

## Adaptive Backstepping Controller Design for Stochastic Jump Systems

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**Abstract**—In this technical note, we improve the results in a paper by Shi *et al.*, in which problems of stochastic stability and sliding mode control for a class of linear continuous-time systems with stochastic jumps were considered. However, the system considered is switching stochastically between different subsystems, the dynamics of the jump system can not stay on each sliding surface of subsystems forever, therefore, it is difficult to determine whether the closed-loop system is stochastically stable. In this technical note, the backstepping techniques are adopted to overcome the problem in a paper by Shi *et al.*. The resulting closed-loop system is bounded in probability. It has been shown that the adaptive control problem for the Markovian jump systems is solvable if a set of coupled linear matrix inequalities (LMIs) have solutions. A numerical example is given to show the potential of the proposed techniques.

**Index Terms**—Adaptive control, backstepping control, linear matrix inequality, Markovian jump system, stochastic stability.

### I. INTRODUCTION

IT is well known that many physical systems have different structures due to random abrupt changes, which may be caused by random failures and repairs of the components, changes in the interconnections of subsystems, sudden environment changes, modification of the operating point of a linearized model of a nonlinear system, etc. The *hybrid* systems, which involve both time-evolving and event-driven mechanisms, may be employed to model the above problems. One special class of hybrid systems is the so-called Markovian jump linear system (MJLS). A MJLS is a hybrid one with many operation modes, and every mode corresponds to a deterministic system. The system mode switching is governed by a Markov process. A number of control problems related to MJLS systems has been analyzed by several authors; see, e.g., [1]–[10] and the references therein.

Moreover, the sliding-mode control (SMC) has received relatively a lot of attention since it has various attractive features such as fast response, good transient performance, order-reduction and so on. In particular, SMC laws are robust with respect to the so-called matched uncertainty, see, e.g., [11]–[18]. Recently, the sliding mode control is proposed to stabilize MJLS with matched uncertainties and disturbances [1]. However, system is switching stochastically between different subsystems, the dynamics of the jump systems can not stay on

each sliding surface of subsystems forever, therefore, it can not be determined whether the closed-loop system is stochastically stable. This motivated us to study the above systems with Markovian jumps further.

In this technical note, we consider the problem of adaptive backstepping controller design for stochastic jump systems with matched uncertainties and disturbances. The jumping parameters are treated as continuous-time, discrete-state Markov process. Note that backstepping method is one of the most popular techniques of nonlinear control design [19]–[27]. In [22], the backstepping method is proposed to design a memoryless state feedback controller for a class of uncertain time-delay systems, but it can not solve the control problem for system with matched disturbances and Markov jumping. In this technical note, adaptive backstepping controller for the system will be designed. Unknown upper bounds of uncertainties and disturbances can be estimated by adaptive control method ([28]–[30]). The above problems are solved in terms of a finite set of coupled linear matrix inequalities (LMIs). Finally, a numerical example is included to demonstrate the effectiveness of the theoretical results obtained.

**Notations:** The notation used in this technical note is quite standard. In the sequel, the Euclidean norm is used for vectors. We use  $W^T$ ,  $W^{-1}$ ,  $\lambda(W)$ ,  $Tr(W)$  and  $\|W\|$  to denote, respectively, the transpose, the inverse, the eigenvalues, the trace and the induced norm of any square matrix  $W$ . We use  $W > 0$  ( $\geq, <, \leq 0$ ) to denote a symmetric positive definite (positive semi-definite, negative, negative semi-definite) matrix  $W$  with  $\lambda_{\min}(W)$  and  $\lambda_{\max}(W)$  being the minimum and maximum eigenvalues of  $W$  and  $I$  to denote the  $n \times n$  identity matrix.  $C^k$  denotes the space of  $k$ -times continuously differentiable functions. The Lebesgue space  $\mathcal{L}_2[0, T]$  consists of square-integrable functions on the interval  $[0, T]$  equipped with the norm  $\|\cdot\|_2$ .  $\mathcal{E}[\cdot]$  stands for mathematical expectation. Given a probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  where  $\Omega$  is the sample space,  $\mathcal{F}$  is the algebra of events and  $\mathbf{P}$  is the probability measure defined on  $\mathcal{F}$ . Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

### II. PROBLEM FORMULATION AND PRELIMINARIES

We consider a class of stochastic systems with Markovian jump parameters in a fixed probability space  $(\Omega, \mathcal{F}, \mathbf{P})$

$$\dot{x}(t) = A(\eta_t)x(t) + B(\eta_t)[u(t) + F(\eta_t)w(x, t)], \quad \eta_0 = i, \quad t \geq 0 \quad (1)$$

where  $x(t) \in R^n$  is the state vector;  $u(t) \in R^m$  is the control input,  $w \in R^l$  is the disturbance, while  $\{\eta_t, t \in [0, T]\}$  is a finite-state Markovian process having a state space  $S \triangleq \{1, 2, \dots, \nu\}$ , generator  $(\alpha_{ij})$  with transition probability from mode  $i$  at time  $t$  to mode  $j$  at time  $t + \delta$ ,  $i, j \in S$

$$p_{ij} = Pr(\eta_{t+\delta} = j \mid \eta_t = i) = \begin{cases} \alpha_{ij}\delta + o(\delta), & \text{if } i \neq j \\ 1 + \alpha_{ii}\delta + o(\delta), & \text{if } i = j \end{cases} \quad (2)$$

$$\alpha_{ii} = - \sum_{m=1, m \neq i}^{\nu} \alpha_{im}, \quad \alpha_{ij} \geq 0 \quad \forall i, j \in S, i \neq j \quad (3)$$

where  $\delta > 0$  and  $\lim_{\delta \rightarrow 0} o(\delta)/\delta = 0$ .

For  $\mathcal{V}(t, x) \in C^1$ , let us introduce the weak infinitesimal operator  $\mathfrak{F}_k^x[\cdot]$  of the process  $\{x(t), \eta_t, t \geq 0\}$  at the point  $\{t, x, k\}$  [31], [32]

$$\mathfrak{F}_k^x[\mathcal{V}] = \frac{\partial \mathcal{V}}{\partial t} + \frac{\partial \mathcal{V}}{\partial x} \dot{x}(t) + \sum_{j=1}^{\nu} \alpha_{kj} \mathcal{V}(x, j). \quad (4)$$

For each possible value  $\eta_t = k$ ,  $k \in S$ , we will denote the system matrices associated with mode  $i$  by

Manuscript received December 18, 2007; revised November 17, 2008. This work was supported by the National Natural Science Foundation of China under Grant 60504020, the Program for New Century Excellent Talents in University of People's Republic of China, NCET, and the Excellent young scholars Research Fund of Beijing Institute of Technology 2008YS0104, respectively, and by the Engineering and Physical Sciences Research Council, UK (EP/F029195). Recommended by Associate Editor V. Krishnamurthy.

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Digital Object Identifier 10.1109/TAC.2009.2033131

$$A(\eta_t) \triangleq A(k), \quad B(\eta_t) \triangleq B(k), \quad F(\eta_t) \triangleq F(k)$$

where  $A(k)$ ,  $B(k)$  and  $F(k)$  are known real constant matrices of appropriate dimensions which describe the nominal system.

*Assumption 2.1:* The matched uncertainties  $w(x, t)$  are assumed to satisfy the following condition

$$\|F(\eta_t)w(x, t)\| \leq c + \kappa\|x(t)\| = \rho \quad (5)$$

where  $c$  and  $\kappa$  are constants, but it may not be easily obtained due to the complexity of the structure of the uncertainty.

*Remark 2.1:* The model of the form (1) is a hybrid system in which one state  $x(t)$  takes values continuously and another state  $\eta_t$ , referred to as the mode or operating form, takes values discretely in  $\mathcal{S}$ . This kind of system can be used to represent many important physical systems subject to random failures and structure changes, such as electric power systems [33], control systems of a solar thermal central receiver [34], communications systems [35], aircraft flight control [36], control of nuclear power plants [37] and manufacturing systems [38], [39].

For convenience, it is assumed that

$$B(\eta_t) = \begin{bmatrix} 0_{(n-m) \times m} \\ B_2(\eta_t) \end{bmatrix}$$

where  $B_2(\eta_t) \in R^{m \times m}$  is nonsingular. Let

$$\begin{aligned} y_1(t) &= [I_{n-m} \quad 0] x(t) \\ y_2(t) &= -K(\eta_t) [I_{n-m} \quad 0] x(t) + [0 \quad I_m] x(t) \end{aligned} \quad (6)$$

that is

$$\begin{aligned} y(t) &= \begin{bmatrix} I_{n-m} & 0 \\ -K(\eta_t) & I_m \end{bmatrix} x(t) \\ x(t) &= \begin{bmatrix} I_{n-m} & 0 \\ K(\eta_t) & I_m \end{bmatrix} y(t) \end{aligned} \quad (7)$$

where  $K(\eta_t)$  is the virtual control input matrix to be determined later.

Let us recall the definition proposed in [40].

*Definition 2.1:* A stochastic process  $y(t)$  is said to be bounded in probability if the random variables  $|y(t)|$  are bounded in probability uniformly in  $t$ , i.e.,

$$\lim_{r \rightarrow \infty} \sup_{t > 0} P\{|y(t)| > r\} = 0. \quad (8)$$

The criterion for boundedness in probability is given as follows.

*Lemma 2.1:* Assume that there exists a function  $V \in C^2$  and parameters  $d_c > 0$  such that

$$\mathcal{E}V(y) \leq d_c \quad (9)$$

$$R \rightarrow \infty \implies V_R = \inf_{|y| > R} V(y(t)) \rightarrow \infty \quad (10)$$

Then for any  $y_0 \in \mathbb{R}^n$  and  $i_0 \in \mathcal{S}$ ,  $y(t)$  is bounded in probability.

*Proof:* By Lemma 1.4.1 of [40], from (9), it follows that

$$P\{|y(t)| > R\} \leq \frac{\mathcal{E}V(y(t))}{\inf_{|y| > R} V(y(t))} \leq \frac{d_c}{V_R} \quad (11)$$

which, together with (10), means that (8) holds.  $\blacksquare$

### III. MAIN RESULTS

In this section, the design results of backstepping controller will be presented.

Taking a symmetric positive-definite matrix variable  $P(\eta(t)) \in R^{(n-m) \times (n-m)}$  and choosing the Lyapunov function candidate as

$$\mathcal{V}_1(x, \eta_t) = y_1^T(t)P(\eta(t))y_1(t). \quad (12)$$

In order to show the stochastic stability of system, two steps will be presented.

*Step 1:* To solve the virtual control  $K(k)$ ,  $k \in \mathcal{S}$ .

Letting (13) and (14), as shown at the bottom of the page.

*Lemma 3.1:* For given positive definite matrices  $\bar{R}(k)$  and  $\bar{Q}(k)$ , if there exist positive definite matrices  $\bar{P}(k)$  and general matrices  $C(k)$  such that the following coupled set of LMIs hold for each  $k \in \mathcal{S}$

$$\begin{bmatrix} \Theta(k) & A_{12}(k)\bar{P}(k) & \mathcal{W}_k(\bar{P}) \\ \bar{P}(k)A_{12}^T(k) & -\bar{Q}(k) & 0 \\ \mathcal{W}_k^T(\bar{P}) & 0 & -\mathcal{X}_k(\bar{P}) \end{bmatrix} < 0 \quad (15)$$

where  $\Theta(k) = A_{11}(k)\bar{P}(k) + A_{12}(k)C(k) + [A_{11}(k)\bar{P}(k) + A_{12}(k)C(k)]^T + \bar{R}(k) + \alpha_{kk}\bar{P}(k)$ , then,  $K(k) = C(k)\bar{P}^{-1}(k)$ , and the weak infinitesimal operator  $\mathfrak{F}_k^x[\cdot]$  of the process  $\{x(t), \eta_t, t \geq 0\}$  at the point  $\{t, x, k\}$

$$\begin{aligned} \mathfrak{F}_k^x[\mathcal{V}_1] &\leq -y_1^T(t)\bar{P}^{-1}(k)\bar{R}(k)\bar{P}^{-1}(k)y_1(t) \\ &\quad + y_2^T(t)\bar{P}^{-1}(k)\bar{Q}(k)\bar{P}^{-1}(k)y_2(t). \end{aligned} \quad (16)$$

*Proof:* Taking

$$\mathcal{V}_1(x, \eta_t) = y_1^T(t)P(\eta_t)y_1(t) \quad (17)$$

applying (4) yields that (18), as shown at the bottom of the page. From (6), it follows that we have (19), as shown at the bottom of the next page, where

$$\begin{aligned} A(k) &= \begin{bmatrix} A_{11}(k) & A_{12}(k) \\ A_{21}(k) & A_{22}(k) \end{bmatrix} \\ \Pi(k) &= \begin{bmatrix} \Pi_{11}(k) & P(k)A_{12}(k) \\ A_{12}^T(k)P(k) & -Q(k) \end{bmatrix} \\ \Pi_{11}(k) &= P(k)(A_{11}(k) + A_{12}(k)K(k)) \end{aligned}$$

$$\mathcal{W}_k(\bar{P}) = [\sqrt{\alpha_{k1}}\bar{P}(k) \quad \sqrt{\alpha_{k2}}\bar{P}(k) \quad \cdots \quad \sqrt{\alpha_{k(k-1)}}\bar{P}(k) \quad \sqrt{\alpha_{k(k+1)}}\bar{P}(k) \quad \cdots \quad \sqrt{\alpha_{k\nu}}\bar{P}(k)]^T \quad (13)$$

$$\mathcal{X}_k(\bar{P}) = \text{diag}\{\bar{P}(1), \bar{P}(2), \dots, \bar{P}(k-1), \bar{P}(k+1), \dots, \bar{P}(\nu)\} \quad (14)$$

$$\mathfrak{F}_k^x[\mathcal{V}_1] = y_1^T(t)P(k)[I_{n-m} \quad 0] \dot{x}(t) + [[I_{n-m} \quad 0] \dot{x}(t)]^T P(k)y_1(t) + y_1^T(t) \left( \sum_{j=1}^{\nu} \alpha_{kj} P(j) \right) y_1(t) \quad (18)$$

$$\begin{aligned}
& + (A_{11}(k) + A_{12}(k)K(k))^T P(k) \\
& + R(k) + \sum_{j=1}^{\nu} \alpha_{kj} P(j) \\
Q(k) &= \bar{P}^{-1}(k) \bar{Q}(k) \bar{P}^{-1}(k), R(k) \\
&= \bar{P}^{-1}(k) \bar{R}(k) \bar{P}^{-1}(k). \tag{20}
\end{aligned}$$

Letting  $\bar{P}(k) = P^{-1}(k)$ , pre- and post-multiplying  $\Pi(k)$  by  $\begin{bmatrix} \bar{P}(k) & 0 \\ 0 & \bar{P}(k) \end{bmatrix}$  gives

$$\begin{aligned}
\Psi(k) &:= \begin{bmatrix} \bar{P}(k) & 0 \\ 0 & \bar{P}(k) \end{bmatrix} \Pi(k) \begin{bmatrix} \bar{P}(k) & 0 \\ 0 & \bar{P}(k) \end{bmatrix} \\
&= \begin{bmatrix} \Psi_{11}(k) & A_{12}(k) \bar{P}(k) \\ \bar{P}(k) A_{12}^T(k) & -\bar{Q}(k) \end{bmatrix} \tag{21}
\end{aligned}$$

where

$$\begin{aligned}
\Psi_{11}(k) &= \Pi_{11}(k) \\
&= (A_{11}(k) + A_{12}(k)K(k)) \bar{P}(k) \\
&\quad + \bar{P}(k) (A_{11}(k) + A_{12}(k)K(k))^T + \bar{R}(k) \\
&\quad + \bar{P}(k) \left( \sum_{j=1}^{\nu} \alpha_{kj} \bar{P}^{-1}(k) \right) \bar{P}(k). \tag{22}
\end{aligned}$$

Letting  $C(k) = K(k) \bar{P}(k)$ , and noting that (14),  $\Psi(k) < 0$ ,  $k \in \mathcal{S}$  are equivalent (15) based on Schur complement formula. It follows from (4) that

$$\mathfrak{F}_k^x[\mathcal{V}_1] \leq -y_1^T(t) R(k) y_1(t) + y_2^T(t) Q(k) y_2(k). \tag{23}$$

I) *Step 2:* To obtain control  $u$  in the following theorem. Let

$$\begin{aligned}
&\begin{bmatrix} \Omega_{11}(k) & \Omega_{12}(k) \\ \Omega_{12}^T(k) & 0 \end{bmatrix} \\
&= \left( \begin{bmatrix} I_{n-m} & 0 \\ K(k) & I_m \end{bmatrix} \right)^T \\
&\quad \times \left( \sum_{j=1}^{\nu} \alpha_{kj} \begin{bmatrix} -K^T(j) \\ I \end{bmatrix} \begin{bmatrix} -K(j) & I \end{bmatrix} \right) \\
&\quad \times \begin{bmatrix} I_{n-m} & 0 \\ K(k) & I_m \end{bmatrix} \\
&\begin{bmatrix} \Lambda_{11}(k) & \Lambda_{12}(k) \end{bmatrix} \\
&= \begin{bmatrix} -K(k) & I \end{bmatrix} A(k) \begin{bmatrix} I_{n-m} & 0 \\ K(k) & I_m \end{bmatrix}. \tag{24}
\end{aligned}$$

**Theorem 3.1:** Assume the condition in Lemma 3.1 holds, i.e., inequalities (15) have solutions  $\bar{P}(k) \in R^{m \times m}$ ,  $K(k) \in R^{m \times (n-m)}$ ,  $k \in \mathcal{S}$ ,  $Q(k) \in R^{m \times (n-m)}$ ,  $k \in \mathcal{S}$ , and there exists a constant scalar  $\lambda > 0$  such that the following inequalities hold

$$R(k) - \Omega_{11}(k) - \sum_{j=1}^{\nu} \alpha_{kj} P(j) > \lambda P(k) \tag{25}$$

then the following control makes the closed system is bounded in probability

$$u = -B_2^{-1}(k) [(\Lambda_{11}(k) + \Omega_{12}(k)) y_1(t) + (\Lambda_{12}(k) + Q(k)) y_2(t)] + u_N \tag{26}$$

$$u_N = \begin{cases} -\frac{B_2^T y_2(t)}{\|B_2^T y_2(t)\|} \hat{\rho}, & \text{if } \|B_2^T y_2(t)\| \hat{\rho} > \epsilon \\ -\frac{B_2^T y_2(t)}{\epsilon} \hat{\rho}^2, & \text{if } \|B_2^T y_2(t)\| \hat{\rho} \leq \epsilon \end{cases} \tag{27}$$

and the adaptation laws as

$$\hat{\rho} = \hat{c}(y(t), t) + \hat{\kappa}(y(t), t) \|y(t)\| \tag{28}$$

$$\dot{\hat{c}}(t, y) = q_1(-\epsilon_0 \hat{c} + \|B_2^T y_2(t)\|) \tag{29}$$

$$\dot{\hat{\kappa}}(t, y) = q_2(-\epsilon_1 \hat{\kappa} + \|B_2^T y_2(t)\| \|y(t)\|) \tag{30}$$

where  $q_1, q_2, \epsilon_0$  and  $\epsilon_1$  are design parameters.

*Proof:* Let us consider the function

$$\mathcal{V}(x, \hat{c}, \hat{\kappa}, \eta_t) = y_2^T(t) y_2(t) + \mathcal{V}_1(y_1(t), \eta_t) + \frac{1}{2q_1} \hat{c}^2 + \frac{1}{2q_2} \hat{\kappa}^2 \tag{31}$$

where  $\tilde{c} = c - \hat{c}(y(t))$  and  $\tilde{\kappa} = \kappa - \hat{\kappa}(y(t))$ .

Upon applying (4) to (31) yields

$$\begin{aligned}
\mathfrak{F}_k^{x, \tilde{c}, \tilde{\kappa}}[\mathcal{V}] &= x^T(t) \begin{bmatrix} -K(k) & I \end{bmatrix}^T \begin{bmatrix} -K(k) & I \end{bmatrix} \dot{x}(t) \\
&\quad + \dot{x}^T(t) \begin{bmatrix} -K(k) & I \end{bmatrix}^T \begin{bmatrix} -K(k) & I \end{bmatrix} x(t) \\
&\quad + x^T(t) \left( \sum_{j=1}^{\nu} \alpha_{kj} \begin{bmatrix} -K^T(j) \\ I \end{bmatrix} \begin{bmatrix} -K(j) & I \end{bmatrix} \right) x(t) \\
&\quad + y_1^T(t) P(k) \begin{bmatrix} I_{n-m} & 0 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} I_{n-m} & 0 \end{bmatrix} \dot{x}(t)^T P(k) y_1(t) \\
&\quad + y_1^T(t) \left( \sum_{j=1}^{\nu} \alpha_{kj} P(j) \right) y_1(t) - \frac{1}{q_1} \tilde{c} \dot{\hat{c}} - \frac{1}{q_2} \tilde{\kappa} \dot{\hat{\kappa}}. \tag{32}
\end{aligned}$$

Based on system (1), (7), and inequality (16), it follows that we have (33), as shown at the bottom of the page, it follows from (24) that we get (34), also shown at the bottom of the next page.

If  $\|B_2^T y_2(t)\| \hat{\rho} > \epsilon$ ,  $y_2^T(t) B_2 u_N = -y_2^T B_2 B_2^T y_2(t) / \|B_2^T y_2(t)\| \hat{\rho} = -\|B_2^T y_2(t)\| (\hat{\kappa} \|z\| + \hat{c})$ , with the control law defined in (26) and adaptation laws defined in (29)–(30), we have (35), as shown at the bottom of the page.

Note that for any positive scalars  $\delta_0 > 1/2$  and  $\delta_1 > 1/2$ , the following inequalities hold:

$$\epsilon_0 \tilde{c} \hat{c} = \epsilon_0 \tilde{c} (-\tilde{c} + c)$$

$$\begin{aligned}
\mathfrak{F}_k^x[\mathcal{V}_1] &= 2y_1^T(t) P(k) \begin{bmatrix} A_{11}(k) & A_{12}(k) \end{bmatrix} \begin{bmatrix} y_1(t) \\ K(k) y_1 + y_2 \end{bmatrix} + y_1^T(t) \left( \sum_{j=1}^{\nu} \alpha_{kj} P(j) \right) y_1(t) \\
&= y_1^T(t) \left[ P(k) (A_{11}(k) + A_{12}(k)K(k)) + (A_{11}(k) + A_{12}(k)K(k))^T P(k) + \left( \sum_{j=1}^{\nu} \alpha_{kj} P(j) \right) \right] \\
&\quad \times y_1(t) + 2y_1^T P(k) A_{12}(k) y_2 \\
&= \begin{bmatrix} y_1^T(t) & y_2^T(t) \end{bmatrix} \Pi(k) \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} - y_1^T(t) R(k) y_1(t) + y_2^T(t) Q(k) y_2(k) \tag{19}
\end{aligned}$$

$$\begin{aligned}
&= \epsilon_0(-\tilde{c}^2 + \tilde{c}\hat{c}) \\
&\leq \epsilon_0 \left( -\tilde{c}^2 + \frac{1}{2\delta_0}\tilde{c}^2 + \frac{\delta_0}{2}c^2 \right) \\
&= \frac{-\epsilon_0(2\delta_0 - 1)}{2\delta_0}\tilde{c}^2 + \frac{\epsilon_0\delta_0}{2}c^2 \\
\epsilon_1\tilde{\kappa}\hat{\kappa} &\leq \frac{-\epsilon_1(2\delta_1 - 1)}{2\delta_1}\tilde{\kappa}^2 + \frac{\epsilon_1\delta_1}{2}\kappa^2.
\end{aligned} \tag{36}$$

From (25), letting  $\beta = \lambda_{\min}(Q(k))$ ,  $k \in \mathcal{S}$ ,  $\gamma = \min(\lambda, \beta)$ ,  $q_1 = \delta_0\gamma/\epsilon_0(2\delta_0 - 1)$ ,  $q_2 = \delta_1\gamma/\epsilon_1(2\delta_1 - 1)$  and a constant  $\sigma_1 = \epsilon_0\delta_0/2c^2 + \epsilon_1\delta_1/2\kappa^2$  and then

$$\mathfrak{F}_k^{x, \tilde{c}, \tilde{\kappa}}[\mathcal{V}] \leq -\lambda\mathcal{V}_1(y_1(t), k) - \beta y_2^T y_2 - \frac{\gamma}{2q_1}\tilde{c}^2 - \frac{\gamma}{2q_2}\tilde{\kappa}^2 + \sigma_1 \tag{38}$$

we have (39), as shown at the bottom of the page. If  $\|B_2^T y_2(t)\|\hat{\rho}^2 \leq \epsilon$ , with the control law defined in (26) and adaptation laws defined in (29)–(30), we obtain (40), as shown at the bottom of the page. From (36) and (37), we have (41), shown at the bottom of the next page. Letting a constant  $\sigma_2 = \epsilon/4 + \epsilon_0\delta_0/2c^2 + \epsilon_1\delta_1/2\kappa^2$ , we have (42), as shown at the bottom of the next page. Based on inequalities (39) and (42), we have

$$\begin{aligned}
\mathfrak{F}_k^{x, \tilde{c}, \tilde{\kappa}}[\mathcal{V}] &\leq 2y_2^T(t)[-K(k) \quad I] \left[ A(k) \begin{bmatrix} I_{n-m} & 0 \\ K(k) & I_m \end{bmatrix} y(t) + B(k)[u(t) + F(k)w(x, t)] \right] \\
&\quad + \left( \begin{bmatrix} I_{n-m} & 0 \\ K(k) & I_m \end{bmatrix} y(t) \right)^T \left( \sum_{j=1}^{\nu} \alpha_{kj} \begin{bmatrix} -K^T(j) \\ I \end{bmatrix} [-K(j) \quad I] \right) \begin{bmatrix} I_{n-m} & 0 \\ K(k) & I_m \end{bmatrix} y(t) \\
&\quad - y_1^T(t)R(k)y_1(t) + y_2^T(t)Q(k)y_2(t) + \sum_{j=1}^{\nu} \alpha_{kj}y_1^T P(j)y_1(t) - \frac{1}{q_1}\tilde{c}\dot{\tilde{c}} - \frac{1}{q_2}\tilde{\kappa}\dot{\tilde{\kappa}}
\end{aligned} \tag{33}$$

$$\begin{aligned}
\mathfrak{F}_k^{x, \tilde{c}, \tilde{\kappa}}[\mathcal{V}] &\leq y_2^T(t)[\Lambda_{11}(k)y_1(t) + \Lambda_{12}(k)y_2(t) + \Omega_{12}(k)y_1(t) + B_2(k)(u(t) + F(k)w(x(t), t)) \\
&\quad + [\Lambda_{11}(k)y_1(t) + \Lambda_{22}(k)y_2(t) + \Omega_{12}(k)y_1(t) + B_2(k)(u(t) + F(k)w(x(t), t))]^T y_2(t) \\
&\quad + y_1^T(t)R(k)y_1(t) + y_2^T(t)Q(k)y_2(t) + y_1^T(t)\Omega_{11}(k)y_1(t) \\
&\quad + \sum_{j=1}^{\nu} \alpha_{kj}y_1^T(t)P(j)y_1(t) - \frac{1}{q_1}\tilde{c}\dot{\tilde{c}} - \frac{1}{q_2}\tilde{\kappa}\dot{\tilde{\kappa}}
\end{aligned} \tag{34}$$

$$\begin{aligned}
\mathfrak{F}_k^{x, \tilde{c}, \tilde{\kappa}}[\mathcal{V}] &\leq -y_1^T(t)(R(k) - \Omega_{11}(k))y_1(t) - y_2^T(t)Q(k)y_2(t) + \sum_{j=1}^{\nu} \alpha_{kj}y_1^T P(j)y_1(t) \\
&\quad - y_2^T(t)B_2 u_N + y_2^T(t)B_2 F(k)w(y(t), t) - \tilde{c}(-\epsilon_0\hat{c} + \|B_2^T y_2(t)\|) \\
&\quad - \tilde{\kappa}(-\epsilon_1\hat{\kappa} + \|B_2^T y_2(t)\|\|y\|) \\
&\leq -y_1^T(t) \left( R(k) - \Omega_{11}(k) - \sum_{j=1}^{\nu} \alpha_{kj}P(j) \right) y_1(t) - y_2^T(t)Q(k)y_2(t) \\
&\quad + \|B_2^T y_2(t)\|(\kappa\|y\| + c) - \|B_2^T y_2(t)\|(\hat{\kappa}\|y\| + \hat{c}) - \tilde{c}(-\epsilon_0\hat{c} + \|B_2^T y_2(t)\|) \\
&\quad - \tilde{\kappa}(-\epsilon_1\hat{\kappa} + \|B_2^T y_2(t)\|\|y\|) \\
&= -y_1^T(t) \left( R(k) - \Omega_{11}(k) - \sum_{j=1}^{\nu} \alpha_{kj}P(j) \right) y_1(t) - y_2^T(t)Q(k)y_2(t) + \epsilon_0\tilde{c}\hat{c} + \epsilon_1\tilde{\kappa}\hat{\kappa}
\end{aligned} \tag{35}$$

$$\mathfrak{F}_k^{x, \tilde{c}, \tilde{\kappa}}[\mathcal{V}] \leq -\gamma \left( \mathcal{V}_1(y_1(t), k) + y_2^T y_2 + \frac{1}{2q_1}\tilde{c}^2 + \frac{1}{2q_2}\tilde{\kappa}^2 \right) + \sigma_1 \tag{39}$$

$$\begin{aligned}
\mathfrak{F}_k^{x, \tilde{c}, \tilde{\kappa}}[\mathcal{V}] &\leq -(\gamma\mathcal{V}_1(t, y, k) + \beta y_2^T y_2) - \frac{\|B_2^T y_2(t)\|^2}{\epsilon}\hat{\rho}^2 + \|B_2^T y_2(t)\|(\kappa\|y\| + c) \\
&\quad - \tilde{c}(-\epsilon_0\hat{c} + \|B_2^T y_2(t)\|) - \tilde{\kappa}(-\epsilon_1\hat{\kappa} + \|B_2^T y_2(t)\|\|y\|) \\
&= -\gamma(\mathcal{V}_1(y_1(t), k) + y_2^T y_2) - \frac{\|B_2^T y_2(t)\|^2}{\epsilon}\hat{\rho}^2 + \|B_2^T y_2(t)\|\hat{\rho} + \epsilon_0\tilde{c}\hat{c} + \epsilon_1\tilde{\kappa}\hat{\kappa} \\
&= -\gamma(\mathcal{V}_1(y_1(t), k) + y_2^T y_2) - \left( \frac{\|B_2^T y_2(t)\|^2}{\sqrt{\epsilon}}\hat{\rho} - \frac{\sqrt{\epsilon}}{2} \right)^2 + \frac{\epsilon}{4} + \epsilon_0\tilde{c}\hat{c} + \epsilon_1\tilde{\kappa}\hat{\kappa}
\end{aligned} \tag{40}$$

$$\mathfrak{F}_k^{x, \tilde{c}, \tilde{\kappa}}[\mathcal{V}] \leq -\gamma \mathcal{V} + \sigma_3 \quad (43)$$

where  $\sigma_3 = \max(\sigma_1, \sigma_2)$ , it results in

$$\frac{d\mathcal{E}(V)}{dt} \leq -\gamma \mathcal{E}(V) + \sigma_3. \quad (44)$$

It follows from Lemma 2.1, we know that the solution of the closed-loop system is bounded in probability.

*Remark 3.1:* Note that Theorem 3.1 provides a solution to the problem of adaptive control for stochastic system. It is worth mentioning that the work conducted in this technical note is the attempt to overcome the problem arising in the sliding mode control for Markov jumping systems and adopt adaptive backstepping controller for systems with Markovian jump parameters. The results obtained in this technical note could be extended to general systems with other forms of stochastic jumps.

#### IV. A NUMERICAL EXAMPLE

Let us consider the following system with generator for the Markov process governing the mode switching being

$$\mathfrak{G} = \begin{bmatrix} -1.4 & 1.4 \\ 1.2 & -1.2 \end{bmatrix}.$$

For the two operating conditions (modes), the associated data is:

Mode 1

$$A(1) = \begin{bmatrix} -2.9 & 0.3 & 0.4 & 1.2 \\ -0.1 & -0.2 & 0.6 & 1.5 \\ 1.2 & 2.1 & 2.8 & 3.4 \\ 1 & -2 & -2.5 & -2.5 \end{bmatrix}$$

$$B(1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1.0 & 0.5 \\ -0.1 & 0.2 \end{bmatrix}$$

$$w(t) = 0.1 + 0.01 * \|y(t)\|.$$

Mode 2

$$A(2) = \begin{bmatrix} -1.3 & -0.1 & 0.21 & 0.3 \\ -0.8 & 0 & 0.2 & 1.2 \\ -0.6 & 0.2 & 1.4 & -0.9 \\ 0.5 & 0.5 & 0.3 & 1.2 \end{bmatrix},$$

$$B(2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.1 & 0.1 \\ 0.5 & -0.2 \end{bmatrix},$$

$$w(t) = 0.1 + 0.01 * \|y(t)\|.$$

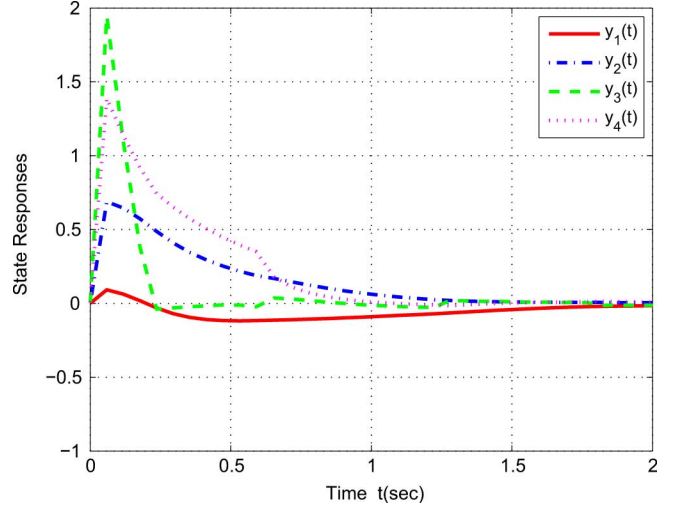


Fig. 1. States  $(y_1(t), y_2(t), y_3(t), y_4(t))$ .

Using Theorem 3.1 and LMI tool box in Matlab, we have

$$\begin{aligned} \bar{P}(1) &= \begin{bmatrix} 0.7041 & -0.1596 \\ -0.1596 & 0.3376 \end{bmatrix} \\ C(1) &= \begin{bmatrix} -2.7156 & 0.0526 \\ 1.4592 & -1.1077 \end{bmatrix} \\ \bar{P}(2) &= \begin{bmatrix} 0.8125 & -0.0660 \\ -0.0660 & 0.4975 \end{bmatrix} \\ C(2) &= \begin{bmatrix} -4.1284 & -1.03950 \\ 1.6207 & -1.2202 \end{bmatrix}. \end{aligned}$$

Then

$$\begin{aligned} K(1) &= \begin{bmatrix} -4.2799 & -1.8670 \\ 1.4882 & -2.5778 \end{bmatrix} \\ K(2) &= \begin{bmatrix} -5.3081 & -2.7934 \\ 1.8150 & -2.2118 \end{bmatrix}. \end{aligned}$$

Taking  $F(1) = F(2) = [0.7071 \ 0.7071]^T$ , it can be shown that  $\|Fw(y, t)\| \leq 0.1 + 0.01\|y(t)\|$ . Letting  $\lambda = 0.5$ ,  $\epsilon = 0.35$ ,  $\epsilon_0 = 0.0844$ ,  $\epsilon_1 = 0.011$ ,  $q_1 = 0.8$ ,  $q_2 = 13$ , we have the following simulation results.

The closed-loop dynamic responses are given in Figs. 1–3 under the following initial conditions  $y = [-0.0003 \ 0.0036 \ 0.0049 \ 0.0083]^T$ ,  $c(0) = 0.5$  and  $k(0) = 0.1$ . Fig. 1 shows that the transformed system states are bounded in probability. Fig. 2 depicts the input control signal. The adaptive parameters are shown in Fig. 3, it can be shown that  $\hat{c}$  and  $\hat{\kappa}$  convergent to the upper bounds of disturbances and uncertainties  $c = 0.1$  and  $\kappa = 0.01$ , respectively. Moreover, it should be pointed

$$\mathfrak{F}_k^{x, \tilde{c}, \tilde{\kappa}}[\mathcal{V}] \leq -\gamma \left( \mathcal{V}_1(y_1(t), k) + y_2^T y_2 + \frac{1}{2q_1} \tilde{c}^2 + \frac{1}{2q_2} \tilde{\kappa}^2 \right) + \frac{\epsilon}{4} + \frac{\epsilon_0 \delta_0}{2} c^2 + \frac{\epsilon_1 \delta_1}{2} \kappa^2 \quad (41)$$

$$\mathfrak{F}_k^{x, \tilde{c}, \tilde{\kappa}}[\mathcal{V}] \leq -\gamma \left( \mathcal{V}_1(y_1(t), k) + y_2^T y_2 + \frac{1}{2q_1} \tilde{c}^2 + \frac{1}{2q_2} \tilde{\kappa}^2 \right) + \sigma_2 \quad (42)$$

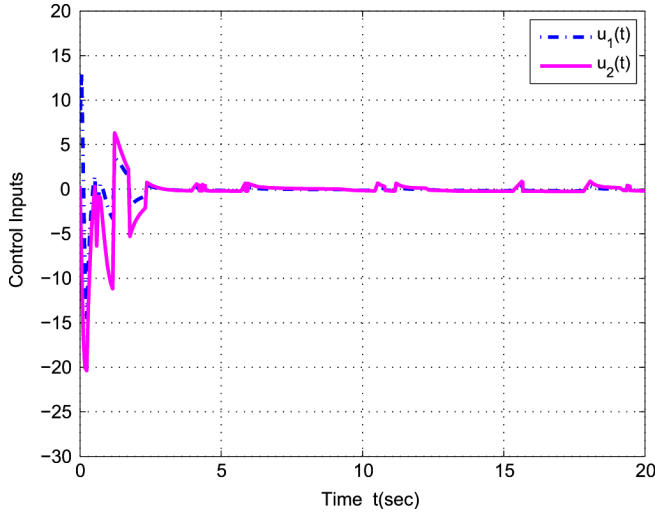


Fig. 2. Control  $(u_1(t), u_2(t))$ .

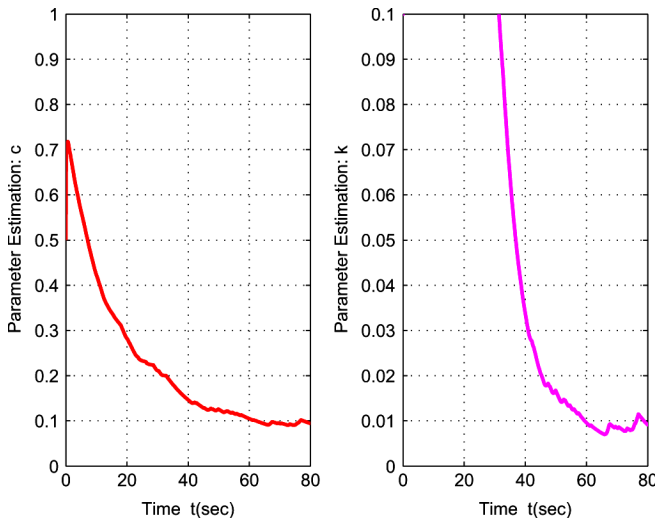


Fig. 3. Adaptive parameters  $c$  and  $\kappa$ .

out that the Markov switching is generated on-line. From the above figures, we can see that the proposed control methods work well.

## V. CONCLUSION

In this technical note, the well-known backstepping method is used to overcome the problem in [1]. The adaptive backstepping controller design problem is investigated by using LMI technique and adaptive control approach. Numerical example has been given to demonstrate the applicability of the theoretical results obtained in this technical note.

## ACKNOWLEDGMENT

The authors would like to thank the reviewers for their very helpful comments and suggestions which have improved the presentation of the technical note.

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