

# Fuzzy $\mathcal{H}_\infty$ Output Feedback Control of Nonlinear Systems Under Sampled Measurements

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## Abstract

This paper studies the problem of designing an  $\mathcal{H}_\infty$  fuzzy feedback control for a class of nonlinear systems. A nonlinear systems is first described by a continuous-time fuzzy system model under sampled output measurements. The premise variables of the fuzzy system model are allowed to be unavailable. We develop a technique for designing an  $\mathcal{H}_\infty$  fuzzy feedback control which globally stabilises this class of fuzzy system models. A design algorithm for constructing the  $\mathcal{H}_\infty$  fuzzy feedback controller is given. A numerical simulation example is given to show the potential of the proposed techniques.

**Keywords:** Fuzzy systems, sampled measurements,  $\mathcal{H}_\infty$  control, Output feedback.

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# 1 Introduction

There has been some substantial interest over the past few years in the direct design of digital controllers using continuous-time performance measures. One of the interesting approaches is the hybrid optimal  $\mathcal{H}_\infty$  control approach. So far a number of different techniques have been proposed to provide solutions to the hybrid optimal  $\mathcal{H}_\infty$  control problems. The techniques include: 1) lifting technique [1, 2, 3, 4] which consists of transforming the original sampled-data system into an equivalent LTI discrete-time system with infinite-dimensional input-output signal space. Then  $\mathcal{L}_2$  induced norm of the sampled-data system is shown to be less than one if and only if the  $\mathcal{H}_\infty$  norm of this equivalent discrete system is less than one; 2) descriptor system technique [5] where the system is first represented by a hybrid state space model and the solution to the  $\mathcal{H}_\infty$  sampled-data problem is then characterised by the solution of certain associated Hamilton-Jacobi equation; 3) technique based on linear systems with jumps [6]-[17] which is a direct characterisation of the problem in the similar terms to those of standard LTI  $\mathcal{H}_\infty$  control problems, and leads to a pair of Riccati equations. Recently, linear  $\mathcal{H}_\infty$  sampled-data results have been extended to nonlinear systems under sampled measurement. In [18]-[22], solutions to the nonlinear  $\mathcal{H}_\infty$  sampled-data control problem have been obtained in terms Hamilton-Jacobi equation (HJE). However, until now, it is still very difficult to solve for a global solution to the Hamilton-Jacobi equation (HJE).

To design a model-based controller for a given process, a mathematical model which captures all the relevant characteristics of the process is required. Many practical systems are very complex, a suitable mathematical model that describes the dynamics of processes is very difficult, if not impossible to obtain. However, many of these systems can be expressed in some form of mathematical model locally or as an aggregation of a set of mathematical models. Based on this idea, Takagi, Sugeno and Kang have proposed a fuzzy inference system known as the TSK model in fuzzy system literature. For the representative work on this topic, we refer readers to the papers of [23]-[32]. This modelling approach provides a powerful tool for modelling complex nonlinear systems. Unlike conventional modelling where a single model is used to describe the global behavior of a systems, TSK modelling is essentially a multimodel approach in which simple submodels (typically linear models) are combined to describe the global behavior of the system.

Typically, a continuous-time Takagi-Sugeno fuzzy dynamic model is locally described by a set of linear models and is represented by fuzzy IF-THEN rules that have the form

Plant Rule  $i$ :

IF  $\nu_1(t)$  is  $M_{i1}$  and  $\dots$  and  $\nu_{\vartheta}(t)$  is  $M_{i\vartheta}$  THEN

$$\dot{x}(t) = A_i x(t) + B_i u(t), \quad i = 1, 2, \dots, r$$

where  $\nu_1(t), \dots, \nu_{\vartheta}(t)$  are the premise variables,  $M_{ij}(j = 1, 2, \dots, \vartheta)$  are fuzzy sets that are characterised by membership functions,  $x(t) \in \mathbb{R}^n$  is the state vector,  $u(t) \in \mathbb{R}^m$  is the input, the matrices  $A_i$  and  $B_i$  are of appropriate dimensions and  $r$  is the number of IF-THEN rules.

Given a pair  $[x(t) \ u(t)]$ , by using a singleton fuzzifier, product fuzzy inference and weighted average defuzzifier, the final state of the fuzzy system is inferred as follows:

$$\begin{aligned} \dot{x}(t) &= \frac{\sum_{i=1}^r J_i(\nu(t)) [A_i x(t) + B_i u(t)]}{\sum_{i=1}^r J_i(x(t))} \\ &= \sum_{i=1}^r \mu_i(\nu(t)) [A_i x(t) + B_i u(t)] \end{aligned} \quad (1.1)$$

where  $J_i(\nu(t))$  is the weight of each rule and it is calculated as follows:

$$J_i(\nu(t)) = \prod_{j=1}^{\vartheta} M_{ij}(\nu_j(t)), \quad \mu_i(\nu(t)) = \frac{J_i(\nu(t))}{\sum_{i=1}^r J_i(\nu(t))}$$

$M_{ij}(\nu_j(t))$  is the grade of membership of  $\nu_j(t)$  in  $M_{ij}$ . It is assumed in this paper that

$$J_i(\nu(t)) \geq 0, \quad i = 1, 2, \dots, r; \quad \sum_{i=1}^r J_i(\nu(t)) > 0$$

for all  $t$ . Therefore

$$\mu_i(\nu(t)) \geq 0, \quad i = 1, 2, \dots, r; \quad \sum_{i=1}^r \mu_i(\nu(t)) = 1$$

for all  $t$ . For the convenience of notations, let  $J_i = J_i(\nu(t))$  and  $\mu_i = \mu_i(\nu(t))$ ; then the final state of the fuzzy system can be represented as

$$\dot{x}(t) = \sum_{i=1}^r \mu_i A_i x(t) + \sum_{i=1}^r \mu_i B_i u(t). \quad (1.2)$$

For the fuzzy controller design, it is supposed that the fuzzy system is locally controllable. First, the local state feedback controllers are designed as follows, based on the pairs  $(A_i, B_i)$ :

Controller Rule  $i$ :

IF  $\nu_1(t)$  is  $M_{i1}$  and  $\dots$  and  $\nu_{\vartheta}(t)$  is  $M_{i\vartheta}$  THEN

$$u(t) = -K_i x(t), \quad i = 1, 2, \dots, r$$

then, the final fuzzy controller is

$$u(t) = - \sum_{i=1}^r \mu_i K_i x(t).$$

In practice, not all the state are available. Indeed, for a continuous-time systems, the output measurement are often available at discrete points, i.e., measured at sampled points. Therefore, it is necessary and practical useful to design an observer to estimate the system state. In [33, 34], by restricting the premise variables  $(\nu_1, \dots, \nu_\vartheta)$  to be measurable, a fuzzy observer has been developed. This restriction enables the authors in [33, 34] to select the fuzzy sets of the fuzzy observer to be the same as the fuzzy sets of the plant. Hence, the development of the separation property of controller and filter is possible. In general, however, the premise variables for a general TSK model can be unavailable. In this case, the premise variables of the fuzzy observer can not be selected to be the same as the premise variables of the plant. Hence, the results given in [33, 34] can not be applied. What we intend in this paper is to design an  $\mathcal{H}_\infty$  output feedback controller by allowing the premise variables of the plant to be unavailable.

**Notation.** Most of the notations used in this paper are fairly standard.  $\mathbb{R}^n$  and  $\mathbb{R}^{n \times m}$  denote respectively, the  $n$  dimensional Euclidean space and the set of all  $n \times m$  real matrices. The superscript “ $t$ ” denotes matrix transposition and the notation  $X \geq Y$  (respectively,  $X > Y$ ) where  $X$  and  $Y$  are symmetric matrices, means that  $X - Y$  is positive semi-definite (respectively, positive definite).  $L_2[0, T]$  stands for the space of square integrable vector functions over  $[0, T]$ ,  $l_2(0, T)$  is the space of square summable vector sequences over  $(0, T)$ ,  $\|\cdot\|_{[0, T]}$  will refer to the  $L_2[0, T]$  norm over  $[0, T]$  and  $\|\cdot\|_{(0, T)}$  is the  $l_2(0, T)$  norm over  $(0, T)$ .  $T$  is allowed to be  $\infty$  and in this case by the notation  $[0, T]$  we mean  $[0, \infty)$ .  $F(\theta^-)$  and  $F(\theta^+)$  stand for the left limit and right limit of a function  $F(\theta)$ , respectively.

## 2 System Description and Definition

The class of nonlinear sampled-data systems under consideration is described by the following fuzzy system model:

Plant Rule  $i$ :

IF  $\nu_1(t)$  is  $M_{i1}$  and  $\dots$  and  $\nu_\vartheta(t)$  is  $M_{i\vartheta}$  THEN, for  $i = 1, 2, \dots, r$ :

$$\dot{x}(t) = A_i x(t) + B_1 w(t) + B_{2i} u(t), t \neq mh, \quad x(0) = x_0 \quad (2.1)$$

$$z(t) = C_1 x(t), t \neq mh \quad (2.2)$$

$$z_d(mh) = C_d x(mh), \quad (2.3)$$

$$y(mh) = C_{2i} x(mh) + D_{21} v(mh), \quad (2.4)$$

where  $M_{ij} (j = 1, 2, \dots, \vartheta)$  are fuzzy sets,  $x(t) \in \mathfrak{R}^n$  is the state,  $x_0$  is an unknown initial state,  $w(t) \in \mathfrak{R}^p$  is the disturbance input,  $u(t) \in \mathfrak{R}^m$  is control input,  $y \in \mathfrak{R}^\ell$  is the sampled measurement,  $v \in \mathfrak{R}^q$  is the measurement noise,  $z \in \mathfrak{R}^r$  is the controlled continuous output,  $z_d \in \mathfrak{R}^s$  is the controlled discrete output,  $0 < h \in \mathfrak{R}$  is the sampling period,  $m$  is a positive integer,  $A_i, B_1, B_{2i}, C_1, C_{2i}, C_d$  and  $D_{21}$  are known real time-varying bounded matrices of appropriate dimensions with  $A_i, B_1, B_{2i}, C_1$  and  $D_{12}$  being piecewise continuous, and  $r$  is the number of IF-THEN rules.

Throughout this paper, we adopt the following standard  $\mathcal{H}_\infty$  assumptions.

**Assumption 2.1**

$$D_{21} [B_1^t \ D_{21}^t] = [0 \ I]. \quad (2.5)$$

**Assumption 2.2**  $(e^{A_i h}, C_{2i})$  are observable and  $(A_i, B_{2i})$  are controllable.

The resulting fuzzy system model is inferred as the weighted average of the local models and has the form

$$\dot{x}(t) = \sum_{i=1}^r \mu_i A_i x(t) + B_1 w(t) + \sum_{i=1}^r \mu_i B_{2i} u(t), t \neq mh, \ x(0) = x_0 \quad (2.6)$$

$$z(t) = C_1 x(t), t \neq mh \quad (2.7)$$

$$z_d(mh) = C_d x(mh) \quad (2.8)$$

$$y(mh) = \sum_{i=1}^r \mu_i C_{2i} x(mh) + D_{21} v(mh). \quad (2.9)$$

We are concerned with designing a fuzzy  $\mathcal{H}_\infty$  output feedback control law  $\mathcal{G}$  for (2.6)-(2.9), based on the sampled output measurements of (2.9) such that the controller  $\mathcal{G}$  reduces  $z$  uniformly for any  $w$  and  $v$  in the sense that given a scalar  $\gamma > 0$ , the worst-case performance measure of closed-loop system of (2.6)-(2.9) with the controller  $\mathcal{G}$ , defined by:

$$\int_0^T z^T(t) z(t) dt + \sum_{m=1}^k z_d^T(mh) z_d(mh) \leq \gamma^2 \left\{ \int_0^T w^T(t) w(t) dt + \sum_{m=1}^k v^T(mh) v(mh) \right\} \quad (2.10)$$

is satisfied with  $k$  be the largest integer in  $[0, T]$ . In this situation, the closed-loop system of (2.6)-(2.9) with  $\mathcal{G}$  is said to have an  $H_\infty$  performance  $\gamma$  over the horizon  $[0, T]$ .

The control problem we address in this paper is as follows: *Given a scalar  $\gamma > 0$ , design a fuzzy controller ( $\mathcal{G}$ ) based on the sampled measurements,  $y(mh)$ , such that (2.10) holds:*

Note that the performance measure in (2.10) is in terms of not only of the controlled signals at the sampling instants but also of the continuous-time controlled output between the sampling instants. This allows the intersampling behaviour to be taken into account in the control design. When only the controlled continuous output is considered, (2.10) will reduce to the performance measure used in [8].

**Remark 2.1** *It should be remarked that (2.8)-(2.9) can be viewed as a “mixed  $L_2/\ell_2$ ” output signals. In real environmental systems, we always face continuous-time systems, discrete-time systems, sampled-data systems and hybrid systems, i.e., systems with both continuous- and discrete-time states. The study of this kind of systems is motivated by robust sampled-data control, filtering and loop transfer recovery of sampled-data systems [14].*

In this paper, we consider the following  $\mathcal{H}_\infty$  fuzzy output feedback controller,  $\mathcal{G}$ :

Controller Rule  $i$ :

IF  $\hat{\nu}_1(t)$  is  $M_{i1}$  and  $\dots$  and  $\hat{\nu}_\vartheta(t)$  is  $M_{i\vartheta}$  THEN

$$\begin{aligned} \dot{\hat{x}}(t) &= a_i \hat{x}(t) + b_i u(t), t \neq mh \\ \hat{x}(mh) &= \hat{x}(mh^-) + L_i \left[ y(mh) - \hat{y}(mh) \right] \\ \hat{y}(t) &= C_{2i} \hat{x}(t) \\ u(t) &= K_i \hat{x}(t) \end{aligned} \quad \text{for } i = 1, 2, \dots, r \quad (2.11)$$

where  $\hat{\nu}_i(t)$  are the premise variables of the controller,  $\hat{x}(t) \in \mathbb{R}^n$  is the controller state vector,  $\hat{y}(t) \in \mathbb{R}^\ell$  is the controller output,  $a_i$  are the controller matrices,  $b_i$  are the input matrices,  $L_i$  are the observer gains,  $K_i$  are the controller gains, and  $r$  is the number of IF-THEN rules.

The final  $\mathcal{H}_\infty$  fuzzy output feedback controller is inferred as follows:

$$\begin{aligned} \dot{\hat{x}}(t) &= \sum_{i=1}^r \hat{\mu}_i a_i \hat{x}(t) + \sum_{i=1}^r \hat{\mu}_i b_i u, t \neq mh \\ \hat{x}(mh^+) &= \hat{x}(mh) + \sum_{i=1}^r \hat{\mu}_i L_i \left[ y(mh) - \hat{y}(mh) \right] \\ \hat{y}(t) &= \sum_{i=1}^r \hat{\mu}_i C_{2i} \hat{x}(t) \\ u(t) &= \sum_{i=1}^r \hat{\mu}_i K_i \hat{x}(t). \end{aligned} \quad (2.12)$$

**Remark 2.2** *In [33, 34], the premise variables of the fuzzy output feedback controller are assumed to be the same as the premise variables of the fuzzy systems model. This actually means that the premise variables of the fuzzy system model are assumed to be measurable.*

However, in general, it is extremely difficult to derive an accurate fuzzy systems model by imposing that all the premise variables are measurable. In this paper, we do not impose that condition, we choose the premise variables of the controller to be different from the premise variables of the fuzzy system model of the plant.

Using (2.12), the control problem can be reformulated as follows:

*Problem Formulation:* Given a scalar  $\gamma > 0$ , design an  $\mathcal{H}_\infty$  fuzzy output feedback controller of the form (2.12) such that the inequality (2.10) holds.

In the sequel, without loss of generality, we assume  $\gamma = 1$ . Let us denote the estimation error as

$$e(t) = x(t) - \hat{x}(t). \quad (2.13)$$

By differentiating (2.13), we get

$$\begin{aligned} \dot{e}(t) &= \dot{x}(t) - \dot{\hat{x}}(t) \\ &= \sum_{i=1}^r \mu_i A_i x(t) + B_1 w(t) + \sum_{i=1}^r \mu_i B_{2i} u(t) - \sum_{i=1}^r \hat{\mu}_i a_i \hat{x}(t) - \sum_{i=1}^r \hat{\mu}_i b_i u(t) \\ &= \sum_{i=1}^r (\mu_i - \hat{\mu}_i) A_i x(t) + \sum_{i=1}^r \sum_{j=1}^r (\mu_i - \hat{\mu}_i) \hat{\mu}_j B_{2i} K_j [x(t) - e(t)] + \sum_{i=1}^r \hat{\mu}_i A_i x(t) \\ &\quad - \sum_{i=1}^r \hat{\mu}_i a_i [x(t) - e(t)] + B_1 w + \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j \{B_{2i} - b_i\} K_j [x(t) - e(t)], t \neq mh \\ &= \sum_{i=1}^r (\mu_i - \hat{\mu}_i) A_i x(t) + \sum_{i=1}^r \sum_{j=1}^r (\mu_i - \hat{\mu}_i) \hat{\mu}_j B_{2i} K_j [x(t) - e(t)] \\ &\quad + \sum_{i=1}^r \hat{\mu}_i \hat{\mu}_j \left[ A_i - a_i - b_i K_j + B_{2i} K_j \right] x(t) + B_1 w \\ &\quad + \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j \left\{ a_i + b_i - B_{2i} \right\} K_j [x(t) - e(t)], t \neq mh \\ e(mh^+) &= e(mh) - \sum_{i=1}^r \hat{\mu}_i \sum_{j=1}^r \hat{\mu}_j L_i \left[ C_{2j} x(t) + D_{21} v(mh) - C_{2j} \hat{x}(t) \right] \\ &= e(mh) - \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j L_i C_{2j} e(mh) - \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i (\mu_j - \hat{\mu}_j) L_i C_{2j} \left[ x(mh) - e(mh) \right] \\ &\quad + \sum_{i=1}^r \hat{\mu}_i L_i D_{21} v(mh). \end{aligned} \quad (2.14)$$

The system (2.6) with (2.12) can be represented as follows:

$$\begin{aligned}
\dot{x}(t) &= \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j [A_i + B_{2i} K_j] x(t) - \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j B_{2i} K_j e(t) \\
&\quad + \sum_{i=1}^r \sum_{j=1}^r (\mu_i - \hat{\mu}_i) \hat{\mu}_j B_{2i} K_j [x(t) - e(t)] \\
&\quad + \sum_{j=1}^r (\mu_i - \hat{\mu}_i) A_i x(t) + B_1 w(t), t \neq mh \\
x(mh^+) &= x(mh).
\end{aligned} \tag{2.15}$$

Using (2.14) and (2.15), we get the augmented system of the following form:

$$\begin{aligned}
\dot{\tilde{x}}(t) &= \begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} \\
&= \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j \begin{bmatrix} A_i + B_{2i} K_j & -B_{2i} K_j \\ A_i + B_{2i} K_j - a_i - b_i K_j & a_i - B_{2i} K_j + b_i K_j \end{bmatrix} \tilde{x}(t) \\
&\quad + \sum_{i=1}^r \sum_{j=1}^r (\mu_i - \hat{\mu}_i) \hat{\mu}_j \begin{bmatrix} A_i + B_{2i} K_j & -B_{2i} K_j \\ A_i + B_{2i} K_j & -B_{2i} K_j \end{bmatrix} \tilde{x}(t) + \sum_{i=1}^r \hat{\mu}_i \begin{bmatrix} B_1 \\ B_1 \end{bmatrix} w(t) \\
&= \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j [A_{ij} \tilde{x}(t) + \Psi_i w(t)] + \sum_{j=1}^r \hat{\mu}_j f_j(x(t)) \tilde{x}(t), t \neq mh
\end{aligned} \tag{2.16}$$

$$\begin{aligned}
\tilde{x}(mh^+) &= \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j \begin{bmatrix} I & 0 \\ 0 & I - L_i C_{2j} \end{bmatrix} \tilde{x}(mh) \\
&\quad + \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i (\mu_j - \hat{\mu}_j) \begin{bmatrix} 0 & 0 \\ -L_i C_{2j} & L_i C_{2j} \end{bmatrix} \tilde{x}(mh) \\
&\quad + \sum_{i=1}^r \hat{\mu}_i \begin{bmatrix} 0 \\ -L_i D_{21} \end{bmatrix} v(mh) \\
&= \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j [(\bar{A}_{ij} + H_i \Delta F E) \tilde{x}(mh) + \Upsilon_i v(mh)]
\end{aligned} \tag{2.17}$$

where

$$A_{ij} = \begin{bmatrix} A_i + B_{2i} K_j & -B_{2i} K_j \\ A_i + B_{2i} K_j - a_i - b_i K_j & a_i - B_{2i} K_j + b_i K_j \end{bmatrix} \tag{2.18}$$

$$\bar{A}_{ij} = \begin{bmatrix} I & 0 \\ 0 & I - L_i C_{2j} \end{bmatrix}, \quad H_i = \begin{bmatrix} 0 \\ L_i \end{bmatrix} \tag{2.19}$$

$$f_j(\tilde{x}(t)) = \begin{bmatrix} \Delta A & 0 \\ \Delta A & 0 \end{bmatrix} + \begin{bmatrix} \Delta B K_j & -\Delta B K_j \\ \Delta B K_j & -\Delta B K_j \end{bmatrix} \tag{2.20}$$



$$\Psi_i = \begin{bmatrix} B_1 \\ B_1 \end{bmatrix}, \quad \Upsilon_i = \begin{bmatrix} 0 \\ -L_i D_{21} \end{bmatrix}, \quad E = \begin{bmatrix} -C_{21} & C_{21} \\ \vdots & \vdots \\ -C_{2r} & C_{2r} \end{bmatrix} \quad (2.21)$$

$$\Delta F = [(\mu_1 - \hat{\mu}_1) \quad \cdots \quad (\mu_r - \hat{\mu}_r)], \quad \Delta B = \sum_{i=1}^r (\mu_i - \hat{\mu}_i) B_i, \quad \Delta A = \sum_{i=1}^r (\mu_i - \hat{\mu}_i) A_i. \quad (2.22)$$

### 3 Fuzzy Output Feedback Control Design

In this section, we convert the problem of  $\mathcal{H}_\infty$  fuzzy output feedback control to the solvability of differential Riccati inequalities with jumps.

**Theorem 3.1** *Given the augmented system (2.16)-(2.17) satisfying Assumptions 2.1 and 2.2, if there exists a positive definite symmetric solution  $P$  such that for  $i, j = 1, 2, \dots, r$ , the following differential Riccati matrix inequalities with jumps hold:*

$$\dot{P}(t) + A_{ij}^T P + P A_{ij} + P \Psi_i \Psi_i^T P + 4\Phi_j + 4\Xi + Q \leq 0 \quad (3.1)$$

$$\left[ I - H_i^T P(mh^+) H_i \right] > 0 \quad (3.2)$$

$$\begin{aligned} \tilde{A}_{ii}^T P(mh^+) \tilde{A}_{ii} + \tilde{A}_{ii}^T P(mh^+) H_i \left[ I - H_i^T P(mh^+) H_i \right]^{-1} H_i^T P(mh^+) \tilde{A}_{ii} + C_d^T C_d + 2\hat{E}^T \hat{E} \\ - \bar{P}(mh) \leq 0 \end{aligned} \quad (3.3)$$

$$\begin{aligned} (\tilde{A}_{ij} + \tilde{A}_{ji})^T P(mh^+) (\tilde{A}_{ij} + \tilde{A}_{ji}) + (\tilde{A}_{ij} + \tilde{A}_{ji})^T H_i P(mh^+) \left[ I - H_i^T P(mh^+) H_i \right]^{-1} \times \\ H_i^T P(mh^+) (\tilde{A}_{ij} + \tilde{A}_{ji}) + 4C_d^T C_d + 8\hat{E}^T \hat{E} - 4\bar{P}(mh) \leq 0 \quad \text{for } i < j \end{aligned} \quad (3.4)$$

where

$$\begin{aligned} \Phi_j &= \begin{bmatrix} \sum_{s=1}^r K_j^T B_s^T B_s K_j & 0 \\ 0 & \sum_{s=1}^r K_j^T B_s^T B_s K_j \end{bmatrix}, \quad \Xi = \begin{bmatrix} \sum_{s=1}^r A_s^T A_s & 0 \\ 0 & \sum_{s=1}^r A_s^T A_s \end{bmatrix} \\ Q &= \begin{bmatrix} C_1^T C_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{P}(mh) = \begin{bmatrix} P(mh) & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{E} = [E \ 0], \quad \tilde{A}_{ij} = [\bar{A}_{ij} \ \Upsilon_i]. \end{aligned}$$

Then the  $\mathcal{H}_\infty$  control performance of (2.10) is guaranteed.

**Proof:** Let us choose a Lyapunov function for the augmented system (2.16)-(2.17) as

$$V(\tilde{x}(t), t) = \tilde{x}^T(t)P(t)\tilde{x}(t) \quad (3.5)$$

For  $\tau \in (mh^+, mh + h)$ ,

$$\int_{mh^+}^{\tau} \frac{d}{dt} \{V(\tilde{x}(t))\} dt = \tilde{x}^T(\tau)P(\tau)\tilde{x}(\tau) - \tilde{x}^T(mh^+)P(mh^+)\tilde{x}(mh^+). \quad (3.6)$$

First let us consider and denote the left hand side of (3.6) as

$$\begin{aligned} \Theta(\tilde{x}(\tau)) &= \int_{mh^+}^{\tau} \frac{d}{dt} \{V(\tilde{x}(t))\} dt = \int_{mh^+}^{\tau} \tilde{x}^T(t)\dot{P}(t)\tilde{x}(t) + \dot{\tilde{x}}^T(t)P(t)\tilde{x}(t) + \tilde{x}^T(t)P(t)\dot{\tilde{x}}(t) dt \\ &= \int_{mh^+}^{\tau} \left[ \left( \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j [A_{ij}\tilde{x}(t) + \Psi_i w(t)] + \sum_{i=1}^r \hat{\mu}_j f_j(x(t))\tilde{x}(t) \right)^T P(t)\tilde{x}(t) \right. \\ &\quad \left. + \tilde{x}^T(t)P(t) \left( \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j [A_{ij}\tilde{x}(t) + \Psi_i w(t)] + \sum_{i=1}^r \hat{\mu}_j f_j(x(t))\tilde{x}(t) \right) + \tilde{x}^T(t)\dot{P}(t)\tilde{x}(t) \right] dt \\ &= \int_{mh^+}^{\tau} \left[ \tilde{x}^T(t)P(t) \left( \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j A_{ij}\tilde{x}(t) \right) + \left( \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j A_{ij}\tilde{x}(t) \right)^T P(t)\tilde{x}(t) \right. \\ &\quad \left\{ w^T(t) \sum_{i=1}^r \hat{\mu}_i \Psi_i^T P(t)\tilde{x}(t) + \tilde{x}^T(t)P(t) \sum_{i=1}^r \hat{\mu}_i \Psi_i w(t) \right. \\ &\quad \left. - w^T(t)w(t) - \tilde{x}^T(t)P(t) \left( \sum_{i=1}^r \hat{\mu}_i \Psi_i \right) \left( \sum_{i=1}^r \hat{\mu}_i \Psi_i \right)^T P(t)\tilde{x}(t) \right\} \\ &\quad + w^T(t)w(t) + \tilde{x}^T(t)P(t) \left( \sum_{i=1}^r \hat{\mu}_i \Psi_i \right) \left( \sum_{i=1}^r \hat{\mu}_i \Psi_i \right)^T P(t)\tilde{x}(t) + \tilde{x}^T(t)\dot{P}(t)\tilde{x}(t) \\ &\quad \left. + \tilde{x}^T(t) \left( \sum_{i=1}^r \hat{\mu}_i h_i(x(t))\tilde{x}(t) \right)^T P(t)\tilde{x}(t) + \tilde{x}^T(t)P(t) \left( \sum_{i=1}^r \hat{\mu}_j f_j(x(t))\tilde{x}(t) \right) \right] dt \\ &\leq \int_{mh^+}^{\tau} \left[ \left( \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j A_{ij}\tilde{x}(t) \right)^T P(t)\tilde{x}(t) + \tilde{x}^T(t)P(t) \left( \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j A_{ij}\tilde{x}(t) \right) \right. \\ &\quad \left. - \left( \sum_{i=1}^r \hat{\mu}_i \Psi_i^T P(t)\tilde{x}(t) - w(t) \right)^T \left( \sum_{i=1}^r \hat{\mu}_i \Psi_i^T P(t)\tilde{x}(t) - w(t) \right) + \tilde{x}^T(t)\dot{P}(t)\tilde{x}(t) \right. \\ &\quad \left. + w^T(t)w(t) + \tilde{x}^T(t)P(t) \left( \sum_{i=1}^r \hat{\mu}_i \Psi_i \right) \left( \sum_{i=1}^r \hat{\mu}_i \Psi_i \right)^T P(t)\tilde{x}(t) + \tilde{x}(t)P(t)P(t)\tilde{x}(t) \right] dt \end{aligned}$$

$$\begin{aligned}
& \left( \sum_{i=1}^r \hat{\mu}_j h_j(x(t)) \tilde{x}(t) \right)^T \left( \sum_{i=1}^r \hat{\mu}_j f_j(x(t)) \tilde{x}(t) \right) \Big] dt \\
& \leq \int_{mh^+}^{\tau} \left[ \left( \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j A_{ij} \tilde{x}(t) \right)^T P(t) \tilde{x}(t) + \tilde{x}^T(t) P(t) \left( \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j A_{ij} \tilde{x}(t) \right) \right. \\
& \quad + w^T(t) w(t) + \sum_{i=1}^r \hat{\mu}_i \tilde{x}^T(t) P(t) \Psi_i \Psi_i^T P(t) \tilde{x}(t) + \tilde{x}(t) P(t) P(t) \tilde{x}(t) \\
& \quad \left. + \tilde{x}^T(t) \dot{P}(t) \tilde{x}(t) + \left( \sum_{i=1}^r \hat{\mu}_j f_j(x(t)) \tilde{x}(t) \right)^T \left( \sum_{i=1}^r \hat{\mu}_j f_j(x(t)) \tilde{x}(t) \right) \right] dt. \tag{3.7}
\end{aligned}$$

Let us examine the last term of (3.7).

$$\begin{aligned}
\left( \sum_{j=1}^r \hat{\mu}_j f_j(x(t)) \tilde{x}(t) \right)^T \left( \sum_{j=1}^r \hat{\mu}_j f_j(x(t)) \tilde{x}(t) \right) & \leq \sum_{j=1}^r \hat{\mu}_j \tilde{x}^T(t) f_j^T(x(t)) f_j(x(t)) \tilde{x}(t) \\
& = 2 \sum_{j=1}^r \hat{\mu}_j \tilde{x}^T(t) \left\{ \begin{bmatrix} \Delta A^T \Delta A & \Delta A^T \Delta A \\ 0 & \Delta A^T \Delta A \end{bmatrix} \tilde{x}(t) \right. \\
& \quad \left. + \begin{bmatrix} K_j^T \Delta B^T \Delta B K_j & -K_j^T \Delta B^T \Delta B K_j \\ -K_j^T \Delta B^T \Delta B K_j & K_j^T \Delta B^T \Delta B K_j \end{bmatrix} \tilde{x}(t) \right\} \\
& \leq 4 \sum_{j=1}^r \hat{\mu}_j \tilde{x}^T(t) \left\{ \begin{bmatrix} \Delta A^T \Delta A & 0 \\ 0 & \Delta A^T \Delta A \end{bmatrix} \tilde{x}(t) \right. \\
& \quad \left. + \begin{bmatrix} K_j^T \Delta B^T \Delta B K_j & 0 \\ 0 & K_j^T \Delta B^T \Delta B K_j \end{bmatrix} \tilde{x}(t) \right\} \\
& \leq 4 \tilde{x}^T(t) \Xi \tilde{x}(t) + 4 \sum_{j=1}^r \hat{\mu}_j \tilde{x}^T(t) \Phi_j \tilde{x}(t) \tag{3.8}
\end{aligned}$$

where  $\Xi$  and  $\Phi_j$  are given in Theorem 3.1.

Employing (3.8), then inequality (3.7) becomes

$$\begin{aligned}
\Theta(\tilde{x}(\tau)) & = \int_{mh^+}^{\tau} \left[ \left( \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j A_{ij} \tilde{x}(t) \right)^T P(t) \tilde{x}(t) + \tilde{x}^T(t) P(t) \left( \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j A_{ij} \tilde{x}(t) \right) \right. \\
& \quad + w^T(t) w(t) + \sum_{i=1}^r \hat{\mu}_i \tilde{x}^T(t) P(t) \Psi_i P(t) \tilde{x}(t) + \tilde{x}(t) P(t) P(t) \tilde{x}(t) + 4 \tilde{x}^T(t) \Xi \tilde{x}(t) \\
& \quad \left. + 4 \sum_{j=1}^r \hat{\mu}_j \tilde{x}^T(t) \Phi_j \tilde{x}(t) + \tilde{x}^T(t) \dot{P}(t) \tilde{x}(t) \right] dt
\end{aligned}$$

$$\begin{aligned}
&= \int_{mh^+}^{\tau} \left[ \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j \tilde{x}^T(t) \left\{ A_{ij}^T P(t) + P(t) A_{ij} + P(t) \Psi_i \Psi_i^T P(t) + 4\Phi_j + 4\Xi + \right. \right. \\
&\quad \left. \left. P(t) P(t) \right\} \tilde{x}(t) + w^T(t) w(t) + \tilde{x}^T(t) \dot{P}(t) \tilde{x}(t) \right] dt. \tag{3.9}
\end{aligned}$$

Using (3.1), we get

$$\Theta(\tilde{x}(\tau)) \leq - \int_{mh^+}^{\tau} \left[ z^T(t) z(t) + w^T(t) w(t) \right] dt. \tag{3.10}$$

Now let us consider at the sampling instant

$$V(\tilde{x}(t))|_{mh}^{mh^+} = V(\tilde{x}(mh^+), mh^+) - V(\tilde{x}(mh), mh). \tag{3.11}$$

Let us denote the left hand side of (3.11) as

$$\begin{aligned}
\Theta(\tilde{x}(mh)) &= V(\tilde{x}(t))|_{mh}^{mh^+} \\
&= \tilde{x}^T(mh^+) P(mh^+) \tilde{x}(mh^+) - \tilde{x}^T(mh) P(mh) \tilde{x}(mh) \\
&= \left( \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j [\hat{A}_{ij} x(mh) + \Upsilon_i v(mh)] \right)^T P(mh^+) \times \\
&\quad \left( \sum_{i=1}^k \sum_{l=1}^r \hat{\mu}_k \hat{\mu}_l [\hat{A}_{ij} \tilde{x}(mh) + \Upsilon_i v(mh)] \right) - \tilde{x}^T(mh) P(mh) \tilde{x}(mh) \tag{3.12}
\end{aligned}$$

where  $\hat{A}_{ij} = \bar{A}_{ij} + H_i \Delta F E$ .

Rewrite (3.12) as

$$\begin{aligned}
\Theta(\tilde{x}(mh)) &= \frac{1}{4} \left( \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j (\hat{A}_{ij} + \hat{A}_{ji}) \tilde{x}(mh) + \Upsilon_i w(mh) \right)^T P(mh^+) \times \\
&\quad \left( \sum_{i=1}^k \sum_{l=1}^r \hat{\mu}_k \hat{\mu}_l (\hat{A}_{kl} + \hat{A}_{lk}) \tilde{x}(mh) + \Upsilon_i v(mh) \right) - \tilde{x}^T(mh) P(mh) \tilde{x}(mh) \\
&\leq \frac{1}{4} \sum_{i=1}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j \left[ \left( (\hat{A}_{ij} + \hat{A}_{ji}) \tilde{x}(mh) + \Upsilon_i v(mh) \right)^T P(mh^+) \times \right. \\
&\quad \left. \left( (\hat{A}_{ij} + \hat{A}_{ji}) \tilde{x} + \Upsilon_i v(mh) \right) \right] - \tilde{x}^T(mh) P(mh) \tilde{x}(mh) \\
&= \sum_{i=1}^r \hat{\mu}_i^2 \left[ \left( \hat{A}_{ii} \tilde{x}(mh) + \Upsilon_i w(mh) \right)^T P(mh^+) \left( \hat{A}_{ii} \tilde{x}(mh) + \Upsilon_i v(mh) \right) \right]
\end{aligned}$$

$$\begin{aligned}
& -\tilde{x}^T(mh)P(mh)\tilde{x}(mh) \Big] + 2 \sum_{i < j}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j \left[ \left( \left( \frac{\hat{A}_{ij} + \hat{A}_{ji}}{2} \right) \tilde{x}(mh) + \Upsilon_i v(mh) \right)^T \times \right. \\
& \left. P(mh^+) \left( \left( \frac{\hat{A}_{ij} + \hat{A}_{ji}}{2} \right) \tilde{x}(mh) + \Upsilon_i v(mh) \right) - \tilde{x}^T(mh)P(mh)\tilde{x}(mh) \right].
\end{aligned}$$

Letting  $\bar{x}^T(mh) = [\tilde{x}^T(mh) \ v^T(mh)]$ , we have

$$\begin{aligned}
\Theta(\tilde{x}(mh)) & \leq \sum_{i=1}^r \hat{\mu}_i^2 \bar{x}^T(mh) \left( [\hat{A}_{ii} \ \Upsilon_i]^T P(mh^+) [\hat{A}_{ii} \ \Upsilon_i] - \bar{P}(mh) \right) \bar{x}(mh) \\
& + 2 \sum_{i < j}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j \bar{x}^T(mh) \left( \frac{1}{4} \left[ (\hat{A}_{ij} + \hat{A}_{ji}) \ 2\Upsilon_i \right]^T P(mh^+) \left[ (\hat{A}_{ij} + \hat{A}_{ji}) \ 2\Upsilon_i \right] \right. \\
& \left. - P(mh^+) \right) \tilde{x}(mh) \\
& = \sum_{i=1}^r \hat{\mu}_i^2 \bar{x}^T(mh) \left( [\tilde{A}_{ii} + H_i \Delta F \hat{E}]^T P(mh^+) [\tilde{A}_{ii} + H_i \Delta F \hat{E}] - \bar{P}(mh) \right) \bar{x}(mh) \\
& + 2 \sum_{i < j}^r \sum_{j=1}^r \hat{\mu}_i \hat{\mu}_j \bar{x}^T(mh) \left( \left[ \frac{1}{4} (\tilde{A}_{ij} + \tilde{A}_{ji} + 2H_i \Delta F \hat{E})^T \times \right. \right. \\
& \left. \left. P(mh^+) (\tilde{A}_{ij} + \tilde{A}_{ji} + 2H_i \Delta F \hat{E}) \right] - \bar{P}(mh) \right) \tilde{x}(mh) \tag{3.13}
\end{aligned}$$

where  $\bar{P}(mh)$ ,  $\hat{E}$  and  $\tilde{A}_{ij}$  are given in Theorem 3.1.

Notice that

$$\begin{aligned}
& [\tilde{A}_{ii} + H_i \Delta F \hat{E}]^T P(mh^+) [\tilde{A}_{ii} + H_i \Delta F \hat{E}] \leq \tilde{A}_{ii}^T P(mh^+) \tilde{A}_{ii} \\
& + \tilde{A}_{ii}^T P(mh^+) H_i \left( I - H_i^T P(mh^+) H_i \right)^{-1} H_i^T P(mh^+) \tilde{A}_{ii} + 2\hat{E}^T \hat{E} \tag{3.14}
\end{aligned}$$

and

$$\begin{aligned}
& (\tilde{A}_{ij} + \tilde{A}_{ji} + 2H_i \Delta F \hat{E})^T P(mh^+) (\tilde{A}_{ij} + \tilde{A}_{ji} + 2H_i \Delta F \hat{E}) \leq (\tilde{A}_{ij} + \tilde{A}_{ji})^T P(mh^+) \times \\
& (\tilde{A}_{ij} + \tilde{A}_{ji}) + (\tilde{A}_{ij} + \tilde{A}_{ji})^T H_i P(mh^+) \left( I - H_i^T P(mh^+) H_i \right)^{-1} H_i^T P(mh^+) (\tilde{A}_{ij} + \tilde{A}_{ji}) \\
& + 8\hat{E}^T \hat{E}. \tag{3.15}
\end{aligned}$$

Using (3.14), (3.15), (3.3), (3.4) and (3.2), we have from (3.13)

$$\Theta(\tilde{x}(mh)) \leq z_d^T(mh) z_d(mh) - v^T(mh) v(mh). \tag{3.16}$$

By combining (3.10) and (3.16) over all possible  $t$  on  $[0, T]$ , one has

$$\begin{aligned} V(\tilde{x}(T), T) - V(0, 0) &\leq \int_0^T [w^t(t)w(t) - z^t(t)z(t)] dt + \sum_{m=1}^k v^t(mh)v(mh) \\ &\quad - \sum_{m=1}^k z_d^t(mh)z_d(mh) dt. \end{aligned} \quad (3.17)$$

Knowing that  $V(0, t) = 0$  and  $V(x(t), t) > 0, \forall x(t) \neq 0$ , we obtain

$$\begin{aligned} \sum_{m=1}^k z_d^t(mh)z_d(mh) + \int_0^T z^t(t)z(t) dt &\leq \int_0^T w^t(t)w(t) dt + \sum_{m=1}^k v^t(mh)v(mh) - V(x(T), T) \\ &\leq \int_0^T w^t(t)w(t) dt + \sum_{m=1}^k v^t(mh)v(mh). \end{aligned} \quad (3.18)$$

Therefore, the  $\mathcal{H}_\infty$  control performance (2.10) is achieved.  $\nabla \nabla \nabla$

In the same spirit as the linear  $\mathcal{H}_\infty$  sampled-data results, if we choose

$$P = \begin{bmatrix} P_{11}(t) & 0 \\ 0 & P_{22}(t) \end{bmatrix} \quad (3.19)$$

$a_i = A_i + B_i B_i^T P_{11}(t)$ ,  $b_i = B_{2i}$ ,  $K_j = -B_{2i}^T P_{11}(t)$  and  $L_i = P_{22}^{-1}(mh^+)C_{2i}$ , then we have the following corollary.

**Corollary 3.1** *Given the closed loop system (2.16)-(2.17) satisfying Assumptions 2.1 and 2.2, if there exist positive definite symmetric solutions  $P_{11}(t)$  and  $P_{22}(t)$  such that for  $i, j = 1, 2, \dots, r$ , the following differential Riccati matrix inequalities with jumps hold*

1)

$$\begin{aligned} A_i^T P_{11} + P_{11} A_i - \frac{1}{2} P_{11} B_{2j} B_{2i}^T P_{11} - \frac{1}{2} P_{11} B_{2i} B_{2j}^T P_{11} + P_{11} B_1 B_1^T P + C_1^T C_1 \\ + 4 P_{11} B_{2j} \sum_{s=1}^r B_s^T B_s B_{2j}^T P_{11} + 4 \sum_{s=1}^r A_s^T A_s \leq 0 \end{aligned} \quad (3.20)$$

$$P_{11}(mh^+) \leq P_{11}(mh) - C_d^T C_d - 4 \sum_{s=1}^r C_k^T C_k \quad (3.21)$$

2)

$$P_{22}(A_i + B_1 B_1^T P_{11}) + (A_i + B_1 B_1^T P_{11})^T P_{22} + \frac{1}{2} P_{11} B_{2j} B_{2i}^T P_{11}$$

$$+\frac{1}{2}P_{11}B_{2i}B_{2j}^TP_{11}+P_{22}B_1B_1^TP_{22}+4P_{11}B_{2j}\sum_{s=1}^rB_s^TB_sB_{2j}^TP_{11}+4\sum_{s=1}^rA_s^TA_s\leq 0 \quad (3.22)$$

$$\left(I-C_{2i}P_{22}^{-1}(mh^+)C_{2i}^T\right)>0 \quad (3.23)$$

$$P_{22}(mh^+)\leq P_{22}(mh)+C_{2i}^TC_{2j}+C_{2j}^TC_{2i}-C_{2j}^TC_{2j}-C_d^TC_d-4\sum_{s=1}^rC_{2s}^TC_{2s}. \quad (3.24)$$

Then the  $\mathcal{H}_\infty$  control performance of (2.10) is guaranteed with the following controller:

$$\begin{aligned} \dot{\hat{x}}(t) &= \sum_{i=1}^r\hat{\mu}_i\{A_i+B_1B_1^TP_{11}\}\hat{x}(t)+\sum_{i=1}^r\hat{\mu}_iB_{2i}u, t\neq mh \\ \hat{x}(mh^+) &= \hat{x}(mh)+\sum_{i=1}^r\hat{\mu}_iP_{22}^{-1}(mh^+)C_{2i}^T\left[y(mh)-\sum_{j=1}^rC_{2j}\hat{x}(mh)\right] \\ u(t) &= -\sum_{j=1}^rB_{2j}^TP_{11}\hat{x}(t). \end{aligned} \quad (3.25)$$

If  $C_{2i}=C_2$  and  $B_{2i}=B_2$  for all  $i=1,2,\dots,r$ , we have the following corollary.

**Corollary 3.2** *Given the closed loop system (2.16)-(2.17) satisfying Assumptions 2.1 and 2.2, if there exist positive definite symmetric solutions  $P_{11}(t)$  and  $P_{22}(t)$  such that for  $i,j=1,2,\dots,r$ , the following differential Riccati matrix inequalities with jumps hold*

1)

$$A_i^TP_{11}+P_{11}A_i-P_{11}B_2B_2^TP_{11}+P_{11}B_1B_1^TP_{11}+C_1^TC_1+4\sum_{s=1}^rA_s^TA_s\leq 0 \quad (3.26)$$

$$P_{11}(mh^+)\leq P_{11}(mh)-C_d^TC_d \quad (3.27)$$

2)

$$P_{22}(A_i+B_1B_1^TP_{11})+(A_i+B_1B_1^TP_{11})^TP_{22}+P_{11}B_2B_2^TP_{11}+P_{22}B_1B_1^TP_{22}+4\sum_{s=1}^rA_s^TA_s\leq 0 \quad (3.28)$$

$$\left(I-C_2P_{22}^{-1}(mh^+)C_2^T\right)>0 \quad (3.29)$$

$$P_{22}(mh^+)\leq P_{22}(mh)+C_2^TC_2. \quad (3.30)$$

Then the  $\mathcal{H}_\infty$  control performance of (2.10) is guaranteed with the following controller:

$$\begin{aligned}\dot{\hat{x}}(t) &= \sum_{i=1}^r \hat{\mu}_i \left\{ A_i + B_1 B_1^T P_{11} - B_2 B_2^T P_{11} \right\} \hat{x}(t), t \neq mh \\ \hat{x}(mh^+) &= \hat{x}(mh) + P_{22}^{-1}(mh^+) C_2^T \left[ y(mh) - C_2 \hat{x}(mh) \right] \\ u(t) &= -B_2^T P_{11} \hat{x}(t).\end{aligned}\tag{3.31}$$

## 4 A Simulation Example

The following model is used in this simulation:

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) - x_2(t) - \sin(x_1(t)) + 0.002w + u(t) \\ \dot{x}_2(t) &= x_1(t) \\ z(t) &= 15x_1(t) + 15x_2(t) \\ y(mh) &= x_1(mh) + x_2(mh) + v(mh).\end{aligned}\tag{4.1}$$

A fuzzy system model under sampled output measurements for the above system is given as follows:

Rule 1: If  $x_1(t)$  is  $M_1$  THEN

$$\begin{aligned}\dot{x}(t) &= A_1 x(t) + B_1 w + B_{21} u(t) \\ z(t) &= C_1 x(t) \\ y(mh) &= C_{21} x(mh) + D_{21} v(mh)\end{aligned}\tag{4.2}$$

Rule 2: If  $x_1(t)$  is  $M_2$  THEN

$$\begin{aligned}\dot{x}(t) &= A_2 x(t) + B_1 w + B_{22} u(t) \\ z(t) &= C_1 x(t) \\ y(mh) &= C_{22} x(mh) + D_{21} v(mh)\end{aligned}\tag{4.3}$$

where  $x(t) = [x_1(t) \ x_2(t)]^T$ , the membership functions  $M_1$  and  $M_2$  are  $\frac{\sin(x_1(t))}{x_1(t)}$ , and  $\frac{x_1(t) - \sin(x_1(t))}{x_1(t)}$ , respectively,

$$A_1 = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0.002 \\ 0 \end{bmatrix}, B_{21} = B_{22} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$C_1 = [15 \ 15], C_d = [0 \ 0], D_{21} = 1, C_{21} = C_{22} = [1 \ 1]$$



. Note that the premise variable of the above fuzzy system model is  $x_1(t)$  which is unavailable. Hence the method proposed in [33, 34] can not be employed here. Applying Corollary 3.2, we have the following stationary fuzzy  $\mathcal{H}_\infty$  output feedback controller:

Rule 1: If  $x_2(t)$  is  $M_1$  THEN

$$\dot{\hat{x}}(t) = \left\{ A_1 + B_1 B_1^T P_{11} - B_2 B_2^T P_{11} \right\} \hat{x}(t), t \neq mh \quad (4.4)$$

$$\hat{x}(mh^+) = \hat{x}(mh) + P_{22}^{-1} C_2^T \left[ y(mh) - C_2 \hat{x}(mh) \right] \quad (4.5)$$

$$u(t) = -B_2^T P_{11} \hat{x}(t). \quad (4.6)$$

Rule 2: If  $x_2(t)$  is  $M_2$  THEN

$$\dot{\hat{x}}(t) = \left\{ A_2 + B_1 B_1^T P_{11} - B_2 B_2^T P_{11} \right\} \hat{x}(t), t \neq mh \quad (4.7)$$

$$\hat{x}(mh^+) = \hat{x}(mh) + P_{22}^{-1} C_2^T \left[ y(mh) - C_2 \hat{x}(mh) \right] \quad (4.8)$$

$$u(t) = -B_2^T P_{11} \hat{x}(t) \quad (4.9)$$

where  $P_{11} = \begin{bmatrix} 100 & 50 \\ 50 & 150 \end{bmatrix}$  and  $P_{22} = \begin{bmatrix} 60000 & 3000 \\ 3000 & 90000 \end{bmatrix}$ .

**Remark 4.1** *Simulation results for the ratio  $\frac{\left\{ \|z\|_{[0,T]}^2 + \|z_d\|_{(0,T)}^2 \right\}}{\left\{ \|w\|_{[0,T]}^2 + \|v\|_{(0,T)}^2 \right\}}$  obtained by using the fuzzy  $\mathcal{H}_\infty$  controller for system (4.1) is depicted in Fig. 1. The graphs in Fig. 2 and Fig. 3, respectively, only show the first second of the input disturbance signals  $w(t)$  and  $v(mh)$  which were used during the simulation. The sampling time used in the simulation was 0.01 sec. From Fig. 1, we can see that after 1200 seconds the ratio  $\frac{\left\{ \|z\|_{[0,T]}^2 + \|z_d\|_{(0,T)}^2 \right\}}{\left\{ \|w\|_{[0,T]}^2 + \|v\|_{(0,T)}^2 \right\}}$  tends to a constant value which is about 0.018. So the  $\mathcal{L}_2$  gain from  $\|w\|_{[0,T]} + \|v\|_{(0,T)}$  to  $\|z\|_{[0,T]} + \|z_d\|_{(0,T)}$  is about  $\sqrt{0.018} = 0.134$ , which is less than the prescribed value 1.*

## 5 Conclusion

This paper has investigated the problem of stabilising a class of fuzzy system models under sampled measurement using an  $\mathcal{H}_\infty$  fuzzy output feedback controller. A nonlinear system is first approximated by a Takagi-Sugeno fuzzy model. Then based on the well-known  $\mathcal{H}_\infty$

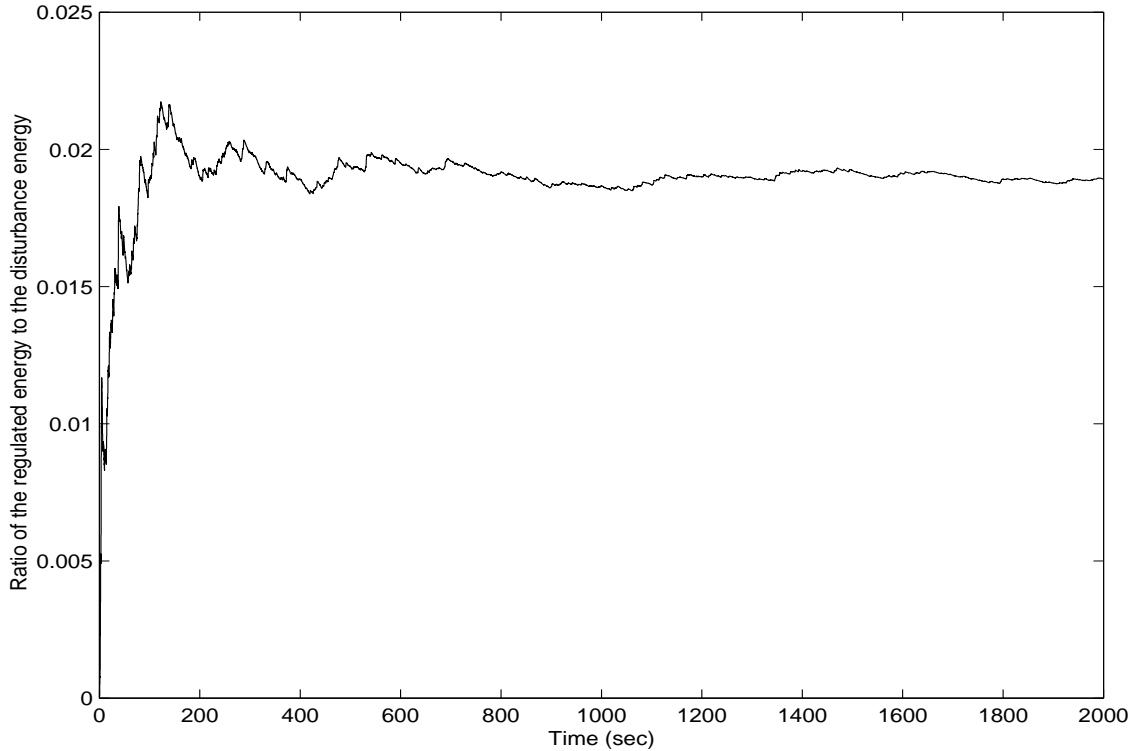


Figure 1: Ratio of the regulated energy to the disturbance energy.

theory, a technique for designing an  $\mathcal{H}_\infty$  fuzzy output feedback control law which globally stabilises this class of nonlinear systems under sampled measurement has been developed. In contrast to the results given in [33, 34], the premise variables of the  $\mathcal{H}_\infty$  fuzzy output feedback controller are allowed to be different from the premise variables of the Takagi-Sugeno fuzzy model of the plant.

## References

- [1] B. Bamieh, J. Pearson, B. Francis, and A. Tannenbaum, “A lifting technique for linear periodic systems with applications to sampled-data control,” *Systems & Control Letters*, vol. 17, pp. 78–88, 1991.
- [2] H. T. Toivonen, “Sampled-data control of continuous-time system with an  $\mathcal{H}_\infty$  optimality criterion,” *Automatica*, vol. 28, pp. 45–54, 1992.
- [3] B. Bamieh and J. B. Pearson, “A general framework for linear periodic systems with application to  $\mathcal{H}_\infty$  sampled-data control,” *IEEE Trans. Automat. Contr.*, vol. 37, pp. 418–

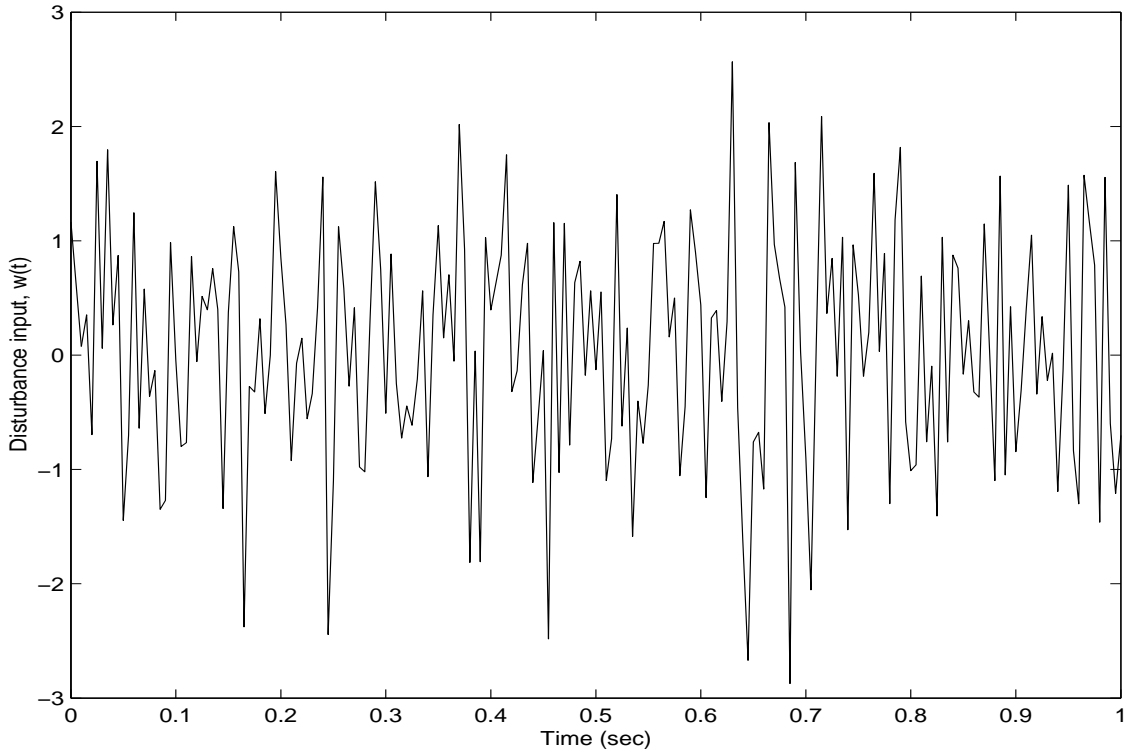


Figure 2: The disturbance input,  $w(t)$ .

435, 1992.

- [4] Y. Yamamoto, "On the state space and frequency domain characterization of  $\mathcal{H}_\infty$  norm of sampled-data systems," *Systems & Control Letters*, vol. 21, pp. 163–172, 1991.
- [5] S. Hara and P. T. Kabamba, "Worst case analysis and design of sampled-data control systems," in *Proc. of 29th IEEE Conf. Decision Contr., Honolulu, HI*, pp. 202–203, 1990.
- [6] P. Shi, "Robust filtering for uncertain systems with sampled measurements," *Int. J. Systems Science*, vol. 27, no. 12, pp. 1403–1415, 1996.
- [7] P. Shi, "Filtering on sampled-data systems with parametric uncertainty," *IEEE Trans. Automat. Control*, vol. 43, 1998. to appear.
- [8] W. Sun, K. M. Nagpal, and P. P. Khargonekar, " $\mathcal{H}_\infty$  control and filtering for sampled-data systems," *IEEE Trans. Automat. Contr.*, vol. 38, pp. 1162–1175, 1993.
- [9] P. Shi, "Issues in robust filtering and control of sampled-data systems," tech. rep., The University of Newcastle, 1994. Ph.D Thesis.

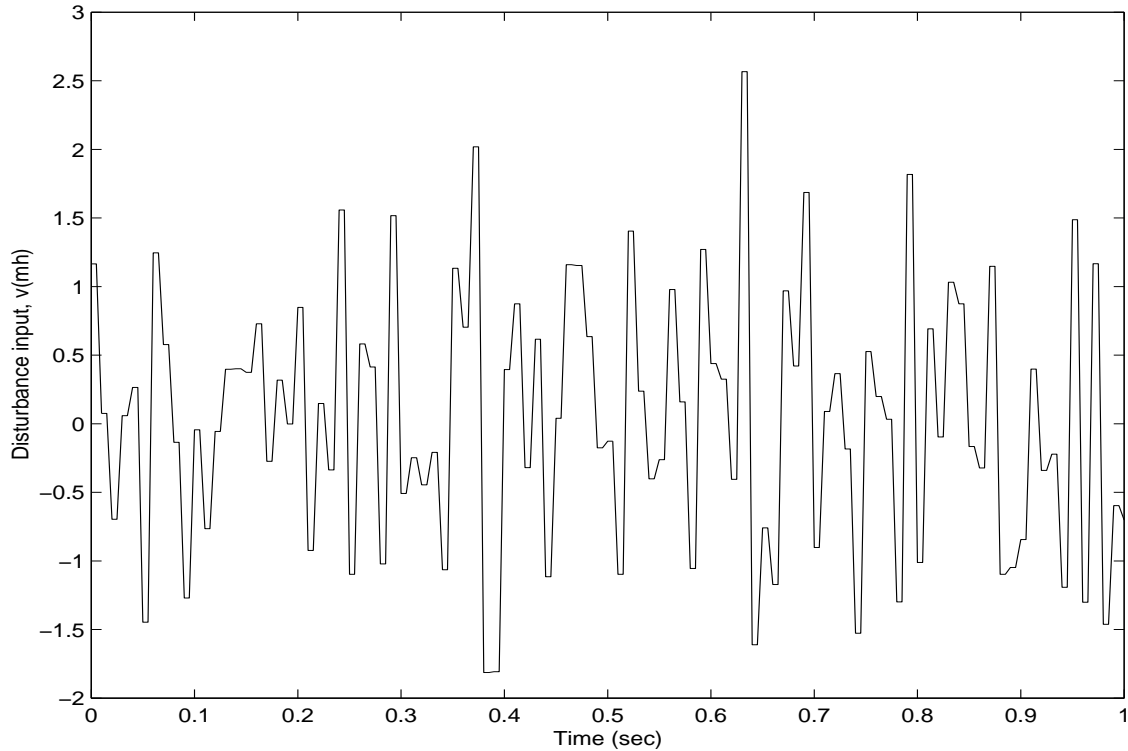


Figure 3: The disturbance input,  $v(mh)$ .

- [10] N. Sivashankar and P. Khargonekar, “Characterization of the  $\mathcal{L}_2$ -induced norm for linear systems with jumps with application to sampled-data systems,” *SIAM J. Control and Optimisation*, vol. 32, pp. 1128–1150, 1994.
- [11] P. Shi, L. Xie, and C. de Souza, “Output feedback control of sampled-data systems with parametric uncertainties,” in *Proc. 31st IEEE Conf. Decision & Control*, (Tucson, AZ), pp. 2876–2881, 1992.
- [12] L. Xie, P. Shi, and C. E. de Souza, “On designing controllers for a class of uncertain sampled-data nonlinear systems,” *IEE Proc.-Control Theory Appl.*, vol. 140, pp. 119–126, 1993.
- [13] P. Shi, “Controller design for uncertain systems under sampled measurements,” *IMA J. Mathe. Contr. Inform.*, vol. 15, no. 3, pp. 1–21, 1998.
- [14] Shi, P., M. Fu, and C. E. de Souza, (1996). Loop transfer recovery for systems under sampled measurements. *IEE Proc.-Control Theory Appl.*, **143**(4), 333–337.

- [15] Shi, P., (1998). Controller design for uncertain systems under sampled measurements. *IMA J. Mathe. Contr. Inform.*, **15**, 133–153.
- [16] Shi, P., E. K. Boukas and R. K. Agarwal, (1999). Control of Markovian jump discrete-time systems with norm bounded uncertainty and unknown delays. *IEEE Trans. Automat. Control*, **44**(11), 2139–2144.
- [17] Tadmor, G., (1992).  $H_\infty$  optimal sampled-data control in continuous time systems. *Int. J. Control*, **56**(1), 99–141.
- [18] A. Isidori and A. Astolfi, “Disturbance attenuation and  $H_\infty$ - control via measurement feedback in nonlinear systems,” *IEEE Trans. Automat. Contr.*, vol. 37, pp. pp. 1283–1293, 1992.
- [19] S. Suzuki, A. Isidori, and T. J. Tarn, “ $\mathcal{H}_\infty$  control of nonlinear systems with sampled measurements,” *J. Math. Systems, Estimation, and Control*, vol. 5, pp. 1–12, 1995.
- [20] H. Guillard, “On nonlinear  $\mathcal{H}_\infty$  control under sampled measurements,” *IEEE Trans. Automat. Contr.*, vol. 42, pp. 880–885, 1997.
- [21] Nguang, S. K. and P. Shi, (2000). Nonlinear  $\mathcal{H}_\infty$  filtering of sampled-data systems. *Automatica*, **36**(2), 303–310.
- [22] Nguang, S. K. and P. Shi, (2000). On design filters for uncertain sampled-data nonlinear systems. *Systems & Contr. Letts*, **41**(2), 305–316.
- [23] Cao, S. G., N. W. Ree and G. Feng, (1996). Quadratic stability analysis and design of continuous-time fuzzy control systems. *Int. J. Syst. Sci.*, **27**, 193–203.
- [24] Chen, C., P. C. Chen and C. K. Chen, (1993). Analysis and design of fuzzy control system. *Fuzzy Sets Sys.*, **57**, 125–140.
- [25] Nguang, S. K. and P. Shi.  $H_\infty$  control of fuzzy system models using linear output controller. Technical report. submitted for publication, 2000.
- [26] Takagi, T. and M. Sugeno, (1985). Fuzzy identification of systems and its applications to modelling and control. *IEEE Trans. Syst. Man. Cybern.*, **15**, 116–132.
- [27] Tanaka, K. and M. Sugeno, (1992). Stability analysis and design of fuzzy control systems. *Fuzzy Sets Sys.*, **45**, 135–156.

- [28] Tanaka, K., (1995). Stability and stabilizability of fuzzy neural linear control systems. *IEEE Trans. Fuzzy Syst.*, **3**, 438–447.
- [29] Tanaka, K., T. Ikeda and H. O. Wang, (1996). Robust stabilization of a class of uncertain nonlinear systems via fuzzy control: Quadratic stabilizability,  $H_\infty$  control theory, and linear matrix inequality. *IEEE Trans. Fuzzy Syst.*, **4**, 1–13.
- [30] Teixeira, M. and S. H. Zak, (1999). Stabilizing controller design for uncertain nonlinear systems using fuzzy models. *IEEE Trans. Fuzzy Syst.*, **7**, 133–142.
- [31] Wang, H. O., K. Tanaka and M. F. Griffin, (1996). An approach to fuzzy control of nonlinear systems: Stability and design issues. *IEEE Trans. Fuzzy Syst.*, **4**, 14–23.
- [32] Wang, L., (1997). *A course in fuzzy systems and control*. Englewood Cliffs, NJ: Prentice-Hall.
- [33] Ma, X. J., Z. Q. Sun and Y. Y. He, (1998). Analysis and design of fuzzy controller and fuzzy observer. *IEEE Trans. Fuzzy Syst.*, **6**, 41–51.
- [34] B. S. Chen, C. S. Tseng and H. J. Uang, “Mixed  $H_2/H_\infty$  fuzzy output feedback control design for nonlinear dynamic systems: An LMI approach”, *IEEE Trans. Fuzzy Syst.*, Vol. 8, (2000)249-265.