RGMIA Research Report Collection, Vol. 1, No. 2, 1998
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# AN OSTROWSKI TYPE INEQUALITY FOR MAPPINGS WHOSE SECOND DERIVATIVES ARE BOUNDED AND APPLICATIONS

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Abstract. An integral inequality of Ostrowski's type for mappings whose second derivatives are bounded is proved. Applications in Numerical Integration and for special means are pointed out.

## 1 Introduction

In [1], S.S. Dragomir and S. Wang obtained the following Ostrowski type inequality [2, p. 468]:

**Theorem 1.1.** *Let $f : [a,b] \to \mathbf{R}$ be continuous on $[a,b]$ and a differentiable on $(a,b)$. If $f' \in L_1(a,b)$ and there exists the constants $\gamma, \Gamma$ so that*

$$\gamma \le f'(x) \le \Gamma \quad \text{for all } x \in (a,b), \tag{1.1}$$

*then we have the inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \le \frac{1}{4}(b-a)(\Gamma-\gamma) \tag{1.2}$$

*for all $x \in [a,b]$.*

The proof used essentially the identity

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{b-a}\int_a^b p(x,t)\,f'(t)\,dt \tag{1.3}$$

for all $x \in [a,b]$, where $f$ is as above and the kernel $p(\cdot,\cdot) : [a,b]^2 \to \mathbf{R}$ is given by

$$p(x,t) := \begin{cases} t-a & \text{if } t \in [a,x] \\ t-b & \text{if } t \in (x,b] \end{cases} \tag{1.4}$$

and Grüss' integral inequality which says (see for example [1]) that:

$$\left| \frac{1}{b-a}\int_a^b g(x)h(x)\,dx - \frac{1}{b-a}\int_a^b g(x)\,dx \cdot \frac{1}{b-a}\int_a^b h(x)\,dx \right| \le \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma) \tag{1.5}$$

provided $g, h : [a,b] \to \mathbf{R}$ are integrable and

$$\varphi \le g(x) \le \Phi, \qquad \gamma \le h(x) \le \Gamma \tag{1.6}$$

for all $x \in [a,b]$.

The main aim of this paper is to point out a new estimation of the left membership of (1.2) and to apply it for special means and in Numerical Integration.

*Date.* December, 1998
*1991 Mathematics Subject Classification.* Primary 26 D 15; Secondary 41 A 55.
*Key words and phrases.* Ostrowski's Inequality, Numerical Integration, Special Means.

## 2 A New Integral Inequality

The following results holds:

**Theorem 2.1.** *Let $f : [a,b] \to \mathbf{R}$ be continuous on $[a,b]$ and twice differentiable on $(a,b)$, whose second derivative $f'' : (a,b) \to \mathbf{R}$ is bounded on $(a,b)$. Then we have the inequality*

(2.1)
$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right|$$
$$\le \frac{1}{2}\left\{\left[\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} + \frac{1}{4}\right]^2 + \frac{1}{12}\right\}(b-a)^2\,\|f''\|_\infty$$
$$\le \frac{\|f''\|_\infty}{6}(b-a)^2$$

*for all $x \in [a,b]$.*

*Proof.* For the sake of completness, we give a short proof of the identity (1.3) which will be used in the sequel.

Integrating by parts, we have

$$\int_a^x (t-a) f'(t)\,dt = (x-a) f(x) - \int_a^x f(t)\,dt$$

and

$$\int_x^b (t-b) f'(t)\,dt = (b-x) f(x) - \int_x^b f(t)\,dt.$$

Adding these two equalities, we get

$$\int_a^x (t-a) f'(t)\,dt + \int_x^b (t-b) f'(t)\,dt = (b-a) f(x) - \int_a^b f(t)\,dt$$

which is equivalent to (1.3).

Applying the identity (1.3) for $f'(\cdot)$ we can state

$$f'(t) = \frac{1}{b-a}\int_a^b f'(s)\,ds + \frac{1}{b-a}\int_a^b p(t,s) f''(s)\,ds$$

which is equivalent to

$$f'(t) = \frac{f(b)-f(a)}{b-a} + \frac{1}{b-a}\int_a^b p(t,s) f''(s)\,ds.$$

Substituting $f'(t)$ in the right membership of (1.3) we get

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt$$
$$+\frac{1}{b-a}\int_a^b p(x,t)\left[\frac{f(b)-f(a)}{b-a} + \frac{1}{b-a}\int_a^b p(t,s) f''(s)\,ds\right]dt$$

$$= \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{f(b)-f(a)}{(b-a)^2}\int_a^b p(x,t)\,dt$$

$$+\frac{1}{(b-a)^2}\int_a^b\int_a^b p(x,t)\,p(t,s)\,f''(s)\,dsdt$$

and as

$$\int_a^b p(x,t)\,dt = \int_a^x (t-a)\,dt + \int_x^b (t-b)\,dt$$

$$= (b-a)\left(x-\frac{a+b}{2}\right)$$

we get the integral identity:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right) \tag{2.2}$$

$$+\frac{1}{(b-a)^2}\int_a^b\int_a^b p(x,t)\,p(t,s)\,f''(s)\,dsdt$$

for all $x\in[a,b]$.

Now, using the identity (2.2) we get

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x-\frac{a+b}{2}\right)\right| \tag{2.3}$$

$$\leq \frac{1}{(b-a)^2}\int_a^b\int_a^b |p(x,t)\,p(t,s)|\,|f''(s)|\,dsdt$$

$$\leq \frac{\|f''\|_\infty}{(b-a)^2}\int_a^b\int_a^b |p(x,t)|\,|p(t,s)|\,dsdt.$$

We have

$$\int_a^b |p(t,s)|\,ds = \frac{(t-a)^2+(b-t)^2}{2}.$$

Also

$$A := \int_a^b |p(x,t)|\left[\frac{(t-a)^2+(b-t)^2}{2}\right]dt$$

$$= \frac{1}{2}\left[\int_a^x (t-a)\left[(t-a)^2+(b-t)^2\right]dt + \int_x^b (b-t)\left[(t-a)^2+(b-t)^2\right]dt\right]$$

$$= \frac{1}{2}\left[\int_a^x \left[(t-a)^3 + (t-a)(b-t)^2\right] dt + \int_x^b \left[(t-a)^2(b-t) + (b-t)^3\right] dt\right].$$

Note that

$$\int_a^x (t-a)^3 \, dt = \frac{(x-a)^4}{4},$$

$$\int_a^x (t-a)(b-t)^2 \, dt$$

$$= -\frac{1}{3}(b-x)^3(x-a) - \frac{1}{12}(b-x)^4 + \frac{1}{12}(b-x)^4;$$

$$\int_x^b (t-b)(t-a)^2 \, dt$$

$$= \frac{1}{3}(x-a)^3(b-x) - \frac{1}{12}(b-a)^4 + \frac{1}{12}(x-a)^4;$$

$$\int_x^b (t-b)^3 \, dt = \frac{(x-b)^4}{4}.$$

Consequently, we have

$$A = \frac{1}{2}\left[\frac{(x-a)^4}{4} - \frac{1}{3}(b-x)^3(x-a) - \frac{1}{12}(b-x)^4 + \frac{1}{12}(b-a)^4\right.$$

$$\left. -\frac{1}{3}(x-a)^3(b-x) + \frac{1}{12}(b-a)^4 - \frac{1}{12}(x-a)^4 + \frac{(x-b)^4}{4}\right]$$

$$= \frac{1}{12}\left[(x-a)^4 - 2(b-x)^3(x-a) - 2(x-a)^3(b-x)\right.$$

$$\left. + (b-x)^4 + (b-a)^4\right].$$

Now, observe that,

$$(x-a)^4 + (b-x)^4 = \left[(x-a)^2 + (b-x)^2\right]^2 - 2(x-a)^2(b-x)^2$$

and

$$-2(b-x)^3(x-a) - 2(x-a)^3(b-x)$$

$$= -2(x-a)(b-x)\left[(x-a)^2 + (b-x)^2\right]$$

then

$$B:=12A=\left[(x-a)^{2}+(b-x)^{2}\right]^{2}-2(x-a)(b-x)\left[(x-a)^{2}+(b-x)^{2}\right]$$

$$-2(x-a)^{2}(b-x)^{2}+(b-a)^{4}$$

$$=\left[(x-a)^{2}+(b-x)^{2}-(x-a)(b-x)\right]^{2}-3(x-a)^{2}(b-x)^{2}+(b-a)^{4}.$$

But a simple calculation shows that

$$(x-a)^{2}+(b-x)^{2}=\frac{1}{2}(b-a)^{2}+2\left(x-\frac{a+b}{2}\right)^{2}$$

and as

$$(x-a)^{2}+(b-x)^{2}+2(x-a)(b-x)=(b-a)^{2}$$

we get

$$2(x-a)(b-x)=(b-a)^{2}-\left[(x-a)^{2}+(b-x)^{2}\right]$$

i.e.,

$$(x-a)(b-x)=\frac{1}{2}(b-a)^{2}-\frac{1}{2}\left[(x-a)^{2}+(b-x)^{2}\right]$$

$$=\frac{1}{4}(b-a)^{2}-\left(x-\frac{a+b}{2}\right)^{2}.$$

Consequently,

$$B=\left[\frac{1}{2}(b-a)^{2}+2\left(x-\frac{a+b}{2}\right)^{2}-\frac{1}{4}(b-a)^{2}+\left(x-\frac{a+b}{2}\right)^{2}\right]^{2}-$$

$$-3\left[\frac{1}{4}(b-a)^{2}-\left(x-\frac{a+b}{2}\right)^{2}\right]^{2}+(b-a)^{4}$$

$$=6\left(x-\frac{a+b}{2}\right)^{2}+3(b-a)^{2}\left(x-\frac{a+b}{2}\right)^{2}+\frac{7}{8}(b-a)^{4}$$

and then

$$A=\frac{1}{12}\left[6\left(x-\frac{a+b}{2}\right)^{4}+3(b-a)^{2}\left(x-\frac{a+b}{2}\right)^{2}+\frac{7}{8}(b-a)^{4}\right].$$

Now, using the inequality (2.3) and simple algebraic manipulations, we get the first result in (2.1).

The second part is obvious by the fact that

$$\left|x-\frac{a+b}{2}\right|\leq\frac{b-a}{2}$$

for all $x\in[a,b]$. ∎

## 3 Applications in Numerical Integration

Let $I_h : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$, $\xi_i \in [x_i, x_{i+1}]$ $(i = 0,1,...,n-1)$ a sequence of intermediate points and $h_i := x_{i+1} - x_i$ $(i = 0,1,...,n-1)$. As in [1], consider the perturbed Riemann's sum defined by

$$A_G(f, I_h, \boldsymbol{\xi}) := \sum_{i=0}^{n-1} f(\xi_i) h_i - \sum_{i=0}^{n-1} \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)(f(x_{i+1}) - f(x_i)). \tag{3.1}$$

In that paper Dragomir and Wang proved the following result:

**Theorem 3.1.** *Let $f : [a,b] \to \mathbf{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)$, whose derivative $f' : (a,b) \to \mathbf{R}$ is bounded on $(a,b)$ and assume that*

$$\gamma \leq f'(x) \leq \Gamma \quad \text{for all } x \in (a,b). \tag{3.2}$$

*Then we have the quadrature formula:*

$$\int_a^b f(x)\,dx = A_G(f, I_h, \boldsymbol{\xi}) + R_G(f, I_h, \boldsymbol{\xi}) \tag{3.3}$$

*and the remainder $R_G(f, I_h, \boldsymbol{\xi})$ satisfies the estimation*

$$|R_G(f, I_h, \boldsymbol{\xi})| \leq \frac{1}{4}(\Gamma - \gamma) \sum_{i=0}^{n-1} h_i^2, \tag{3.4}$$

*for all $\boldsymbol{\xi} = (\xi_0, ..., \xi_{n-1})$ as above.*

Here, we prove another type of estimation for the remainder $R_G(f, I_h, \boldsymbol{\xi})$ in the case when $f$ is twice differentiable.

**Theorem 3.2.** *Let $f : [a,b] \to \mathbf{R}$ be continuous on $[a,b]$ and twice differentiable on $(a,b)$, whose second derivative $f'' : (a,b) \to \mathbf{R}$ is bounded on $(a,b)$. Denote $\|f''\|_\infty := \sup_{t \in (a,b)} |f''(t)| < \infty$. Then we have the quadrature formula (3.3) and the remainder $R_G(f, I_h, \boldsymbol{\xi})$ satisfies the estimation:*

$$\begin{aligned} |R_G(f, I_h, \boldsymbol{\xi})| &\leq \frac{\|f''\|_\infty}{2} \sum_{i=0}^{n-1} \left\{ \left[ \frac{\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2}{h_i^2} + \frac{1}{4} \right]^2 + \frac{1}{12} \right\} h_i^3 \\ &\leq \frac{\|f''\|_\infty}{6} \sum_{i=0}^{n-1} h_i^3 \end{aligned} \tag{3.5}$$

*for all $\xi_i$ as above.*

*Proof.* Apply Theorem 2.1 on the interval $[x_i, x_{i+1}]$ $(i = 0, \ldots, n-1)$ to obtain

$$\begin{aligned} &\left| f(\xi_i) h_i - \int_{x_i}^{x_{i+1}} f(t)\,dt - \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)(f(x_{i+1}) - f(x_i)) \right| \\ &\leq \frac{\|f''\|_\infty}{2} \left[ \left[ \frac{\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2}{h_i^2} + \frac{1}{4} \right]^2 + \frac{1}{12} \right] h_i^3 \leq \frac{\|f''\|_\infty}{6} h_i^3 \end{aligned}$$

for all $\xi_i \in [x_i, x_{i+1}]$ and $i \in \{0, \ldots, n-1\}$.

Summing over $i$ from 0 to $n-1$ and using the generalized triangle inequality, we get the desired inequality (3.5).

We omit the details. ∎

## 4 Applications for Special Means

Recall the following means :

($a$) The arithmetic mean

$$A = A(a,b) := \frac{a+b}{2}, \qquad a,b \geq 0;$$

($b$) The geometric mean:

$$G = G(a,b) := \sqrt{ab}, \qquad a,b \geq 0;$$

($c$) The harmonic mean:

$$H = H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}, \qquad a,b \geq 0;$$

($d$) The logarithmic mean:

$$L = L(a,b) := \begin{cases} a & \text{if } a = b \\ \dfrac{b-a}{\ln b - \ln a} & \text{if } a \neq b \end{cases} \quad a,b > 0;$$

($e$) The identric mean:

$$I := I(a,b) = \begin{cases} a & \text{if } a = b \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a \neq b \end{cases} \quad a,b > 0;$$

($f$) The $p$-logarithmic mean:

$$L_p = L_p(a,b) := \begin{cases} \left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a \neq b; \\ a & \text{if } a = b \end{cases}$$

where $p \in \mathbf{R} \backslash \{-1, 0\}$ and $a,b > 0$.

The following simple relationships are well known in the literature

$$H \leq G \leq L \leq I \leq A \tag{4.1}$$

and

(4.2) $L_p$ is monotonically increasing in $p \in \mathbf{R}$ with $L_0 := I$ and $L_{-1} := L$.

1. Consider the mapping $f(x) = x^p$ $(p \geq 2)$ on $[a,b] \subset (0,\infty)$.
   Applying the inequality (2.1) for $f(x) = x^p$, we get:

$$\left| x^p - L_p^p - pL_{p-1}^{p-1}(x-A) \right| \tag{4.3}$$

$$\leq \frac{p(p-1)b^{p-2}}{2}\left\{\left[\frac{(x-A)^2}{(b-a)^2}+\frac{1}{4}\right]^2+\frac{1}{12}\right\}(b-a)^2$$

$$\leq \frac{p\,(p-1)\,b^{p-2}}{6}\,(b-a)^2$$

for all $x \in [a,b]$.

Choosing in(4.3), $x = A$, we get

$$0 \leq L_p^p - A^p \leq \frac{7}{96} p\,(p-1)\,b^{p-2}\,(b-a)^2 . \tag{4.4}$$

2. Consider the mapping $f(x) = \frac{1}{x}$ $(x \in [a,b] \subset (0,\infty))$.

   Applying the inequality (2.1) for this mapping we get:

$$\left| \frac{1}{x} - \frac{1}{L} - \frac{x-A}{G^2} \right| \tag{4.5}$$

$$\leq \frac{1}{3a^3} \left\{ \left[ \frac{(x-A)^2}{(b-a)^2} + \frac{1}{4} \right]^2 + \frac{1}{12} \right\} (b-a)^2 \leq \frac{1}{3a^3}\,(b-a)^2$$

   for all $x \in [a,b]$.

   Choosing in (4.5) $x = A$, we get

$$0 \leq \frac{A-L}{AL} \leq \frac{7}{48a^3}\,(b-a)^2 . \tag{4.6}$$

   Also, choosing in (4.5) $x = L$, we get

$$0 \leq \frac{A-L}{G^2} \leq \frac{1}{3a^3} \left\{ \left[ \frac{(x-A)^2}{(b-a)^2} + \frac{1}{4} \right]^2 + \frac{1}{12} \right\} (b-a)^2 \tag{4.7}$$

$$\leq \frac{1}{3a^3}\,(b-a)^2 .$$

3. Finally, let us consider the mapping $f(x) = -\ln x$ $(x \in [a,b] \subset (0,\infty))$. Then, by (2.1), we get:

$$\left| \ln \left( \frac{I \left( \frac{b}{a} \right)^{\frac{x-A}{b-a}}}{x} \right) \right| \leq \frac{1}{2a^2} \left\{ \left[ \frac{(x-A)^2}{(b-a)^2} + \frac{1}{4} \right]^2 + \frac{1}{12} \right\} (b-a)^2 \leq \frac{1}{6a^2}\,(b-a)^2 \tag{4.8}$$

   for all $x \in [a,b]$.

   Putting $x = A$ in (4.8) we get

$$1 \leq \frac{A}{I} \leq \exp\left[ \frac{7}{96a^2}\,(b-a)^2 \right] . \tag{4.9}$$

**Acknowledgement.** The authors would like to thank *Professor* Peter Cerone for his suggestions in improving this paper.

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