)} \left[h_{b,\boldsymbol{\nu}}(1) - h_{b,\boldsymbol{\nu}}(t)\right] - \sum_{k=0}^{n} \sum_{|\boldsymbol{\nu}|=k} \frac{D^{\boldsymbol{\nu}} f(a)}{\prod_{i=1}^{m} (\nu_i!)} h_{a,\boldsymbol{\nu}}(t) \\
&\leqslant \sum_{|\boldsymbol{\nu}|=n+1} \frac{N_{n+1}(\boldsymbol{\nu}) h_{b,\boldsymbol{\nu}}(1) - M_{n+1}(\boldsymbol{\nu}) h_{b,\boldsymbol{\nu}}(t)}{\prod_{i=1}^{m} (\nu_i!)} + \sum_{|\boldsymbol{\nu}|=n+1} \frac{M_{n+1}(\boldsymbol{\nu})}{\prod_{i=1}^{m} (\nu_i!)} h_{a,\boldsymbol{\nu}}(t);
\end{aligned}$$

(ii) *if n is an odd,*

(6)
$$\begin{aligned}
&\sum_{|\boldsymbol{\nu}|=n+1} \frac{N_{n+1}(\boldsymbol{\nu}) h_{b,\boldsymbol{\nu}}(1) - M_{n+1}(\boldsymbol{\nu}) h_{b,\boldsymbol{\nu}}(t)}{\prod_{i=1}^{m} (\nu_i!)} + \sum_{|\boldsymbol{\nu}|=n+1} \frac{N_{n+1}(\boldsymbol{\nu})}{\prod_{i=1}^{m} (\nu_i!)} h_{a,\boldsymbol{\nu}}(t) \\
&\leqslant \int_{Q_m} w(x) f(x)\,dx - \sum_{k=0}^{n} \sum_{|\boldsymbol{\nu}|=k} \frac{D^{\boldsymbol{\nu}} f(b)}{\prod_{i=1}^{m} (\nu_i!)} \left[h_{b,\boldsymbol{\nu}}(1) - h_{b,\boldsymbol{\nu}}(t)\right] - \sum_{k=0}^{n} \sum_{|\boldsymbol{\nu}|=k} \frac{D^{\boldsymbol{\nu}} f(a)}{\prod_{i=1}^{m} (\nu_i!)} h_{a,\boldsymbol{\nu}}(t) \\
&\leqslant \sum_{|\boldsymbol{\nu}|=n+1} \frac{M_{n+1}(\boldsymbol{\nu}) h_{b,\boldsymbol{\nu}}(1) - N_{n+1}(\boldsymbol{\nu}) h_{b,\boldsymbol{\nu}}(t)}{\prod_{i=1}^{m} (\nu_i!)} + \sum_{|\boldsymbol{\nu}|=n+1} \frac{M_{n+1}(\boldsymbol{\nu})}{\prod_{i=1}^{m} (\nu_i!)} h_{a,\boldsymbol{\nu}}(t).
\end{aligned}$$

2. Proof of Main Theorem

Let $t \in (0,1)$ be a parameter, and write

$$\int_{Q_m} w(x) f(x)\,dx = \int_{Q_m(t)} w(x) f(x)\,dx + \int_{Q_m \setminus Q_m(t)} w(x) f(x)\,dx. \tag{7}$$

The well-known Taylor's formula for a multivariable function states that

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} \left(\sum_{i=1}^{m} (x_i - a_i) \frac{\partial}{\partial x_i} \right)^k f(a) + R_n(x), \tag{8}$$

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} \left(\sum_{i=1}^{m} (x_i - b_i) \frac{\partial}{\partial x_i} \right)^k f(b) + r_n(x), \tag{9}$$

where

$$R_n(x) = \frac{1}{(n+1)!}\left(\sum_{i=1}^{m}(x_i-a_i)\frac{\partial}{\partial x_i}\right)^{n+1} f(a+\theta(x-a)), \quad \theta\in(0,1), \tag{10}$$

$$r_n(x) = \frac{1}{(n+1)!}\left(\sum_{i=1}^{m}(x_i-b_i)\frac{\partial}{\partial x_i}\right)^{n+1} f(b+\mu(x-b)), \quad \mu\in(0,1). \tag{11}$$

Since

$$\left(\sum_{i=1}^{m} q_i\right)^k = k!\sum_{|\boldsymbol{\nu}|=k}\prod_{i=1}^{m}\frac{q_i^{\nu_i}}{\nu_i!}, \tag{12}$$

integrating on both sides of (8) over $Q_m(t)$ gives us

$$\begin{aligned}
&\int\limits_{Q_m(t)} w(x)f(x)\,dx\\
&=\sum_{k=0}^{n}\frac{1}{k!}\int\limits_{Q_m(t)} w(x)\left(\sum_{i=1}^{m}(x_i-a_i)\frac{\partial}{\partial x_i}\right)^k f(a)\,dx+\int\limits_{Q_m(t)} w(x)R_n(x)\,dx\\
&=\sum_{k=0}^{n}\sum_{|\boldsymbol{\nu}|=k}\frac{1}{\prod\limits_{i=1}^{m}(\nu_i!)}\int\limits_{Q_m(t)} w(x)\prod_{i=1}^{m}\left((x_i-a_i)\frac{\partial}{\partial x_i}\right)^{\nu_i} f(a)\,dx\\
&\quad+\sum_{|\boldsymbol{\nu}|=n+1}\frac{1}{\prod\limits_{i=1}^{m}(\nu_i!)}\int\limits_{Q_m(t)} w(x)\prod_{i=1}^{m}\left((x_i-a_i)\frac{\partial}{\partial x_i}\right)^{\nu_i} f(a+\theta(x-a))\,dx\\
&=\sum_{k=0}^{n}\sum_{|\boldsymbol{\nu}|=k}\frac{1}{\prod\limits_{i=1}^{m}(\nu_i!)}\frac{\partial^k f(a)}{\prod\limits_{i=1}^{m}\partial x_i^{\nu_i}}\int\limits_{Q_m(t)} w(x)\prod_{i=1}^{m}(x_i-a_i)^{\nu_i}\,dx\\
&\quad+\sum_{|\boldsymbol{\nu}|=n+1}\frac{1}{\prod\limits_{i=1}^{m}(\nu_i!)}\int\limits_{Q_m(t)} w(x)\prod_{i=1}^{m}(x_i-a_i)^{\nu_i}\cdot\frac{\partial^{n+1}f(a+\theta(x-a))}{\prod\limits_{i=1}^{m}\partial x_i^{\nu_i}}\,dx\\
&=\sum_{k=0}^{n}\sum_{|\boldsymbol{\nu}|=k}\frac{D^{\boldsymbol{\nu}}f(a)}{\prod\limits_{i=1}^{m}(\nu_i!)}h_{a,\boldsymbol{\nu}}(t)\\
&\quad+\sum_{|\boldsymbol{\nu}|=n+1}\frac{1}{\prod\limits_{i=1}^{m}(\nu_i!)}\int\limits_{Q_m(t)} w(x)\prod_{i=1}^{m}(x_i-a_i)^{\nu_i}D^{\boldsymbol{\nu}}f(a+\theta(x-a))\,\mathrm{d}x.
\end{aligned} \tag{13}$$

Using inequality (2) and computing directly yields

$$
\begin{aligned}
&\sum_{|\boldsymbol{\nu}|=n+1}\frac{N_{n+1}(\boldsymbol{\nu})}{\prod_{i=1}^{m}(\nu_i!)}h_{a,\boldsymbol{\nu}}(t)\\
&\le\sum_{|\boldsymbol{\nu}|=n+1}\frac{1}{\prod_{i=1}^{m}(\nu_i!)}\int_{Q_m(t)}w(x)\prod_{i=1}^{m}(x_i-a_i)^{\nu_i}D^{\boldsymbol{\nu}}f(a+\theta(x-a))\,dx\\
&\le\sum_{|\boldsymbol{\nu}|=n+1}\frac{M_{n+1}(\boldsymbol{\nu})}{\prod_{i=1}^{m}(\nu_i!)}h_{a,\boldsymbol{\nu}}(t).
\end{aligned}
\tag{14}
$$

The combination of (13) and (14) leads to

$$
\begin{aligned}
&\sum_{|\boldsymbol{\nu}|=n+1}\frac{N_{n+1}(\boldsymbol{\nu})}{\prod_{i=1}^{m}(\nu_i!)}h_{a,\boldsymbol{\nu}}(t)\\
&\le\int_{Q_m(t)}w(x)f(x)\,dx-\sum_{k=0}^{n}\sum_{|\boldsymbol{\nu}|=k}\frac{D^{\boldsymbol{\nu}}f(a)}{\prod_{i=1}^{m}(\nu_i!)}h_{a,\boldsymbol{\nu}}(t)\\
&\le\sum_{|\boldsymbol{\nu}|=n+1}\frac{M_{n+1}(\boldsymbol{\nu})}{\prod_{i=1}^{m}(\nu_i!)}h_{a,\boldsymbol{\nu}}(t).
\end{aligned}
\tag{15}
$$

Integrating (9) on the domain $Q_m \setminus Q_m(t)$, we arrive at

$$
\begin{aligned}
&\int\limits_{Q_m \setminus Q_m(t)} w(x) f(x)\, dx \\
=&\sum_{k=0}^{n} \frac{1}{k!} \int\limits_{Q_m \setminus Q_m(t)} w(x) \left(\sum_{i=1}^{m} (x_i - b_i) \frac{\partial}{\partial x_i} \right)^k f(b)\, dx + \int\limits_{Q_m \setminus Q_m(t)} r_n(x)\, dx \\
=&\sum_{k=0}^{n} \sum_{|\boldsymbol{\nu}|=k} \frac{1}{\prod_{i=1}^{m} (\nu_i!)} \int_{Q_m} w(x) \prod_{i=1}^{m} \left((x_i - b_i) \frac{\partial}{\partial x_i} \right)^{\nu_i} f(b)\, dx \\
&- \sum_{k=0}^{n} \sum_{|\boldsymbol{\nu}|=k} \frac{1}{\prod_{i=1}^{m} (\nu_i!)} \int_{Q_m(t)} w(x) \prod_{i=1}^{m} \left((x_i - b_i) \frac{\partial}{\partial x_i} \right)^{\nu_i} f(b)\, dx \\
&+ \sum_{|\boldsymbol{\nu}|=n+1} \frac{1}{\prod_{i=1}^{m} (\nu_i!)} \int_{Q_m} w(t) \prod_{i=1}^{m} \left((x_i - b_i) \frac{\partial}{\partial x_i} \right)^{\nu_i} f(b + \mu(x-b))\, dx \\
&- \sum_{|\boldsymbol{\nu}|=n+1} \frac{1}{\prod_{i=1}^{m} (\nu_i!)} \int_{Q_m(t)} w(t) \prod_{i=1}^{m} \left((x_i - b_i) \frac{\partial}{\partial x_i} \right)^{\nu_i} f(b + \mu(x-b))\, dx \\
=&\sum_{k=0}^{n} \sum_{|\boldsymbol{\nu}|=k} \frac{D^{\boldsymbol{\nu}} f(b)}{\prod_{i=1}^{m} (\nu_i!)} \left[h_{b,\boldsymbol{\nu}}(1) - h_{b,\boldsymbol{\nu}}(t) \right] \\
&+ \sum_{|\boldsymbol{\nu}|=n+1} \frac{1}{\prod_{i=1}^{m} (\nu_i!)} \int_{Q_m} w(t) \prod_{i=1}^{m} \left((x_i - b_i) \frac{\partial}{\partial x_i} \right)^{\nu_i} f(b + \mu(x-b))\, dx \\
&- \sum_{|\boldsymbol{\nu}|=n+1} \frac{1}{\prod_{i=1}^{m} (\nu_i!)} \int_{Q_m(t)} w(t) \prod_{i=1}^{m} \left((x_i - b_i) \frac{\partial}{\partial x_i} \right)^{\nu_i} f(b + \mu(x-b))\, dx.
\end{aligned} \tag{16}
$$

Similar to the deduction of (14), if n is an odd, we have

$$
\begin{aligned}
&\sum_{|\boldsymbol{\nu}|=n+1} \frac{N_{n+1}(\boldsymbol{\nu})}{\prod_{i=1}^{m} (\nu_i!)} h_{b,\boldsymbol{\nu}}(t) \\
\leqslant &\sum_{|\boldsymbol{\nu}|=n+1} \frac{1}{\prod_{i=1}^{m} (\nu_i!)} \int_{Q_m(t)} w(t) \prod_{i=1}^{m} \left((x_i - b_i) \frac{\partial}{\partial x_i} \right)^{\nu_i} f(b + \mu(x-b))\, dx \\
= &\sum_{|\boldsymbol{\nu}|=n+1} \frac{1}{\prod_{i=1}^{m} (\nu_i!)} \int_{Q_m(t)} w(x) \prod_{i=1}^{m} (x_i - b_i)^{\nu_i} D^{\boldsymbol{\nu}} f(b + \mu(x-b))\, dx \\
\leqslant &\sum_{|\boldsymbol{\nu}|=n+1} \frac{M_{n+1}(\boldsymbol{\nu})}{\prod_{i=1}^{m} (\nu_i!)} h_{b,\boldsymbol{\nu}}(t);
\end{aligned} \tag{17}
$$

if n is even, the reversed inequalities in (17) hold. Note that $Q_m(1) = Q_m$.

Substituting (17) into (16) we have that, if n is an odd number, then

$$\begin{aligned} &\sum_{|\boldsymbol{\nu}|=n+1} \frac{N_{n+1}(\boldsymbol{\nu}) h_{b,\boldsymbol{\nu}}(1) - M_{n+1}(\boldsymbol{\nu}) h_{b,\boldsymbol{\nu}}(t)}{\prod_{i=1}^{m} (\nu_i!)} \\ &\le \int_{Q_m \setminus Q_m(t)} w(x) f(x)\,dx - \sum_{k=0}^{n} \sum_{|\boldsymbol{\nu}|=k} \frac{D^{\boldsymbol{\nu}} f(b)}{\prod_{i=1}^{m} (\nu_i!)} [h_{b,\boldsymbol{\nu}}(1) - h_{b,\boldsymbol{\nu}}(t)] \\ &\le \sum_{|\boldsymbol{\nu}|=n+1} \frac{M_{n+1}(\boldsymbol{\nu}) h_{b,\boldsymbol{\nu}}(1) - N_{n+1}(\boldsymbol{\nu}) h_{b,\boldsymbol{\nu}}(t)}{\prod_{i=1}^{m} (\nu_i!)}; \end{aligned} \tag{18}$$

if n is an even number, then the inequalities in (18) are reversed.

By addition of inequalities (15) and (18), the Main Theorem was proved. ∎

Remark 1. It is noted that we also can consider the similar estimates for the weighted multiple integral $\int_{Q_m} w(x) f(x)\,dx$ on the m-dimensional ball centered at a with radius $|b-a|$, that is, $Q_m = B_a(|b-a|)$, $a, b \in \mathbb{R}^m$.

Remark 2. In the Main Theorem, if we take $m = 1$, we can obtain the results in [14]; if we set $m = 1$ and $w(x) = 1$, then we get the results in [12]; if we let $w(x) = 1$, we have the results in [13]. In particular, if we take $w(x) = 1$, $m = 1$ and $n = 0$, the Iyengar inequality [6] is deduced, which has been generalized by many mathematicians in [1, 2, 3, 4, 5, 8, 11, 15] (also see [7, 9, 10]).

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