

A class of partially solvable two-dimensional quantum models with periodic potentials

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Abstract

The supersymmetric approach is used to analyse a class of two-dimensional quantum systems with periodic potentials. In particular, the method of SUSY-separation of variables allowed us to find a part of the energy spectra and the corresponding wave functions (partial solvability) for several models. These models are not amenable to conventional separation of variables, and they can be considered as two-dimensional generalizations of Lamé, associated Lamé, and trigonometric Razavy potentials. All these models have the symmetry operators of fourth order in momenta, and one of them (the Lamé potential) obeys the property of self-isospectrality.

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1 Introduction

The supersymmetric (SUSY) approach has provided a powerful impulse for new developments in analytical studies in Quantum Mechanics. To date, most new results have been obtained for one-dimensional quantum systems (see the monographs and reviews [1]). Over the last two decades, indubitable progress has been achieved also for higher-dimensional systems within the framework of SUSY Quantum Mechanics [2], [3], [4], [5]. The most essential results have been found for two-dimensional Quantum Mechanics. In particular, the new

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approach for the construction of completely integrable systems with symmetry operators of fourth order in momenta was proposed [3]. Two new methods - SUSY-separation of variables and two-dimensional shape-invariance [5], [4] - have allowed the problem of solvability to be tackled for two-dimensional quantum systems, beyond the standard separation of variables.

The method of SUSY-separation of variables has been applied successfully to investigate the spectra and wave functions of some models on the whole plane $\vec{x} = (x_1, x_2)$, which are not amenable to conventional separation of variables: the Morse potential [5], [6], [4], the Pöschl-Teller potential [7], [8], and some others [9], [6].

The starting point of this approach is the analysis of solutions of the intertwining relations between a pair of two-dimensional Hamiltonians of the Schrödinger form:

$$H(\vec{x})Q^- = Q^-\widetilde{H}(\vec{x}); \quad Q^+H(\vec{x}) = \widetilde{H}(\vec{x})Q^+; \quad (1)$$

$$H = -\Delta^{(2)} + V(\vec{x}); \quad \widetilde{H} = -\Delta^{(2)} + \widetilde{V}(\vec{x}). \quad (2)$$

In general, the intertwining operators Q^\pm are the operators of second order in momenta, i.e. in derivatives. In [7] (see also [8]), the particular class of intertwining operators Q^\pm was considered: operators with twisted reducibility

$$Q^- = (Q^+)^\dagger = (-\partial_l + \partial_l\chi(\vec{x}))(\sigma_3)_{lk}(+\partial_k + \partial_k\tilde{\chi}(\vec{x})), \quad \partial_l \equiv \frac{\partial}{\partial x_l} \quad (3)$$

where $\chi(\vec{x}), \tilde{\chi}(\vec{x})$ are two different functions (superpotentials), σ_3 is the Pauli matrix, and summation over $k, l = 1, 2$ is implied. In this class of models, both Hamiltonians H and \widetilde{H} are quasifactorized [7]

$$\begin{aligned} H &= (-\partial_l + \partial_l\chi(\vec{x}))(+\partial_l + \partial_l\chi(\vec{x})) = -\Delta^{(2)} + (\partial_l\chi(\vec{x}))^2 - \partial_l\partial_l\chi(\vec{x}); \\ \widetilde{H} &= (-\partial_l + \partial_l\tilde{\chi}(\vec{x}))(+\partial_l + \partial_l\tilde{\chi}(\vec{x})) = -\Delta^{(2)} + (\partial_l\tilde{\chi}(\vec{x}))^2 - \partial_l\partial_l\tilde{\chi}(\vec{x}), \end{aligned} \quad (4)$$

and hence their energy spectra are non-negative.

It was shown in [7], [8] in a general form that Eqs.(1) and (3) lead to the following representation for superpotentials $\chi, \tilde{\chi}$ in terms of four functions $\mu_{1,2,\pm}$:

$$\begin{aligned} \chi &= \mu_1(x_1) + \mu_2(x_2) + \mu_+(x_+) + \mu_-(x_-), \\ \tilde{\chi} &= \mu_1(x_1) + \mu_2(x_2) - \mu_+(x_+) - \mu_-(x_-), \end{aligned}$$

where $x_\pm = (x_1 \pm x_2)/\sqrt{2}$. These functions $\mu_{1,2,\pm}$ satisfy the equation

$$\mu'_1(x_1) [\mu'_+(x_+) + \mu'_-(x_-)] + \mu'_2(x_2) [\mu'_+(x_+) - \mu'_-(x_-)] = 0.$$

By $\phi \equiv \mu'$ substitutions, this becomes a purely functional equation with no derivatives:

$$\phi_1(x_1) [\phi_+(x_+) + \phi_-(x_-)] = -\phi_2(x_2) [\phi_+(x_+) - \phi_-(x_-)]. \quad (5)$$

The solution of Eq.(5) is necessary to build the solutions of intertwining relations (1) for the potentials V , \tilde{V} and for the supercharges Q^\pm . In particular,

$$\begin{aligned} V(\vec{x}) &= \left(\phi_1^2(x_1) - \phi_1'(x_1) \right) + \left(\phi_2^2(x_2) - \phi_2'(x_2) \right) + \left(\phi_+^2(x_+) - \phi_+'(x_+) \right) + \\ &\quad + \left(\phi_-^2(x_-) - \phi_-'(x_-) \right), \quad (6) \\ Q^\pm &= \partial_1^2 - \partial_2^2 \pm \sqrt{2} \left(\phi_+(x_+) + \phi_-(x_-) \right) \partial_1 \mp \sqrt{2} \left(\phi_+(x_+) - \phi_-(x_-) \right) \partial_2 - \\ &\quad - \left(\phi_1^2(x_1) - \phi_1'(x_1) \right) + \left(\phi_2^2(x_2) - \phi_2'(x_2) \right) + 2\phi_+(x_+)\phi_-(x_-). \end{aligned}$$

Thus, in order to find the systems with intertwining (1) by supercharges of the form (3), it is necessary to solve (5). This equation seems to be rather complicated, but it appeared to be solvable in a general form. The two-dimensional generalization of the Pöschl-Teller potential, investigated in [7], [8], was based just on a particular solution of Eq.(5).

In the present paper we focus our attention on the $\phi_{1,2,\pm}$, solutions which are periodic in the variables x_1 and x_2 . These solutions will be further used to build a class of two-dimensional potentials $V(\vec{x})$ of the form (6) - not amenable to standard separation of variables - which are periodic along x_1, x_2 with the same periods. By means of the method of supersymmetric separation of variables [5], [4] we shall derive the partial solvability of these periodic models: several energy eigenvalues and corresponding eigenfunctions will be found. Until now the supersymmetric approach has been used for the analysis of periodic potentials in **one-dimensional case** only (see for example [10], [11], [12]). To the best of our knowledge, this is^a the first attempt to study analytically **two-dimensional periodic** potentials, not amenable to separation of variables, within the framework of SUSY Quantum Mechanics.

It is appropriate to make some remarks concerning the possible spectra of two-dimensional periodic systems. The general statements are well known from textbooks (mainly on solid state theory) [14]: the spectra of d -dimensional ($d \geq 2$) models with periodic potentials in general have a band structure similar to that of the one-dimensional case. However, in contrast to $d = 1$, no

^a See some remarks in [13] (Section 5) and [8].

strict results on the (anti)periodicity properties of the band edge wave functions are known [15], and therefore analysis of the spectra of two-dimensional systems is much more complicated. In some sense, the situation resembles the non-periodic case: no analogues of the oscillation theorem are known for $d \geq 2$ quantum systems. This is a reason why the analysis of multidimensional excited bound states is much more difficult than in the one-dimensional situation.

To imagine the variety of possible structures of the band spectrum, one can consider (contrary to the rest of this paper) the simplest two-dimensional periodic systems which **allow** separation of variables (see, for example, [16] and references therein). After separation of variables, both one-dimensional problems have the band structure of energies $\epsilon_1(k_1)$, $\epsilon_2(k_2)$. Let us assume that they have a finite number (one or two) of band gaps (as in Subsections 4.2 and 4.3 below). Then, the positions of the two-dimensional band edges depend crucially on the parameters of one-dimensional bands ϵ_1, ϵ_2 . In particular, these bands may be overlapped, so that the band gaps (or at least some part of them) of the two-dimensional spectrum, $E = \epsilon_1 + \epsilon_2$, may even disappear. It is natural to consider two limiting cases: of almost free particles (the band gaps are vanishing) and of tight binding particles (the band gaps are very wide).

The structure of this paper is as follows. Section 2 is devoted to the solution of the functional equation (5): the explicit expressions for **all four** functions $\phi_{1,2,\pm}$ will be found. In Section 3 the symmetry properties of the general solution of Eq.(5) are analysed. In Section 4, which is the main part of the paper, a new class of two-dimensional systems, partially solvable by means of SUSY-separation of variables, is constructed. The potentials of these systems are periodic on the plane, and they can be considered as two-dimensional generalizations (not amenable to separation of variables) of fairly well known [17], [10], [11], [12], [18], [19] one-dimensional potentials: the Lamé potential, the associated Lamé potential and the trigonometric Razavy potential. Using the known positions of band edges and corresponding wave functions for these one-dimensional models with a finite number of gaps, some energy eigenvalues and their eigenfunctions for two-dimensional periodic generalizations will be obtained explicitly (partial solvability). Section 5 includes a discussion of certain specific properties of the models of the previous Section and some limiting cases. In particular, all two-dimensional models constructed are integrable: the symmetry operators of fourth order in momenta commute with the Hamiltonians. In addition, the two-dimensional Lamé model obeys the property of self-isospectrality, like its one-dimensional prototype [10].

2 The general solution of the functional equation

In this Section we analyze the functional equation (5), which plays an important role in our approach. This analysis was started in the Appendix of the paper [7], where the necessary conditions for the existence of its solutions were derived. First, we will remind briefly the main steps of this derivation. In the formulas below we shall imply that the functions $\phi_{1,2}$, ϕ_{\pm} depend on the corresponding arguments: $\phi_{1,2} = \phi_{1,2}(x_{1,2})$ and $\phi_{\pm} = \phi_{\pm}(x_{\pm})$, unless otherwise stated.

Acting by $(\partial_1^2 - \partial_2^2)$ on both sides of (5), we obtain:

$$2 \left(\left(\frac{1}{\phi_1} \right)' \phi_1 \partial_1 - \frac{1}{\phi_2} \phi_2' \partial_2 \right) (\phi_- - \phi_+) = \left(\frac{1}{\phi_2} \phi_2'' - \left(\frac{1}{\phi_1} \right)'' \phi_1 \right) (\phi_- - \phi_+). \quad (7)$$

It is now convenient to introduce a new unknown function, Λ :

$$\phi_- - \phi_+ \equiv \phi_1 |\phi_1' \phi_2'|^{-1/2} \Lambda(x_1, x_2).$$

Substitution of this definition into (7) gives:

$$\left(\frac{\phi_1'}{\phi_1} \partial_1 + \frac{\phi_2'}{\phi_2} \partial_2 \right) \Lambda = 0,$$

and therefore Λ depends only on $\left(\int^{x_1} \frac{\phi_1(\xi)}{\phi_1'(\xi)} d\xi - \int^{x_2} \frac{\phi_2(\eta)}{\phi_2'(\eta)} d\eta \right)$, and the general solution of (7) is:

$$\phi_- - \phi_+ = \phi_1 |\phi_1' \phi_2'|^{-1/2} \Lambda \left(\int^{x_1} \frac{\phi_1(\xi)}{\phi_1'(\xi)} d\xi - \int^{x_2} \frac{\phi_2(\eta)}{\phi_2'(\eta)} d\eta \right). \quad (8)$$

From the initial equation (5) we also have that:

$$\phi_- + \phi_+ = \phi_2 |\phi_1' \phi_2'|^{-1/2} \Lambda \left(\int^{x_1} \frac{\phi_1(\xi)}{\phi_1'(\xi)} d\xi - \int^{x_2} \frac{\phi_2(\eta)}{\phi_2'(\eta)} d\eta \right). \quad (9)$$

From Eqs. (8), (9) we could already obtain the solutions for ϕ_{\pm} , but we have to check that these solutions will indeed depend on proper arguments. Thus, the constraints for the function Λ and functions $\phi_{1,2}$ are obtained from the equations:

$$\partial_{\pm} \phi_{\mp} = 0.$$

The result is:

$$\frac{1}{2} \left(\frac{\phi_2''}{\phi_2 \phi_2'} + \frac{\phi_1''}{\phi_1 \phi_1'} \right) \Lambda = \left(\frac{1}{\phi_1'} - \frac{1}{\phi_2'} \right) \Lambda', \quad (10)$$

$$\left(\phi_1' + \phi_2' - \frac{\phi_1 \phi_1''}{2\phi_1'} - \frac{\phi_2 \phi_2''}{2\phi_2'} \right) \Lambda = \left(\frac{\phi_2^2}{\phi_2'} - \frac{\phi_1^2}{\phi_1'} \right) \Lambda'. \quad (11)$$

Here we disregard the trivial solution $\Lambda \equiv 0$, for which $\phi_+ = \phi_- = 0$, $\phi_{1,2}$ are arbitrary, but the potentials (6) are amenable to separation of variables.

Otherwise one can exclude Λ , dividing Eq.(10) by Eq.(11):

$$\frac{\phi_1'' \phi_2^2}{\phi_1} - \frac{\phi_2'' \phi_1^2}{\phi_2} = 2\phi_2'^2 - 2\phi_1'^2 + \phi_1 \phi_1'' - \phi_2 \phi_2''. \quad (12)$$

There is no separation of variables in (12), but it will appear after applying the operator $\partial_1 \partial_2$, so that:

$$\frac{(\phi_1''/\phi_1)'}{(\phi_1^2)'} = \frac{(\phi_2''/\phi_2)'}{(\phi_2^2)'} \equiv 2a = \text{const.}$$

Integrating, multiplying by $\phi_{1,2}'$, integrating again and taking into account (12), one has that $\phi_{1,2}$ must satisfy the equation:

$$(\phi')_{1,2}^2 = a\phi_{1,2}^4 + b\phi_{1,2}^2 + c, \quad (13)$$

where a, b, c are arbitrary real constants. All solutions of this equation can be expressed in terms of elliptic functions, and they are described for different ranges of parameters, for example, in [20].

The subsequent discussion will depend crucially on the sign of the discriminant $D = b^2 - 4ac$. In the present paper we restrict ourselves to the case of $D > 0$, i.e. when the quadratic polynomial $a\phi_{1,2}^4 + b\phi_{1,2}^2 + c$ has two different real roots, which will be denoted as r_1 and r_2 (let $r_1 > r_2$). For positive values of the discriminant, three types of solutions of (13) exist, depending on the relative position of the roots $r_{1,2}$. These solutions are proportional either to $\frac{cn(\alpha x|k)}{sn(\alpha x|k)}$ or to $\frac{dn(\alpha x|k)}{sn(\alpha x|k)}$ or to $\frac{1}{sn(\alpha x|k)}$, where sn, cn, dn are the well-known elliptic Jacobi functions [17].

Equations (13) are the necessary conditions for the functions $\phi_{1,2}$ to satisfy equation (5). But are these conditions also sufficient? To answer this question, we must solve Eqs.(10), (11) for function Λ , with arbitrary elliptic functions $\phi_{1,2}$, satisfying (13). From Eq.(13), the argument of the function Λ can be written in the form:

$$\int^{x_1} \frac{\phi_1(\xi)}{\phi_1'(\xi)} d\xi - \int^{x_2} \frac{\phi_2(\eta)}{\phi_2'(\eta)} d\eta = \frac{1}{2a(r_1 - r_2)} \ln \left| \frac{(\phi_1^2 - r_1)(\phi_2^2 - r_2)}{(\phi_1^2 - r_2)(\phi_2^2 - r_1)} \right| \quad (14)$$

The module sign is not important here, since

$$\nu \equiv \frac{(\phi_1^2 - r_1)(\phi_2^2 - r_2)}{(\phi_1^2 - r_2)(\phi_2^2 - r_1)} \geq 0 \quad (15)$$

because of (13). In terms of this new variable ν , Eq.(10) takes the form

$$\frac{\partial_\nu \Lambda(\nu)}{\Lambda(\nu)} = -\frac{\nu^{\frac{1}{2}} + \sigma}{4\nu(\nu^{\frac{1}{2}} - \sigma)}, \quad (16)$$

where the notation $\sigma \equiv \text{sign}(\phi_1'(x_1)\phi_2'(x_2))$ was used.

The solution of this differential equation is^b:

$$\Lambda(\nu) = \Lambda_0 \frac{\nu^{\frac{1}{4}}}{\nu^{\frac{1}{2}} - \sigma}, \quad \Lambda_0 = \text{const.} \quad (17)$$

Formally substituting (17) into the l.h.s. of Eq.(16), we obtain:

$$\frac{\partial_\nu \Lambda(\nu)}{\Lambda(\nu)} = -\frac{\nu^{\frac{1}{2}} + \sigma - 4\nu\partial_\nu\sigma}{4\nu(\nu^{\frac{1}{2}} - \sigma)}.$$

Let us prove that the extra term in the nominator of the r.h.s. actually vanishes; i.e. that $\nu\partial_\nu\sigma \equiv 0$. The variable ν may be considered as a function of ϕ_1^2 and ϕ_2^2 , and hence

$$\frac{\partial}{\partial \nu} = \frac{\partial \phi_1^2}{\partial \nu} \frac{\partial}{\partial \phi_1^2} + \frac{\partial \phi_2^2}{\partial \nu} \frac{\partial}{\partial \phi_2^2}.$$

In turn, due to Eq.(5), one can rewrite derivatives over ϕ_i^2 in terms of derivatives over ϕ_i' :

$$\frac{\partial}{\partial \phi_i^2} = \frac{d\phi_i'}{d\phi_i^2} \frac{\partial}{\partial \phi_i'} = \frac{a(2\phi_i^2 - r_1 - r_2)}{2\sqrt{a(\phi_i^2 - r_1)(\phi_i^2 - r_2)}} \text{sign}(\phi_i') \frac{\partial}{\partial \phi_i'}.$$

Gathering everything together and using $\frac{\partial}{\partial \phi_i'}\sigma = 2\delta(\phi_i')$, we find that:

$$\nu\partial_\nu\sigma = \frac{(r_1 - r_2)}{2(2\phi_1^2 - r_1 - r_2)} \text{sign}(\phi_1')\phi_1'\delta(\phi_1') + \frac{(r_1 - r_2)}{2(2\phi_2^2 - r_1 - r_2)} \text{sign}(\phi_2')\phi_2'\delta(\phi_2'),$$

which is zero for all values of $x_{1,2}$ since at the points where $\delta(\phi_i) \neq 0$, the denominator has no singularity (it tends to $\pm(r_1 - r_2)$ for $\phi_i' \rightarrow 0$).

^b This solution coincides with the solution in the simpler case of a constant (not depending on ν) value of σ .

Now, according to Eqs.(8), (9) solution (17) gives ϕ_{\pm} in terms of functions $\phi_{1,2}$:

$$\phi_{\pm} = \Lambda_0 \frac{\phi_2 \mp \phi_1}{2|\phi'_1 \phi'_2|^{\frac{1}{2}}} \cdot \frac{\nu^{\frac{1}{4}}}{\nu^{\frac{1}{2}} - \sigma}. \quad (18)$$

The important fact is that without any new constraints on $\phi_{1,2}(x_{1,2})$ (besides Eq.(13)) the functions ϕ_{\pm} depend on the proper arguments x_{\pm} :

For any pair of functions $\phi_{1,2}$ satisfying (13) there exists a pair of functions ϕ_{\pm} given by (18), such that they are solutions of (5).

Closing this Section we shall derive an alternative expression for ϕ_{\pm} , to be used later (see Section 4). Let us define the function

$$G \equiv \frac{2\phi'_1 \phi'_2 + 2a\phi_1^2 \phi_2^2 + b\phi_1^2 + b\phi_2^2 + 2c}{(\phi_2 - \phi_1)^2}.$$

One can check by straightforward calculations, that

$$\frac{\partial_- G}{G} = 2\sqrt{2} \frac{\phi'_1 \phi_2 + \phi_1 \phi'_2}{\phi_2^2 - \phi_1^2}. \quad (19)$$

Then, taking into account

$$\phi'_- = \sqrt{2} \phi_- \frac{\phi'_1 \phi_2 + \phi_1 \phi'_2}{\phi_2^2 - \phi_1^2},$$

Eq.(19) can be rewritten in compact form as:

$$\partial_- \left(\frac{G}{\phi_-^2} \right) = 0.$$

Hence, $\phi_-^2 = Const \cdot G$, where the constant can be calculated by considering this expression in some specific point: $Const = \Lambda_0^2 / (4(b^2 - 4ac))$, with arbitrary constant Λ_0 . We therefore obtain the desired alternative expression for ϕ_- :

$$\phi_-^2 = \frac{\Lambda_0^2}{b^2 - 4ac} \cdot \frac{2\phi'_1 \phi'_2 + 2a\phi_1^2 \phi_2^2 + b\phi_1^2 + b\phi_2^2 + 2c}{(\phi_2 - \phi_1)^2}. \quad (20)$$

By a completely analogous calculation, or directly from Eq.(5), the expression for ϕ_+ can be derived:

$$\phi_+^2 = \frac{\Lambda_0^2}{b^2 - 4ac} \cdot \frac{2\phi'_1 \phi'_2 + 2a\phi_1^2 \phi_2^2 + b\phi_1^2 + b\phi_2^2 + 2c}{(\phi_2 + \phi_1)^2}. \quad (21)$$

Formulas (20) and (21) are very convenient to prove that ϕ_{\pm} satisfy the same

Eq.(13), but with different coefficients. Direct calculations show that:

$$(\phi'_{\pm})^2 = \frac{2}{\Lambda_0^2}(b^2 - 4ac)\phi_{\pm}^4 - b\phi_{\pm}^2 + \frac{\Lambda_0^2}{8} \equiv \tilde{a}\phi_{\pm}^4 + \tilde{b}\phi_{\pm}^2 + \tilde{c}. \quad (22)$$

Let us note that although the discriminant $D = b^2 - 4ac$ was chosen to be positive, the analogous discriminant \tilde{D} for Eq.(22), which is equal to $\tilde{D} = 4ac$, can be non-positive as well.

3 Symmetries of the functional equation

As pointed out in a previous paper [7] (Subsection 3.2), the functional equation (5) has two discrete symmetries: S_1, S_2 . After the change of variables $y_{1,2} = x_{\pm}$, $y_{\pm} \equiv (y_1 \pm y_2)/\sqrt{2} = x_{1,2}$, equation (5) becomes:

$$\phi_1(y_+)[\phi_+(y_1) + \phi_-(y_2)] = -\phi_2(y_-)[\phi_+(y_1) - \phi_-(y_2)],$$

which, by rearrangement of terms, can be brought to:

$$\phi_+(y_1)[\phi_1(y_+) + \phi_2(y_-)] = -\phi_-(y_2)[\phi_1(y_+) - \phi_2(y_-)].$$

The last form coincides with (5) up to interchanged $\phi_{1,2}$ and ϕ_{\pm} . Hence, if $(\phi_1, \phi_2, \phi_+, \phi_-)$ is a solution of (5), then the set $(\phi_+, \phi_-, \phi_1, \phi_2)$ is also a solution (this discrete symmetry was called S_1). By writing the four functions in brackets, we mean that the first of them should be put in the place of ϕ_1 in (5), the second one in the place of ϕ_2 , and the last two in the place of ϕ_+ and ϕ_- , correspondingly.

This symmetry can be observed explicitly from the general solutions (20), (21). Namely, if we calculate $\phi_{1,2}$ from ϕ_{\pm} with the use of analogues of (20), (21):

$$\tilde{\phi}_{1,2}^2 = \frac{\tilde{\Lambda}_0^2}{b^2 - 4\tilde{a}\tilde{c}} \cdot \frac{2\phi'_+ \phi'_- + 2\tilde{a}\phi_+^2 \phi_-^2 + \tilde{b}\phi_+^2 + \tilde{b}\phi_-^2 + 2\tilde{c}}{(\phi_- \pm \phi_+)^2} \quad (23)$$

then we must arrive at the same functions $\phi_{1,2}$. Indeed, calculating the r.h.s. of (23), one obtains $\tilde{\phi}_1^2 = \tilde{\Lambda}_0 \phi_1^2 / (8c)$, and by proper choice of the arbitrary constant $\tilde{\Lambda}_0$ (which can be imaginary) one has $\tilde{\phi}_1 = \phi_1$. Thus, $\tilde{\phi}_2(y_2) = \phi_1(y_1)(\phi_+(y_+) + \phi_-(y_-)) / (\phi_-(y_-) - \phi_+(y_+))$, and by comparing it with (5) we see that $\tilde{\phi}_2 = \phi_2$ too.

There is another symmetry S_2 . If $(\phi_1, \phi_2, \phi_+, \phi_-)$ is a solution of (5), then $(\phi_1, -\phi_2, \frac{1}{\phi_+}, \frac{1}{\phi_-})$ is also a solution of (5). This was shown in [7] directly from the functional equation (5), and now we shall derive it from the general solu-

tion (18). It can be rewritten in the form:

$$\phi_{\pm} = \frac{\Lambda_0}{2\sqrt{|b|}} \left(\left| \frac{\nu^{\frac{1}{2}} + \sigma}{\nu^{\frac{1}{2}} - \sigma} \right| \right)^{\frac{1}{2}} \left(\left| \frac{\phi_2 \mp \phi_1}{\phi_2 \pm \phi_1} \right| \right)^{\frac{1}{2}}.$$

If one performs the change $(\phi_1, \phi_2) \rightarrow (\phi_1, -\phi_2)$ then $\sigma \rightarrow -\sigma$, and

$$\phi_{\pm} \rightarrow \frac{\Lambda_0}{2\sqrt{|b|}} \left(\left| \frac{\nu^{\frac{1}{2}} - \sigma}{\nu^{\frac{1}{2}} + \sigma} \right| \right)^{\frac{1}{2}} \left(\left| \frac{-\phi_2 \mp \phi_1}{-\phi_2 \pm \phi_1} \right| \right)^{\frac{1}{2}} = \frac{Const}{\phi_{\pm}},$$

which completes the proof.

It should be noted that these symmetries may play an important role. In particular, the first one - S_1 - was applied in [7] to combine the shape-invariance with SUSY-separation of variables in the 2D Pöschl-Teller model.

4 SUSY-separation of variables for some models with periodic potentials

4.1 The algorithm of SUSY-separation

In the general expressions (6) for the potential $V(\vec{x})$ one sees the very special dependence on coordinates (x_1, x_2) : there are two terms that do not mix x_1 and x_2 , and two mixing terms that depend on the "light-cone" variables x_{\pm} :

$$V(\vec{x}) = V_1(x_1) + V_2(x_2) + V_+(x_+) + V_-(x_-); \quad V_{1,2,\pm} = \phi_{1,2,\pm}^2 - \phi'_{1,2,\pm}, \quad (24)$$

all terms being represented in "supersymmetric" form. Just the last two terms of $V(\vec{x})$ show that separation of variables in the Schrödinger equation is not possible.

The method of SUSY-separation of variables (see details in [5], [4], [7]) allows us to **partially** solve some two-dimensional models, i.e. to find **a part** of their spectra and corresponding wave functions. Due to the intertwining relations (1), the subspace spanned by zero modes $\Omega_n(\vec{x})$ is closed under the action of the Hamiltonian H . Therefore, the wave functions $\Psi(\vec{x})$ can be built as linear combinations of zero modes $\Omega_n(\vec{x})$ of the supercharge Q^+ . In turn, the zero modes $\Omega_n(\vec{x})$ can be found through the specific similarity ("gauge") transformation of the supercharge (3) to the separated form:

$$\mathfrak{q}^+ = e^{-\kappa(\vec{x})} Q^+ e^{\kappa(\vec{x})} = \partial_1^2 - \partial_2^2 - (\phi_1^2 - \phi_1') + (\phi_2^2 - \phi_2'), \quad (25)$$

where the function κ is given by:

$$\kappa(\vec{x}) \equiv - \left[\int^{x_+} \phi_+(\xi) d\xi + \int^{x_-} \phi_-(\eta) d\eta \right]. \quad (26)$$

This transformation allows us to reduce the two-dimensional quantum problem for $\Omega_n(\vec{x})$ to a pair of one-dimensional problems:

$$\Omega_n(\vec{x}) = e^{\kappa(\vec{x})} \omega_n(\vec{x}); \quad \omega_n(\vec{x}) \equiv \eta_n(x_1)\rho_n(x_2),$$

where ρ_n and η_n are the eigenfunctions of one-dimensional Schrödinger equations:

$$- \eta_n'' + (\phi_1^2 - \phi_1')\eta_n = \epsilon_n \eta_n, \quad (27)$$

$$- \rho_n'' + (\phi_2^2 - \phi_2')\rho_n = \epsilon_n \rho_n \quad (28)$$

(ϵ_n are the constants of separation). Let us note that the one-dimensional potentials in (27), (28) coincide exactly with the first two terms in the two-dimensional potential (24). Moreover, they take the form typical of potentials within the framework of one-dimensional SUSY Quantum Mechanics [1]. In this sense **two-dimensional** potentials (24) can be thought of as generalizations of one-dimensional models (27), (28).

Our goal in this Section is to build a new class of partially solvable models within the framework of the approach described above. In practice, this means that we have to present a new class of functions $\phi_{1,2}(x_{1,2})$, which obey the following special requirements:

- they must obey the functional equation (5), i.e. Eq.(13).
- the one-dimensional models (27) and (28) with superpotentials $\phi_{1,2}$ must be exactly solvable in order to provide for the normalizable zero modes Ω_n .

In such a way we hope to find ϕ_{\pm} by means of (18) (or by means of (20), (21)), and hence to build the two-dimensional potential (6). The model with this potential will be partially solvable by means of SUSY-separation of variables.

At the last stage, one has to build the "gauge-transformed" Hamiltonian:

$$h(\vec{x}) \equiv e^{-\kappa(\vec{x})} H(\vec{x}) e^{\kappa(\vec{x})} = -\partial_1^2 - \partial_2^2 + \sqrt{2}(\phi_+ + \phi_-)\partial_1 + \sqrt{2}(\phi_+ - \phi_-)\partial_2 + \phi_1^2 - \phi_1' + \phi_2^2 - \phi_2', \quad (29)$$

and, by direct calculations, find the matrix \hat{C} , such that

$$h\vec{\omega} = \hat{C}\vec{\omega}; \quad \vec{\omega} \equiv (\omega_0, \omega_1, \dots, \omega_N).$$

This can be done by the action of the operator h on zero modes ω_n :

$$h\omega_n = [2\epsilon_n + \sqrt{2}(\phi_+ + \phi_-)\partial_1 + \sqrt{2}(\phi_+ - \phi_-)\partial_2]\omega_n \quad (30)$$

after expressing the r.h.s. as a linear combination of ω_k .

According to the prescriptions of SUSY-separation of variables (see more details in [5], [4], [7]), the eigenvalues E_k of the matrix \hat{C} give part of the spectrum of the Hamiltonian H . The corresponding wave functions $\Psi_k(\vec{x})$ are obtained as:

$$\vec{\Psi}_k(\vec{x}) = \hat{B}\vec{\Omega}(\vec{x}),$$

where the matrix \hat{B} is a solution of the matrix equation

$$\hat{B}\hat{C} = \hat{\Lambda}\hat{B} \quad (31)$$

with the diagonal matrix $\hat{\Lambda}$. Its diagonal elements coincide with E_k .

Up to now, this program was performed successfully for the following cases.

1) A two-dimensional generalization of the Morse potential [5], [4] with

$$\phi_1(x) = \phi_2(x) \sim \exp(\alpha x).$$

2) A two-dimensional generalization of the Pöschl-Teller potential [7], [8] with

$$\phi_1(x) = -\phi_2(x) \sim \sinh^{-1}(\alpha x).$$

In both models, the functions ϕ_{\pm} , which were originally guessed directly from Eq.(5), can now be obtained by means of Eqs.(18), (20), (21).

Below we shall consider new specific choices of functions $\phi_{1,2}$ that will satisfy all the aforementioned conditions. The common feature of these new models is that the functions $\phi_{1,2}$ are periodic and therefore lead to partially solvable two-dimensional potentials, $V(\vec{x})$, periodic on the plane (x_1, x_2) .

4.2 The two-dimensional Lamé potential

From the variety of elliptic functions $\phi_{1,2}$, which give solutions of Eq.(13), we have to choose the subclass leading to exactly solvable one-dimensional Schrödinger equations (27) and (28) with periodic potentials. Let us start from the particular $2K$ -periodic solutions^c

$$\phi_1(x) = \phi_2(x) = k^2 \frac{sn(x|k)cn(x|k)}{dn(x|k)}, \quad (32)$$

^c One can check straightforwardly that this expression satisfies Eq.(13). It can be transformed to the third type of solutions of (13), mentioned in Section 2, by the Landen transformation [17] accompanied by a suitable shift of argument.

where the standard notations [17] for Jacobi elliptic functions $sn(x|k)$, $cn(x|k)$, $dn(x|k)$ with modulus $k \in (0, 1]$ and the complete elliptic integral $K(k) \equiv \int_0^{\pi/2} d\theta / \sqrt{1 - k^2 \sin^2 \theta}$ were used. Later on, for simplicity we skip the argument k as long as no confusion appears.

Thus, the one-dimensional potentials $V_{1,2}$ of (24), which coincide with potentials in (27) and (28), have the form of the Lamé equation:

$$V_{1,2}(x) = 2k^2 sn^2(x) - k^2 \quad (33)$$

We notice that (33) is only the simplest $l = 1$ case of the general Lamé potential $V = l(l + 1)k^2 sn^2(x)$ with arbitrary integer l . For higher values, $l > 1$, the superpotentials $\phi_{1,2}$ do not satisfy the basic Eq.(13), and for this reason are not considered here. Since the elliptic function $sn(x)$ is $4K$ -periodic, and $2K$ -antiperiodic $sn(x + 2K) = -sn(x)$, the potential (33) is periodic with the period $2K$.

The spectrum and the wave functions for the Lamé potential (33) are known exactly [17]: it has one bound band with energies $\epsilon \in (0, 1 - k^2)$ and the continuous band with energies $\epsilon \in (1, \infty)$, with the band gap between them. The band edge eigenfunctions are

$$\begin{aligned} \epsilon_0 = 0; & & \psi_0(x) &= dn(x); \\ \epsilon_1 = 1 - k^2; & & \psi_1(x) &= cn(x); \\ \epsilon_2 = 1; & & \psi_2(x) &= sn(x), \end{aligned} \quad (34)$$

and the wave functions obey the well known [17] (anti)periodicity property of band edge eigenfunctions for one-dimensional periodic potentials: $\psi_{0,1}(x + 2K) = \psi_{0,1}(x)$, $\psi_2(x + 2K) = -\psi_2(x)$.

According to formulas (20), (21), the choice of (32) for $\phi_{1,2}(x_{1,2})$ leads to very compact explicit expressions for $\phi_{\pm}(x_{\pm})$:

$$\phi_+(x) = -\phi_-(x) = B \frac{cn(\sqrt{2}x)}{sn(\sqrt{2}x)}, \quad (35)$$

where the new constant $B = \Lambda_0 / (4\sqrt{1 - k^2})$.

The two-dimensional potential $V(\vec{x})$ can be obtained from (6):

$$\begin{aligned} V(\vec{x}) = & 2k^2(sn^2(x_1) + sn^2(x_2) - 1) + \\ & + \frac{B(B + \sqrt{2}dn(x_1 + x_2))}{sn^2(x_1 + x_2)} + \frac{B(B - \sqrt{2}dn(x_1 - x_2))}{sn^2(x_1 - x_2)} - 2B^2. \end{aligned} \quad (36)$$

This can be considered as a two-dimensional generalization (not amenable to separation of variables) of the Lamé potential (33). The potential (36) is

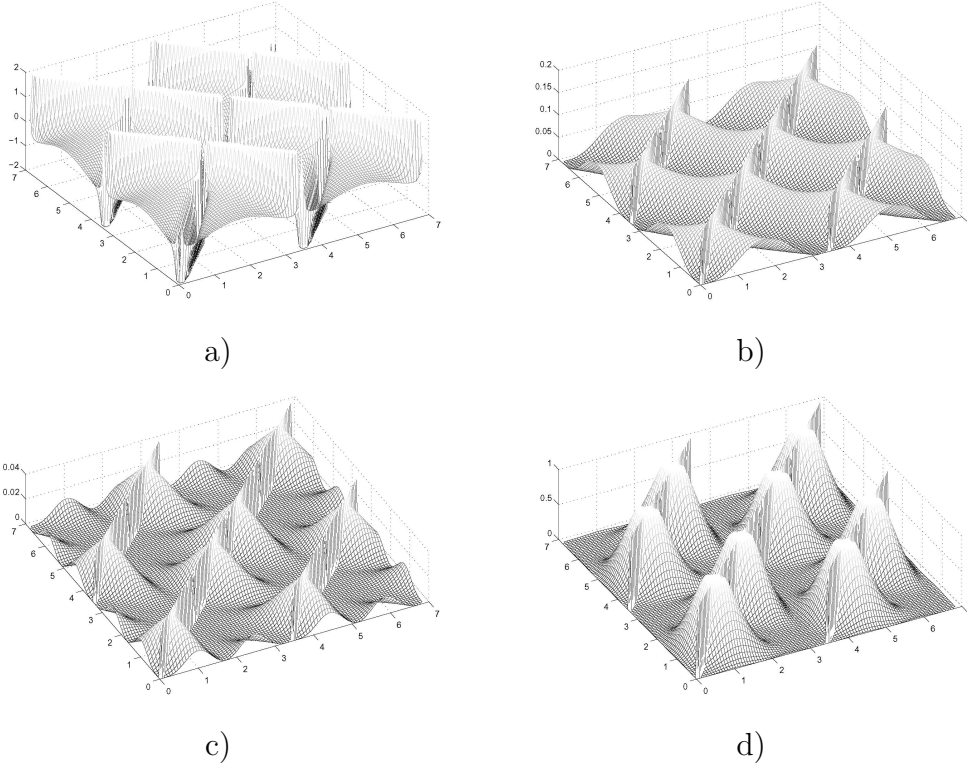


Fig. 1. Panel a): The two-dimensional Lamé potential (36) for the parameters $k^2 = 0.30$ and $B = 0.05$. The period of potential along x_1, x_2 is $2K(k) = 3.42$. We used a cut-off of the potential at ± 2.0 .

Panels b), c), d): The squares of three known wave functions $\Psi_0^2, \Psi_1^2, \Psi_2^2$ (see Eqs.(38)) for two-dimensional Lamé potential with the same values of parameters and suitable cut-offs. The energy eigenvalues are $E_0 = 0, E_1 = 1.38, E_2 = 2.02$.

periodic under the shifts of $x_{1,2}$ with the periods $2K$. The coefficients of both terms in (36), singular at $(x_1 \pm x_2) \rightarrow 0$, are such that no fall to the center occurs for arbitrary value of parameter B . The plot of the two-dimensional potential (36) is represented in Fig.1a for a specific choice of parameter values.

To start the procedure of SUSY-separation of variables, it is easy to calculate the function $\kappa(\vec{x})$ by (26):

$$\kappa(\vec{x}) = \frac{B}{2\sqrt{2}} \ln \left(\frac{(1 - dn(x_1 - x_2))(1 + dn(x_1 + x_2))}{(1 + dn(x_1 - x_2))(1 - dn(x_1 + x_2))} \right). \quad (37)$$

The functions ω_n for the one-dimensional energies ϵ_n at band edges are:

$$\omega_0(\vec{x}) = dn(x_1)dn(x_2); \quad \omega_1(\vec{x}) = cn(x_1)cn(x_2); \quad \omega_2(\vec{x}) = sn(x_1)sn(x_2).$$

We must act through the operator h (see (29), (30)) on all ω_n in order to find the matrix \hat{C} and its eigenvalues. For this model, the action of the "gauge-

transformed" Hamiltonian h on ω_n can be written as:

$$h\omega_n = 2\epsilon_n\omega_n - \frac{2\sqrt{2}B}{sn^2(x_1) - sn^2(x_2)} \left(sn(x_2)cn(x_2)dn(x_1)\partial_1 - sn(x_1)cn(x_1)dn(x_2)\partial_2 \right) \omega_n.$$

One can calculate that $h\omega_0 = 0$, i.e. $\Psi_0(\vec{x}) = \Omega_0(\vec{x}) = e^{-\kappa(\vec{x})}\omega_0$, is the lowest eigenfunction of H with energy $E_0 = 0$. Further calculations show that:

$$h(\vec{x}) \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 2(1-k^2); & 2\sqrt{2}B(1-k^2) \\ 2\sqrt{2}B; & 2 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

Diagonalizing the matrix on the r.h.s. and following the algorithm of Subsection 4.1, one obtains the eigenfunctions $\Psi_n(\vec{x})$ for (36):

$$\begin{aligned} \Psi_0 &= e^{-\kappa(\vec{x})} dn(x_1)dn(x_2); \\ \Psi_1 &= e^{-\kappa(\vec{x})} \left(cn(x_1)cn(x_2) + \frac{E_1-2(1-k^2)}{2\sqrt{2}B} sn(x_1)sn(x_2) \right); \\ \Psi_2 &= e^{-\kappa(\vec{x})} \left(cn(x_1)cn(x_2) + \frac{E_2-2(1-k^2)}{2\sqrt{2}B} sn(x_1)sn(x_2) \right), \end{aligned}$$

and their energy eigenvalues: $E_0 = 0$, $E_{1,2} = 2 - k^2 \mp \sqrt{k^4 + 8B^2(1-k^2)}$. At first sight, this result may be in conflict with the obvious non-negativeness of the spectrum of the Hamiltonian H due to its quasi-factorizability (4): the energy E_1 above seems to be negative for some values of B . The resolution of this paradox is quite simple, since for the wave functions of interest each possible singularity must be normalizable. The wave functions above possess the power singularities at $(x_1 \pm x_2) \rightarrow 0$ via the multiplier $e^{-\kappa}$ according to expression (37). These singularities are normalizable for $B^2 < 1/2$, and just for only these values of B is the eigenvalue E_1 positive. The squared wave functions $\Psi_0(\vec{x})^2$, $\Psi_1(\vec{x})^2$, $\Psi_2(\vec{x})^2$ are given in Fig.1b-1d for the same parameter values as in Fig.1a.

The periodicity properties of elliptic Jacobi functions allow us to check that all three Bloch eigenfunctions [14] have vanishing quasi-momenta, i.e. they are $2K$ -periodic:

$$\Psi_n(x_1+2Km_1, x_2+2Km_2) = \exp(ik_1(\epsilon_n) \cdot 2Km_1 + ik_2(\epsilon_n) \cdot 2Km_2) \Psi_n(x_1, x_2),$$

where m_1, m_2 are arbitrary integer numbers, and the quasi-momentum vector $\vec{k}(\epsilon_n) = (k_1(\epsilon_n), k_2(\epsilon_n)) = \vec{0}$. Although we cannot find the whole structure of the spectrum of the model, the zero values of the quasi-momenta hint that perhaps the states obtained correspond to the band edges (in analogy with the one-dimensional case). In any case, we analytically find three eigenfunctions of the generalized two-dimensional Lamé Hamiltonian (36) for $B^2 < 1/2$, and

one of them, $\Psi_0(\vec{x})$, is certainly the ground state; i.e., the lower edge of the bound band.

4.3 The two-dimensional associated Lamé potential

Since the functional equation (5) is homogeneous, one can consider the more general form of (32) and (35), multiplying the functions $\phi_{1,2}$ by an arbitrary parameter l and keeping ϕ_{\pm} unchanged:

$$\phi_1(x) = \phi_2(x) = lk^2 \frac{sn(x)cn(x)}{dn(x)}; \quad (38)$$

$$\phi_+(x) = -\phi_-(x) = B \frac{cn(\sqrt{2}x)}{sn(\sqrt{2}x)}. \quad (39)$$

Thus, the one-dimensional potentials $V_{1,2}$ are:

$$V_{1,2}(x_{1,2}) = l(l+1)k^2 sn^2(x_{1,2}) + l(l-1)k^2 \frac{cn^2(x_{1,2})}{dn^2(x_{1,2})} - l^2 k^2. \quad (40)$$

These potentials coincide with the so-called associated Lamé potential [17], with well known band structure [11], [12] for different (not only integer) values of the parameter l . Here we consider only^d the simplest case $l = 2$, where the spectrum has two bound bands and the continuous band with the following energy values of the band edges and analytical expressions for the corresponding wave functions with necessary (anti)periodic conditions under $x_i \rightarrow x_i + 2K$ [11]:

$$\begin{aligned} \epsilon_0 &= 0; & \rho_0 &= \eta_0 = dn^2(x); \\ \epsilon_1 &= 5 - 3k^2 - 2\sqrt{4 - 3k^2}; & \rho_1 &= \eta_1 = \frac{cn(x)}{dn(x)} (3k^2 sn^2(x) + \alpha_1); \\ \epsilon_2 &= 5 - 2k^2 - 2\sqrt{4 - 5k^2 + k^4}; & \rho_2 &= \eta_2 = \frac{sn(x)}{dn(x)} (3k^2 sn^2(x) + \beta_1); \\ \epsilon_3 &= 5 - 2k^2 + 2\sqrt{4 - 5k^2 + k^4}; & \rho_3 &= \eta_3 = \frac{sn(x)}{dn(x)} (3k^2 sn^2(x) + \beta_2); \\ \epsilon_4 &= 5 - 3k^2 + 2\sqrt{4 - 3k^2}; & \rho_4 &= \eta_4 = \frac{cn(x)}{dn(x)} (3k^2 sn^2(x) + \alpha_2), \end{aligned}$$

where $\alpha_{1,2} = -2 \mp \sqrt{4 - 3k^2}$, $\beta_{1,2} = -2 - k^2 \mp \sqrt{4 - 5k^2 + k^4}$. The zero modes of \mathbf{q}^+ are again $\omega_n = \rho_n(x_1)\eta_n(x_2)$.

^d This approach seems to be applicable to the higher values of $l > 2$ as well, although further calculations will be much more complicated.

The two-dimensional potential is:

$$V(\vec{x}) = 6k^2(sn^2(x_1) + sn^2(x_2)) + 2k^2 \left(\frac{cn^2(x_1)}{dn^2(x_1)} + \frac{cn^2(x_2)}{dn^2(x_2)} \right) - 8k^2 + \frac{B(B + \sqrt{2}dn(x_1 + x_2))}{sn^2(x_1 + x_2)} + \frac{B(B - \sqrt{2}dn(x_1 - x_2))}{sn^2(x_1 - x_2)} - 2B^2, \quad (41)$$

which is the two-dimensional generalization of the associated Lamé potential (40). Due to properties of elliptic Jacobi functions, it is again periodic under $x_i \rightarrow x_i + 2K \cdot m_i$ with an arbitrary integer m_i .

In order to find a part of spectrum of the system (41), we must study the action of the operator h on the functions ω_n found above. Again, the matrix \hat{C} is block-diagonal since $h\omega_0 = 0$, and therefore the lowest energy eigenvalue (the lower edge of the first bound band) vanishes. Other zero modes ω_n are mixed with each other by the action of h :

$$h(\vec{x}) \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix} = \begin{pmatrix} 2E_1 & 2\sqrt{2}Ba_{11} & 2\sqrt{2}Ba_{12} & 0 \\ 2\sqrt{2}Bb_{11} & 2E_2 & 0 & 2\sqrt{2}Bb_{12} \\ 2\sqrt{2}Bb_{21} & 0 & 2E_3 & 2\sqrt{2}Bb_{22} \\ 0 & 2\sqrt{2}Ba_{21} & 2\sqrt{2}Ba_{22} & 2E_4 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix}, \quad (42)$$

where the coefficients a_{ij}, b_{ij} have a rather involved form:

$$b_{ij} = (-1)^j \frac{\beta_i(\beta_i - \alpha_{3-j})}{\alpha_j(\alpha_2 - \alpha_1)}; \quad a_{ij} = (-1)^j \frac{M_i - L_i\beta_{3-j}}{\beta_j(\beta_2 - \beta_1)};$$

$$M_i = (1 - k^2)\alpha_i^2 + 6k^4(3 + \alpha_i); \quad L_i = (1 - k^2)\alpha_i^2 - 2k^2(3 + \alpha_i).$$

Diagonalizing the matrix on the r.h.s. of (42), one obtains the energy eigenvalues E_n ; $n = 1, 2, 3, 4$ and eigenfunctions $\Psi_n(\vec{x})$ for (41) analogously to the previous Subsection. In this case also the quasi-momenta $\vec{k}(\epsilon_n)$ for $n = 1, 2, 3, 4$ are zero.

4.4 The two-dimensional trigonometric Razavy potential

Although the two models studied in the previous Subsections lead to very different potentials - the two-dimensional Lamé and associated Lamé potentials - the forms of the initial solutions $\phi_{1,2}$ are very similar to each other, and the functions ϕ_{\pm} simply coincide. The difference can be described by the parameter γ :

$$\phi_{1,2}(x) = \gamma k^2 \frac{sn(x|k)cn(x|k)}{dn(x|k)}, \quad (43)$$

where $\gamma = 1$ for the Lamé system, and $\gamma = 2$ for the associated Lamé.

In this Subsection, we consider the limiting case when

$$\gamma \equiv \frac{2\beta}{k^2}; \quad k \rightarrow 0, \quad (44)$$

β being a new arbitrary **finite** parameter, and ϕ_{\pm} being the same as in (35), but with $k \rightarrow 0$. Taking into account that in this limit

$$sn(x|k) \rightarrow \sin x; \quad cn(x|k) \rightarrow \cos x; \quad dn(x|k) \rightarrow 1,$$

we obtain:

$$\phi_{1,2}(x) = \beta \sin(2x); \quad \phi_+(x) = -\phi_-(x) = B \cot(\sqrt{2}x), \quad (45)$$

and the one-dimensional potentials $V_{1,2}(x)$ takes the form:

$$V_{1,2}(x) = \frac{\beta^2}{2}(1 - \cos 4x) - 2\beta \cos 2x. \quad (46)$$

These potentials coincide with the periodic potentials used by M. Razavy [19] for the description of torsional oscillations of certain molecules. One must choose $\xi = -2\beta$; $n = 0$ in Eq.(4) of [19] in order to identify the trigonometric Razavy potential with our Eq.(46). The trigonometric Razavy potential admits [21] partial solvability: a few of its band edge levels (eigenvalues and wave functions) were found analytically for different values of the integer parameter n . Actually, in our case of $n = 0$ only the lowest band edge eigenfunction with $\epsilon_0 = 0$ was found:

$$\eta_0(x) = \rho_0(x) = \exp\left(-\frac{\beta}{2} \cos 2x\right). \quad (47)$$

Our choice for $\phi_{1,2,\pm}$ leads to the following two-dimensional periodic potential (24), which can be considered as a generalization of the one-dimensional trigonometric Razavy potential:

$$\begin{aligned} V(\vec{x}) = & -\frac{\beta^2}{2}(\cos 4x_1 + \cos 4x_2) - 2\beta(\cos 2x_1 + \cos 2x_2) + \\ & + \frac{B(B + \sqrt{2})}{\sin^2(\sqrt{2}x_+)} + \frac{B(B - \sqrt{2})}{\sin^2(\sqrt{2}x_-)} + \beta^2 - 2B^2. \end{aligned} \quad (48)$$

For this potential, the method of SUSY-separation of variables provides the analytical expression for the lowest ($E_0 = 0$) band edge wave function:

$$\begin{aligned} \Psi_0(\vec{x}) = & \exp(\kappa(\vec{x}))\eta_0(x_1)\rho_0(x_2) = \\ = & \left(\left| \frac{\sin(x_1 - x_2)}{\sin(x_1 + x_2)} \right| \right)^{\frac{B}{\sqrt{2}}} \exp\left(-\frac{\beta}{2}(\cos 2x_1 + \cos 2x_2)\right), \end{aligned} \quad (49)$$

where $\kappa(\vec{x})$ was calculated directly from (26), and η_0, ρ_0 - from (47).

5 Discussions and conclusions

In Section 3 two different symmetries (S_1, S_2) of the functional equation (5) were presented. Here we apply them to the Lamé and associated Lamé potentials discussed above.

First, S_1 symmetry applied to *any* solution $(\phi_1, \phi_2, \phi_+, \phi_-)$ leads to the potential

$$V^{(S_1)} = (\phi_+^2(x_1) - \phi_+'(x_1)) + (\phi_-^2(x_2) - \phi_-'(x_2)) + (\phi_1^2(x_+) - \phi_1'(x_+)) + (\phi_2^2(x_-) - \phi_2'(x_-)), \quad (50)$$

which differs from (6) only in the change of variables $\tilde{x}_{1,2} = x_{\pm}$. Since the Laplace operator has the same form in $\tilde{x}_{1,2}$ coordinates, new wave functions $\Psi_n^{(S_1)}$ of (50) are obtained from the old ones Ψ_n of (6) simply as $\Psi_n^{(S_1)}(x_1, x_2) = \Psi_n(x_+, x_-)$.

In contrast to S_1 , the effect of S_2 symmetry could in principle be more promising for the building of new models. Performing the S_2 -transformation of the associated Lamé potential, generated by (38)-(39) with $l = 2$, we obtain a new potential:

$$V^{(S_2)}(\vec{x}) = 6k^2 sn^2(x_1) + 2k^2 sn^2(x_2) + 2k^2 \frac{cn^2(x_1)}{dn^2(x_1)} + 6k^2 \frac{cn^2(x_2)}{dn^2(x_2)} - 8k^2 + \frac{B(B - \sqrt{2}dn(x_1 + x_2))}{cn^2(x_1 + x_2)} + \frac{B(B + \sqrt{2}dn(x_1 - x_2))}{cn^2(x_1 - x_2)} - 2B^2. \quad (51)$$

It is connected to (41) as:

$$V^{(S_2)}(x_1, x_2; B) = V(x_1, x_2 + K; -\frac{B}{\sqrt{1-k^2}}),$$

and the wave functions of $V^{(S_2)}$ can be obtained easily from those of V by the same change of coordinates and parameters. The analogous conclusion is also suitable for the Lamé potential, the case $l = 1$.

In the main text - Section 4 - we in fact considered only one of the superpartner Hamiltonians H, \widetilde{H} from the basic intertwining relations (1). It is well known [1] that in **one-dimensional** SUSY Quantum Mechanics these superpartners are usually almost isospectral, i.e. they have the same spectra but up to normalizable zero modes of the supercharges Q^{\pm} . The absence of such zero

modes in the case of non-periodic potentials means the spontaneous breaking of SUSY. The situation is different [10] for some one-dimensional **periodic** potentials, where the two partner potentials V , \tilde{V} are related by a discrete symmetry, but SUSY is not broken. These potentials were called "self-isospectral", and the particular examples are given by the one-dimensional Lamé (33) and associated Lamé (40) potentials. Another - non-self-isospectral - class of one-dimensional periodic models also exists [11], [12], where (quite the contrary) the SUSY intertwining relations provide a variety of new solvable periodic potentials.

In the non-periodic models on the whole plane [3], [4], [5] both second-order supercharges Q^\pm may have zero modes. It is interesting to consider **the two-dimensional periodic** supersymmetric models from the point of view of their self-isospectrality. For example, the superpartner of the generalized Lamé potential (36) has the form:

$$\begin{aligned} \tilde{V}(\vec{x}) = & 2k^2(sn^2(x_1) + sn^2(x_2) - 1) + \\ & + \frac{B(B - \sqrt{2}dn(x_1 + x_2))}{sn^2(x_1 + x_2)} + \frac{B(B + \sqrt{2}dn(x_1 - x_2))}{sn^2(x_1 - x_2)} - 2B^2, \end{aligned}$$

i.e., it differs from $V(\vec{x})$ only by the signs in front of the functions $dn(x_1 \pm x_2)$ in the nominators. It is easy to check that the reflection $(x_1, x_2) \rightarrow (x_1, -x_2)$ merely turns \tilde{V} into V and vice versa:

$$\tilde{V}(x_1, x_2) = V(x_1, -x_2).$$

Therefore, the spectra of the Hamiltonians H and \tilde{H} coincide, and the two-dimensional generalized Lamé potential (36) obeys the property of self-isospectrality. The analogous proof also works for the two-dimensional associated Lamé potential (41) .

It is significant that while in a certain sense the self-isospectrality for one-dimensional models (with first-order supercharges) renders supersymmetry useless, this is not the case in the two-dimensional situation (with second order supercharges). Indeed, since all the two-dimensional models considered above satisfy the supersymmetric intertwining relations (1) with second-order supercharges Q^\pm , the corresponding Hamiltonians H (**irrespective** of properties of the superpartners \tilde{H}) commute with the operators

$$R = Q^- Q^+,$$

where Q^\pm are given by (3). This means that all these systems are completely integrable - R plays the role of the symmetry operators. It is clear that these symmetry operators R annihilate the wave functions constructed in Section 4, since they were built simply as linear combinations of zero modes of Q^+ . However, the action of the operators R on other (as yet unknown) wave functions

of H may be very nontrivial.

In this paper the one-dimensional systems with a finite number of bands were represented by the **band edge** wave functions only. The wave functions inside the bands for the Lamé potential, however, are also known. Two linearly independent wave functions with energy ϵ are given by:

$$\Psi_{\pm}(x) = \frac{H(x \pm \alpha)}{\Theta(x)} e^{\mp x Z(\alpha)}; \quad \epsilon \equiv dn^2(\alpha), \quad (52)$$

where the Jacobi theta-functions, H , Θ , and the Jacobi zeta-function, Z , are defined in the theory of elliptic functions (see [17]). One could use these wave functions, instead of band edge functions above, to construct the additional eigenvalues of the two-dimensional periodic models. This task seems to be much more difficult technically and will be considered elsewhere. Here we wish to illustrate how the wave functions (52) coincide with the band edge wave functions (34) in the limits $\epsilon \rightarrow 0$, $(1 - k^2)$, 1 . Indeed, these limits correspond to the following values of α : $K + iK'$, K , 0 , where K' is the associated complete elliptic integral $K'(k) = K(k') = K(\sqrt{1 - k^2})$. One then has to substitute these limiting values into H , Θ , Z in (52). As a result, just Eqs.(34) are derived.

Let us also mention the limit $k \rightarrow 1$ of the two-dimensional Lamé and associated Lamé systems (36) and (41). In this limit, $sn(x|k = 1) = \tanh x$; $cn(x|k = 1) = \operatorname{sech} x$; $dn(x|k = 1) = \operatorname{sech} x$. Therefore, for $k \rightarrow 1$ Eqs.(38) and (39) lead to

$$\phi_{1,2}(x) = l \cdot \tanh x; \quad \phi_+(x) = -\phi_-(x) = \frac{B}{\sinh(\sqrt{2}x)},$$

where $l = 1$ for the Lamé and $l = 2$ for the associated Lamé systems. Substitution of these expressions into two-dimensional potentials Eqs.(36), (41) gives exactly the potential of the two-dimensional generalization of Pöschl-Teller model. This was studied in detail in [7], [8], where its partial solvability and complete integrability were demonstrated.

We should notice in conclusion that even in one-dimensional quantum mechanics very limited number of exactly solvable periodic problems is known [10], [11], [12]. There is no need to stress the interest of finding analytically solvable higher dimensional models like those described in this paper. Such models would be very desirable both as a basis for perturbation theory and for further study of general properties. The importance of the investigation of this kind of model is increasing owing to the development of modern physical technologies, which have led to the manufacture of new materials: a variety of superlattices, films, quantum two-dimensional dots etc. A study of these materials and the corresponding devices should be based on two-dimensional (and three-dimensional) Schrödinger equations with periodic potentials.

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