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AN INEQUALITY OF OSTROWSKI-GRÜSS TYPE FOR TWICE DIFFERENTIABLE MAPPINGS AND APPLICATIONS IN NUMERICAL INTEGRATION

P. CERONE, S.S. DRAGOMIR AND J. ROUMELIOTIS

Abstract. In this paper we derive a new integral inequality of Ostrowski-Grüss type and apply it to estimate the error bounds for some numerical quadrature rules.

1 Introduction

The following integral inequality which establishes a connection between the integral of the product of two functions and the product of the integrals of the two functions is well known in the literature as Grüss' inequality (see for example [1, p. 296]).

Theorem 1.1. *Let $f, g : [a, b] \to \mathbf{R}$ be two integrable functions such that $\varphi \le f(x) \le \Phi$ and $\gamma \le g(x) \le \Gamma$ for all $x \in [a, b]$, φ, Φ, γ and Γ are constants. Then we have:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx \cdot \frac{1}{b-a}\int_a^b g(x)\,dx\right| \le \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma) \tag{1.1}$$

The constant $\frac{1}{4}$ is sharp.

Another celebrated integral inequality which provides an approximation of the integral mean $\frac{1}{b-a}\int_a^b f(t)\,dt$ in terms of the values of f at a certain point $x \in [a, b]$, is Ostrowski's inequality [2, p. 468]:

Theorem 1.2. *Let $f : (a, b) \to \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b), whose derivative $f' : [a, b] \to \mathbf{R}$ is bounded on (a, b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then we have the inequality*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\,\|f'\|_\infty \tag{1.2}$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is sharp.

In the recent paper [3], S.S. Dragomir and S. Wang proved the following Ostrowski type inequality in terms of the lower and upper bounds of the first derivative.

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Theorem 1.3. *Let $f : [a,b] \to \mathbf{R}$ be continuous on $[a,b]$ and differentiable on (a,b), whose derivative satisfies the condition:*

$$\gamma \leq f'(x) \leq \Gamma \quad \text{for all } x \in (a,b). \tag{1.3}$$

Then we have the inequality:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \leq \frac{1}{4}(b-a)(\Gamma-\gamma) \tag{1.4}$$

for all $x \in [a,b]$.

In that paper [3], the authors applied this result in Numerical Integration and for special means (identric mean, logarithmic mean, Stolarski's mean etc...).

The main aim of this article is to point out a similar result to (1.4) for twice differentiable mappings in terms of the upper and lower bounds of the second derivative.

Some applications in Numerical Analysis are also given.

2 The Results

The following result holds

Theorem 2.1. *Let $f : [a,b] \to \mathbf{R}$ be continuous on $[a,b]$ and twice differentiable on (a,b), and assume that the second derivative $f'' : (a,b) \to \mathbf{R}$ satisfies the condition:*

$$\varphi \leq f'' \leq \Phi \quad \text{for all } x \in (a,b). \tag{2.1}$$

Then we have the inequality

$$\begin{aligned} &\Bigg| f(x) - \left(x - \frac{a+b}{2}\right) f'(x) \\ &+ \left[\frac{(b-a)^2}{24} + \frac{1}{2}\left(x - \frac{a+b}{2}\right)^2\right]\frac{f'(b)-f'(a)}{b-a} \\ &- \frac{1}{b-a}\int_a^b f(t)\,dt \Bigg| \\ &\leq \frac{1}{8}(\Phi-\varphi)\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^2 \end{aligned} \tag{2.2}$$

for all $x \in [a,b]$.

Proof. Let us prove the following integral identity:

$$\int_a^b f(t)\,dt = (b-a) f(x) - (b-a)\left(x - \frac{a+b}{2}\right) f'(x) + \int_a^b K(x,t) f''(t)\,dt \tag{2.3}$$

where the kernel $K : [a,b]^2 \to \mathbf{R}$ is defined by

$$K(x,t) := \begin{cases} \dfrac{(t-a)^2}{2} & \text{if } t \in [a,x] \\ \dfrac{(t-b)^2}{2} & \text{if } t \in (x,b]. \end{cases}$$

We have successively

$$\int_a^b K(x,t) f''(t)\,dt = \int_a^x \frac{(t-a)^2}{2} f''(t)\,dt + \int_x^b \frac{(t-b)^2}{2} f''(t)\,dt,$$

and integrating by parts twice and simplifying gives

$$\int_a^b K(x,t) f''(t)\,dt$$

$$= \frac{1}{2}\left[(x-a)^2 - (b-x)^2\right] f'(x) - (x-a) f(x)$$

$$+ \int_a^x f(t)\,dt + (x-b) f(x) + \int_x^b f(t)\,dt$$

$$= (b-a)\left(x - \frac{a+b}{2}\right) f'(x) - (b-a) f(x) + \int_a^b f(t)\,dt$$

and the identity (2.3) is proved.

Now, let us observe that the Kernel K satisfies the estimation

$$0 \le K(x,t) \le \begin{cases} \dfrac{(b-x)^2}{2}, & x \in \left[a, \frac{a+b}{2}\right) \\[2ex] \dfrac{(x-a)^2}{2}, & x \in \left[\frac{a+b}{2}, b\right] \end{cases} \tag{2.4}$$

for all $t \in [a,b]$.

Applying Grüss' integral inequality for the mappings $f''(\cdot)$ and $K(x,\cdot)$ we get

$$\left| \frac{1}{b-a}\int_a^b K(x,t) f''(t)\,dt - \frac{1}{b-a}\int_a^b K(x,t)\,dt \cdot \frac{1}{b-a}\int_a^b f''(t)\,dt \right| \tag{2.5}$$

$$\le \frac{1}{4}(\Phi - \varphi) \times \begin{cases} \dfrac{(b-x)^2}{2}, & x \in \left[a, \frac{a+b}{2}\right) \\[2ex] \dfrac{(x-a)^2}{2}, & x \in \left[\frac{a+b}{2}, b\right]. \end{cases}$$

Now, let us observe that

$$\int_a^b K(x,t)\,dt = \int_a^x \frac{(t-a)^2}{2} dt + \int_x^b \frac{(b-x)^2}{2dt} dt = \frac{1}{6}\left[(x-a)^3 + (b-x)^3\right].$$

Also, a simple computation shows that

$$\begin{aligned}
(x-a)^3+(b-x)^3 &= (b-a)\left[(x-a)^2+(b-x)^2-(x-a)(b-x)\right] \\
&= (b-a)\left[(b-a)^2-3(x-a)(b-x)\right] \\
&= (b-a)\left[(b-a)^2+3\left[x^2-(a+b)x+ab\right]\right] \\
&= (b-a)\left[(b-a)^2+3\left[\left(x-\frac{a+b}{2}\right)^2-\left(\frac{b-a}{2}\right)^2\right]\right] \\
&= (b-a)\left[\frac{(b-a)^2}{4}+3\left(x-\frac{a+b}{2}\right)^2\right].
\end{aligned}$$

Consequently

$$\int_a^b K(x,t)\,dt = (b-a)\left[\frac{(b-a)^2}{24}+\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2\right].$$

Using (2.5) we can state

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b K(x,t)\,f''(t)\,dt\right. \\
&\left.-\left[\frac{(b-a)^2}{24}+\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2\right]\cdot\frac{f'(b)-f'(a)}{b-a}\right| \\
&\leq \frac{1}{4}(\Phi-\varphi)\times\begin{cases}\dfrac{(b-x)^2}{2}, & \text{if } x\in\left[a,\frac{a+b}{2}\right) \\[2ex] \dfrac{(x-a)^2}{2}, & \text{if } x\in\left[\frac{a+b}{2},b\right].\end{cases}
\end{aligned} \tag{2.6}$$

Using the identity (2.3), then by (2.6) we get

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-f(x)+\left(x-\frac{a+b}{2}\right)f'(x)\right. \\
&\left.-\left[\frac{(b-a)^2}{24}+\frac{1}{2}\left(x-\frac{a+b}{2}\right)^2\right]\cdot\frac{f'(b)-f'(a)}{b-a}\right| \\
&\leq \frac{1}{4}(\Phi-\varphi)\times\begin{cases}\dfrac{(b-x)^2}{2}, & \text{if } x\in\left[a,\frac{a+b}{2}\right) \\[2ex] \dfrac{(x-a)^2}{2}, & \text{if } x\in\left[\frac{a+b}{2},b\right].\end{cases}
\end{aligned}$$

Now, let us observe that

$$\max\left\{\frac{(b-x)^2}{2},\frac{(x-a)^2}{2}\right\}=\begin{cases}\dfrac{(b-x)^2}{2}, & \text{if } x\in\left[a,\frac{a+b}{2}\right)\\[2ex] \dfrac{(x-a)^2}{2}, & \text{if } x\in\left[\frac{a+b}{2},b\right].\end{cases}$$

On the other hand,

$$\begin{aligned}&\max\left\{\frac{(b-x)^2}{2},\frac{(x-a)^2}{2}\right\}\\ &= \frac{1}{2}\left[\frac{(b-x)^2+(x-a)^2}{2}+\frac{1}{2}\left|(b-x)^2-(x-a)^2\right|\right]\\ &= \frac{1}{2}\left[\frac{(b-a)^2}{4}+\left(x-\frac{a+b}{2}\right)^2+(b-a)\left|x-\frac{a+b}{2}\right|\right]\\ &= \frac{1}{2}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^2\end{aligned}$$

and the inequality (2.2) is proved. ∎

Corollary 2.2. *Let f be as in Theorem 2.1. Then we have the perturbed midpoint inequality:*

$$\left|f\left(\frac{a+b}{2}\right)+\frac{1}{24}(b-a)\left(f'(b)-f'(a)\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{32}(\Phi-\varphi)(b-a)^2. \tag{2.7}$$

Remark 2.1. *Let us recall the classical midpoint inequality which says that*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{24}(b-a)^2\left\|f''\right\|_\infty \tag{2.8}$$

where

$$\left\|f''\right\|_\infty:=\sup_{t\in(a,b)}\left|f''(t)\right|.$$

Note that if $\Phi-\varphi\leq\frac{4}{3}\left\|f''\right\|_\infty$, then the estimation provided by (2.7) *is better than the estimation given in* (2.8). *A sufficient condition for the assumption $\Phi-\varphi\leq\frac{4}{3}\left\|f''\right\|_\infty$ to be true is $0\leq\varphi\leq\Phi$. Indeed, if $0\leq\varphi\leq\Phi$, then $\Phi-\varphi\leq\left\|f''\right\|_\infty<\frac{4}{3}\left\|f''\right\|_\infty$.*

Corollary 2.3. *Let f be as in Theorem 2.1 Then we have the following perturbed trapezoid inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{12}(b-a)\left(f'(b)-f'(a)\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq\frac{1}{8}(\Phi-\varphi)(b-a)^2. \tag{2.9}$$

Proof. Put in (2.2) $x=a$ and $x=b$ to get

$$\left| f(a)+\frac{b-a}{2}f'(a)+\frac{1}{6}(b-a)(f'(b)-f'(a))-\frac{1}{b-a}\int_a^b f(t)\,dt \right|$$

$$\le \frac{1}{8}(\Phi-\varphi)(b-a)^2$$

and

$$\left| f(b)-\frac{b-a}{2}f'(b)+\frac{1}{6}(b-a)(f'(b)-f'(a))-\frac{1}{b-a}\int_a^b f(t)\,dt \right|$$

$$\le \frac{1}{8}(\Phi-\varphi)(b-a)^2$$

respectively.

Summing the above two inequalities, using the triangle inequality and dividing by 2 we get

$$\left| \frac{f(a)+f(b)}{2}-\frac{b-a}{4}(f'(b)-f'(a))+\frac{1}{6}(b-a)(f'(b)-f'(a))-\frac{1}{b-a}\int_a^b f(t)\,dt \right|$$

$$\le \frac{1}{8}(\Phi-\varphi)(b-a)^2$$

i.e., (2.9). ∎

Remark 2.2. *The classical trapezoid inequality states that*

$$\left| \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \frac{1}{12}\|f''\|_\infty (b-a)^2 . \tag{2.10}$$

Now, if we assume that $\Phi-\varphi\le\frac{2}{3}\|f''\|_\infty$, *and this condition holds if we assume that the infimum and supremum of the second derivative* f'' *are close enough, then the estimation provided by* (2.9) *is better than the estimation in the classical trapezoid inequality* (2.10).

3 Applications for Composite Quadrature Formulae

Let $I_n : a=x_0<x_1<...<x_{n-1}<x_n=b$ be a division of the interval $[a,b]$ and $\boldsymbol{\xi}=(\xi_0,...,\xi_{n-1})$ a sequence of intermediate points, $\xi_i\in[x_i,x_{i+1}]$ $(i=0,...,n-1)$.

The following result holds.

Theorem 3.1. *Let* $f:[a,b]\to\mathbf{R}$ *be continuous on* $[a,b]$ *and twice differentiable on* (a,b), *whose second derivative* $f'':(a,b)\to\mathbf{R}$ *satisfies the assumption:*

$$\varphi\le f''(x)\le\Phi \qquad \text{for all } x\in(a,b). \tag{3.1}$$

Then we have

$$\int_a^b f(t)\,dt = A(f,f',I_n,\boldsymbol{\xi})+R(f,f',I_n,\boldsymbol{\xi}) \tag{3.2}$$

where

$$A\left(f,f',I_n,\boldsymbol{\xi}\right) := \sum_{i=0}^{n-1} h_i f\left(\xi_i\right) - \sum_{i=0}^{n-1}\left(\xi_i - \frac{x_i+x_{i+1}}{2}\right) f'\left(\xi_i\right) h_i \tag{3.3}$$

$$+\sum_{i=0}^{n-1}\left[\frac{h_i^2}{24}+\frac{1}{2}\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2\right]\left(f'\left(x_{i+1}\right)-f'\left(x_i\right)\right).$$

The remainder $R\left(f,f',I_n,\boldsymbol{\xi}\right)$ *satisfies the estimation:*

$$\left|R\left(f,f',I_n,\boldsymbol{\xi}\right)\right| \le \frac{1}{8}\left(\Phi-\varphi\right)\sum_{i=0}^{n-1} h_i\left[\frac{1}{2}h_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^2 \tag{3.4}$$

$$\le \frac{1}{8}\left(\Phi-\varphi\right)\sum_{i=0}^{n-1} h_i^3$$

for any choice $\boldsymbol{\xi}$ *of the intermediate points.*

Proof. Apply Theorem 2.1 on the interval $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ to get:

$$\left|h_i f\left(\xi_i\right)-\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right) f'\left(\xi_i\right) h_i+\left[\frac{h_i^2}{24}+\frac{1}{2}\left(\xi_i-\frac{x_i+x_{i+1}}{2}\right)^2\right]\right.$$

$$\left.\times\left(f'\left(x_{i+1}\right)-f'\left(x_i\right)\right)-\int_{x_i}^{x_{i+1}} f(t)\,dt\right|$$

$$\le \frac{1}{8}\left(\Phi-\varphi\right) h_i\left[\frac{1}{2}h_i+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^2 \le \frac{1}{8}\left(\Phi-\varphi\right) h_i^3$$

as

$$\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right| \le \frac{h_i}{2} \qquad \text{for all } i \in \{0,...,n-1\}$$

for any choice ξ_i of the intermediate points.

Summing the above inequalities over i from 0 to $n-1$, and using the generalized triangle inequality, we get the desired estimation (3.4). ∎

Corollary 3.2. *The following perturbed midpoint rule holds:*

$$\int_a^b f(x)\,dx = M\left(f,f',I_n\right)+R_M\left(f,f',I_n\right) \tag{3.5}$$

where

$$M\left(f,f',I_n\right) := \sum_{i=0}^{n-1} f\left(\frac{x_i+x_{i+1}}{2}\right) h_i+\frac{1}{24}\sum_{i=0}^{n-1} h_i^2\left(f'\left(x_{i+1}\right)-f'\left(x_i\right)\right) \tag{3.6}$$

and the remainder term $R_M\left(f,f',I_n\right)$ *satisfies the estimation.*

$$\left|R_M\left(f,f',I_n\right)\right| \le \frac{1}{32}\left(\Phi-\varphi\right)\sum_{i=0}^{n-1} h_i^3. \tag{3.7}$$

Corollary 3.3. *The following perturbed trapezoid rule holds.*

$$\int_a^b f(x)\,dx = T(f, f', I_n) + R_T(f, f', I_n) \tag{3.8}$$

where

$$T(f, f', I_n) := \frac{1}{2}\sum_{i=0}^{n-1} [f(x_i) + f(x_{i+1})]\, h_i \tag{3.9}$$

$$-\frac{1}{12}\sum_{i=0}^{n-1} h_i^2 (f'(x_{i+1}) - f'(x_i))$$

and the remainder term $R_T(f, f', I_n)$ satisfies the estimation:

$$|R_T(f, f', I_n)| \le \frac{1}{8}(\Phi - \varphi)\sum_{i=0}^{n-1} h_i^3. \tag{3.10}$$

Remark 3.1. *Note that both perturbed midpoint formula* (3.6) *and perturbed trapezoid formula* (3.9) *can offer better approximations of the integral* $\int_a^b f(x)\,dx$ *for general classes of mappings as discussed in Remark 2.1 and Remark 2.2. Moreover, if we consider equidistant partitioning of $[a, b]$, then the perturbed term in both formulae will involve the calculation for f' only at the endpoints a and b, which is a good advantage for practical applications.*

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School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC, Victoria 8001, Australia.

E-mail address: {pc, sever, johnr}@matilda.vut.edu.au