THE UNIFIED TREATMENT OF TRAPEZOID, SIMPSON AND OSTROWSKI TYPE INEQUALITY FOR MONOTONIC MAPPINGS AND APPLICATIONS

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Abstract. We give new trapezoid inequality as well as Simpson and Ostrowski type inequalities for monotonic functions. We provide their applications in Probability Theory, Numerical Analysis and for Special Means.

1. Introduction

In [1], S.S. Dragomir established the following Ostrowski's type inequality for monotonic mappings.

Theorem 1. *Let $f : [a, b] \to R$ be a monotonic nondecreasing mapping on $[a, b]$. Then for all $x \in [a, b]$, we have the inequality:*

$$\begin{aligned}(1.1)\qquad & \left| f(x) - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ \leq\ & \frac{1}{b-a}\left\{ [2x-(a+b)]\, f(x) + \int_a^b \operatorname{sgn}(t-x) f(t)dt \right\} \\ \leq\ & \frac{1}{b-a}\left[(x-a)\left(f(x)-f(a)\right) + (b-x)\left(f(b)-f(x)\right) \right] \\ \leq\ & \left[\frac{1}{2} + \frac{\left| x - \frac{a+b}{2} \right|}{b-a} \right] \left(f(b) - f(a) \right).\end{aligned}$$

All the inequalities in (1.1) *are sharp and the constant* $\frac{1}{2}$ *is the best possible one.*

In the present paper we shall obtain a generalization of this result which also contains trapezoid and Simpson type inequalities. For recent results in these topics see the papers [4]-[13].

2. Main Result

We shall start with the following result:

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Theorem 2. *Let $f : [a,b] \to R$ be a monotonic nondecreasing mapping on $[a,b]$ and $t_1, t_2, t_3 \in (a,b)$ be such that $t_1 < t_2 < t_3$. Then*

$$\begin{aligned}
(2.1)\qquad & \left| \int_a^b f(x)dx - [(t_1 - a) f(a) + (b - t_3) f(b) + (t_3 - t_1) f(t)] \right| \\
\leq\ & (b - t_3) f(b) + (2t_2 - t_1 - t_3) f(t_2) - (t_1 - a) f(a) + \int_a^b T(x) f(x)dx \\
\leq\ & (b - t_3)(f(b) - f(t_3)) + (t_3 - t_2)(f(t_3) - f(t_2)) \\
& + (t_2 - t_1)(f(t_2) - f(t_1)) + (t_1 - a)(f(t_1) - f(t_2)) \\
\leq\ & \max\{t_1 - a, t_2 - t_1, t_3 - t_2, b - t_3\}(f(b) - f(a))
\end{aligned}$$

where

$$T(x) = \begin{cases} \operatorname{sgn}(t_1 - x), & \text{for } x \in [a, t_2] \\ \operatorname{sgn}(t_3 - x), & \text{for } x \in [t_2, b] \end{cases}.$$

Proof. Using integration by parts formula for Riemann-Stieltjes integral, we have

$$\int_a^b s(x)df(x) = (t_1 - a) f(a) + (b - t_3) f(b) + (t_3 - t_1) f(t_2) - \int_a^b f(x)d(x)$$

where

$$s(x) = \begin{cases} x - t_1, & x \in [a, t_2] \\ x - t_3, & x \in [t_2, b] \end{cases}.$$

Indeed,

$$\begin{aligned}
\int_a^b s(x)df(x) &= \int_a^{t_2} (x - t_1)df(x) + \int_{t_2}^b (x - t_3)df(x) \\
&= (x - t_1) f(x)|_a^{t_2} + (x - t_3) f(t)|_{t_2}^b - \int_a^b f(x)d(x) \\
&= (t_1 - a) f(a) + (b - t_3) f(b) + (t_3 - t_1) f(t_2) - \int_a^b f(x)dx.
\end{aligned}$$

Assume that $A_n : a = x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b$ is a sequence of divisions with $\nu(A_n) \to 0$ as $n \to \infty$, where $\nu(A_n) := \max_{i=0,\dots,n-1} \left(x_{i+1}^{(n)} - x_i^{(n)} \right)$ and $\xi_i^{(n)} \in \left[x_i^{(n)}, x_{i+1}^{(n)} \right]$. If $p : [a,b] \to R$ is a continuous mapping on $[a,b]$ and v is monotonic nondecreasing on $[a,b]$, then

$$\begin{aligned}
(2.2)\qquad & \left|\int_a^b p(x)dv(x)\right| \\
&= \left|\lim_{\nu(An)\to\infty}\sum_{i=0}^{n-1} p(\xi_i^{(n)})\left[v(x_{i+1}^{(n)}) - v(x_i^{(n)})\right]\right| \\
&\le \lim_{\nu(An)\to\infty}\sum_{i=0}^{n-1}\left|p(\xi_i^{(n)})\right|\left|v(x_{i+1}^{(n)}) - v(x_i^{(n)})\right| \\
&= \lim_{\nu(An)\to\infty}\sum_{i=0}^{n-1}\left|p(\xi_i^{(n)})\right|\left(v(x_{i+1}^{(n)}) - v(x_i^{(n)})\right) = \int_a^b |p(x)|\,dv(x).
\end{aligned}$$

Applying the inequality (2.2) for $p(x) = s(x)$ and $v(x) = f(x)$, $x \in [a,b]$ we can state:

$$\begin{aligned}
&\left|\int_a^b s(x)df(x)\right| \\
&\le \int_a^b |s(x)|\,df(x) \\
&= \int_a^{t_1}(t_1 - x)df(x) + \int_{t_1}^{t_2}(x - t_1)\,df(x) + \int_{t_2}^{t_3}(t_3 - x)df(x) + \int_{t_3}^{b}(x - t_3)\,df(x) \\
&= (t_1 - x)f(x)\big|_a^{t_1} + \int_a^{t_1} f(x)dx + (x - t_1)f(x)\big|_{t_1}^{t_2} - \int_{t_1}^{t_2} f(x)dx + \\
&+ (t_3 - x)f(x)\big|_{t_2}^{t_3} + \int_{t_2}^{t_3} f(x)dx + (x - t_3)f(x)\big|_{t_3}^{b} + \int_{t_3}^{b} f(x)dx \\
&= -(t_1 - a)\,f(a) + (t_2 - t_1)\,f(t_2) - (t_3 - t_2)\,f(t_2) \\
&+ (b - t_3)\,f(b) + \int_a^b T(x)f(x)dx.
\end{aligned}$$

what is the first inequality in (2.1).

If $f : [a,b] \to R$ is monotonic nondecreasing in $[a,b]$, we can also state that

$$\begin{aligned}
\int_a^{t_1} f(x)dx &\le f(t_1)(t_1 - a), & \int_{t_1}^{t_2} f(x)dx \ge f(t_2)(t_2 - t_1), \\
\int_{t_2}^{t_3} f(x)dx &\le f(t_3)(t_3 - t_2), & \int_{t_3}^{b} f(x)dx \ge f(t_3)(b - t_3).
\end{aligned}$$

So,

$$\begin{aligned}
&\int_a^b T(x)f(x)dx \\
=\ & \int_a^{t_1} f(x)dx - \int_{t_1}^{t_2} f(x)dx + \int_{t_2}^{t_3} f(x)dx - \int_{t_3}^{b} f(x)dx \\
\leq\ & f(t_1)(t_1-a) - f(t_2)(t_2-t_1) + f(t_3)(t_3-t_2) - f(t_3)(b-t_3).
\end{aligned}$$

We have

$$\begin{aligned}
&-(t_1-a)f(a) + (t_2-t_1)f(t_2) - (t_3-t_2)f(t_2) \\
&-(b-t_3)f(b) + \int_a^b T(x)f(x)dx \\
\leq\ & -(t_1-a)f(a) + (t_2-t_1)f(t_2) - (t_3-t_2)f(t_2) + (b-t_3)f(b) \\
&+(t_1-a)f(t_1) - (t_2-t_1)f(t_1) + (t_3-t_2)f(t_3) - (b-t_3)f(t_3) \\
=\ & (t_1-a)(f(t_1)-f(a)) + (t_2-t_1)(f(t_2)-f(t_1)) \\
&+(t_3-t_2)(f(t_3)-f(t_2)) + (b-t_3)(f(b)-f(t_3)) \\
\leq\ & \max\{t_1-a, t_2-t_1, t_3-t_2, b-t_3\}(f(b)-f(a)),
\end{aligned}$$

and the theorem is thus proved. ∎

Remark 1. *For $t_1 = 0$, $t_2 = x$, $t_3 = b$ we get Theorem 1 from the above Theorem.*

For $t_1 = t_2 = t_3 = x$, Theorem 2 becomes:

Corollary 1. *Let f be defined as in Theorem 2. Then*

$$\begin{aligned}
(2.3)\qquad & \left|\int_a^b f(t)dt - [(x-a)f(a) + (b-x)f(b)]\right| \\
\leq\ & (b-x)f(b) - (x-a)f(a) + \int_a^b \operatorname{sgn}(x-t)f(t)dt \\
\leq\ & (b-x)(f(b)-f(x)) + (x-a)(f(x)-f(a)) \\
\leq\ & \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right](f(b)-f(a)).
\end{aligned}$$

All the inequalities in (2.3) are sharp and the constant $\frac{1}{2}$ is the best possible one.

Proof. We only need to prove that the constant $\frac{1}{2}$ is the best possible one. Choose the mapping $f_0 : [a,b] \to R$ given by

$$f_0(x) = \begin{cases} 0, & \text{if } x \in [a,t]; \\ 1, & \text{if } x = b. \end{cases}$$

Then, f_0 is monotonic nondecreasing on $[a,b]$, and for $x=a$ we have

$$\begin{aligned}
&\left|\int_a^b f(t)dt-[(x-a)f(a)+(b-x)f(b)]\right| \\
&= (b-x)f(b)-(x-a)f(a)+\int_a^b \operatorname{sgn}(t-x)f(t)dt \\
&= (b-x)\left(f(b)-f(x)\right)+(x-a)\left(f(x)-f(a)\right) \\
&= (b-a) \\
&\leq \left[C(b-a)+\left|x-\frac{a+b}{2}\right|\right]\left(f(b)-f(a)\right) \\
&= \left(C+\frac{1}{2}\right)(b-a)
\end{aligned}$$

which prove the sharpness of the first two inequalities and the fact that C shouldn't be less than $\frac{1}{2}$. ∎

For $x=\frac{a+b}{2}$ we get trapezoid inequality.

Corollary 2. *Let $f:[a,b]\to R$ be a monotonic nondecreasing mapping on $[a,b]$. Then*

$$\begin{aligned}
&\left|\int_a^b f(t)dt-\frac{f(a)+f(b)}{2}(b-a)\right| \qquad (2.4)\\
&\leq \frac{1}{2}(b-a)(f(b)-f(a))-\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)f(t)dt \\
&\leq \frac{1}{2}(b-a)(f(b)-f(a)).
\end{aligned}$$

The constant factor $\frac{1}{2}$ is the best in both inequalities.

Corollary 3. *Let f be as in Theorem 2 and $p,q\in R_+$ with $p>q$. Then*

$$\begin{aligned}
&\left|\int_a^b f(x)dx-\frac{q}{p+q}(b-a)\left[f(a)+f(b)+\frac{p-q}{q}f\left(\frac{a+b}{2}\right)\right]\right| \\
&\leq \frac{q}{p+q}(b-a)(f(b)-f(a))+\int_a^b T_1(x)f(x)dx \\
&\leq \frac{q}{p+q}(b-a)(f(b)-f(a)) \\
&\quad +\frac{p-3q}{2(p+q)}(b-a)\left[f\left(\frac{pb+qa}{p+q}\right)-f\left(\frac{pa+qb}{p+q}\right)\right] \\
&\leq \max\left\{q,\frac{p-q}{2}\right\}\frac{b-a}{p+q}(f(b)-f(a))
\end{aligned}$$

where

$$T_1(x) = \begin{cases} \operatorname{sgn}\left(\frac{pa+qb}{p+q} - x\right), & x \in \left[a, \frac{a+b}{2}\right] \\ \operatorname{sgn}\left(\frac{pb+qa}{p+q} - x\right), & \text{if } x \in \left(\frac{a+b}{2}, b\right] \end{cases}.$$

Proof. Set in Theorem 2: $t_1 = \frac{pa+qb}{p+q}$, $t_2 = \frac{a+b}{2}$, $t_3 = \frac{qa+pb}{p+q}$. ∎

Remark 2. *Of special interest is the case $p = 5$ and $q = 1$ where we get from Corollary 3 the following result of Simpson type;*

$$\begin{aligned} & \left| \int_a^b f(x)dx - \frac{1}{3}(b-a)\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] \right| \\ \leq\ & \frac{b-a}{6}(f(b)-f(a)) + \int_a^b T_2(x) f(x)dx \\ \leq\ & \frac{b-a}{6}\left[f(b) - f(a) + f\left(\frac{5b+a}{6}\right) - f\left(\frac{5a+b}{6}\right)\right] \\ \leq\ & \frac{1}{3}(b-a)(f(b)-f(a)), \end{aligned}$$

where

$$T_2(x) = \begin{cases} \operatorname{sgn}\left(\frac{5a+b}{3} - x\right), & x \in \left[a, \frac{a+b}{2}\right] \\ \operatorname{sgn}\left(\frac{a+5b}{3} - x\right), & x \in \left(\frac{a+b}{2}, b\right]. \end{cases}$$

Remark 3. *For $p \to q$ we get Corollary 2 from Corollary 3.*

Remark 4. *For some related results see* [3].

3. An Inequality for Cumulative Distribution Function

Let X be a random variable taking values in the finite interval $[a, b]$, with cumulative distributions function $F(X) = \Pr(X \leq x)$.

The following result from [2] can be obtained from Theorem 1 and from Theorem 2.

Theorem 3. *Let X and F be as above. Then we have the inequalities*

$$\begin{aligned} & \left| \Pr(X \leq x) - \frac{b - E(x)}{b-a} \right| \qquad (3.1) \\ \leq\ & \frac{1}{b-a}\left[[2x-(a+b)]\Pr(X \leq x) + \int_a^b \operatorname{sgn}(t-x)\, F(t)dt\right] \\ \leq\ & \frac{1}{b-a}\left[(b-x)\Pr(X \geq x) + (x-a)\Pr(X \leq x)\right] \\ \leq\ & \frac{1}{2} + \frac{\left|x - \frac{a+b}{2}\right|}{b-a}, \end{aligned}$$

for all $x \in [a, b]$.

All the inequalities in (3.1) are sharp and the constant $\frac{1}{2}$ is the best possible.

Now we shall prove the following result.

Theorem 4. *Let X and F be as above. Then we have the inequalities*

$$
\begin{aligned}
|E(X)-x| &\leq b-x+\int_a^b \operatorname{sgn}(x-t)\,F(t)dt \\
&\leq (b-x)\Pr(X\geq x)+(x-a)\Pr(X\leq x) \\
&\leq \frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|
\end{aligned}
\tag{3.2}
$$

for all $x\in[a,b]$.

All the inequalities in (3.2) are sharp and the constant $\frac{1}{2}$ is the best possible.

Proof. Apply Corollary 1 for the monotonic nondecreasing mapping $f(t):=F(t), t\in[a,b]$ to get

$$
\begin{aligned}
&\left|\int_a^b F(t)dt-[(x-a)F(a)+(b-x)F(b)]\right| \\
&\leq (b-x)F(b)+(x-a)F(a)+\int_a^b \operatorname{sgn}(x-t)\,F(t)dt \\
&\leq (b-x)(F(b)-F(x))+(x-a)\,(F(x)-F(a)) \\
&\leq \left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right](F(b)-F(a))
\end{aligned}
\tag{3.3}
$$

and as

$$F(a)=0, F(b)=1$$

by integration by parts formula for Riemann-Stieltjes integral, we have

$$
\begin{aligned}
E(X) &= \int_a^b t\,dF(t)=tF(t)|_a^b-\int_a^b F(t)dt \\
&= bF(b)-aF(a)-\int_a^b F(t)dt \\
&= b-\int_a^b F(t)dt
\end{aligned}
$$

i.e.,

$$\int_a^b F(t)dt=b-E(X).$$

The inequalities (3.3) give the desired estimation (3.2). ∎

Corollary 4. *Let X be a random variable taking values in the finite interval $[a,b]$, with cumulative distribution function $F(x) = \Pr(X \le x)$ and the expectation $E(X)$. Then we have the inequality*

$$\left| E(X) - \frac{a+b}{2} \right| \le \frac{1}{2}(b-a) - \int_a^b sgn(t - \frac{a+b}{2})F(t)dt \le \frac{1}{2}(b-a).$$

The factor constant $\frac{1}{2}$ is the best in both inequalities.

4. Application for Quadrature Formulae

By using our Corollary 3 we can give a general result for Simpson's type of quadrature formula. Moreover, we shall give applications of Corollary 2, i.e., the corresponding result for classical trapezoidal rule.

Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a partitioning of the interval $[a,b]$, put $h_i := x_{i+1} - x_i$ $(i = 0,..,n-1)$ and $\nu(h) := \max\{h_i | i = 0,...,n-1\}$ the norm of the division. Define the trapezoid formula associated with the division I_n and with a mapping $f : [a,b] \to R$

$$T_n(f, I_n) := \sum_{i=0}^{n-1} \frac{f(x_i) + f(x_{i+1})}{2} \cdot h_i.$$

The following theorem contains an evaluation of the remainder in Trapezoid rule.

Theorem 5. *Let $f : [a,b] \to R$ be a monotonic nondecreasing mapping on $[a,b]$. Then*

$$\int_a^b f(t)dt = T_n(f, I_n) + R_n(f, I_n)$$

where $T_n(f, I_n)$ is the trapezoid formula and the remainder $R_n(f, I_n)$ satisfies the estimation

$$(4.1) \qquad \begin{aligned} |R_n(f, I_n)| &\le \frac{1}{2}\sum_{i=0}^{n-1} (f(x_{i+1}) + f(x_i))\, h_i - \int_a^b K_n(f, I_n, t) f(t)dt \\ &\le \frac{1}{2}\sum_{i=0}^{n-1} (f(x_{i+1}) + f(x_i))\, h_i \\ &\le \frac{\nu(h)}{2}(f(b) - f(a)) \end{aligned}$$

where

$$K_n(f, I_n, t) := \begin{cases} sgn\left(t - \frac{a+x_1}{2}\right) & if\ t \in [a, x_1) \\ sgn\left(t - \frac{x_1+x_2}{2}\right) & if\ t \in [x_1, x_2) \\ \dots\dots\dots\dots & \\ sgn\left(t - \frac{x_{n-2}+x_{n-1}}{2}\right) & if\ t \in [x_{n-2}, x_{n-1}) \\ sgn\left(t - \frac{x_{n-1}+b}{2}\right) & if\ t \in [x_{n-1}, b]. \end{cases}$$

Proof. Write the inequality (2.4) on the intervals $[x_i, x_{i+1}]$ to get

$$\left|\int_{x_i}^{x_{i+1}} f(t)dt - \frac{f(x_i)+f(x_{i+1})}{2}h_i\right| \tag{4.2}$$
$$\leq \frac{1}{2}h_i\left(f(x_{i+1})+f(x_i)\right) - \int_{x_i}^{x_{i+1}} sgn\left(t-\frac{x_i+x_{i+1}}{2}\right)f(t)dt$$
$$\leq \frac{1}{2}h_i\left(f(x_{i+1})+f(x_i)\right),$$

for all $i \in \{0, ..., n-1\}$.
Summing over i from 0 to $n-1$ and using the generalized triangle inequality, we get the first two inequalities in (4.1).
The last inequality is obvious, observing that

$$\begin{aligned}\sum_{i=0}^{n-1}(f(x_{i+1})-f(x_i))h_i &= \nu(h)\sum_{i=0}^{n-1}(f(x_{i+1})-f(x_i)) \\ &= \nu(h)(f(b)-f(a)).\end{aligned}$$

Hence, the theorem is proved. ∎

Corollary 5. *Let* f *be as above and* I_n *are equidistant partitioning of* $[a,b]$*, i.e.,* $I_n : x_i = a + i \cdot \frac{b-a}{n}$*,* $i = 0, ..., n$*. Then*

$$\int_a^b f(t)dt = T_n(f) + R_n(f)$$

where

$$T_n(f) = \frac{b-a}{2n}\sum_{i=0}^{n-1}\left[f\left(a+i\cdot\frac{b-a}{n}\right)+f\left(a+(i-1)\cdot\frac{b-a}{n}\right)\right].$$

The remainder satisfies the estimation.

$$|R_n(f)| \leq \frac{b-a}{2n}(f(b)-f(a)) - \int_a^b K_n(f,t)f(t)dt \leq \frac{b-a}{2n}(f(b)-f(a))$$

where

$$K_n(f,t) := \begin{cases} sgn\left(t-\frac{(2n-1)a+b}{2n}\right) & if\ t \in \left[a, \frac{(n-1)a+b}{n}\right] \\ \qquad\qquad .... & \\ sgn\left(t-\frac{a+(2n-1)b}{2n}\right) & if\ t \in \left(\frac{a+(n-1)b}{n}, b\right] \end{cases}$$

5. Application for Special Means

As in Section 4 we shall give application of Corollary 3. It is clear that similar applies from of the other results from Section 2.

Let us recall the following means

1. *The arithmetic mean*
$$A = A(a,b) := \frac{a+b}{2}, a,b \geq 0;$$
2. *The geometric mean*
$$G = G(a,b) := \sqrt{ab}, a,b \geq 0;$$
3. *The harmonic mean*
$$H = H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}, a,b \geq 0;$$
4. *The logarithmic mean*
$$L = L(a,b) := \begin{cases} a \text{ if } a = b \\ \frac{b-a}{\ln b - \ln a} \text{ if } a \neq b \end{cases}, a,b \geq 0;$$
5. *The identric mean*
$$I = I(a,b) := \begin{cases} a \text{ if } a = b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} \text{ if } a \neq b \end{cases}, a,b \geq 0;$$
6. *The p-logarithmic mean*
$$L_p = L_p(a,b) := \begin{cases} a \text{ if } a = b \\ \left[\frac{b^{p-1}-a^{p-1}}{(p-1)(b-a)}\right]^{\frac{1}{p}} \text{ if } a \neq b, \end{cases}$$
where $p \in R \backslash \{-1, 0\}$ and $a, b > 0$.

The following simple relationships are known in the literature
$$H \leq G \leq L \leq I \leq A.$$
It is also known that L_p is monotonically increasing over $p \in R$ with $L_0 := I$ and $L_{-1} := L$.

We are going to use inequality (2.4) in the following equivalent version:
$$\left|\frac{1}{b-a}\int_a^b f(t)dt - \frac{f(a)+f(b)}{2}\right| \tag{5.1}$$
$$\leq \frac{1}{2}(f(b)-f(a)) - \frac{1}{b-a}\int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) f(t)dt$$
$$\leq \frac{1}{2}(f(b)-f(a)),$$
where $f : [a,b] \to R$ is monotonic nondecreasing on $[a,b]$.

5.1. **Mapping $f(x) = x^p$.** Consider the mapping $f : [a,b] \subset (0,\infty) \to R$, $f(x) = x^p$, $p > 0$. Then
$$\frac{1}{b-a}\int_a^b f(t)dt = L_p^p(a,b);$$
$$\frac{f(a)+f(b)}{2} = A(a^p, b^p);$$
$$f(b) - f(a) = p(b-a)L_{p-1}^{p-1}(a,b);$$

$$
\begin{aligned}
& -\frac{1}{b-a}\int_a^b sgn\left(t-\frac{a+b}{2}\right)f(t)dt \\
&= \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} f(t)dt - \int_{\frac{a+b}{2}}^b f(t)dt\right] \\
&= \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} t^p dt - \int_{\frac{a+b}{2}}^b t^p dt\right] \\
&= \frac{2}{(b-a)(p+1)}\left[A^{p+1}(a,b) - A(a^{p+1},b^{p+1})\right]
\end{aligned}
$$

and then, by (5.1), we get

$$
\begin{aligned}
(5.2)\qquad & \left|L_p^p(a,b) - A(a^p,b^p)\right| \\
&\le \frac{1}{2}p(b-a)L_{p-1}^{p-1}(a,b) + \frac{2}{(b-a)(p+1)}\left[A^{p+1}(a,b) - A(a^{p+1},b^{p+1})\right] \\
&\le \frac{1}{2}p(b-a)L_{p-1}^{p-1}(a,b).
\end{aligned}
$$

5.2. **Mapping** $f(x) = -\frac{1}{x}$**.** Consider the mapping $f : [a,b] \subset (0,\infty) \to R$, $f(x) = -\frac{1}{x}$. Then

$$
\begin{aligned}
\frac{1}{b-a}\int_a^b f(t)dt &= -L^{-1}(a,b); \\
\frac{f(a)+f(b)}{2} &= -\frac{A(a,b)}{G^2(a,b)}; \\
f(b)-f(a) &= \frac{b-a}{G^2(a,b)};
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{b-a}\int_a^b sgn\left(t-\frac{a+b}{2}\right)f(t)dt \\
&= \frac{1}{b-a}\left[-\int_a^{\frac{a+b}{2}} \frac{dt}{t} + \int_{\frac{a+b}{2}}^b \frac{dt}{t}\right] \\
&= \frac{2}{b-a}\left[\ln G(a,b) - \ln A(a,b)\right];
\end{aligned}
$$

and then, by (5.1), we get

$$
\begin{aligned}
\left|\frac{A(a,b)}{G^2(a,b)} - \frac{1}{L(a,b)}\right| &\le \frac{1}{2}\frac{b-a}{G^2(a,b)} - \frac{2}{b-a}\left[\ln A(a,b) - \ln G(a,b)\right] \\
&\le \frac{1}{2}\frac{b-a}{G^2(a,b)}.
\end{aligned}
$$

If we multiply this inequality with $L(a,b)G^2(a,b)$, we get

$$0 \leq AL - G^2 \leq \frac{1}{2}(b-a)L - 2G^2L \cdot \frac{\ln A - \ln G}{A-G} \cdot \frac{A-G}{b-a} \leq \frac{1}{2}(b-a)L.$$

But

$$\frac{\ln A - \ln G}{A-G} = L^{-1}(G,A)$$

and then we get

$$0 \leq AL - G^2 \leq \frac{1}{2}(b-a)L - 2G^2L \cdot \frac{A-G}{b-a} \cdot \frac{1}{L(G,A)} \leq \frac{1}{2}(b-a)L. \tag{5.3}$$

5.3. **Mapping $f(x) = \ln x$.** Consider the mapping $f : [a,b] \subset (0,\infty) \to R$, $f(x) = \ln x$. Then

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(t)dt &= \ln I(a,b); \\ \frac{f(a)+f(b)}{2} &= \ln G(a,b); \\ f(b)-f(a) &= \frac{b-a}{L(a,b)}; \end{aligned}$$

$$\begin{aligned} & -\frac{1}{b-a}\int_a^b sgn\left(t+\frac{a+b}{2}\right)f(t)dt \\ = \; & \frac{1}{b-a}\left[\int_a^{\frac{a+b}{2}} \ln t dt - \frac{1}{b-a}\int_{\frac{a+b}{2}}^b \ln t dt\right] \\ = \; & \frac{1}{2}\left[\ln I(a,\frac{a+b}{2}) - \ln I(\frac{a+b}{2},b)\right] = \ln\left(\frac{I(a,A)}{I(A,b)}\right)^{\frac{1}{2}}; \end{aligned}$$

and then, by (5.1), we get

$$|\ln I(a,b) - \ln G(a,b)| \leq \frac{1}{2}\frac{b-a}{L(a,b)} + \ln\left(\frac{I(a,A)}{I(A,b)}\right)^{\frac{1}{2}} \leq \frac{1}{2}\frac{b-a}{L(a,b)}$$

from where we deduce

$$1 \leq \frac{I}{G} \leq \left[\frac{I(a,A)}{I(A,b)}\right]^{\frac{1}{2}} \exp\left(\frac{1}{2}\frac{b-a}{L(a,b)}\right) \leq \exp\left[\frac{1}{2}\frac{b-a}{L(a,b)}\right]. \tag{5.4}$$

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