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A NOTE ON BOUNDS FOR THE ESTIMATION ERROR VARIANCE OF A CONTINUOUS STREAM WITH STATIONARY VARIOGRAM

N.S. BARNETT AND S.S. DRAGOMIR

Abstract. In this paper, by the use of an Ostrowski type integral inequality for double integrals, we establish an upper bound for the estimation error variance of a continuous stream with stationary variogram.

1 Introduction

In [1], the authors considered $X(t)$ as defining the quality of a product at time t where $X(t)$ is a continuous time stochastic process which may be non-stationary. Typically, $X(t)$ represents a continuous stream industrial process such as is common in many areas of the chemical industry. The paper was concerned with issues related to sampling the stream with a view to estimating the mean quality characteristic of the flow, $\bar{X}$, over the interval $[0,d]$. Specifically, focus was on obtaining the sampling location, said to be optimal, which minimizes the estimation error variance, $E\left[\left(\bar{X}-X(t)\right)^2\right]$, $0<t<d$.

Given that t is as specified, the problem is to find the value of t (the sampling location) that minimizes $E\left[\left(\bar{X}-X(t)\right)^2\right]$. It is shown that for constant stream flows the optimal sampling point is the mid point of $[0,d]$ for situations where the process variogram,

$$V(u)=\frac{1}{2}E\left[\left(X(t)-X(t+u)\right)^2\right],$$

$$V(0)=0, V(-u)=V(u),$$

is stationary (note that variogram stationarity is not equivalent to process stationarity).

The paper continues to consider optimal sampling locations for situations where the stream flow rate varies. The optimal sampling location is seen to depend on both the flow rate function and the form of the process variogram - some examples are given.

In this note, rather than focusing on the optimal sampling point, we focus on the actual value of the estimation error variance itself. In particular, we focus on obtaining an upper bound for its value. To do this, we use a result obtained in [2], an inequality of the Ostrowski type. For other results of this type see Chapter XV of [7] and [3, 4, 5, 6].

2 An Ostrowski Inequality for Double Integrals

Let $f:[a,b]\times[c,d]\to\mathbf{R}$ be so that $f(\cdot,\cdot)$ is continuous on $[a,b]\times[x,d]$, $f''_{x,y}=\frac{\partial^2 f}{\partial x\partial y}$ exists on $(a,b)\times(c,d)$ and is bounded, i.e.,

$$\left\|f''_{s,t}\right\|_\infty:=\sup_{(x,y)\in(a,b)\times(c,d)}\left|\frac{\partial^2 f(x,y)}{\partial x\partial y}\right|<\infty.$$

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Then we have the inequality:

$$\left|\int_a^b\int_c^d f(s,t)\,dsdt-\left[(b-a)\int_c^d f(x,t)\,dt+(d-c)\int_a^b f(s,y)\,ds\right.\right.$$
$$\left.\left.-(d-c)(b-a)f(x,y)\right]\right|$$
$$\leq\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\times\left[\frac{1}{4}(d-c)^2+\left(y-\frac{c+d}{2}\right)^2\right]\left\|f''_{s,t}\right\|_\infty \tag{2.1}$$

for all $(x,y)\in[a,b]\times[c,d]$.

For the sake of completeness we give here a short proof of this inequality.

Integrating by parts successively, we have the equality:

$$\int_a^x\int_c^y(s-a)(t-c)f''_{s,t}(s,t)\,dtds \tag{2.2}$$
$$=(y-c)(x-a)f(x,y)-(y-c)\int_a^x f(s,y)\,ds$$
$$-(x-a)\int_c^y f(x,t)\,dt+\int_a^x\int_c^y f(s,t)\,dsdt.$$

Also, by similar computations we have

$$\int_a^x\int_y^d(s-a)(t-d)f''_{s,t}(s,t)\,dsdt \tag{2.3}$$
$$=(x-a)(d-y)f(x,y)-(d-y)\int_a^x f(s,y)\,ds$$
$$-(x-a)\int_y^d f(x,t)\,dt+\int_a^x\int_c^y f(s,t)\,dsdt.$$

Now,

$$\int_x^b\int_y^d(s-b)(t-d)f''_{s,t}(s,t)\,dsdt \tag{2.4}$$
$$=(d-y)(b-x)f(x,y)-(d-y)\int_x^b f(s,y)\,ds$$

$$-(b-x)\int_{y}^{d} f(x,t)\,dt+\int_{x}^{b}\int_{y}^{d} f(s,t)\,dsdt$$

and finally

$$\int_{x}^{b}\int_{c}^{y}(s-b)(t-c)\,f''_{s,t}(s,t)\,dsdt \tag{2.5}$$

$$=(y-c)(b-x)\,f(x,y)-(y-c)\int_{x}^{b} f(s,y)\,ds$$

$$-(b-x)\int_{c}^{y} f(x,t)\,dt+\int_{x}^{b}\int_{c}^{y} f(s,t)\,dsdt.$$

If we add the equalities $(2.2)-(2.5)$ we get, in the right membership:

$$[(y-c)(x-a)+(x-a)(d-y)$$

$$+(d-y)(b-x)+(y-c)(b-x)]\,f(x,y)$$

$$-(d-c)\int_{a}^{x} f(s,y)\,ds-(d-c)\int_{x}^{b} f(s,y)\,ds-(b-a)\int_{c}^{y} f(x,t)\,dt$$

$$-(b-a)\int_{y}^{d} f(x,t)\,dt+\int_{a}^{x}\int_{c}^{y} f(s,t)\,dsdt+\int_{a}^{x}\int_{y}^{d} f(s,t)\,dsdt$$

$$+\int_{x}^{b}\int_{y}^{d} f(s,t)\,dsdt+\int_{x}^{b}\int_{c}^{y} f(s,t)\,dsdt$$

$$=(d-c)(b-a)\,f(x,y)-(d-c)\int_{a}^{b} f(s,y)\,ds$$

$$-(b-a)\int_{c}^{b} f(x,t)\,dt+\int_{a}^{b}\int_{c}^{d} f(s,t)\,dsdt.$$

For the first membership, let us define the kernels: $p:[a,b]^2\to\mathbf{R}$, $q:[c,d]^2\to\mathbf{R}$ given by:

$$p(x,s):=\begin{cases} s-a & \text{if } s\in[a,x] \\ s-b & \text{if } s\in(x,b] \end{cases}$$

and

$$q(y,t) := \begin{cases} t-c & \text{if } t \in [c,y] \\ t-d & \text{if } t \in (y,d] \end{cases}.$$

Now, using this notation, we deduce that the left membership can be represented as :

$$\int_a^b \int_c^d p(x,s)\, q(y,t)\, f''_{s,t}(s,t)\, dsdt.$$

Consequently, we get the identity

$$\int_a^b \int_c^d p(x,s)\, q(y,t)\, f''_{s,t}(s,t)\, dsdt \tag{2.6}$$

$$= (d-c)(b-a) f(x,y) - (d-c) \int_a^b f(s,y)\, ds$$

$$- (b-a) \int_c^d f(x,t)\, dt + \int_a^b \int_c^d f(s,t)\, dsdt$$

for all $(x,y) \in [a,b] \times [c,d]$.

Now, using the identity (2.6) we get,

$$\left| \int_a^b \int_c^d f(s,t)\, dsdt - \left[(b-a) \int_c^d f(x,t)\, dt + (d-c) \int_a^b f(s,y)\, ds \right. \right.$$

$$\left. \left. - (d-c)(b-a) f(x,y) \right] \right|$$

$$\leq \int_a^b \int_c^d |p(x,s)|\, |q(y,t)|\, \left| f''_{s,t}(s,t) \right| dsdt$$

$$\leq \left\| f''_{s,t} \right\|_\infty \int_a^b \int_c^d |p(x,s)|\, |q(y,t)|\, dsdt.$$

Observe that

$$\int_a^b |p(x,s)|\, ds = \frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2$$

and,

$$\int_c^d |q(y,t)|\, dt = \frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2} \right)^2.$$

Finally, using (2.6), we get the desired inequality (2.1).

3 Bound on Estimation Error Variance. Constant Flow Rate

¿From [1], using an identity given in [8], it can be shown that

$$E\left[\left(\bar{X}-X(t)\right)^2\right] \tag{3.1}$$

$$=-\frac{1}{d^2}\int_0^d\int_0^d V(v-u)dudv+\frac{2}{d}\left\{\int_0^t V(u)du+\int_0^{d-t}V(u)du\right\}.$$

Assume that V is twice differentiable on $(-d,d)$ and having the second derivative V bounded on that interval.

Applying inequality [1] for the mapping $f(u,v)=V(v-u)$ we can state the inequality

$$\left|\int_0^d\int_0^d V(v-u)dudv-\left[d\int_0^d V(v-x)dv+d\int_0^d V(y-u)du-d^2V(y-x)\right]\right|$$

$$\le\left[\frac{1}{4}d^2+\left(x-\frac{d}{2}\right)^2\right]\left[\frac{1}{4}d^2+\left(y-\frac{d}{2}\right)^2\right]\|V''\|_\infty$$

for all $x,y\in[0,d]$.

Let $x=y=t$. Then we get

$$\left|\int_0^d\int_0^d V(v-u)dudv-\left[d\int_0^d V(v-t)dv+d\int_0^d V(t-u)du\right]\right| \tag{3.2}$$

$$\le\left[\frac{1}{4}d^2+\left(t-\frac{d}{2}\right)^2\right]^2\|V''\|_\infty$$

as $V(0)=0$.

Now, observe that

$$\int_0^d V(v-t)dv=\int_0^t V(u)du+\int_0^{d-t}V(u)du$$

and

$$\int_0^d V(t-u)du=\int_0^t V(u)du+\int_0^{d-t}V(u)du.$$

By the inequality (3.2) we get that

$$\left|\int_0^d\int_0^d V(v-u)dudv-2d\left[\int_0^t V(v)dv+\int_0^{d-t}V(v)dv\right]\right|$$

$$\le\left[\frac{1}{4}d^2+\left(t-\frac{d}{2}\right)^2\right]^2\|V''\|_\infty$$

and dividing by d^2

$$\left|\frac{1}{d^2}\int_0^d\int_0^d V(v-u)dudv-\frac{2}{d}\left[\int_0^t V(v)dv+\int_0^{d-t}V(v)dv\right]\right|$$

$$\leq\left[\frac{1}{4}+\frac{(t-\frac{d}{2})^2}{d^2}\right]^2 d^2\left\|V''\right\|_\infty .$$

Using the equation (3.1) we conclude that the following inequality for the variance $E\left[\left(\bar{X}-X\left(t\right)\right)^2\right]$ holds

$$E\left[\left(\bar{X}-X\left(t\right)\right)^2\right]\leq\left[\frac{1}{4}+\frac{\left(t-\frac{d}{2}\right)^2}{d^2}\right]^2 d^2\left\|V''\right\|_\infty . \tag{3.3}$$

Note that the best inequality we can get from (3.3) is that one for which $t=t_o=\frac{d}{2}$ giving the bound

$$E\left[\left(\bar{X}-X\left(t_o\right)\right)^2\right]\leq\frac{d^2}{16}\|V''\|_\infty .$$

Remark 3.1. *It should be noted that this result requires double differentiability of V in $(-d,d)$ and that this condition does not hold for the case of a linear variogram, i.e.,*

$$V(u)=a\mid u\mid,\quad u\in\mathbf{R}.$$

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School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC, Victoria 8001, Australia.
E-mail address: {neil, sever}@matilda.vut.edu.au

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