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On the Jacobi-Metric Stability Criterion

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Abstract

We investigate the exact relation existing between the stability equation for the solutions of a mechanical system and the geodesic deviation equation of the associated geodesic problem in the Jacobi metric constructed via the Maupertuis-Jacobi Principle. We conclude that the dynamical and geometrical approaches to the stability/instability problem are not equivalent.

1 Introduction

In recent years, several authors [1], [2], [3], [4], have formulated geometrical criteria of (local) stability/instability in mechanical systems using different "geometrization" techniques (Maupertuis-Jacobi Principle, Eisenhart metric, etc). The main idea is to interpret the local instability problem, understood in terms of sensitive dependence on initial conditions, as the study of an appropriate geodesic deviation equation. As a principal application, chaotic behaviors in Hamiltonian mechanical systems that appears in cosmological models have been described using these results. Most of these works are constructed using the Maupertuis-Jacobi principle for natural mechanical systems, both in the very well known Riemannian case, but also in the recent generalization to the non-Riemanian one [2].

The Maupertuis-Jacobi principle establishes, in his classical formulation, the equivalence between the resolution of the Euler-Lagrange equations of a natural Hamiltonian dynamical system (hence the Newton equations), for a given value of the mechanical energy, and the calculation of the geodesic curves in an associated Riemannian manifold. Throughout the time this equivalence has been used for different purposes, as the mentioned description of chaotic situations, but also in the analysis of ergodic systems [3], [5], non-integrability problems [7], determination of stability properties of solitons [8], [9], etcetera.

The linealization of the geodesic equations in a given manifold gives in a natural way the so-called Jacobi equation, or geodesic deviation equation, that allows to compute the stability/instability of a given geodesic curve in terms of the sign of the curvature tensor over the geodesic (in fact, for two-dimensional manifolds, the problem reduces simply to the computation of the sign of the gaussian curvature along the geodesic, see for instance [6]).

The geometrization of the mechanical problem provides, as mentioned, a possible criterion of stability of the solutions of in terms of the geodesic deviation equation of the Jacobi metric associated to the system, via the Maupertuis-Jacobi principle, that we will call Jacobi-metric stability criterion.

From the point of view of the Variational Calculus applied to geodesics, a similar result is obtained for the problem of calculation of fixed-endpoints geodesics, where the sign of the second variation functional is determined by the geodesic deviation operator.

In this work we analyze the exact relation existing between this Jacobi-metric criterion and the direct analysis of the stability of the solutions without using the geometrization principle. The linealization of the Euler-Lagrange equation (in this case, Newton equations) lead to a Jacobi-like equation that generalizes the geodesic deviation one to the case of natural mechanical systems. In fact, this equation is also called Jacobi equation in the context of second-order ordinary differential equations theory or KCC theory (Kosambi-Cartan-Chern), [10], [11].

As we will see, the two approaches (geometrical and dynamical) are not equivalent in general, and the Jacobi-metric criterion do not provide exactly the same result as the standard (or dynamical) one.

The structure of the paper is as follows: in section 2 we present the concepts involved in the work; Section 3 is dedicated to Jacobi-metric stability criterion and its relation with the dynamical one. In Section 4, the analysis is extended to the variational point of view for fixed end-points problems. Finally, an Appendix is included with several technical formulas (more or less well known) about the behavior of covariant derivatives and curvature tensor under conformal transformations and reparametrizations of curves.

2 Preliminaries and Notation

We treat in this work with natural Hamiltonian dynamical systems, i.e., the triple (M, g, L), where (M, g) is a Riemannian manifold, and L is a natural Lagrangian function: $L : TM \to \mathbb{R}$, $L = T - U$,

$$T = \frac{1}{2}\langle\dot\gamma, \dot\gamma\rangle = \frac{1}{2} g_{ij}\dot q^i \dot q^j$$

in a system of local coordinates $(q^1, \dots, q^n)$ in M, U is a given smooth function $U : M \to \mathbb{R}$, $\gamma(t) \equiv (q^1(t), \dots, q^n(t))$ is a smooth curve on M, and g_{ij} are the components of the metric g in this coordinate system (Einstein convention about sum in repeated indices will be used along the paper).

The solutions (trajectories) of the system are the extremals of the action functional $S[\gamma]$, defined in the space of smooth curves on M: $\gamma : [t_0, t_1] \to M$, (we assume that γ is at least of class C^2 in the interval (t_0, t_1)).

$$S[\gamma] = \int_{t_0}^{t_1} L(\gamma, \dot\gamma)\, dt \qquad (1)$$

where $\dot\gamma \in \Gamma(TM)$ stands for the tangent vector field $\frac{d\gamma}{dt}$, i.e. $\dot\gamma(t) \equiv \frac{d\gamma}{dt}(t) \in T_{\gamma(t)}M$.

Euler-Lagrange equations associated to this functional are Newton equations for the system:

$$\delta S = 0 \Rightarrow \nabla_{\dot\gamma}\dot\gamma = -\mathrm{grad}\,U \tag{2}$$

where $\nabla_{\dot\gamma}$ stands for the covariant derivative along $\gamma(t) \equiv (q^i(t))$:

$$\nabla_{\dot\gamma}\dot\gamma \equiv \frac{D\dot q^i}{dt} = \frac{d\dot q^i}{dt} + \Gamma^i_{jk}\dot q^j\dot q^k$$

being Γ^i_{jk} the Christofell symbols of the Levi-Civitta connection associated to the metric g.

$$\Gamma^l_{ij} = \frac{1}{2}g^{kl}\left(\frac{\partial g_{jk}}{\partial q^i} + \frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{ij}}{\partial q^k}\right)$$

$\mathrm{grad}\,U$ is the vector field with components: $(\mathrm{grad}\,U)^i = g^{ij}\frac{\partial U}{\partial q^j}$. Equation (2) is thus written in local coordinates as the following system of ordinary differential equations:

$$\frac{D\dot q^i}{dt} = \ddot q^i + \Gamma^i_{jk}\dot q^j\dot q^k = -g^{ij}\frac{\partial U}{\partial q^j} \tag{3}$$

Natural Hamiltonian dynamical systems over Riemannian manifolds satisfy Legendre's condition in an obvious way, and thus the Legendre transformation is regular, i.e. there exists a diffeomorphism between the tangent and cotangent bundles of M in such a way that the Euler-Lagrange equations are equivalent to the Hamilton (or canonical) equations.

$$\dot p_i = -\frac{\partial H}{\partial q^i}; \qquad \dot q^j = \frac{\partial H}{\partial p_j} \tag{4}$$

where

$$p_j = \frac{\partial L}{\partial \dot q^j} = g_{ij}\dot q^i; \qquad H = \frac{1}{2}g^{ij}p_ip_j + U$$

and g^{ij} denotes the components of the inverse of g.

This kind of systems are autonomous, thus the mechanical energy is a first integral of the system:

$$E = \frac{1}{2}g_{ij}\dot q^i\dot q^j + U(q^1, \dots, q^n)$$

Stability of the solutions of (3), understood in terms of sensitive dependence on initial conditions, is interpreted as follows: The trajectory $\gamma(t)$, solution of (3), is said to be stable if all trajectories with sufficiently close initial conditions at t_0 remains close to the trajectory $\gamma(t)$ for later times $t > t_0$.

Let $\gamma(t;\alpha) = (q^1(t;\alpha), \dots, q^n(t;\alpha))$ be a family of solutions of equations (3), with $\gamma(t) \equiv \gamma(t;0)$, and given initial conditions $q^i(t_0;\alpha)$, $\dot q^i(t_0;\alpha)$. Let us assume that the initial conditions are analytic in the parameter α. Then: $\gamma(t) = (q^i(t))$ is a stable trajectory iff for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $|q^i(t;\alpha) - q^i(t)| < \varepsilon$ for $t > t_0$ and for all trajectories $q(t;\alpha) = (q^i(t;\alpha))$ satisfying both $|q^i(t_0;\alpha) - q^i(t_0)| < \delta$ and $|\dot q^i(t_0;\alpha) - \dot q^i(t_0)| < \delta$.

Assuming that g is smooth and considering that $\gamma(t;\varepsilon)$ are analytic in ε (they are solutions of an analytic system of differential equations), we can write, for ε sufficiently small:

$$q^i(t;\varepsilon) = q^i(\tau) + \varepsilon v^i(t) + o(\varepsilon^2) \qquad ; \qquad v^i(t) = \left.\frac{\partial q^i(t;\varepsilon)}{\partial\varepsilon}\right|_{\varepsilon=0} \tag{5}$$

In a similar way, we can write:

$$\Gamma^i_{jk}(q(t;\varepsilon)) = \Gamma^i_{jk}(q(t)) + \varepsilon\frac{\partial\Gamma^i_{jk}}{\partial q^l}(q(t))\,v^l(t) + o(\varepsilon^2) \tag{6}$$

$$g^{ij}(q(t;\varepsilon)) = g^{ij}(q(t)) + \varepsilon\frac{\partial g^{ij}}{\partial q^l}(q(t))\,v^l(t) + o(\varepsilon^2) \tag{7}$$

$$\partial_j U(q(t;\varepsilon)) = \partial_j U(q(t)) + \varepsilon\,\partial_l\partial_j U(q(t))\,v^l(t) + o(\varepsilon^2) \tag{8}$$

where $\partial_j U = \frac{\partial U}{\partial q^j}$.

Thus equations (3) become:

$$\ddot v^i + 2\Gamma^i_{jk}\dot v^j\dot q^k = -g^{ip}v^l\left(\partial_l\partial_p U + \Gamma^j_{lp}\partial_j U\right) + g^{jp}\Gamma^i_{lp}\partial_j U\,v^l \tag{9}$$

where all functions are taken at $\gamma(t)$. Taking into account the expression of the second order covariant derivatives:

$$\frac{D^2v^i}{dt^2} = \ddot v^i + \partial_l\Gamma^i_{jk}\dot q^l\dot q^j v^k + 2\Gamma^i_{jk}\dot v^j\dot q^k + \Gamma^i_{jk}v^j\ddot q^k + \Gamma^i_{lp}\Gamma^l_{jk}\dot q^j\dot q^p v^k$$

and the components of the Riemann curvature tensor: $R(X,Y)Z = -\nabla_X(\nabla_Y Z) + \nabla_Y(\nabla_X Z) + \nabla_{[X,Y]}Z$, $\forall X,Y,Z \in \mathfrak{T}(M)$:

$$R^i_{lkj} = \Gamma^i_{kp}\Gamma^i_{pl} - \Gamma^i_{pl}\Gamma^p_{jk} + \partial_k\Gamma^i_{jl} - \partial_l\Gamma^i_{jk}$$

we finally arrive to the expression:

$$\frac{D^2v^i}{dt^2} + R^i_{ljk}\dot q^l\dot q^j v^k = -g^{ij}\left(\partial_l\partial_j U - \Gamma^r_{jl}\partial_r U\right)v^l$$

that can be written as a vector equation:

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}V + K_{\dot\gamma}(V) + \nabla_V\,\mathrm{grad}U = 0 \tag{10}$$

where $V = V(t) \equiv (v^i(t))$, and we have used the sectional curvature tensor:

$$K_X(Y) = R(X,Y)X\,; \quad \forall X,Y \in \mathfrak{T}(M)$$

and the Hessian of the potential energy U: $H(U) = \nabla dU$

$$\nabla dU = \left(\partial_j\partial_l U - \partial_k U\,\Gamma^k_{jl}\right)dq^j \otimes dq^l$$

in such a way that $\forall X,Y \in \mathfrak{T}(M)$

$$\nabla dU(X,Y) = \langle\nabla_X\mathrm{grad}(U),Y\rangle = \langle\nabla_Y\mathrm{grad}(U),X\rangle$$

Solutions of equation (10) determine the behavior of the family of solutions $\gamma(t;\alpha)$ with respect to the selected solution $\gamma(t)$. Thus typical solutions of linear equations (trigonometric functions, exponentials, etc.) will prescribe the stability/instability situations. In several contexts equation (10) is usually called Jacobi equation, by analogy with the geodesic case. In fact, in the so-called KCC theory on second order differential equations, equation (11) is nothing but the Jacobi equation for the special case of Newton differential equations. In order to avoid confusions we will denote Hessian operator for the mechanical system to:

$$\mathcal{H}V = \nabla_{\dot\gamma}\nabla_{\dot\gamma}V + K_{\dot\gamma}(V) + \nabla_V \operatorname{grad} U$$

and thus we reserve the term Jacobi operator (and equation) to the geodesic case, i.e. to the geodesic deviation equation.

In the special case of fixed starting point for the family of solutions $\gamma(t;\alpha)$, i.e. $\gamma(t_0;\alpha) = \gamma(t_0)$, an equivalent approach to equation (11) can be considered. The first variational derivative of functional (1) lead to Euler-Lagrange equations (3), and thus the second variation functional (or Hessian functional) will determine (together obviously with the Legendre straightness condition, automatically satisfied for this kind of systems, see [12]) the local minimum/maximum character of a solution of (3). The second-variation functional of the action S, for the case of proper variations ($V \in \Gamma(TM)$ such that $V(t_0) = V(t_1) = 0$) is:

$$\delta^2 S[\gamma(t)] = \int_{t_0}^{t_1} dt\, \langle \nabla_{\dot\gamma}\nabla_{\dot\gamma}V + K_{\dot\gamma}(V) + \nabla_V \operatorname{grad} U, V\rangle = \int_{t_0}^{t_1} dt\, \langle \mathcal{H}V, V\rangle \qquad (11)$$

and thus the positive or negative definiteness of the $\mathcal{H}$ operator determines the character of the solution $\gamma(t)$.

3 The Jacobi-Metric Stability Criterion

The Maupertuis-Jacobi Principle establishes the equivalence between the resolution of the Newton equations (3) of the natural system and the calculation of the geodesic curves in an associated Riemannian manifold. The crucial point of the Principle is the existence of the mechanical energy as first integral for equations (3). Solutions of (3) corresponding to a fixed value $E = T + U$ will be in one to one correspondence with the solutions of the equations of geodesics in the manifold M with the so-called Jacobi metric: $h = 2(E - U)g$, associated to the E value.

Geodesics in the Riemannian manifold $\mathbb{M} \equiv (M, h)$[1] can be viewed as extremals of the free-action

[1] We will call $s \equiv s_h$, i.e.: $ds_g^2 = g_{ij}dq^i dq^j$, $ds^2 = h_{ij}dq^i dq^j$, and $h_{ij} = 2(E - U)g_{ij}$. We will also write ∇^J for the covariant derivative with respect to h, and, for any vector fields $X, Y \in \Gamma(TM)$: $h(X,Y) = \langle X, Y\rangle^J$, and $\|X\|^J = \sqrt{\langle X, X\rangle^J}$.

functional S_0 or of the Length functional L:

$$S_0[\gamma]=\int_{t_0}^{t_1}\frac{1}{2}(\|\dot\gamma(t)\|^J)^2\,dt;\qquad L[\gamma]=\int_{t_0}^{t_1}\|\dot\gamma(t)\|^J\,dt \tag{12}$$

for any differentiable curve $\gamma:[t_0,t_1]\to M$ connecting the points $\gamma(t_0)=P$ and $\gamma(t_1)=Q$, $P,Q\in M$. The extremal conditions, $\delta S_0=0$ and $\delta L=0$, lead us to the Euler-Lagrange equations (equations of the geodesics in M):

$$\delta S_0=0\Rightarrow\nabla^J_{\dot\gamma}\dot\gamma=0;\qquad \delta L=0\Rightarrow\nabla^J_{\dot\gamma}\dot\gamma=\lambda(t)\dot\gamma;\qquad \lambda(t)=\frac{d^2t}{ds^2}\left(\frac{ds}{dt}\right)^2 \tag{13}$$

$\delta L=0$ leads to the equations of the geodesics parametrized with respect to an arbitrary parameter t (often called pre-geodesics) as a natural consequence of the invariance under reparametrizations of the Length functional, whereas $\delta S_0=0$ produces the equations of affinely parametrized geodesics. If we restrict to the arc-length parametrization and we will denote, as usual, $'=\frac{d}{ds}$, equations (13) are written as: $\nabla^J_{\gamma'}\gamma'=0$, or explicitly, in terms of Christoffel symbols $\tilde\Gamma^i_{jk}$ of the Levi-Civitta connection of h, as:

$$\frac{D(q^i)'}{ds}=(q^i)''+\tilde\Gamma^i_{jk}(q^j)'(q^k)'=0 \tag{14}$$

The Maupertuis-Jacobi Principle can be formulated in the following form:

Theorem of Jacobi. The extremal trajectories of the variational problem associated to the functional (1) with mechanical energy E, are pre-geodesics of the manifold (M,h), where h is the Jacobi metric: $h=2(E-U)g$.

From an analytic point of view, the theorem simply establishes that the Newton equations (3) for the action S, are written as the geodesic equations in (M,h): $\nabla^J_{\gamma'}\gamma'=0$, when the conformal transformation: $h=2(E-U)g$, and a reparametrization (from the dynamical time t to the arc-length parameter s in (M,h)) are performed.

Moreover, the dependence between the two parameters is determined over the solutions by the equation:

$$\frac{ds}{dt}=2\sqrt{E-U(\gamma(s))T}=2(E-U(\gamma(s))) \tag{15}$$

The proof of this theorem can be viewed in several references (see for instance [6], see also [12] for a general version of the Principle). However, a very simple proof of the theorem can be carried out by the explicit calculation of equations (14) in terms of the original metric g, making use of Lemmas 1 and 2 of the Appendix, that detail the behavior of the covariant derivatives under conformal transformations and re-parametrizations. $\nabla^J_{\gamma'}\gamma'=0$ turns out to be

$$\nabla_{\gamma'}\gamma'+h\langle\mathrm{grad}(\ln(2(E-U))),\gamma'\rangle\gamma'-\frac{1}{2}h\langle\gamma',\gamma'\rangle\mathrm{grad}(\ln(2(E-U)))=0 \tag{16}$$

in terms of the ∇ derivative. By applying now Lemma 2 to (16) we obtain, after the corresponding reparametrization and simplifications, the equation

$$\nabla_{\dot\gamma}\dot\gamma + \mathrm{grad}(U) = 0$$

i.e. the Newton equations of the mechanical system.

This result allows to define the Jacobi-metric criterion for stability of the mechanical solutions in terms of the corresponding geodesics of the Jacobi metric.

In an analogous way to the previous section, one can linearize the equations (14) of the geodesics in (M,h) by considering a family of geodesics $\gamma(s;\varepsilon)$:

$$\gamma(t;\varepsilon) = \gamma(t) + \varepsilon V + o(\varepsilon^2)$$

with $V(s) = \frac{\partial\gamma(s;\varepsilon)}{\partial\varepsilon}\Big|_{\varepsilon=0}$. Following the same steps, one finally arrives to the expression

$$\nabla^J_{\gamma'}\nabla^J_{\gamma'}V + K^J_{\gamma'}(V) = 0 \tag{17}$$

where $V = V(s) \equiv (v^i(s))$, and K^J is the sectional curvature tensor of the h metric.

Equation (17) is the Geodesic Deviation Equation, or Jacobi Equation, for a given geodesic $\gamma(s)$ of (M,h). We will denote Jacobi Operator, or Geodesic Deviation Operator to:

$$\mathcal{J}^J V = \nabla^J_{\gamma'}\nabla^J_{\gamma'}V + K^J_{\gamma'}(V) \tag{18}$$

Thus stability of a solution of Newton equations $\gamma(t)$ will be determined, in this criterion, if the corresponding geodesic $\gamma(s)$ is stable, that finally leads to equation (17).

In order to determine the exact relation existing between the Jacobi-metric criterion and the dynamical o standard one, we will analyze now equation (17), by using the results about conformal transformations and re-parametrizations included in the Appendix.

Applying Lemma 1 and Lemma 3 (see Appendix) to the Jacobi operator (18) and simplifying the expressions, equation (17) is written as:

$$\begin{aligned}
\mathcal{J}^J V \;=\;& \nabla_{\gamma'}\nabla_{\gamma'}V + K_{\gamma'}(V) + \frac{1}{2}\langle F, V\rangle\nabla_{\gamma'}\gamma' + \langle F,\gamma'\rangle\nabla_{\gamma'}V - \frac{1}{2}\langle\gamma',\gamma'\rangle\nabla_V F + \\
&+ \left(\langle F, \nabla_{\gamma'}V\rangle + \frac{1}{2}\langle F,V\rangle\langle F,\gamma'\rangle + \langle\nabla_V F,\gamma'\rangle\right)\gamma' + \\
&+ \left(\frac{1}{2}\langle F, \nabla_{\gamma'}\gamma'\rangle + \frac{1}{2}\langle F,\gamma'\rangle^2 - \frac{1}{4}\langle\gamma',\gamma'\rangle\langle F,F\rangle\right)V + \\
&+ \left(\frac{1}{2}\langle\nabla_{\gamma'}\gamma', V\rangle - \langle\gamma',\nabla_{\gamma'}V\rangle - \frac{1}{2}\langle F,\gamma'\rangle\langle\gamma',V\rangle\right)F
\end{aligned} \tag{19}$$

depending only on the metric g, and where F denotes: $F = \mathrm{grad}\ln(2(E-U))$. Re-parametrization of $\gamma(s)$ in terms of the t-parameter:

$$\gamma'(s) = \frac{1}{2(E-U(\gamma(t)))}\dot\gamma(t); \quad \nabla_{\gamma'}X = \frac{1}{2(E-U(\gamma(t)))}\nabla_{\dot\gamma}X$$

and application of Lemma 2 to (19) lead to:

$$
\begin{aligned}
{}^{J}V = \frac{1}{(2(E-U))^2}\Big[& \nabla_{\dot\gamma}\nabla_{\dot\gamma}V + K_{\dot\gamma}(V) + \frac12\langle F,V\rangle\nabla_{\dot\gamma}\dot\gamma - \frac12\langle\dot\gamma,\dot\gamma\rangle\nabla_V F + \\
& + (\langle F,\nabla_{\dot\gamma}V\rangle + \langle\nabla_V F,\dot\gamma\rangle)\dot\gamma + \\
& + \left(\frac12\langle F,\nabla_{\dot\gamma}\dot\gamma\rangle - \frac14\langle\dot\gamma,\dot\gamma\rangle\langle F,F\rangle\right)V + \\
& + \left(\frac12\langle V,\nabla_{\dot\gamma}\dot\gamma\rangle - \langle\dot\gamma,\nabla_{\dot\gamma}V\rangle\right)F\Big]
\end{aligned}
\tag{20}
$$

Expression (20) is written in terms of quantities depending only on the metric g and the t-parameter. In order to relate this expression with the Hessian operator $\mathcal{H}$ we need to remember that $\gamma(t)$ is a solution of the Newton equations (3) of energy E, and thus: $\nabla_{\dot\gamma}\dot\gamma = -\mathrm{grad}U$, $\langle\dot\gamma,\dot\gamma\rangle = 2(E-U(\gamma(t)))$. Using these facts and simplifying we arrive to:

$$
{}^{J}V = \frac{1}{(2(E-U))^2}\left[\mathcal{H}V - \frac{d}{dt}\left(\frac{\langle V,\mathrm{grad}U\rangle}{E-U}\right)\dot\gamma + \frac{\langle\mathrm{grad}U,V\rangle + \langle\dot\gamma,\nabla_{\dot\gamma}V\rangle}{E-U}\mathrm{grad}U\right] \tag{21}
$$

where we have used the identity: $\langle\dot\gamma,\nabla_V\mathrm{grad}U\rangle = \langle V,\nabla_{\dot\gamma}\mathrm{grad}U\rangle$.

Obviously, the two operators do not coincide, and correspondingly solutions of the Jacobi equation ${}^{J}V = 0$ and the equation $\mathcal{H}V = 0$ do not so. The two criteria of stability are not equivalent. In order to investigate equation (21) to determine the reasons of this non-equivalence between the two criteria, we have to remark that whereas all the geodesics $\gamma(s;\epsilon)$ considered in the calculation of ^{J} correspond to mechanical solutions of energy E (they are solutions of the equation of geodesics in (M,h), with $h = 2(E-U)g$), the solutions $\gamma(t;\epsilon)$ are in principle of energy:

$$
E_\epsilon = \frac12\dot q^i(t;\epsilon)g_{ij}(\gamma(t;\epsilon))\dot q^j(t;\epsilon) + U(q(t;\epsilon)) \tag{22}
$$

But a correct comparison between two stability criteria is only well established if the criteria act over the same objects. Thus the comparison is only licit if one restricts the family $\gamma(t;\epsilon)$ to verify: $E_\epsilon = E$. Expanding (22) in ϵ we find:

$$
E_\epsilon = E + \epsilon(\langle\dot\gamma,\nabla_{\dot\gamma}V\rangle + \langle\mathrm{grad}U,V\rangle) + o(\epsilon^2) \tag{23}
$$

And thus the requirement $E_\epsilon = E$ reduces to the verification of: $\langle\dot\gamma,\nabla_{\dot\gamma}V\rangle = -\langle\mathrm{grad}U,V\rangle$.

Thus the relation between the Jacobi operator and the hessian operator restricted to equal-energy variations is:

$$
{}^{J}V = \frac{1}{(2(E-U))^2}\left[\mathcal{H}V - \frac{d}{dt}\left(\frac{\langle V,\mathrm{grad}U\rangle}{E-U}\right)\dot\gamma\right] \tag{24}
$$

and the two operators are not equivalent, even considering the equal-energy restriction.

4 The Variational point of view

As it has been explained in the Introduction of this work, we will apply now the above obtained results to the special case of fixed end-points, i.e. we will restrict our analysis to the situation where the conditions: $\gamma(t_0) = P$ and $\gamma(t_1) = Q$, with P and Q fixed, are imposed. From the mechanical point of view, this is exactly the case of the calculation of solitonic solutions in Field Theories (see for instance [8]) where asymptotic conditions determine the starting and ending points. Using the Maupertuis-Jacobi Principle, this situation is translated to the problem of calculating the geodesics connecting two fixed points in the manifold M. We thus use the framework of the Variational Calculus for fixed end-points problems.

The minimizing character (local minimum) of a geodesic $\gamma(s)$ connecting two fixed points is determined by the second variation functional:

$$\delta^2 S_0 = \int_{s_0}^{s_1} \langle \Delta^J V, V\rangle\, ds\,; \quad \delta^2 L = \int_{s_0}^{s_1} \left\langle \Delta^J V^\perp, V^\perp \right\rangle ds \tag{25}$$

where Δ^J is the geodesic deviation operator of h:

$$\Delta^J V = \nabla^J_{\gamma'}\nabla^J_{\gamma'} V + R^J(\gamma', V)\gamma' = \nabla_{\gamma'}\nabla_{\gamma'} V + K^J_{\gamma'}(V)$$

where $V \in \Gamma(TM)$ denotes any proper variation and $V^\perp$ is the orthogonal component of V to the geodesic.

We will show now two theorems, in the first one it is established the difference between the second variation functional of the dynamical problem and the corresponding one to the free-action functional associated to the Jacobi metric. In the second one, a similar analysis is carried out for the Length functional.

Theorem 1. Let $\gamma(t)$ be an extremal of the functional $S[\gamma] = \int_{t_0}^{t_1}\left(\frac{1}{2}\langle\dot\gamma,\dot\gamma\rangle - U(\gamma)\right) dt$, and let $S_0^J[\gamma] = \int_{s_0}^{s_1}\frac{1}{2}\langle\gamma',\gamma'\rangle^J ds$ be the free-action functional of the Jacobi metric associated to $S[\gamma]$ and corresponding to a fixed value, E, of the mechanical energy, then the corresponding Hessian functionals verify:

$$\delta^2 S_0^J[\gamma] = \delta^2 S[\gamma] + \int_{t_0}^{t_1} dt\, 2\langle\dot\gamma, \nabla_{\dot\gamma} V\rangle\langle F, V\rangle \tag{26}$$

where $F = \mathrm{grad}\ln(2(E-U))$.

Theorem 2. Let $\gamma(t)$ be an extremal of the $S[\gamma] = \int_{t_0}^{t_1}\left(\frac{1}{2}\langle\dot\gamma,\dot\gamma\rangle - U(\gamma)\right) dt$ functional and let $L^J[\gamma] = \int_{s_0}^{s_1}\|\gamma'\| ds$ be the length functional of the Jacobi metric associated to $S[\gamma]$ and corresponding to a fixed value, E, of the mechanical energy, then the corresponding hessian functionals verify:

$$\delta^2 L^J[\gamma] = \delta^2 S[\gamma] - \int_{t_0}^{t_1}\frac{dt}{2(E-U)}\left[\langle\nabla\dot\gamma, V\rangle - s\langle\dot\gamma, \nabla_{\dot\gamma} V\rangle\right]^2 \tag{27}$$

From (27) it is obvious that minimizing geodesics are equivalent to minimizing (stable) solutions of the dynamical system, i.e. a positive definiteness of $\delta^2 L^J$ implies the same behaviour for $\delta^2 S$, but it is not necessarily true the reciprocal statement.

If we restrict the variations to the orthogonal ones, $V = V^\perp$, (27) can be re-written as:

$$\delta^2 S\big|_{V=V^\perp} = \delta^2 L^J + \int_{s_0}^{s_1} ds\, \langle F^J, V^\perp\rangle_J^{\,2}$$

The proofs of these two theorems are based on the behaviour of the covariant derivatives and the curvature tensor under reparametrizations and conformal transformations of the metric tensor. We thus use the technical results included in the Appendix.

Proof of Theorem 1. We start with equation (25) particularized to the case of the Jacobi metric:

$$\delta^2 S_0^J[\gamma] = \int_{s_0}^{s_1} ds\, \langle \mathcal{J}^J V, V\rangle^J$$

with $\mathcal{J}^J V = \nabla^J_{\gamma'}\nabla^J_{\gamma'}V + K^J_{\gamma'}(V)$.

Using expression (21), deduced in the previous section after changing the metric and re-parametrizing, we can write:

$$\langle \nabla^J_{\gamma'}\nabla^J_{\gamma'}V + K^J_{\gamma'}(V), V\rangle^J = \frac{1}{2(E-U)}\langle \nabla_{\dot\gamma}\nabla_{\dot\gamma}V + K_{\dot\gamma}(V) + \nabla_V \mathrm{grad}U, V\rangle + \tag{28}$$

$$+\frac{1}{2(E-U)}\frac{\partial}{\partial t}(\langle F,V\rangle\langle\dot\gamma,V\rangle) + \frac{1}{(E-U)^2}\langle\dot\gamma,\nabla_{\dot\gamma}V\rangle\langle\mathrm{grad}U,V\rangle$$

And thus, the second variation functional is written as:

$$\begin{aligned}
\frac{d^2 S_0^J[\gamma]}{d\alpha^2}(0) &= \int_{s_0}^{s_1} ds\, \langle \nabla^J_{\gamma'}\nabla^J_{\gamma'}V + K^J_{\gamma'}(V), V\rangle^J = \\
&= \int_{t_0}^{t_1} dt \langle \nabla_{\dot\gamma}\nabla_{\dot\gamma}V + K_{\dot\gamma}(V) + \nabla_V \mathrm{grad}U, V\rangle + \\
&+ \int_{t_0}^{t_1} dt\, 2\langle\dot\gamma,\nabla_{\dot\gamma}V\rangle\langle F,V\rangle - \langle F,V\rangle\langle\dot\gamma,V\rangle\Big|_{t_0}^{t_1}
\end{aligned}$$

For proper variations: $V(t_1) = V(t_2) = 0$

$$\frac{d^2 S_0^J[\gamma]}{d\alpha^2}(0) = \frac{d^2 S[\gamma]}{d\alpha^2}(0) + \int_{t_1}^{t_2} dt\, 2\langle\dot\gamma,\nabla_{\dot\gamma}V\rangle\langle F,V\rangle$$

with $F = \mathrm{grad}\,\mathrm{Ln}(2(E-U)) = -\frac{1}{E-U}\mathrm{grad}U$.

Q.E.D.

Proof of Theorem 2: For the Length functional we have:

$$\frac{d^2 L^J[\gamma]}{d\alpha^2}(0) = \int_{s_1}^{s_2} ds \left\langle \nabla^J_{\gamma'}\nabla^J_{\gamma'}V^\perp + K^J_{\gamma'}(V^\perp), V^\perp\right\rangle_J$$

where

$$V^{\perp} = V - \left\langle \frac{\gamma'}{\|\gamma'\|_J}, V \right\rangle_J \frac{\gamma'}{\|\gamma'\|_J} = V - \langle \gamma', V \rangle \gamma'$$

and thus:

$$\frac{d^2 L^J[\gamma]}{d\epsilon^2}(0) = \frac{d^2 S_0^J[\gamma]}{d\epsilon^2}(0) - \int_{s_1}^{s_2} ds \, \langle \gamma', \nabla^J_{\gamma'} V \rangle_J^2$$

By using Theorem 1, we have that

$$\begin{aligned}
\frac{d^2 L^J[\gamma]}{d\epsilon^2}(0) &= \frac{d^2 S_0^J[\gamma]}{d\epsilon^2}(0) - \int_{s_1}^{s_2} ds \, \langle \gamma', \nabla^J_{\gamma'} V \rangle_J^2 = \\
&= \frac{d^2 S[\gamma]}{d\epsilon^2}(0) + \int_{t_1}^{t_2} 2\langle \dot\gamma, \nabla_{\dot\gamma} V\rangle \langle F, V\rangle dt + \int_{t_1}^{t_2} A(t) dt
\end{aligned}$$

where:

$$\int_{t_1}^{t_2} A(t)dt = -\int_{s_1}^{s_2} ds \, \langle \gamma', \nabla^J_{\gamma'} V \rangle_J^2 = -\int_{t_1}^{t_2} dt (2(E-U))^3 \langle \gamma', \nabla^J_{\gamma'} V \rangle^2$$

From Lemma 1 and Newton equations, we have

$$\int_{t_1}^{t_2} A(t)dt = -\int_{t_1}^{t_2} dt \frac{1}{2(E-U)} \left(\langle \dot\gamma, \nabla_{\dot\gamma} V\rangle + (E-U)\langle F, V\rangle\right)^2$$

Finally

$$\frac{d^2 L^J[\gamma]}{d\epsilon^2}(0) = \frac{d^2 S[\gamma]}{d\epsilon^2}(0) - \int_{t_1}^{t_2} dt \frac{1}{2(E-U)} \left[\langle \dot\gamma, \nabla_{\dot\gamma} V\rangle - \langle \nabla_{\dot\gamma}\dot\gamma, V\rangle\right]^2$$

Q.E.D.

5 Appendix

Lemma 1. Given a conformal transformation in a riemannian manifold, $(M,g) \to (M,\tilde g); \tilde g = f(x) g$, $f(x) \neq 0, \forall x \in M$, let ∇ and $\tilde\nabla$ be the associated Levi-Civita connections respectively. Then, for all $X, Y, Z \in \Gamma(TM)$ it is verified that:

$$\tilde\nabla_X Y = \nabla_X Y + \frac{1}{2}\langle F, Y\rangle X + \frac{1}{2}\langle F, X\rangle Y - \frac{1}{2}\langle X, Y\rangle F \tag{29}$$

$$\begin{aligned}
\tilde\nabla_X \tilde\nabla_Y Z &= \nabla_X \nabla_Y Z + \frac{1}{2}\langle F, Z\rangle \nabla_X Y + \frac{1}{2}\langle F, Y\rangle \nabla_X Z - \frac{1}{2}\langle Y, Z\rangle \nabla_X F + \frac{1}{2}\langle F, X\rangle \nabla_Y Z + \\
&+ \left(\frac{1}{2}\langle F, \nabla_Y Z\rangle + \frac{1}{2}\langle F, Z\rangle\langle F, Y\rangle - \frac{1}{4}\langle Y, Z\rangle\langle F, F\rangle\right) X + \\
&+ \left(\frac{1}{2}\langle \nabla_X F, Z\rangle + \frac{1}{2}\langle F, \nabla_X Z\rangle + \frac{1}{4}\langle F, Z\rangle\langle F, X\rangle\right) Y + \\
&+ \left(\frac{1}{2}\langle \nabla_X F, Y\rangle + \frac{1}{2}\langle F, \nabla_X Y\rangle + \frac{1}{4}\langle F, X\rangle\langle F, Y\rangle\right) Z + \\
&+ \left(-\frac{1}{2}\langle \nabla_X Y, Z\rangle - \frac{1}{2}\langle Y, \nabla_X Z\rangle - \frac{1}{2}\langle X, \nabla_Y Z\rangle - \frac{1}{4}\langle F, Z\rangle\langle X, Y\rangle - \frac{1}{4}\langle F, Y\rangle\langle X, Z\rangle\right) F
\end{aligned} \tag{30}$$

where the scalar products are taken with respect to the metric g and $F = \mathrm{grad}(\ln f)$ (grad stands for the gradient with respect the metric g).

Proof: By direct calculation. Let us consider the expression of the Christoffel symbols of the $\tilde g$ metric:

$$\tilde\Gamma^i_{jk} = \frac{1}{2}\tilde g^{ir}\left(\partial_k \tilde g_{jr} - \partial_r \tilde g_{jk} + \partial_j \tilde g_{rk}\right)$$

and substitute $\tilde g_{ij} = f g_{ij}$, $\tilde g^{ij} = \frac{1}{f} g^{ij}$. So

$$\tilde\Gamma^i_{jk} = \Gamma^i_{jk} + \frac{1}{2}\left(\mathrm{grad}(\ln f)^m\left(\delta^i_j g_{mk} + \delta^i_k g_{mj}\right) - g_{jk}\,\mathrm{grad}(\ln f)^i\right)$$

and the covariant derivative will be

$$\tilde\nabla_X Y = X^j\tilde\nabla_j Y = X^j\left(\frac{\partial Y^i}{\partial x^j} + \tilde\Gamma^i_{jk}Y^k\right)\frac{\partial}{\partial x^i} = \nabla_X Y + A^i_{jk}X^jY^k\frac{\partial}{\partial x^i}$$

where A^i_{jk} stands for:

$$A^i_{jk}X^jY^k = \frac{1}{2}\left(\mathrm{grad}(\ln f)^m\left(\delta^i_j g_{mk} + \delta^i_k g_{mj}\right) - g_{jk}\,\mathrm{grad}(\ln f)^i\right)X^jY^k =$$

$$= \frac{1}{2}\left(\langle \mathrm{grad}(\ln f), Y\rangle X^i + \langle \mathrm{grad}(\ln f), X\rangle Y^i\right) - \frac{1}{2}\langle X, Y\rangle\,\mathrm{grad}(\ln f)^i$$

Finally, simplifying

$$\tilde\nabla_X Y = \nabla_X Y + \frac{1}{2}\langle \mathrm{grad}(\ln f), Y\rangle X + \frac{1}{2}\langle \mathrm{grad}(\ln f), X\rangle Y - \frac{1}{2}\langle X, Y\rangle\,\mathrm{grad}(\ln f)$$

and similarly for (30).

Q.E.D.

Lemma 2. Given a (differentiable) curve $\gamma : [t_1, t_2] \to M$ on M, let $\sigma(s) = \gamma(t(s))$ be an admissible re-parametrization of γ, $ds = f(x(t))dt$ ($f(x(t)) \neq 0, \forall t \in [t_1, t_2]$). Then $\forall X \in \Gamma(TM)$:

$$\nabla_{\sigma'} X = \frac{1}{f(x)}\nabla_{\dot\gamma} X \tag{31}$$

$$\nabla_{\sigma'}\sigma' = \frac{1}{f(x)^2}\left(\nabla_{\dot\gamma}\dot\gamma - \langle \mathrm{grad}(\ln f), \dot\gamma\rangle\dot\gamma\right) \tag{32}$$

$$\nabla_{\sigma'}\nabla_{\sigma'} X = \frac{1}{f(x)^2}\left(\nabla_{\dot\gamma}\nabla_{\dot\gamma} X - \langle \mathrm{grad}(\ln f), \dot\gamma\rangle\nabla_{\dot\gamma} X\right) \tag{33}$$

where $\dot\gamma(t) = \frac{d\gamma(t)}{dt}$ and $\sigma'(s) = \frac{d\sigma(s)}{ds}$.

Proof: Again by direct calculation

$$\nabla_{\sigma'} X = \left(\frac{dX^i}{ds} + \Gamma^i_{jk}x'^jX^k\right)\frac{\partial}{\partial x^i} = \left(\frac{dX^i}{dt}\frac{dt}{ds} + \Gamma^i_{jk}\dot x^j\frac{dt}{ds}X^k\right)\frac{\partial}{\partial x^i} = \frac{1}{f}\nabla_{\dot\gamma} X$$

$$\nabla_{\sigma'}\sigma' = \frac{1}{f}\nabla_{\dot\gamma}\sigma' = \frac{1}{f}\left(\frac{dx'^i}{dt} + \Gamma^i_{jk}\dot x^j x'^k\right)\frac{\partial}{\partial x^i} = \left(\frac{1}{f}\frac{d}{dt}\left(\frac{\dot x^i}{f}\right) + \Gamma^i_{jk}\dot x^j\dot x^k\frac{1}{f^2}\right)\frac{\partial}{\partial x^i} =$$

$$= \frac{1}{f^2}\left(\nabla_{\dot\gamma}\dot\gamma - \partial_k \ln f\,\dot x^k\dot x^i\frac{\partial}{\partial x^i}\right) = \frac{1}{f^2}\left(\nabla_{\dot\gamma}\dot\gamma - \langle \mathrm{grad}(\ln f), \dot\gamma\rangle\dot\gamma\right)$$

$$\nabla_{\sigma'}\nabla_{\sigma'} X = \nabla_{\sigma'}\left(\frac{1}{f}\nabla_{\dot\gamma} X\right) = \nabla_{\sigma'}\left(\frac{1}{f}\right)\nabla_{\dot\gamma} X + \frac{1}{f^2}\nabla_{\dot\gamma}\nabla_{\dot\gamma} X =$$

$$= \frac{dt}{ds}\frac{d}{dt}\left(\frac{1}{f}\right)\nabla_{\dot\gamma} X + \frac{1}{f^2}\nabla_{\dot\gamma}\nabla_{\dot\gamma} X = \frac{1}{f^2}\left(\nabla_{\dot\gamma}\nabla_{\dot\gamma} X - \langle \mathrm{grad}(\ln f), \dot\gamma\rangle\nabla_{\dot\gamma} X\right)$$

Q.E.D.

Lemma 3. Given a conformal transformation in a Riemannian manifold: $(M,g) \to (M,\tilde{g})$, $\tilde{g} = f(x)g$, let R and $\tilde{R}$ be the associated curvature tensors respectively. Then, for any $X,Y,Z \in \mathfrak{X}(TM)$, it is verified that:

$$
\begin{aligned}
\tilde{R}(X,Y)Z &= R(X,Y)Z - \frac{1}{2}\langle X,Z\rangle \nabla_Y F + \frac{1}{2}\langle Y,Z\rangle \nabla_X F + \\
&+ \left(\frac{1}{2}\langle \nabla_Y F, Z\rangle - \frac{1}{4}\langle F,Z\rangle\langle F,Y\rangle + \frac{1}{4}\langle Y,Z\rangle\langle F,F\rangle\right) X + \\
&+ \left(-\frac{1}{2}\langle \nabla_X F, Z\rangle + \frac{1}{4}\langle F,Z\rangle\langle F,X\rangle - \frac{1}{4}\langle X,Z\rangle\langle F,F\rangle\right) Y + \\
&+ \left(\frac{1}{2}\langle \nabla_Y F, X\rangle - \frac{1}{2}\langle \nabla_X F, Y\rangle\right) Z + \qquad (34)\\
&+ \left(\frac{1}{4}\langle F,Y\rangle\langle X,Z\rangle - \frac{1}{4}\langle F,X\rangle\langle Y,Z\rangle\right) F
\end{aligned}
$$

where ∇ is the Levi-Civita connection associated to g, $F = \mathrm{grad}(\ln f)$ and the scalar products and the gradient are taken with respect to the metric g.

Proof: Apply Lemma 1 to the formula: $\tilde{R}(X,Y)Z = -\tilde{\nabla}_X(\tilde{\nabla}_Y Z) + \tilde{\nabla}_Y(\tilde{\nabla}_X Z) + \tilde{\nabla}_{[X,Y]}Z$, and simplify.
Q.E.D.

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