Generalized Zeta Functions and One-loop Corrections to Quantum Kink Masses

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Abstract

A method for describing the quantum kink states in the semi-classical limit of several (1+1)-dimensional field theoretical models is developed. We use the generalized zeta function regularization method to compute the one-loop quantum correction to the masses of the kink in the sine-Gordon and cubic sinh-Gordon models and another two $P(\phi)_2$ systems with polynomial self-interactions.

1 Introduction

BPS states arising both in extended supersymmetric gauge theories [1] and string/M theory [2] play a crucial rôle in the understanding of the dualities between the different regimes of the system. In this framework, domain walls appear as extended states in $N = 1$ SUSY gluodynamics and the Wess-Zumino model [3]. This circumstance prompted the question of whether or not these topological defects saturate the quantum Bogomolny bound. A return to the study of quantum corrections to the masses of (1+1)-dimensional solitons has thus been unavoidable. These subtle matters were first addressed in the classical papers of Dashen, Hasslacher and Neveu, [4], and Faddeev and Korepin, [5], for the purely bosonic $[\phi]^4_2$ and sine-Gordon theories, and then in Reference [6] for the super-symmetric extension of these theories. Analysis of the ultraviolet cut-off regularization procedure in the presence of a background is the main concern in the papers of Reference [7]: the authors carefully distinguish between using a cut-off either in the energy or in the number of modes. The second method leads to the same result as in the computation performed by DHN for bosonic fluctuations. Another point of view is taken in Reference [8], where SUSY boundary conditions related more to infrared behaviour, are carefully chosen. On this basis, and by using higher-derivative ultraviolet regularization (SUSY preserving), the authors demonstrated an anomaly in the central charge that compensates for the extra (quantum) contribution to the classical mass.

In this paper we shall confine ourselves to purely bosonic theories and leave the treatment of fermionic fluctuations for future research. We address the quantization of non-linear waves relying on the generalized zeta function regularization method to control the infinite quantities arising in the quantum theory. This procedure has been used previously in computing Casimir energies and the quantum corrections to kink masses, see [11]. We shall develop this topic, however, in a completely general way, also offering a comparison with other approaches. As well as obtaining exact results, we also shall explain how asymptotic methods lead to a very good approximation of the right answer. We believe that the novel application of the asymptotic method should be very useful in the cubic sinh-Gordon model as well as in multi-component scalar field theory, where the traditional approach is limited by the lack of detailed knowledge of the spectrum of the second-order fluctuation operator (see [13], [14] for extensive work on multi-component kinks and their stability).

The organization of the paper is as follows: In Section §2 the general semi-classical picture of quantum solitons, the zeta function regularization procedure, the zero-point energy and mass renormalization prescriptions, and the asymptotic method are described. In Section §3, we apply the method to the "loop" kinks of the sine-Gordon, $(\phi)^4_2$, and cubic sinh-Gordon models. In the first two paradigmatic cases, it is possible to obtain an exact result, which allows comparison with other methods. Approximate computations by means of the asymptotic expansion of the heat function are also offered to test the goodness of our procedure against the well known exact answers. Section §4 is devoted to the analysis of the "link" kink arising in the $(\phi)^6_2$ model. Finally, Section §5 offers an outlook on further applications of our approach.

2 Semi-classical picture of quantum soliton states

We shall consider (1+1)-dimensional scalar field theories whose classical dynamics is governed by the action

$$S[\phi]=\int d^2y\left\{\frac{1}{2}\frac{\partial\phi}{\partial y^\mu}\frac{\partial\phi}{\partial y_\mu}-U(\phi)\right\}$$

We choose the metric tensor in $T^2(\mathbb{R}^{1,1})$ as $g=\mathrm{diag}(1,-1)$ and the Einstein convention will be used throughout the paper. We shall not use a natural unit system because we wish to keep track of $\hbar$ in our formulas; nevertheless, we choose the speed of light to be $c=1$. Elementary dimensional analysis tells us that $[\hbar]=ML$, $[U(\phi)]=ML^{-1}$ and $[\phi]=M^{\frac{1}{2}}L^{\frac{1}{2}}$ are the dimensions of the important quantities.

The classical configuration space $\mathcal{C}$ is formed by the static configurations $\phi(y)$, for which the energy functional

$$E(\phi)=\int dy\left\{\frac{1}{2}\frac{d\phi}{dy}\frac{d\phi}{dy}+U(\phi)\right\}$$

is finite: $\mathcal{C}=\{\phi(y)/E(\phi)<+\infty\}$. In the Schrödinger picture, quantum evolution is ruled by the functional differential Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\Psi[\phi(y);t]=H\Psi[\phi(y);t].$$

The quantum Hamiltonian operator

$$H=\int dy\left\{-\frac{\hbar^2}{2}\frac{\delta}{\delta\phi(y)}\frac{\delta}{\delta\phi(y)}+E[\phi(y)]\right\}$$

acts on wave functionals $\Psi[\phi(y);t]$ that belong to $L^2(\mathcal{C})$.

We wish to compute the matrix element of the evolution operator in the "field" representation

$$G\left(\phi^{(f)}(y),\phi^{(i)}(y);T\right)=\left\langle\phi^{(f)}(y)\right|e^{-\frac{i}{\hbar}TH}\left|\phi^{(i)}(y)\right\rangle=\int\mathcal{D}[\phi(y,t)]\exp\left\{\frac{i}{\hbar}S[\phi]\right\}\qquad(1)$$

for the choice

$$\phi^{(i)}(y,0)=\phi_K(y)\quad;\quad\phi^{(f)}(y,T)=\phi_K(y)$$

where $\phi_K(y)$ is a kink static solution of the classical field equations. We are, however, only interested in the loop ($\hbar$) expansion of G up to the first quantum correction. Also performing a analytic continuation to "Euclidean" time, $t=-i\tau$, $T=-i\beta$, this is achieved by the steepest-descent method applied to the Feynman integral in (1):

$$G_E(\phi_K(y),\phi_K(y);\beta)=\exp\left\{-\frac{\beta E[\phi_K]}{\hbar}\right\}\mathrm{Det}^{-\frac{1}{2}}\left[-\frac{\partial^2}{\partial\tau^2}+PK\right](1+o(\hbar))$$

where K is the differential operator

$$K=-\frac{\partial^2}{\partial y^2}+\left.\frac{d^2U}{d\phi^2}\right|_{\phi_K};$$

and P is the projector over the strictly positive part of the spectrum of K. Note that, on the mathematical side, the steepest-descent method provides a well defined approximation to the Feynman integral if the spectrum of the quadratic form K is positive definite and, on the physical side, the zero eigenvalue that appears in Spec(K) contributes to the next order in the loop expansion: it is due to neutral equilibrium on the orbit of the kink solution under the action of the spatial translation group. Moreover, in order to avoid the problems that arise in connection with the existence of a continuous spectrum of K, we place the system in a interval of finite but very large length L, i.e. $x \in [-\frac{L}{2};\frac{L}{2}]$, and, eventually -after assuming periodic boundary conditions on the small fluctuations all throughout the paper- we shall let L go to infinity.

>From the spectral resolution of K,

$$K\psi_n(y) = \omega_n^2\psi_n(y) ; \qquad \omega_n^2 \in \mathrm{Spec}(K) = \mathrm{Spec}(PK) + \{0\};$$

we write the functional determinant in the form

$$\mathrm{Det}\left[-\frac{\partial^2}{\partial\tau^2} + K\right] = \prod_n \det\left[-\frac{\partial^2}{\partial\tau^2} + \omega_n^2\right] :$$

All the determinants in the infinite product correspond to harmonic oscillators of frequency $i\omega_n$ and thus, with an appropriate normalization, we obtain for large β

$$G_E(\phi_K(y);\phi_K(y);\beta) = e^{-\frac{\beta}{\hbar}E[\phi_K]}\prod_n\left(\frac{\omega_n}{\pi\hbar}\right)^{\frac12} e^{-\frac{\beta}{2}\sum_n\omega_n(1+o(\hbar))}$$

where the eigenvalue in the kernel of K has been excluded.

Inserting eigen-energy wave functionals

$$H\Psi_j[\phi_K(y)] = \varepsilon_j\Psi_j[\phi_K(y)]$$

we have an alternative expression for G_E for $\beta \to \infty$:

$$G_E(\phi_K(y);\phi_K(y);\beta) = \Psi_0^*[\phi_K(y)]\Psi_0[\phi_K(y)]e^{-\frac{\beta\varepsilon_0}{\hbar}}$$

and, therefore, we obtain

$$\varepsilon_0 = E[\phi_K] + \frac{\hbar}{2}\sum_{\omega_n^2>0}\omega_n + o(\hbar^2)$$

$$|\Psi_0[\phi_K(y)]|^2 = \mathrm{Det}^{\frac14}\left[\frac{PK}{\pi^2\hbar^2}\right] + o(\hbar^2) ;$$

as the kink ground state energy and wave functional up to one-loop order.

We define the generalized zeta function

$$\zeta_{PK}(s) = \mathrm{Tr}(PK)^{-s} = \sum_{\omega_n^2>0}\frac{1}{(\omega_n^2)^s}$$

associated to the differential operator PK. Then,

$$\varepsilon_0^K = E[\phi_K] + \frac{\hbar}{2}\mathrm{Tr}(PK)^{\frac12} + o(\hbar^2) = E[\phi_K] + \frac{\hbar}{2}\zeta_{PK}(-\tfrac12) + o(\hbar^2) \qquad (2)$$

$$|\Psi_0[\phi_K(y)]|^2 = \hbar \exp\{\zeta_{PK}(0)\}\exp\left(\frac{1}{4}\frac{d\zeta_{PK}}{ds}(0)\right) \tag{3}$$

show how to read the energy and wave functional of the quantum kink ground state in terms of the generalized zeta function of the projection of the second variation operator K in the semi-classical limit.

2.1 The generalized zeta function regularization method : zero-point energy and mass renormalizations

The eigen-functions of K form a basis for the quantum fluctuations around the kink background. Therefore, the sum of the associated zero-point energies encoded in $\zeta_{PK}(-\frac{1}{2})$ in formula (2) is infinite and we need to use some regularization procedure. We shall regularize $\zeta_{PK}(-\frac{1}{2})$ by defining the analogous quantity $\zeta_{PK}(s)$ at some point in the s complex plane where $\zeta_{PK}(s)$ does not have a pole. $\zeta_{PK}(s)$ is a meromorphic function of s, such that its residues and poles can be derived from heat kernel methods, see [15]. If $K_K(y;z;\beta)$ is the kernel of the heat equation associated with K,

$$\left(\frac{\partial}{\partial\beta} + K\right) K_K(y;z;\beta) = 0 \quad ; \quad K_K(y;z;0) = \delta(y-z) \tag{4}$$

the Mellin transformation tells us that,

$$\zeta_{PK}(s) = \frac{1}{\Gamma(s)}\int_0^\infty d\beta\, \beta^{s-1} h_{PK}(\beta)$$

where,

$$h_{PK}[\beta] = \mathrm{Tr}\, e^{-\beta PK} = \mathrm{Tr}\, e^{-\beta K} - 1 = -1 + \int dy\, K_K(y;y;\beta)$$

is the heat function $h_{PK}[\beta]$, if K is positive semi-definite and $\dim \mathrm{Ker}(K) = 1$. Thus, the "regularized" kink energy is in the semi-classical limit:

$$\varepsilon_0^K(s) = E[\phi_K] + \frac{\hbar}{2}\mu^{2s+1}\zeta_{PK}(s) + o(\hbar^2) \tag{5}$$

where μ is a unit of length^{-1} dimension, introduced to make the terms in (5) homogeneous from a dimensional point of view. The infiniteness of the bare quantum energy is seen here in the pole that the zeta function develops for $s = -\frac{1}{2}$.

To renormalize ε_0^K we must: A. Subtract the regularized vacuum quantum energy. B. Add counter-terms that will modify the bare masses of the fundamental quanta, also regularized by means of the generalized zeta function. C. Take the appropriate limits.

A. The quantum fluctuations around the vacuum are governed by the Schrodinger operator:

$$V = -\frac{d^2}{dy^2} + \left.\frac{d^2U}{d\phi^2}\right|_{\phi_V}$$

where ϕ_V is a constant minimum of $U[\phi]$. The kernel of the heat equation

$$\left(\frac{\partial}{\partial\beta} + V\right) K_V(y;z;\beta) = 0 \quad ; \quad K_V(y;z;0) = \delta(y-z)$$

provides the heat function $h_V(\beta)$,

$$h_V(\beta) = \mathrm{Tr}\, e^{-\beta V} = \int dy\, K_V(y;y;\beta):$$

We exclude the constant mode and, through the Mellin transformation, we obtain

$$\zeta_V(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\beta\, \beta^{s-1}\, \mathrm{Tr}\, e^{-\beta V}:$$

The regularized kink energy measured with respect to the regularized vacuum energy is thus:

$$\begin{aligned}
\varepsilon^K(s) &= E[\phi_K] + \Delta\varepsilon_1^K(s) + o(\hbar^2) \\
&= E[\phi_K] + \frac{\hbar}{2}\mu^{2s+1}\left[\zeta_{PK}(s) - \zeta_V(s)\right] + o(\hbar^2):
\end{aligned}$$

B. If we now go to the physical limit $\varepsilon^K = \lim_{s\to-\frac{1}{2}} \varepsilon^K(s)$, we still obtain an infinite result. The reason for this is that the physical parameters of the theory have not been renormalized. It is well known that in (1+1)-dimensional scalar field theory normal ordering takes care of all the infinities in the system: the only ultraviolet divergences that occur in perturbation theory come from graphs that contain a closed loop consisting of a single internal line, [16]. From Wick's theorem, adapted to contractions of two fields at the same point in space-time, we see that normal ordering adds the mass renormalization counter-term,

$$\Delta H(\delta m^2) = \frac{\hbar}{2}\int dy\, \delta m^2 : \frac{d^2U}{d\phi^2} : + o(\hbar^2)$$

to the Hamiltonian up to one-loop order. To regularize

$$\delta m^2 = \int \frac{dk}{4\pi} \frac{1}{\sqrt{k^2 + U''(\phi_V)}}$$

we first place the system in a 1D box of length L so that $\delta m^2 = \frac{1}{2L}\zeta_V(\frac{1}{2})$, if the constant eigenfunction of V is not included in ζ_V. Then, we again use the zeta function regularization method and define: $\delta m^2(s) = \frac{1}{L}\frac{\Gamma(s+1)}{\Gamma(s)}\zeta_V(s+1)$. Note that $\delta m^2 = \lim_{s\to-\frac{1}{2}} \delta m^2(s)$. The criterion behind this regularization prescription is the vanishing tadpole condition, which is shown in Appendix B of Reference [12] to be equivalent to the heat kernel subtraction scheme.

The one-loop correction to the kink energy due to $\Delta H(\delta m^2(s))$ is thus

$$\begin{aligned}
\Delta\varepsilon_2^K(s) &= \langle\phi_K|\Delta H(\delta m^2(s))|\phi_K\rangle - \langle\phi_V|\Delta H(\delta m^2(s))|\phi_V\rangle = \\
&= \lim_{L\to\infty} \frac{\hbar}{2L}\mu^{2s+1}\frac{\Gamma(s+1)}{\Gamma(s)}\int_{-\frac{L}{2}}^{\frac{L}{2}} dy\, \zeta_V(s+1)\left[\left.\frac{d^2U}{d\phi^2}\right|_{\phi_K} - \left.\frac{d^2U}{d\phi^2}\right|_{\phi_V}\right] = \\
&= \lim_{L\to\infty} \frac{\hbar}{2L}\mu^{2s+1}\frac{\Gamma(s+1)}{\Gamma(s)}\zeta_V(s+1)\int_{-\frac{L}{2}}^{\frac{L}{2}} dy\, V(y) \qquad (6)
\end{aligned}$$

because the expectation values of normal ordered operators in coherent states are the corresponding c-number-valued functions.

C. The renormalized kink energy is thus

$$\varepsilon^K_R = E[\phi_K] + \Delta M_K + o(\hbar^2) = E[\phi_K] + \lim_{s\to-\frac{1}{2}}\left[\Delta_1\varepsilon^K(s) + \Delta_2\varepsilon^K(s)\right] + o(\hbar^2) \qquad (7)$$

whereas the renormalized wave functional reads

$$\int dx \, \left|\Psi^R_0(\phi_K)\right|^2 = \mathrm{Det}^{\frac{1}{4}}\left(\frac{PK}{2\hbar^2}\right)\mathrm{Det}^{(-\frac{1}{4})}\left(\frac{V}{2\hbar^2}\right)$$

$$= \hbar\exp\{-(\zeta_{PK}(0)-\zeta_V(0))\}\exp\left[\frac{1}{4}\left(\frac{d\zeta_{PK}}{ds}(0)-\frac{d\zeta_V}{ds}(0)\right)\right] :$$

2.2 Asymptotic approximation to semi-classical kink masses

In order to use the asymptotic expansion of the generalized zeta function of the K operator to compute the semi-classical expansion of the corresponding quantum kink mass, it is convenient to use non-dimensional variables. We define non-dimensional space-time coordinates $x^\mu = m_d y^\mu$ and field amplitudes $\phi(x^\mu) = c_d \psi(y^\mu)$, where m_d and c_d are constants with dimensions $[m_d] = L^{-1}$ and $[c_d] = M^{\frac{1}{2}}L^{\frac{1}{2}}$ to be determined in each specific model. Also, we write $U(\phi) = \frac{c_d^2}{m_d^2}U(\psi)$.

The action and the energy can now be written in terms of their non-dimensional counterparts:

$$S[\phi] = \frac{1}{c_d^2}\int d^2x\left\{\frac{1}{2}\frac{\partial\psi}{\partial x^\mu}\frac{\partial\psi}{\partial x_\mu} - U(\psi)\right\} = \frac{1}{c_d^2}S[\psi]$$

$$E[\phi] = \frac{m_d}{c_d^2}\int dx\left\{\frac{1}{2}\frac{d\psi}{dx}\frac{d\psi}{dx} + U(\psi)\right\} = \frac{m_d}{c_d^2}E[\psi]:$$

The important point is that the Hessians at the vacuum and kink configurations now read

$$\mathcal{V} = m_d^2\left(-\frac{d^2}{dx^2} + v^2\right) = m_d^2 V \quad ; \quad \mathcal{K} = m_d^2\left(-\frac{d^2}{dx^2} + v^2 - V(x)\right) = m_d^2 K$$

where $\frac{d^2U}{d\psi^2}\big|_V = v^2$ and $\frac{d^2U}{d\psi^2}\big|_K = v^2 - V(x)$. Therefore,

$$\zeta_{\mathcal{V}}(s) = \frac{1}{m_d^{2s}}\zeta_V(s) \quad ; \quad \zeta_{\mathcal{K}}(s) = \frac{1}{m_d^{2s}}\zeta_K(s):$$

The asymptotic expansion is superfluous if $\mathrm{Tr}\, e^{-\beta PK}$ and $\zeta_{PK}(s)$ are susceptible of an exact computation. If $V(x)$ is a potential well of the Posch-Teller type, see [17], one can completely solve the spectral problem for K and there is no need for any approximation to $\zeta_{PK}(s)$. In general the spectrum of K is not known in full detail, specially in systems with multi-component kinks, and we can only determine $\zeta_{PK}(s)$ by means of an asymptotic expansion. Nevertheless, we shall also compute the asymptotic expansion of $\zeta_{PK}(s)$ in the cases where the exact answer is known in order to estimate the error accepted in this approach.

In the formulas (4), (5), (6) and (7) we replace $\mathcal{V}$, $\mathcal{K}$ and v^2 by V, K and v^2 and write the kernel of the heat equation for K in the form:

$$K_K(x,x';\beta) = K_V(x,x';\beta)A(x,x';\beta);$$

$A(x;x';\beta)$ is thus the solution of the PDE

$$\left[\frac{\partial}{\partial\beta} + \frac{x-x'}{\beta}\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2} - V(x)\right] A(x;x';\beta) = 0 \tag{8}$$

with "initial" condition: $A(x;x';0) = 1$.

For $\beta < 1$, we solve (8) by means of an asymptotic (high-temperature) expansion: $A(x;x';\beta) = \sum_{n=0}^{\infty} a_n(x;x')\beta^n$. Note that there are no half-integer powers of β in this expansion because our choice of boundary conditions with no boundary effects.

In this regime the heat function is given by:

$$\mathrm{Tr}\, e^{-\beta K} = \int_{-\frac{m_d L}{2}}^{\frac{m_d L}{2}} dx\, K_K(x;x;\beta) = \frac{e^{-\beta v^2}}{\sqrt{4\pi\beta}} \sum_{n=0}^{\infty} \int_{-\frac{m_d L}{2}}^{\frac{m_d L}{2}} dx\, a_n(x;x)\beta^n = \frac{e^{-\beta v^2}}{\sqrt{4\pi\beta}} \sum_{n=0}^{\infty} a_n(K)\beta^n \, .$$

It is not difficult to find the coefficients $a_n(x;x)$ by an iterative procedure starting from $a_0(x;x') = 1$. This procedure is explained in the Appendix, which also includes the explicit expression of some of the lower-order coefficients.

The use of the power expansion of $h_{PK}[\beta] = \mathrm{Tr}\, e^{-\beta PK}$ in the formula for the quantum kink mass is quite involved:

1. First, we write the generalized zeta function of V in the form:

$$\zeta_V(s) = \frac{1}{\Gamma(s)} \frac{m_d L}{\sqrt{4\pi}} \int_0^1 d\beta\, \beta^{s-\frac{3}{2}} e^{-\beta v^2} + B_V(s)$$

with

$$B_V(s) = \frac{m_d L}{\sqrt{4\pi}} \frac{\Gamma[s-\frac{1}{2};v^2]}{v^{2s-1}\Gamma[s]} \quad ; \quad \zeta_V(s) = \frac{m_d L}{\sqrt{4\pi}} \frac{\gamma[s-\frac{1}{2};v^2]}{v^{2s-1}\Gamma(s)} + B_V(s)$$

and $\Gamma[s;v^2]$ and $\gamma[s-\frac{1}{2};v^2]$ being respectively the upper and lower incomplete gamma functions, see [18]. It follows that $\zeta_V(s)$ is a meromorphic function of s with poles at the poles of $\gamma[s-\frac{1}{2};v^2]$, which occur when $s-\frac{1}{2}$ is a negative integer or zero. $B_V(s)$, however, is a entire function of s.

2. Second, from the asymptotic expansion of $h_K[\beta]$ we estimate the generalized zeta function of PK:

$$\begin{aligned}\zeta_{PK}(s) &= \frac{1}{\Gamma(s)}\left[\int_0^1 d\beta\, \beta^{s-1} + \frac{1}{\sqrt{4\pi}}\sum_{n<n_0} a_n(K)\int_0^1 d\beta\, \beta^{s+n-\frac{3}{2}} e^{-\beta v^2} + b_{n_0,K}(s)\right] + B_{PK}(s) = \\ &= \frac{1}{s\,\Gamma(s)} + \frac{1}{\Gamma(s)\sqrt{4\pi}}\sum_{n<n_0} a_n(K)\frac{\gamma[s+n-\frac{1}{2};v^2]}{v^{2(s+n-\frac{1}{2})}} + \frac{1}{\Gamma(s)} b_{n_0,K}(s) + B_{PK}(s)\end{aligned}$$

where

$$b_{n_0,K}(s) = \frac{1}{\sqrt{4\pi}}\sum_{n\geq n_0}^{\infty} a_n(K)\frac{\gamma[s+n-\frac{1}{2};v^2]}{v^{2(s+n-\frac{1}{2})}}$$

is holomorphic for $\mathrm{Re}\, s > -n_0 + \frac{1}{2}$, whereas

$$B_{PK}(s) = \frac{1}{\Gamma(s)}\int_{v^2}^{\infty} d\beta\, \mathrm{Tr}\, e^{-\beta PK}\beta^{s-1}$$

is a entire function of s. The values of s where $s+n-\frac{1}{2}$ is a negative integer or zero are the poles of $\zeta_{PK}(s)$ because the poles of $\Gamma[s+n-\frac{1}{2};\nu^2]$ lie at these points in the s-complex plane.

Renormalization of the zero-point energy requires the subtraction of $\zeta_V(s)$ from $\zeta_{PK}(s)$. We find,

$$\zeta_{PK}(s)-\zeta_V(s)\simeq\frac{1}{\Gamma(s)}\left[-\frac{1}{s}+\sum_{n=1}^{\infty}\frac{a_n(K)}{\sqrt{4\pi}}\frac{\Gamma[s+n-\frac{1}{2};\nu^2]}{\nu^{2(s+n-\frac{1}{2})}}\right]$$

and the error in this approximation with respect to the exact result to $\Delta_1\varepsilon^K$ is:

$$\mathrm{error}_1=\frac{\hbar m_d}{2}\left[-\frac{1}{2\sqrt{\pi}}b_{n_0,K}(-\tfrac{1}{2})+B_{PK}(-\tfrac{1}{2})-B_V(-\tfrac{1}{2})\right].$$

Note that the subtraction of $\zeta_V(s)$ exactly cancels the contribution of $a_0(K)$ and hence, the divergence arising at $s=-\frac{1}{2}$, $n=0$. The quadratic ultraviolet divergences appear in this scheme as related to the pole of $\zeta_V(s)$ at $s=-\frac{1}{2}$, $n=0$.

3. Third, $\Delta_1\varepsilon^K$ now reads

$$\Delta_1\varepsilon^K=\frac{\hbar m_d}{\Gamma(-\frac{1}{2})}+\frac{\hbar}{2}\lim_{s\to-\frac{1}{2}}\left(\frac{\mu^2}{m_d^2}\right)^s\frac{a_1(K)}{\sqrt{4\pi}\,\Gamma(s)}\frac{\Gamma[s+\frac{1}{2};\nu^2]}{\nu^{2s+1}}+$$
$$+\frac{\hbar m_d}{2}\sum_{n=2}^{\infty}\frac{a_n(K)}{\sqrt{4\pi}\,\Gamma(-\frac{1}{2})}\frac{\Gamma[n-1;\nu^2]}{\nu^{2n-2}}.$$

The logarithmic ultraviolet divergences, hidden at first sight in the DHN approach, arise here in connection with the pole of $\zeta_{PK}(s)-\zeta_V(s)$ at $s=-\frac{1}{2}$, $n=1$.

The surplus in energy due to the mass renormalization counter-term is,

$$\Delta_2\varepsilon^K=-\lim_{L\to\infty}\frac{\hbar\,a_1(K)}{2L}\lim_{s\to-\frac{1}{2}}\left(\frac{\mu}{m_d}\right)^{2s+1}\frac{\Gamma(s+1)}{\Gamma(s)}\zeta_V(s+1)+o(\hbar^2)$$
$$=-\frac{\hbar m_d}{2\sqrt{4\pi}}a_1(K)\lim_{s\to-\frac{1}{2}}\left(\frac{\mu}{m_d}\right)^{2s+1}\frac{\Gamma[s+\frac{1}{2};\nu^2]}{\nu^{2s+1}\Gamma(s)}+o(\hbar^2)$$

and the deviation from the exact result is

$$\mathrm{error}_2=\lim_{L\to\infty}\frac{\hbar}{4L}a_1(K)B_V(\tfrac{1}{2}).$$

Therefore,

$$\varepsilon_R^K=E[\phi_K]+\Delta M_K+o(\hbar^2)=E[\phi_K]-\frac{\hbar m_d}{2\sqrt{\pi}}\left[1+\frac{1}{\sqrt{4\pi}}\sum_{n=2}^{\infty}a_n(K)\frac{\Gamma[n-1;\nu^2]}{\nu^{2n-2}}\right]+$$
$$+\frac{\hbar m_d}{2\sqrt{4\pi}}a_1(K)\lim_{s\to-\frac{1}{2}}\left(\frac{\mu}{m_d}\right)^{2s+1}\left[\frac{\Gamma[s+\frac{1}{2};\nu^2]}{\nu^{2s+1}\Gamma(s)}-\frac{\Gamma[s+\frac{1}{2};\nu^2]}{\nu^{2s+1}\Gamma(s)}\right]+o(\hbar^2).$$

Note that the contributions proportional to $a_1(K)$ of the poles at $s=-\frac{1}{2}$ in $\Delta_1\varepsilon^K(s)$ and $\Delta_2\varepsilon^K(s)$ cancel.

We are left with the very compact formula:

$$M_K = \hbar m_d \left[\Lambda_0 + D_{n_0}\right] \qquad \begin{cases} \Lambda_0 = -\dfrac{1}{2\sqrt{\pi}} \\[2ex] D_{n_0} = -\displaystyle\sum_{n=2}^{n_0-1} \frac{a_n(K)}{8\pi}\,\frac{\gamma[n-1; v^2]}{v^{2n-2}} \end{cases} \tag{9}$$

In sum, there are only two contributions to semi-classical kink masses obtained by means of the asymptotic method: 1) $\hbar m_d \Lambda_0$ is due to the subtraction of the translational mode; 2) $\hbar m_d D_{n_0}$ comes from the partial sum of the asymptotic series up to the $n_0 - 1$ order. We stress that the merit of the asymptotic method lies in the fact that there is no need to solve the spectral problem of K: all the information is encoded in the potential $V(x)$.

3 Loop kinks

The existence of kinks is guaranteed if the minima of $U(\phi)$ are a discrete set which is the union of orbits of the discrete symmetry group of the system. We shall use the term "loop" kinks to refer to those classical solutions that interpolate between vacua belonging to the same orbit of the symmetry group; otherwise, the solitary waves will be referred to as "link" kinks, see [20]. In this Section we shall discuss three kinks of the "loop" type.

3.1 The quantum sine-Gordon soliton

We first treat the sine-Gordon model by considering the potential energy density: $U[\psi(y)] = \frac{m^4}{\lambda}\left(1 - \cos\frac{\sqrt{\lambda}}{m}\psi\right)$. The dimensions of the parameters m and λ are respectively: $[m] = L^{-1}$ and $[\lambda] = M^{-1}L^{-3}$. Therefore, we choose $m_d = m$ and $c_d = \frac{\sqrt{\lambda}}{m}$ and find: $U[\phi(x;t)] = (1 - \cos\phi)$.

The "internal" symmetry group of the system is the infinite dihedral group $D_\infty = Z_2 \ltimes Z$ generated by internal reflections, $\phi \to -\phi$, and 2π translations, $\phi \to \phi + 2\pi$. The vacuum classical configurations $\phi_V(x;t) = 2\pi n$ form the orbit $M = \frac{D_\infty}{Z_2}$ and there is spontaneous symmetry breakdown of the internal translational symmetry through the choice of vacuum. The moduli space of vacua, however, $\hat{M} = \frac{M}{D_\infty}$, is a single point and all the equivalents kinks of the model,

$$\phi_K(x;t) = \pm 4\arctan e^{x} + 2\pi n \quad ; \quad \psi_K(y;y_0) = \pm\frac{4m}{\sqrt{\lambda}}\arctan e^{my} + \frac{2\pi n m}{\sqrt{\lambda}} \quad ;$$

are loop kinks. It is easy to check that $E[\phi_K] = \frac{8m^3}{\lambda}$ and $E[\phi_V] = 0$.

The second order variation operator around the kink solutions is

$$K = -\frac{d^2}{dy^2} + m^2(1 - 2\,\mathrm{sech}^2\, my) \quad ; \quad K = -\frac{d^2}{dx^2} + 1 - 2\,\mathrm{sech}^2\, x :$$

Note that $K = m^2 K$; henceforth, $\zeta_{PK}(s) = \frac{1}{m^{2s}}\zeta_{PK}(s)$. Similimodo, in the vacuum sector we have:

$$V = -\frac{d^2}{dy^2} + m^2 \quad ; \quad V = -\frac{d^2}{dx^2} + 1 \quad ; \quad V = m^2 V \quad ; \quad \zeta_V(s) = \frac{1}{m^{2s}}\zeta_V(s) :$$

3.1.1 Exact computation of the mass and the wave functional

Generalized zeta function of V:

The spectrum of V acting on functions belonging to $L^2(\mathbb{R})$ is $\mathrm{Spec}\,V = k^2+1$, with $k \in \mathbb{R}$ a real number. There is a half-bound state $f_{k^2=0}(x)=$ constant that we shall not consider because it is paired with the other half-bound state in Spec(K). The spectral density on the interval $I=[-\frac{mL}{2};\frac{mL}{2}]$ with periodic boundary conditions is $\rho_V(k)=\frac{mL}{2\pi}$. The heat function is,

$$\mathrm{Tr}\, e^{-\beta V} = \frac{mL}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-\beta(k^2+1)} = \frac{mL}{\sqrt{4\pi\beta}}\, e^{-\beta}$$

and the generalized zeta function reads:

$$\zeta_V(s) = \frac{mL}{\Gamma(s)\sqrt{4\pi}}\int_0^{\infty} d\beta\, \beta^{s-\frac{3}{2}} e^{-\beta} = \frac{mL}{\sqrt{4\pi}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} :$$

Therefore, $\zeta_V(s)$ (hence $\Gamma(s)\zeta_V(s)$) is a meromorphic function of s with poles at $s=\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \cdots$. The generalized zeta function of the Hessian at the vacuum is, however, also infrared-divergent: it is linearly divergent when $L\to\infty$ even at points $s\in\mathbb{C}$ where $\zeta_V(s)$ is regular.

Generalized zeta function of K:

In this case $\mathrm{Spec}\,K = \{0\}\cup\{k^2+1\}$, $k\in\mathbb{R}$ and the spectral density on I is

$$\rho_K(k) = \frac{mL}{2\pi} + \frac{1}{2\pi}\frac{d\delta(k)}{dk}$$

with phase shifts

$$\delta(k) = 2\arctan\frac{1}{k}$$

because K is the Schrodinger operator that governs the scattering through the first of the "transparent" Posch-Teller potentials, [17]. Thus,

$$\begin{aligned}\mathrm{Tr}\, e^{-\beta K} &= 1+\frac{mL}{2\pi}\int_{-\infty}^{\infty} dk e^{-\beta(k^2+1)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\frac{d\delta(k)}{dk} e^{-\beta(k^2+1)} = \\ &= 1+\frac{mL}{\sqrt{4\pi\beta}} e^{-\beta} - \mathrm{Erfc}\sqrt{\beta}\end{aligned}$$

where $\mathrm{Erfc}\sqrt{\beta}$ is the complementary error function, [18]. Note that K has a zero mode, the eigen-function being the translational mode $\frac{d\phi_K}{dx}=\mathrm{sech}^2 x$, which must be subtracted because it arises in connection with the breaking of the translational symmetry, $x\to x+a$, by the kink solution and does not contribute to the kink mass up to this order in the loop expansion. There is also a half-bound state, $f_{k^2=0}(x)=\tanh x$, that exactly cancels the contribution of the constant half-bound state in Spec V. Therefore, we obtain

$$\mathrm{Tr}\, e^{-\beta P_K} = \mathrm{Tr}\, e^{-\beta K} - 1 = \mathrm{Tr}\, e^{-\beta V} - \mathrm{Erfc}\sqrt{\beta}$$

and

$$\zeta_{P_K}(s) = \zeta_V(s) - \frac{1}{\Gamma(s)}\int_0^{\infty} d\beta\, \mathrm{Erfc}\sqrt{\beta}\,\beta^{s-1} = \zeta_V(s) - \frac{1}{\sqrt{\pi}}\frac{\Gamma(s+\frac{1}{2})}{s\,\Gamma(s)} :$$

$\zeta_{PK}(s)$ (hence $\zeta_{PK}(s)$) is also a meromorphic function of s that shares all the poles with $\zeta_V(s)$, but the residues are different except at $s=\frac{1}{2}$, a pole where the residues of $\zeta_{PK}(s)$ and $\zeta_V(s)$ coincide. The infrared divergence, however, is identical in the kink background and the vacuum.

We can now compute the limit of the regularized quantities that enter in the one-loop correction formula to the kink mass:

$$\begin{aligned}\Delta_1 E^K &= \frac{\hbar}{2}\lim_{s\to-\frac{1}{2}}\left[\left(\frac{\mu^2}{m^2}\right)^s\left(\zeta_{PK}(s)-\zeta_V(s)\right)\right] = -\frac{\hbar m}{2\sqrt{\pi}}\lim_{\varepsilon\to 0}\left(\frac{\mu}{m}\right)^{2\varepsilon}\frac{\Gamma(\varepsilon)}{\Gamma(-\frac{1}{2}+\varepsilon)}\\ &= \frac{\hbar m}{2\pi}\lim_{\varepsilon\to 0}\left[-\frac{1}{\varepsilon}-2\log\frac{2\mu}{m}-\psi(1)+\psi(-\frac{1}{2})+o(\varepsilon)\right]\end{aligned} \qquad (10)$$

and

$$\begin{aligned}\Delta_2 E^K &= \frac{2\hbar}{L}\lim_{s\to-\frac{1}{2}}\left(\frac{\mu}{m}\right)^{2s+1}\frac{\Gamma(s+1)}{\Gamma(s)}\zeta_V(s+1)+o(\hbar^2) = \frac{\hbar m}{\sqrt{\pi}}\lim_{s\to-\frac{1}{2}}\left(\frac{\mu}{m}\right)^{2s+1}\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s)}\\ &= \frac{\hbar m}{\sqrt{\pi}}\lim_{\varepsilon\to 0}\left(\frac{\mu}{m}\right)^{2\varepsilon}\frac{\Gamma(\varepsilon)}{\Gamma(-\frac{1}{2}+\varepsilon)} = \frac{\hbar m}{2\pi}\lim_{\varepsilon\to 0}\left[\frac{1}{\varepsilon}+2\log\frac{2\mu}{m}+\psi(1)-\psi(-\frac{1}{2})+o(\varepsilon)\right]+o(\hbar^2):\end{aligned}$$

where $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$ is the digamma function.

The important point to notice is that the renormalization of the zero-point energy performed by the subtraction of $\zeta_V(-\frac{1}{2})$ still leaves a divergence coming from the $s=-\frac{1}{2}$ poles because the residues are different. The correction due to the mass renormalization counter-term also has a pole. The sum of the contributions of the two poles leaves a finite remainder and we end with the finite answer:

$$\Delta_1 E^K+\Delta_2 E^K=-\frac{\hbar m}{\pi}\quad;\quad E^K_R=E[\phi_K]-\frac{\hbar m}{\pi}+o(\hbar^2)=\frac{8m}{\beta^2}-\frac{\hbar m}{\pi}+o(\hbar^2): \qquad (11)$$

The one-loop quantum correction to the mass of the sine-Gordon soliton obtained by means of the generalized zeta function procedure exactly agrees with the accepted result, see [4], [5], and, henceforth, with the outcome of the mode number regularization method, [7].

The square of the modulus of the ground state wave functional up to one-loop order is given by (3). If $W=\frac{P_K}{C^2}$ and $C=\frac{\hbar}{m_d^2}$, obviously, $\zeta_W(s)=C^{2s}\zeta_{PK}(s)$ ($\zeta_W(0)=\zeta_{PK}(0)$) and we have $\frac{d\zeta_W}{ds}=C^{2s}\zeta_{PK}(s)\log C+C^{2s}\frac{d\zeta_{PK}}{ds}(s)$: Thus,

$$\begin{aligned}\frac{d\zeta_{PK}}{ds} &= \frac{d\zeta_V}{ds}-\frac{1}{\sqrt{\pi}}\frac{\Gamma(s+\frac{1}{2})}{s\,\Gamma(s)}\left[\psi(s+\frac{1}{2})-\frac{1}{s}-\psi(s)\right]\\ \frac{d\zeta_V}{ds} &= \frac{mL}{\sqrt{4\pi}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\left[\psi(s-\frac{1}{2})-\psi(s)\right]\end{aligned} \qquad (12)$$

and

$$\zeta_V(0)=0\quad;\quad\zeta_{PK}(0)=-1\quad;\quad\frac{d\zeta_V}{ds}(0)=-mL\quad;\quad\frac{d\zeta_{PK}}{ds}(0)=-mL+\psi(1)-\psi(\tfrac{1}{2}):$$

Therefore,

$$|\Psi_0(\phi_K(x))|^2=\sqrt{\frac{C}{2}}\exp\left[-\frac{1}{4}mL\right]\quad;\quad|\Psi_0(\phi_V(x))|^2=\exp\left[-\frac{1}{4}mL\right]:$$

Renormalizing the wave functional with respect to the vacuum we obtain

$$\frac{|\Psi_0(\phi_K(x))|^2}{|\Psi_0(\phi_V(x))|^2} = \sqrt{\frac{C}{2}} \tag{13}$$

3.1.2 The asymptotic expansion and quantum corrections

In the sine-Gordon model the exact formulas for $\mathrm{Tr}\, e^{-\beta PK}$ and $\zeta_{PK}(s)$ are readily derived because the spectrum of the Schrodinger operator K is completely known. On the other hand, the series expansion of the complementary error function tells us that

$$\mathrm{Tr}\, e^{-\beta PK} = \left[\frac{mL}{\sqrt{4\pi\beta}} + \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{2^n \beta^{n-\frac{1}{2}}}{1\cdot 3\cdot 5 \cdots (2n-1)} \right] e^{-\beta} - 1$$

and the $a_n(K)$ coefficients can be computed from this exact expression:

$$\mathrm{Tr}\, e^{-\beta PK} = \frac{e^{-\beta}}{\sqrt{4\pi}} \sum_{n=0}^{\infty} a_n(K)\, \beta^{n-\frac{1}{2}} - 1 \quad ; \quad a_0(K) = mL \quad ; \quad a_n(K) = \frac{2^{n+1}}{(2n-1)!!} :$$

One can check by direct calculation that indeed,

$$a_n(K) = \int_{-\frac{mL}{2}}^{\frac{mL}{2}} dx\, a_n(x;x) \quad ; \quad n = 0;1;2;3;\ldots$$

and the $a_n(K)$ are the integrals of the functions defined in Appendix for $V(x) = -2\,\mathrm{sech}^2 x$.

In any case we see from the formula (9) that the comparison with the exact result is satisfactory:

$$\Delta M_K = -0.282095\hbar m - \sum_{n=2}^{n_0-1} a_n(K) d_n \hbar m \quad \text{versus} \quad \Delta M_K = -0.318309\hbar m :$$

The partial sums

$$D_{n_0} = \sum_{n=2}^{n_0-1} a_n(K) d_n = \sum_{n=2}^{n_0-1} a_n(K) \frac{\gamma[n-1;1]}{8\pi}$$

can be estimated with the help of the following Table,

n	$a_n(K)$	$n_0 - 1$	D_{n_0}
2	2.66667	2	-0.0670702
3	1.06667	3	-0.0782849
4	0.30476	4	-0.0802324
5	0.06772	5	-0.0805373
6	0.012324	6	-0.0805803
7	0.0018944	7	-0.0805857
8	0.00025258	8	-0.0805863
9	0.00002972	9	-0.0805863

For instance, choosing $n_0 = 10$ we find that $D_{10} = -0.080586$ and the correction obtained by means of the asymptotic expansion is:

$$E[\phi_K] + \Delta M_K = \frac{8m^3}{\lambda} - 0.362681\hbar m + o(\hbar^2)$$

In fact

$$\frac{\hbar m}{2}\left[B_{PK}\left(-\tfrac12\right) - B_V\left(-\tfrac12\right)\right] + \lim_{L\to\infty}\frac{\hbar m}{L}B_V\left(-\tfrac12\right) = -\frac{\hbar m}{2\sqrt{\pi}}\int_1^\infty d\beta\left(\frac{1}{2}\frac{\mathrm{Erfc}\sqrt{\beta}}{\beta^{\frac32}} + \frac{e^{-\beta}}{\sqrt{\pi}}\right) = -0.044373\hbar m$$

is almost the total error: $0.044372\hbar m$. The difference is:

$$-\frac{\hbar m}{4\sqrt{\pi}}\, b_{10,K}\left(-\tfrac12\right) \simeq 10^{-6}\hbar m \, .$$

Note in the Table that $a_n(K)$ rapidly decreases with increasing n.

3.2 The quantum $(\lambda\phi^4)_2$ kink

We now consider the other prototype of solitary waves in relativistic (1+1)-dimensional field theory: the kink of the $(\lambda\phi^4)_2$ model. The potential energy density is: $U[\phi(y)] = \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2$. We choose, however, $m_d = \frac{m}{\sqrt{2}}$ but keep $c_d = \frac{\sqrt{\lambda}}{m}$ and find: $U[\phi(x,t)] = \frac12(\phi^2 - 1)^2$.

The internal symmetry group is now the $\mathbb{Z}_2$ group generated by the $\phi \to -\phi$ reflections and the orbit of vacuum classical configurations $\phi_V(x,t) = \pm 1 \in \mathcal{M}$ gives rise to a moduli space of vacua $\hat{\mathcal{M}} = \frac{\mathcal{M}}{\mathbb{Z}_2}$ which is a single point. The kink solitary waves are thus loop kinks and read

$$\phi_K(x,t) = \pm\tanh x \qquad ; \qquad \phi_K(y;y_0) = \pm\frac{m}{\sqrt{\lambda}}\tanh\frac{my}{\sqrt{2}} \, .$$

The kink and vacuum solutions have classical energies of $E[\phi_K] = \frac{4}{3}\frac{m^3}{\sqrt{2}\lambda}$ and $E[\phi_V] = 0$ respectively. The Hessian operators for the vacuum and kink solutions are

$$\Delta V = -\frac{d^2}{dy^2} + 2m^2 = \frac{m^2}{2}\left(-\frac{d^2}{dx^2} + 4\right) = \frac{m^2}{2}V$$

$$\Delta K = -\frac{d^2}{dy^2} + 2m^2 - \frac{3m^2}{\cosh^2\frac{my}{\sqrt{2}}} = \frac{m^2}{2}\left(-\frac{d^2}{dx^2} + 4 - \frac{6}{\cosh^2 x}\right) = \frac{m^2}{2}K$$

and the corresponding generalized zeta functions satisfy

$$\zeta_{\Delta PK}(s) = \left(\frac{\sqrt{2}}{m}\right)^{2s}\zeta_{PK}(s) \qquad ; \qquad \zeta_{\Delta V}(s) = \left(\frac{\sqrt{2}}{m}\right)^{2s}\zeta_V(s) \, .$$

3.2.1 Exact computation of the semi-classical mass and wave functional

Generalized zeta function of $V = -\frac{d^2}{dx^2} + 4$.

Acting on the $L^2(\mathbb{R}) \subset \mathbb{C}$ Hilbert space we have that $\mathrm{Spec}\,V = \{k^2 + 4\}, k \in \mathbb{R}$, but the spectral density on the interval $I = [-\frac{mL}{2\sqrt{2}}, \frac{mL}{2\sqrt{2}}]$ of eigen-functions with periodic boundary conditions is $\rho_V(k) = \frac{mL}{2\sqrt{2}\pi}$. From these data, the heat and generalized zeta functions are easily computed:

$$\mathrm{Tr}\, e^{-\beta V} = \frac{mL}{2\sqrt{2}\pi}\int_{-\infty}^{\infty} dk\, e^{-\beta(k^2+4)} = \frac{mL}{\sqrt{8\pi\beta}}\, e^{-4\beta}$$

$$\zeta_V(s) = \frac{mL}{\sqrt{8\pi}}\frac{1}{\Gamma(s)}\int_0^{\infty} d\beta\, \beta^{s-\frac{3}{2}} e^{-4\beta} = \frac{mL}{\sqrt{8\pi}}\frac{1}{4^{s-\frac{1}{2}}}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \qquad (14)$$

and we find that $\zeta_V(s)$ has the same poles and infrared behaviour in the $(\lambda\phi^4)_2$ and the sine-Gordon models.

Generalized zeta function of $K = -\frac{d^2}{dx^2} + 4 - \frac{6}{\cosh^2 x}$.

K is the Schrodinger operator for the second transparent Posch-Teller potential, [17]. Thus, $\mathrm{Spec}\,K = \{0\} \cup \{3\} \cup \{k^2 + 4\}, k \in \mathbb{R}$, and the spectral density on I is

$$\rho_K(k) = \frac{mL}{2\sqrt{2}\pi} + \frac{1}{2\pi}\frac{d\delta(k)}{dk}$$

where the phase shifts are $\delta(k) = -2\arctan\frac{3k}{2-k^2}$, if PBC are considered. Thus, we find

$$\begin{aligned}\mathrm{Tr}\, e^{-\beta PK} &= e^{-3\beta} + \frac{mL}{\sqrt{8}\pi}\int_{-\infty}^{\infty} dk\, e^{-\beta(k^2+4)} + \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \frac{d\delta(k)}{dk} e^{-\beta(k^2+4)} \\ &= \frac{mL}{\sqrt{8\pi\beta}}\, e^{-4\beta} + e^{-3\beta}\left(1 - \mathrm{Erfc}\sqrt{\beta}\right) - \mathrm{Erfc}\, 2\sqrt{\beta} \; .\end{aligned}$$

The Mellin transform immediately provides the generalized zeta function:

$$\zeta_{PK}(s) = \zeta_V(s) + \frac{\Gamma(s+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(s)}\,\frac{2}{3^{s+\frac{1}{2}}}\, {}_2F_1[\tfrac{1}{2}; s+\tfrac{1}{2}; \tfrac{3}{2}; -\tfrac{1}{3}] - \frac{1}{4^s}\frac{1}{s} \qquad (15)$$

where ${}_2F_1[a;b;c;d]$ is the Gauss hypergeometric function, [18].

The power expansion of ${}_2F_1$,

$${}_2F_1[\tfrac{1}{2}; s+\tfrac{1}{2}; \tfrac{3}{2}; -\tfrac{1}{3}] = \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})\,\Gamma(s+\frac{1}{2})}\sum_{l=0}^{\infty}\frac{(-1)^l}{3^l\, l!}\frac{\Gamma(l+\frac{1}{2})\,\Gamma(s+l+\frac{1}{2})}{\Gamma(l+\frac{3}{2})}$$

tells us that, besides the poles of $\zeta_V(s)$, $\zeta_{PK}(s)$ has poles at $s = -\frac{1}{2} + l; -\frac{3}{2} + l; -\frac{5}{2} + l; \cdots$, $l \in \mathbb{Z}^+ \cup \{0\}$; i.e., as in the sG soliton case, $\zeta_V(s)$ and $\zeta_{PK}(s)$ share the same poles except $s = \frac{1}{2}$ but the residues in the $(\lambda\phi^4)_2$ model are increasingly different with larger and larger values of $|\mathrm{Res}|$.

Applying these results to the kink mass formula, we obtain

$$
\begin{aligned}
{}_1\epsilon^K &= \lim_{s\to -\frac12} \frac{\hbar}{2}\left(\frac{2\ ^2}{m^2}\right)^s \left[\ _{PK}(s) \quad _V(s)\right] \\
&= \frac{\hbar m}{2\sqrt{2}} \lim_{\epsilon\to 0} \left(\frac{2\ ^2}{m^2}\right)^{\epsilon} \left[\frac{(\epsilon)}{(\ \frac12 + \epsilon)} \quad \frac{2}{3^{\epsilon}}\, {}_2F_1[\tfrac12;\epsilon;\tfrac32;\ \tfrac13] \quad \frac{1}{(\ \frac12+\epsilon)\,4^{\frac12+\epsilon}}\right] \\
&= \frac{\hbar m}{2\sqrt{2}} \lim_{\epsilon\to 0} \quad \frac{3}{\epsilon} \quad 3\ln\frac{2\ ^2}{m^2} + 2 + \ln\frac{3}{4} \quad {}_2F_1^{0}[\tfrac12;0;\tfrac32;\ \tfrac13] + o(\epsilon)
\end{aligned}
$$

$$
\begin{aligned}
{}_2\epsilon^K &= \lim_{L\to\infty} \frac{6\hbar}{L} \lim_{s\to\ \frac12} \left(\frac{2\ ^2}{m^2}\right)^{s+\frac12} \frac{[s+1]}{[s]}\ _V(s+1) = \frac{3\hbar m}{\sqrt{2}} \lim_{\epsilon\to 0} \left(\frac{2\ ^2}{m^2}\right)^{\epsilon} \frac{4^{\epsilon}\ (\epsilon)}{(\ \frac12+\epsilon)} \\
&\quad + o(\hbar^2) = \frac{3\hbar m}{2\sqrt{2}} \lim_{\epsilon\to 0} \quad \frac{1}{\epsilon} + \ln\frac{2\ ^2}{m^2} \quad \ln 4 + \quad (1) \quad (\ \frac12) + o(\epsilon) \quad + o(\hbar^2)
\end{aligned}
$$

where ${}_2F_1^{0}$ is the derivative of the Gauss hypergeometric function with respect to the second argument. Therefore, ${}_1\epsilon^K + {}_2\epsilon^K = \frac{\hbar m}{2\sqrt{6}} \quad \frac{3\hbar m}{\sqrt{2}}$, and we obtain:

$$
\epsilon_R^K = E[\ _K] + \ M_K = \frac{4}{3}\frac{m^3}{\sqrt{2}} + \hbar m \quad \frac{1}{2\sqrt{6}} \quad \frac{3}{\sqrt{2}} \quad + o(\hbar^2) \quad ;
$$

the same answer as offered by the mode-number regularization method [7].

To compute the norm of the ground state wave functionals we closely follow the procedure applied in sub-Section §3.1. to the sine-Gordon soliton. In the $(\ ^4)_2$ model, we find that

$$
\begin{aligned}
\frac{d\ _{PK}}{ds} &= \frac{d\ _V}{ds} + \frac{1}{\sqrt{\ }} \frac{(s+\frac12)}{(s)}\ 4^{\ s} \quad \frac{1}{s} + \ln 4 + \quad (s) \quad (s+\tfrac12) \\
&\quad 2s\,3^{\ s\ \frac12}\,{}_2F_1[\tfrac12;s+\tfrac12;\tfrac32;\ \tfrac13](\log 3 \quad (s+\tfrac12) + \quad (s)) + 2s\,3^{\ s\ \frac12}\,{}_2F_1^{0}[\tfrac12;s+\tfrac12;\tfrac32;\ \tfrac13]\Big]
\end{aligned}
$$

and

$$
\frac{d\ _V}{ds} = \frac{mL}{\sqrt{8}} \frac{1}{4^{s\ \frac12}} \frac{(s\ \ \frac12)}{(s)} \quad (s\ \ \tfrac12) \quad (s) \quad \log 4 \quad :
$$

from these expressions and formulas (14) and (15) one checks that

$$
_V(0) = 0 \ ; \quad _{PK}(0) = \quad 1 \ ; \quad \frac{d\ _V}{ds}(0) = \quad \sqrt{2}mL \ ; \quad \frac{d\ _{PK}}{ds}(0) = \quad \sqrt{2}mL + \log 48 :
$$

We obtain

$$
j\ _0(\ _K(x))j^2 = \frac12 \quad \frac{C}{\sqrt{3}}^{\ \frac12} \exp\ \frac{mL}{2\sqrt{2}} \quad ; \qquad j\ _0(\ _V(x))j^2 = \exp\ \frac{mL}{2\sqrt{2}} \quad :
$$

The quotient of the probability densities is

$$
\frac{j\ _0(\ _K(x))j^2}{j\ _0(\ _V(x))j^2} = \frac12 \quad \frac{C}{\sqrt{3}}^{\ \frac12} \quad : \tag{16}
$$

3.2.2 The asymptotic expansion and quantum corrections

In the $(\phi^4)_2$ model $\frac{\delta^2 U}{\delta\phi^2}\big|_{\phi_V}(x) = 4$ and $V(x) = \frac{\delta^2 U}{\delta\phi^2}\big|_{\phi_V}(x) - \frac{\delta^2 U}{\delta\phi^2}\big|_{\phi_K}(x) = 6\,\mathrm{sech}^2 x$ are the potentials of the Schrodinger operators that respectively correspond to the Hessians at the vacuum and the kink configurations. The asymptotic expansion of the heat function

$$\mathrm{Tr}\, e^{-\beta P K} = \left[1 + \frac{e^{-4\beta}}{\sqrt{4\pi}} \sum_{n=0}^{\infty} \int_{-\frac{mL}{2\sqrt{2}}}^{\frac{mL}{2\sqrt{2}}} dx\, a_n(x;x)\, \beta^{n}\right] = \left[1 + \frac{e^{-4\beta}}{\sqrt{4\pi}} \sum_{n=0}^{\infty} a_n(K)\, \beta^{n-\frac{1}{2}}\right]$$

can be either obtained as a series expansion of the exact result

$$\mathrm{Tr}\, e^{-\beta P K} = \left[1 + \left(\frac{mL}{\sqrt{8\pi}} + \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{2^n (1 + 2^{2n-1})}{(2n-1)!!}\, \beta^{n-\frac{1}{2}}\right) e^{-4\beta}\right]$$

or from the coefficients defined in the Appendix for $V(x) = 6\,\mathrm{sech}^2 x$

$$a_n(K) = \int_{-\frac{mL}{2\sqrt{2}}}^{\frac{mL}{2\sqrt{2}}} dx\, a_n(x;x) \quad ; \quad a_0(K) = \frac{mL}{\sqrt{2}} \quad ; \quad a_n(K) = \frac{2^{n+1}(1 + 2^{2n-1})}{(2n-1)!!} \, .$$

To compare with the exact result, we apply the formula given in the Appendix and observe that

$$\epsilon_R^K = \frac{4 m^3}{3\sqrt{2}\lambda} - 0.199471\hbar m \sum_{n=2}^{n_0-1} a_n(K) d_n \hbar m + o(\hbar^2)$$

is far from the exact result

$$\epsilon_R^K = \frac{4 m^3}{3\sqrt{2}\lambda} - 0.471113\hbar m + o(\hbar^2)$$

before adding the contribution of the terms between $n = 2$ and $n = n_0 - 1$ in the asymptotic expansion to the contribution coming from the subtraction of the translational mode. The partial sums

$$D_{n_0} = \sum_{n=2}^{n_0-1} a_n(K) d_n = \sum_{n=2}^{n_0-1} a_n(K) \frac{\gamma[n-1;4]}{8\sqrt{2\pi}\, 4^{n-1}}$$

can be estimated up to $n_0 = 11$ with the help of the following Table

n	$a_n(K)$	$n_0 - 1$	D_{n_0}
2	24.0000	2	-0.165717
3	35.2000	3	-0.221946
4	39.3143	4	-0.248281
5	34.7429	5	-0.261260
6	25.2306	6	-0.267436
7	15.5208	7	-0.270186
8	8.27702	8	-0.271317
9	3.89498	9	-0.271748
10	1.63998	10	-0.271900

Choosing $n_0 = 11$, we find that $D_{11} = -0.271900\hbar m$ and the correction obtained by adding $D_{11}\hbar m$ is:

$$\Delta M_K = -0.471371\hbar m + o(\hbar^2)$$

in good agreement with the exact result above. In fact

$$\frac{\hbar m}{2}\left[B_{PK}(\tfrac{1}{2}) - B_V(\tfrac{1}{2})\right] + \frac{3\hbar m}{\sqrt{2}L}B_V(\tfrac{1}{2})$$
$$= -\frac{\hbar m}{2\sqrt{2\pi}}\int_1^\infty d\beta\left(\frac{e^{-3\beta}}{2\beta^{\frac{3}{2}}} + \frac{e^{-3\beta}\,\mathrm{Erfc}\sqrt{\beta}}{2\beta^{\frac{3}{2}}} + \frac{\mathrm{Erfc}\,2\sqrt{\beta}}{2\beta^{\frac{3}{2}}} + \frac{3e^{-4\beta}}{\sqrt{\beta}}\right) \simeq -0.00032792\hbar m$$

is almost the total error: $0.0002580\hbar m$. The deviation is

$$\frac{\hbar m}{4\sqrt{2\pi}}b_{10,K}(\tfrac{1}{2}) \simeq 10^{-4}\hbar m .$$

With respect to the sine-Gordon model there are two differences: a) in the $(\phi^4)_2$ model the error committed by using asymptotic methods is smaller, of the order of $10^{-4}\hbar m$, a 0.07 percent, as compared with $10^{-2}\hbar m$, a 6.00 percent, in the sG case; b) the rejection of the contributions of the $n_0 > 11$ terms and the non-exact computation of the mass counter-term contribution has a cost of approximately $10^{-4}\hbar m$ in the $(\phi^4)_2$ model versus $10^{-6}\hbar m$ in the sG system. Both facts have to do with the larger value of the smaller eigenvalue of the vacuum Hessian in the $(\phi^4)_2$ model with respect to the sG system, 4 versus 1.

3.3 The cubic sinh-Gordon kink

We shall now study a system of the same type where the potential energy density is: $U[\psi(y)] = \frac{m^4}{4\lambda}\left(\sinh^2\frac{\sqrt{\lambda}}{m}\psi - 1\right)^2$. Non-dimensional quantities are defined through the choice $m_d = m$ and $c_d = \frac{\sqrt{\lambda}}{m}$; the Euler-Lagrange equation is

$$\Box\phi(t,x) = -\frac{1}{2}\sinh(2\phi)(\sinh^2\phi - 1) \tag{17}$$

and the justification for the choice of name is clear. We find this model interesting because it reduces to the $(\phi^4)_2$ system if $|\phi(t,x)| < 1$ and is the Liouville model, [19], with opposite sign of the coupling constant, in the $\phi(t,x) = \pm\infty$ ranges. In fact, the potential energy density $U(\phi) = \frac{1}{4}(\sinh^2\phi - 1)^2$, see Figure 1(a), presents two minima at the classical values: $\phi_V = \pm\,\mathrm{arcsinh}\,1$. The two vacuum points are identified by the $\phi \to -\phi$ internal symmetry transformation and the semi-classical vacuum moduli space is a point. For this reason, $U(x)$ has been applied to the study of the quantum theory of diatomic molecules: the solutions of the associated time-independent Schrodinger equation are a good approximation to the eigen-states of a quantum particle that moves under the influence of two centers of force. We deal with the $\nu = 1$ and $M = 3$ member of the Razavy family of quasi-exactly-solvable Schrodinger operators, [25], although we are looking at it from a field-theoretical perspective.

The solutions of the first-order equations

$$\frac{d\phi}{dx} = \mp\frac{1}{\sqrt{2}}(\sinh^2\phi - 1) \quad ; \quad \phi_K(x) = \pm\,\mathrm{arctanh}\frac{\tanh(x+b)}{\sqrt{2}}; \tag{18}$$

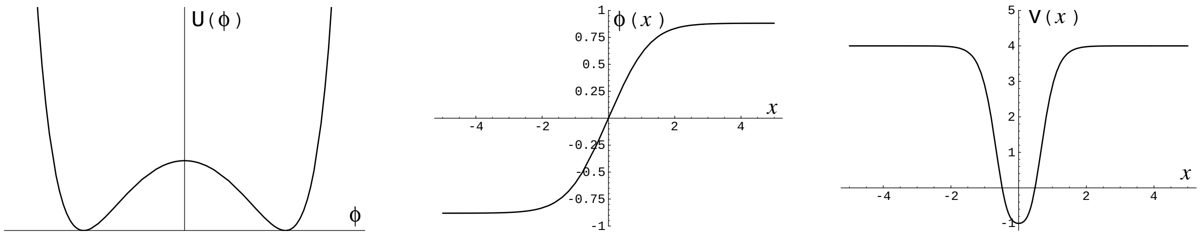

Figure 1: Graphic representation of (a) the potential energy density (b) the kink and (c) the Hessian potential well.

see Figure 1(b) for $b = 0$, are the kink solitary waves of the system. The Hessian operators at the vacuum and kink solutions are respectively:

$$V = \frac{d^2}{dy^2} + 4m^2 = m^2 \left(\frac{d^2}{dx^2} + 4 \right) = m^2 V$$

$$K = \frac{d^2}{dy^2} + 2m^2 + \frac{16m^2}{(1+\mathrm{sech}^2 y)^2} \quad \frac{14m^2}{1+\mathrm{sech}^2 y} = m^2 \left(\frac{d^2}{dx^2} + 2 + \frac{16}{(1+\mathrm{sech}^2 x)^2} \quad \frac{14}{1+\mathrm{sech}^2 x} \right) = m^2 K$$

The mass of the fundamental mesons is thus $2m$. K is an Schrodinger operator:

$$K = \frac{d^2}{dx^2} + 4 \quad V(x) \quad ; \quad V(x) = \frac{2\,\mathrm{sech}^2 x\,(9+\mathrm{sech}^2 x)}{(1+\mathrm{sech}^2 x)^2}$$

where the potential well plotted in Figure 1(c), albeit analytically very different from the s-G and $(^4)_2$ kink potential wells, exhibits a similar shape.

We shall not attempt to solve the spectral problem of K. The only thing that we need to know in order to apply the asymptotic method is that the lowest eigen-state is the unique zero mode:

$$f_0(x) = \frac{d_K}{dx} = \frac{2\sqrt{2}}{(3+\cosh 2x)}$$

Therefore, the energy of the semi-classical kink state is approximately (see formula (9))

$$"^K_R = \frac{m^3}{2} \left(1 \quad 3\sqrt{2}\,\mathrm{arcsinh} 1 \right) \sim m \left(\frac{1}{2\sqrt{2}} + \sum_{n=2}^{n_0 \; 1} a_n(K) \frac{[n \;\; 1;4]}{8 \; 4^{n \; 1}} \right) \tag{19}$$

In the Table below we write the Seeley's coefficients and the partial sums $D_{n_0} = \sum_{n=2}^{n_0 \; 1} a_n(K) \frac{[n \;\; 1;4]}{8 \; 4^{n \; 1}}$ up to $n_0 = 11$:

n	$a_n(K)$	$n_0 \;\; 1$	D_{n_0}
2	29.1604	2	−0.20135
3	39.8523	3	−0.26501
4	42.1618	4	−0.293253
5	36.0361	5	−0.306715
6	25.7003	6	−0.313005
7	15.6633	7	−0.315779
8	8.3143	8	−0.316917
9	3.9033	9	−0.317349
10	1.6590	10	−0.317502

obtaining the approximate answer:

$$M_K = \hbar m\left[\beta_0 + D_{11}\right] = -0.282095\hbar m - 0.317502\hbar m = -0.599597\hbar m$$

We cannot estimate the error but we assume that this result is as good as the answer obtained for the $(\phi^4)_2$ kink because the continuous spectrum of K also starts at 4.

4 Link kinks: the $(\phi^6)_2$ model

Finally, we consider the following potential energy density: $U[\psi(y)] = \frac{\lambda^2}{4m^2}\psi^2\left(\psi^2 - \frac{m^2}{\lambda^2}\right)^2$. The choice of $m_d = \frac{m}{\sqrt{2}}$ and $c_d = \frac{\sqrt{\hbar}\lambda}{m}$ leads to the non-dimensional potential: $U[\phi(x,t)] = \frac{1}{2}\phi^2(\phi^2 - 1)^2$. The moduli space of vacua $\hat{M} = \frac{M}{Z_2}$, made out of two Z_2 orbits, contains two points:

$$\phi_{V_0}(x,t) = 0 \quad ; \quad \phi_{V_\pm}(x,t) = \pm 1 :$$

Quantization around the $\phi_{V_0}(x,t)$ vacuum preserves the $\phi \to -\phi$ symmetry, which is spontaneously broken at the degenerate vacua $\phi_{V_\pm}(x,t)$. The kink solitary waves of the system

$$\phi_K(x,t) = \pm\frac{1}{\sqrt{2}}\sqrt{1 \pm \tanh(x+b)} \quad ; \quad \psi_K(y;y^0) = \pm\frac{m}{\sqrt{2}\lambda}\sqrt{1 \pm \tanh\frac{m}{\sqrt{2}}(y+\tilde{b})}$$

interpolate between $\phi_{V_\pm}(x,t)$ and $\phi_{V_0}(x,t)$, or vice-versa, which are vacua belonging to distinct Z_2 orbits: these solutions are thus link kinks.

The kink and vacuum solutions have classical energies of $E[\phi_K] = \frac{1}{4}\frac{m^3}{\sqrt{2}\lambda^2}$ and $E[\phi_{V_0}] = E[\phi_{V_\pm}] = 0$ respectively. The Hessian operators for the vacuum and kink solutions are

$$\begin{aligned}
V_0 &= -\frac{d^2}{dy^2} + \frac{m^2}{2} = \frac{m^2}{2}\left(-\frac{d^2}{dx^2} + 1\right) = \frac{m^2}{2}V_0 \\
V_\pm &= -\frac{d^2}{dy^2} + 2m^2 = \frac{m^2}{2}\left(-\frac{d^2}{dx^2} + 4\right) = \frac{m^2}{2}V_\pm \\
K &= -\frac{d^2}{dy^2} + \frac{5m^2}{4} \mp \frac{3m^2}{4}\tanh\frac{my}{\sqrt{2}} - \frac{15m^2}{8\cosh^2\frac{my}{\sqrt{2}}} = \\
&= \frac{m^2}{2}\left(-\frac{d^2}{dx^2} + \frac{5}{2} \mp \frac{3}{2}\tanh x - \frac{15}{4\cosh^2 x}\right) = \frac{m^2}{2}K
\end{aligned}$$

The problem of the semi-classical quantization of these and other link kinks have been addressed somewhat unsuccessfully in [21] due to the analytical complexity of the eigen-functions of K as well as the conceptual difficulty of dealing with a QFT on the real line where the asymptotic states far on the left and far on the right correspond to mesons with different masses. This issue has been analyzed in depth in [22]: the main suggestion is that the normal-order prescription should be performed with an arbitrary mass to be fixed in order to avoid the ambiguity induced by the step function background. We now apply the asymptotic expansion of the heat function method in this complex circumstance to find a very natural way of choosing the mass renormalization parameter. Moreover, we improve the approximation obtained in the computation of the quantum kink mass by going farther than first-order in the asymptotic expansion.

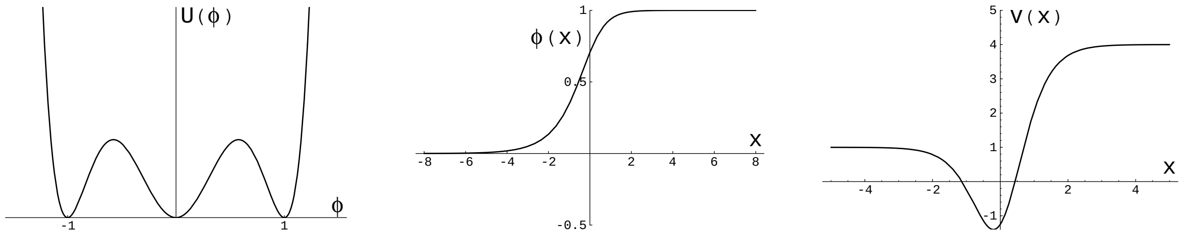

Figure 2: Graphical representation of (a) the potential energy density (b) the kink and (c) the Hessian potential well.

Besides the bound state,

$$f_0(x) = \frac{1}{2\cosh^2 x \quad p \overline{2(1 \quad \tanh x)}};$$

the $!^2 = 0$ translational mode, the spectrum of K includes transmissionless scattering states for $1 \quad !^2 \quad 4$, and states with both non null transmission and re ection coe cients if $!^2 \quad 4$. In the language of QFT, the topological sectors based on link kinks are peculiar in the sense that the N-particle asymptotic states are mesons that have di erent masses at $x = \quad 1$. If the meson energy is less than $2m^2$, the bosons are re ected when coming from the left/right towards the kink. More energetic mesons can either be re ected by or pass through the kink. If the mesons are transmitted there is a conversion from kinetic to "inertial" energy, or vice-versa, in such a way that the poles of the propagators far to the left or far to the right of the kink can only occur at $p^2 = \frac{m^2}{2}$ and $p^2 = 2m^2$.

This is the reason why the subtraction from the Casimir energy of $_K, \tilde{_2}\ _{PK}(\ \frac{1}{2})$, of either the Casimir energy of the $_{V_0}, \tilde{_2}\ _{V_0}(\ \frac{1}{2})$, or the $_V\ , \tilde{_2}\ _V\ (\ \frac{1}{2})$, vacua is hopeless, even after adding the mass renormalization counter-term to the Lagrangian. Therefore, we cannot use the generalized zeta functions $_{V_0}(s)$ and $_V\ (s)$ to renormalize the zero point energy in the kink sector. Instead, we will gauge the kink Casimir energy against the Casimir energies of a family of background eld con gurations that satisfy:

$$5\ {}^4_B(x) \quad 4\ {}^2_B(x) = \frac{1}{2}(1 \quad \tanh\ x); \tag{20}$$

where $\ 2\ R^+$. The rationale behind this choice is that the $\ ! \ 1$ limit is the background used by Lohe, [21]: $_{B_1}(x) = \quad (x)$. The problem with Lohe's choice is that the discontinuity at the origin poses many problems for the algorithm of the asymptotic expansion because a nightmare of delta functions and their derivatives appears at $x = 0$ at orders higher than the rst. Thus, we need some regularization, which is achieved by replacing the sign function by tanh in the formula (20) above. In Figures 3(a) and 3(b) the Hessian potential wells for the backgrounds $_{B_1}$ and $_{B_1}$ are compared.

For any non-zero nite $\ ,\ _B\ (x)$ interpolates smoothly between $\frac{4}{5}$ and $\ 1$ when x varies from $\ 1$ to 1 . The jump from $\ 1$ to 0 occurring at $x = 0$ in $_{B_1}(x)$ becomes a jump from $\frac{4}{5}$ to 0, which therefore takes place at $x = \quad 1 \ !$, followed by the smooth interpolation to $\ 1$. If $\ = 0$ the background con guration is also pathological: $_{B_0}(x) = \frac{2+ p\overline{\frac{13}{2}}}{5}$; $8x$, except at $x = \quad 1$, where there are jumps to 0 and $\ 1$.

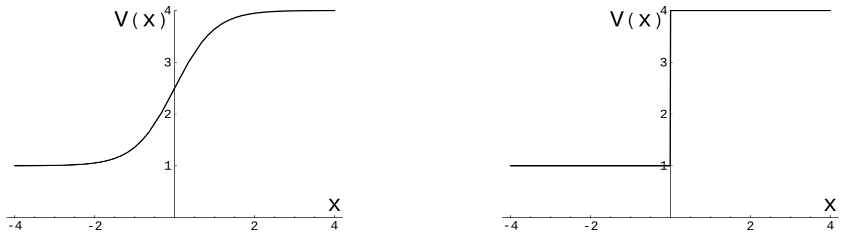

Figure 3: Graphic representation of the potential well produced by (a) the ϕ_{B_1} background (b) the ϕ_{B_1} background as functions of x

The Schrodinger operators

$$B = -\frac{d^2}{dx^2} + \frac{5}{2} - \frac{3}{2}\tanh x$$

$$B_1 = -\frac{d^2}{dx^2} + \frac{5}{2} - \frac{3}{2}\varepsilon(x) \qquad ; \qquad B_0 = -\frac{d^2}{dx^2} + \frac{5}{2}$$

govern the small fluctuations around the background ϕ_B. Thus,

$$\Delta_1 \varepsilon^K(\hbar) = \frac{\hbar m_d}{2} \lim_{s \to -\frac{1}{2}} \left(\frac{\mu^2}{m_d}\right)^{s+\frac{1}{2}} [\zeta_{PK}(s) - \zeta_B(s)]$$

is the Casimir kink energy renormalized with respect to the ϕ_B background. From the asymptotic expansion of both $\zeta_{PK}(s)$ and $\zeta_B(s)$ we obtain:

$$\Delta_1 \varepsilon^K(\hbar) = \frac{\hbar m_d}{2} \left\{ \frac{2}{\Gamma(-\frac{1}{2})} + \lim_{s \to -\frac{1}{2}} \left(\frac{2\mu^2}{5m_d^2}\right)^{s+\frac{1}{2}} \frac{c_1(K)}{\sqrt{4\pi}} \frac{\Gamma[s+\frac{1}{2};\frac{5}{2}]}{\Gamma(s)} + \sum_{n=2}^{\infty} \frac{c_n(K)}{\sqrt{4\pi}} \left(\frac{2}{5}\right)^{n-1} \frac{\Gamma[n-1;\frac{5}{2}]}{\Gamma(-\frac{1}{2})} \right\}$$

where $c_n(K) = a_n(K) - a_n(B)$. The deviation from the exact result is:

$$\text{error}_1 = \frac{\hbar m_d}{2} \left[-\frac{1}{\sqrt{4\pi}} \left(b_{n_0,K}(-\tfrac{1}{2}) - b_{n_0,B}(-\tfrac{1}{2}) \right) + B_{PK}(-\tfrac{1}{2}) - B_B(-\tfrac{1}{2}) \right] :$$

In order to implement the mass renormalization prescription, we assume that virtual mesons running on the loop of the tadpole graph have a mass of $\frac{m}{\sqrt{2}}$ half of the time and a mass of $\sqrt{2}m$ the other half-time on average. The normal-order is thus prescribed for annihilation and creation operators of mesons with $M = \sqrt{\frac{5}{2}}m$ mass; this amounts to considering

$$\delta m^2 = \frac{1}{2m_d L} \zeta_{B_0}(\tfrac{1}{2})$$

as the infinite quantity associated with the single divergent graph of the system. Zeta function regularization plus the asymptotic expansion tell us that the induced counter-term adds

$$\begin{aligned} \Delta_2 \varepsilon^K(\hbar) &= \langle K | H(\delta m^2) | K \rangle - \langle B | H(\delta m^2) | B \rangle \\ &= -\frac{\hbar m_d}{2\sqrt{4\pi}} c_1(K) \lim_{s \to -\frac{1}{2}} \left(\frac{2\mu^2}{5m_d^2}\right)^{s-\frac{1}{2}} \frac{\Gamma[s+\frac{1}{2};\frac{5}{2}]}{\Gamma(s)} \end{aligned}$$

to the one-loop correction to the link kink mass, whereas the error is

$$\text{error}_2 = \lim_{L\to\infty} \frac{\hbar}{4L} c_1(K) B_{B_0}(\tfrac{1}{2}) \, .$$

The sum of the contributions coming from the $s \to -\frac{1}{2}$ poles of $\zeta_1^{K}(s)$ and $\zeta_2^{K}(s)$ vanishes:

$$-\frac{\hbar m_d c_1(K)}{2\sqrt{4\pi}} \lim_{s\to -\frac{1}{2}} \left(\frac{2\mu^2}{5m_d^2}\right)^{s+\frac{1}{2}} \left[\frac{\gamma[s+\frac{1}{2},\frac{5}{2}]}{\Gamma(s)} - \frac{\gamma[s+\frac{1}{2},\frac{5}{2}]}{\Gamma(s)}\right] = 0 \, .$$

The choice of $M = \sqrt{\frac{5}{2}}m$ as a mass renormalization parameter leads to exactly the same result that we encountered in the more conventional systems with loop kinks and we end with the answer:

$$M_K = \hbar m\left[\beta_0 + D_{n_0}(\beta)\right]$$

where $\beta_0 = -\frac{1}{2\sqrt{2\pi}}$ and

$$D_{n_0}(\beta) = \sum_{n=2}^{n_0-1} c_n(K) d_n = -\sum_{n=2}^{n_0-1} c_n(K) \left(\frac{2}{5}\right)^{n-1} \frac{\gamma[n-1,\frac{5}{2}]}{8\sqrt{2\pi}} \, .$$

The coefficients and the partial sums up to $n_0 = 11$ for $\beta = 1$ are shown in the following Table

n	$c_n(K_1)$	$n_0 - 1$	$D_{n_0}(1)$
2	-9.3750	2	0.0968454
3	10.9375	3	0.0617547
4	-10.2567	4	0.0786049
5	7.89397	5	0.0703349
6	-5.12392	6	0.0741904
7	2.86874	7	0.0725233
8	-1.40987	8	0.0731872
9	0.61636	9	0.0729439
10	-0.24186	10	0.0730259

We find:

$$M_{K_1} = \hbar m\left[\beta_0 + D_{11}(1)\right] = -0.199471\hbar m + 0.0730259\hbar m = -0.126445\hbar m$$

as the approximation to the kink Casimir energy measured with respect to the Casimir energy of the $\psi_{B_1}(x)$ background field configuration.

The choice of $\beta = 1$ is optimum in the sense that for smaller values of β a tendency of the quantum correction towards $-\infty$ is observed whereas for β greater than 1 the tendency is toward $+\infty$. In Figure 4, $\beta = 1$ is identified as the inflexion point of a family that interpolates between two background configurations with bad features: too abrupt if $\beta = \infty$ and too smooth if $\beta = 0$.

We end this Section by comparing our renormalization criterion with the prescription used in [22]. Lohe and O'Brien choose a mass renormalization parameter M^0 in such a way that the mass counter-term exactly cancels the difference in vacuum Casimir energies between different points in the vacuum moduli space.

$$\frac{\hbar m}{2\sqrt{2}}\left[\zeta_{\mathcal{V}_0}(-\tfrac{1}{2}) - \zeta_{\mathcal{V}}(-\tfrac{1}{2}) + \frac{3m}{\sqrt{2}} L\, \delta m'^2\right] = 0 \qquad (21)$$

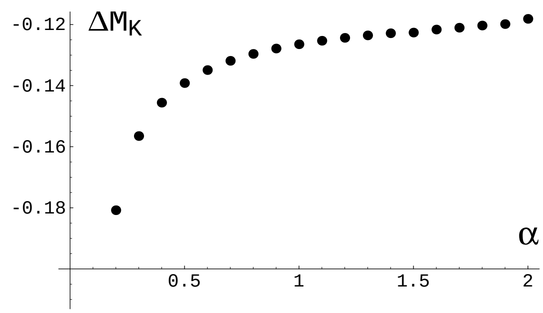

Figure 4: Quantum correction to the kink mass as a function of α in the $(0.4, 2.0)$ interval

The contribution of the tadpole graph must be considered for mesons with a suitable mass to satisfy (21):

$$m'^2 = \frac{1}{\sqrt{2mL}}\,\zeta_V\left(\tfrac{1}{2}\right) \quad ; \quad V = -\frac{d^2}{dx^2} + M'^2$$

and we find $M'^2 = 2.33$, $M' = M'\frac{m}{\sqrt{2}}$, a very close value to M. At the $L \to \infty$ limit

$$\zeta_{B_0}\left(\tfrac{1}{2}\right) - \zeta_V\left(\tfrac{1}{2}\right) = \frac{1}{4}\log\frac{2.33}{2.50}$$

If we had used M' as the mass renormalization parameter, the result would differ by

$$\Delta_{K_1}M' - \Delta_{K_1}M = \frac{\hbar m}{4\sqrt{2}}\frac{c_1(K_1)}{(4\pi)^{\frac{3}{2}}}\log\frac{2.33}{2.50}$$

which is a very small quantity indeed.

5 Outlook

The natural continuation of this work, and the main motivation to develop the asymptotic method, is the computation of quantum kink masses in theories with N-component scalar fields. Nevertheless, explorations in the supersymmetric world along these lines are also interesting.

All the models that we have described admit a supersymmetric extension because the potential energy density always can be written as $U(\phi) = \frac{1}{2}\frac{dW}{d\phi}\frac{dW}{d\phi}$. In non-dimensional variables the superpotential $W(\phi)$ for each model is:

$$W(\phi) = -4\cos\frac{\phi}{2} \quad ; \quad W(\phi) = \left(\phi - \frac{\phi^3}{3}\right)$$

$$W(\phi) = 4\frac{1}{\sqrt{2}}\left(\frac{1}{4}\sinh 2\phi - \frac{3}{2}\phi\right) \quad ; \quad W(\phi) = \frac{\phi^2}{2}\left(\frac{\phi^2}{2} - 1\right):$$

The supersymmetric extension includes also a Majorana spinor field:

$$\psi(x^\mu) = \begin{pmatrix} \psi_1(x^\mu) \\ \psi_2(x^\mu) \end{pmatrix} \quad ; \quad \psi_\alpha^* = \psi_\alpha \quad ; \quad \alpha = 1, 2:$$

Choosing the Majorana representation $\gamma^0 = \sigma^2$; $\gamma^1 = i\sigma^1$; $\gamma^5 = \sigma^3$ of the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and defining the Majorana adjoint $\bar{\psi} = \psi^t\gamma^0$, the action of the supersymmetric model is:

$$S = \frac{1}{2c_d^2}\int dx^2 \left[\partial_\mu\phi\,\partial^\mu\phi + i\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{dW}{d\phi}\frac{dW}{d\phi} - \frac{d^2W}{d\phi^2}\bar{\psi}\psi\right]:$$

The $N=1$ supersymmetry transformation is generated on the space of classical configurations by the Hamiltonian spinor function

$$Q = \int dx \left\{ \gamma^0 \partial\!\!\!/\phi\,\psi + i\gamma^0\psi\frac{dW}{d\phi}\right\}:$$

The components of the Majorana spinorial charge Q close the supersymmetry algebra

$$\{Q,\bar{Q}\} = 2(\gamma^\mu\gamma^0)P_\mu - 2\gamma^1 T: \qquad (22)$$

Their (anti)-Poisson bracket is given in (22) in terms of the momentum P_μ and the topological central charge $T = |\int dW|$.

The chiral projections $Q_\pm = \frac{1\pm\gamma^5}{2}Q$ and $\psi_\pm = \frac{1\pm\gamma^5}{2}\psi$ provide a very special combination of the supersymmetric charges:

$$Q_+ + Q_- = \int dx \left\{(\psi_+ - \psi_-)\frac{d\phi}{dx} - (\psi_- + \psi_+)\frac{dW}{d\phi}\right\}:$$

$Q_+ + Q_-$ is zero for the classical configurations that satisfy $\frac{d\phi}{dx} = \frac{dW}{d\phi}$ and $\psi = 0$ which are thus classical BPS states. One immediately notices that our kinks are such BPS states and besides the small bosonic fluctuations one must take into account the small fermionic fluctuations around the kink for computing the quantum correction to the kink mass in the extended system. The fermionic fluctuations around the kink configuration lead to other solutions of the field equations if the Dirac equation

$$\left[i\gamma^\mu\partial_\mu + \frac{d^2W}{d\phi^2}(\phi_K)\right]\psi_F(x;t) = 0$$

is satisfied. We multiply this equation for the adjoint of the Dirac operator

$$\left[-i\gamma^\mu\partial_\mu + \frac{d^2W}{d\phi^2}(\phi_K)\right]\left[i\gamma^\nu\partial_\nu + \frac{d^2W}{d\phi^2}(\phi_K)\right]\psi_F(x;t) = 0$$

and, due to the time-independence of the kink background, look for solutions of the form: $\psi_F(x;t) = f_F(x;\omega)e^{i\omega t}$. This is tantamount to solving the spectral problem

$$\left[-\frac{d^2}{dx^2} + \frac{d^2W}{d\phi^2}(\phi_K)\frac{d^2W}{d\phi^2}(\phi_K) - i\gamma^1\frac{dW}{d\phi}(\phi_K)\frac{d^3W}{d\phi^3}(\phi_K)\right] f_F(x;\omega) = \omega^2 f_F(x;\omega):$$

Projecting onto the eigen-spinors of $i\gamma^1$,

$$f_F^{(\pm)}(x;\omega) = \frac{1\pm i\gamma^1}{2}f_F(x;\omega) = \frac{1}{2}\begin{pmatrix} f_F^+(x;\omega) \mp f_F^-(x;\omega) \\ f_F^+(x;\omega) \pm f_F^-(x;\omega)\end{pmatrix}$$

we end with the spectral problem:

$$\left[-\frac{d^2}{dx^2} + \frac{d^2W}{d\phi^2}(\phi_K)\frac{d^2W}{d\phi^2}(\phi_K) \mp \frac{dW}{d\phi}(\phi_K)\frac{d^3W}{d\phi^3}(\phi_K)\right] f_F^{(\pm)}(x;\omega) = K^\pm f_F^{(\pm)}(x;\omega) = \omega^2 f_F^{(\pm)}(x;\omega)$$

for the same Schrodinger operator as that governing the bosonic fluctuations.

Therefore, generalized zeta function methods can also be used in supersymmetric models for computing the quantum corrections to the mass of BPS kinks. Great care however, is needed in choosing the boundary conditions on the fermionic fluctuations without spoiling supersymmetry. We look forward to extend this research in this direction.

Acknowledgments

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Appendix

In this Appendix we describe the iterative procedure that gives the coefficients $a_n(x;x)$ used in the text. For alternative descriptions, see [23], [24]. For an interesting interpretation of these coefficients as invariants of the Korteweg-de Vries equation, see [26].

Starting from formula (8) in the text, we write the recurrence relation

$$(n+1)a_{n+1}(x;y) + (x-y)\frac{\partial a_{n+1}(x;y)}{\partial x} - V(x)a_n(x;y) = \frac{\partial^2 a_n(x;y)}{\partial x^2}: \qquad (23)$$

In order to take the limit $y \to x$ properly, we introduce the notation

$$^{(k)}A_n(x) = \lim_{y\to x}\frac{\partial^k a_n(x;y)}{\partial x^k}$$

and, after differentiating (23) k times, we find

$$^{(k)}A_n(x) = \frac{1}{n+k}\left[\,^{(k+2)}A_{n-1}(x) + \sum_{j=0}^{k}\binom{k}{j}\frac{\partial^j V(x)}{\partial x^j}\,^{(k-j)}A_{n-1}(x)\right]:$$

from this equation and $^{(k)}A_0(x) = \lim_{y\to x}\frac{\partial^k a_0}{\partial x^k} = \delta_{k0}$, all the $^{(k)}A_n(x)$ can be generated recursively. Returning to (23), we finally obtain a well-defined recurrence relation

$$a_{n+1}(x;x) = \frac{1}{n+1}\left[\,^{(2)}A_n(x) + V(x)a_n(x;x)\right]$$

suitable for our purposes.

We give the explicit expressions of the first eight $a_n(x;x)$ coefficients. The abbreviated notation is $u_k = \frac{d^kV}{dx^k}(x)$, $u_k^n = \left(\frac{d^kV}{dx^k}(x)\right)^n$:

$$a_1(x;x) = u_0$$

$$a_2(x;x) = \frac{1}{2}u_0^2 + \frac{1}{6}u_2$$

$$a_3(x;x) = \frac{1}{6}u_0^3 + \frac{1}{6}u_2u_0 + \frac{1}{12}u_1^2 + \frac{1}{60}u_4$$

$$a_4(x;x) = \frac{1}{24}u_0^4 + \frac{1}{12}u_2u_0^2 + \frac{1}{12}u_1^2u_0 + \frac{1}{60}u_4u_0 + \frac{1}{40}u_2^2 + \frac{1}{30}u_1u_3 + \frac{1}{840}u_6$$

$$a_5(x;x) = \frac{1}{120}u_0^5 + \frac{1}{36}u_2u_0^3 + \frac{1}{24}u_1^2u_0^2 + \frac{1}{120}u_4u_0^2 + \frac{1}{40}u_2^2u_0 + \frac{1}{30}u_1u_3u_0 + \frac{1}{840}u_6u_0 + \frac{11}{360}u_1^2u_2$$

$$
\begin{aligned}
& + \frac{23}{5040}u_3^2 + \frac{19}{2520}u_2u_4 + \frac{1}{280}u_1u_5 + \frac{1}{15120}u_8
\end{aligned}
$$

$$
\begin{aligned}
a_6(x;x) = {} & \frac{1}{720}u_0^6 + \frac{1}{144}u_2u_0^4 + \frac{1}{72}u_1^2u_0^3 + \frac{1}{360}u_4u_0^3 + \frac{1}{80}u_2^2u_0^2 + \frac{1}{60}u_1u_3u_0^2 + \frac{11}{360}u_1^2u_2u_0 + \frac{1}{280}u_1u_5u_0 \\
& + \frac{1}{288}u_1^4 + \frac{1}{15120}u_8u_0 + \frac{61}{15120}u_2^3 + \frac{43}{2520}u_1u_2u_3 + \frac{23}{5040}u_0u_3^2 + \frac{5}{1008}u_1^2u_4 + \frac{19}{2520}u_0u_2u_4 \\
& + \frac{23}{30240}u_4^2 + \frac{19}{15120}u_3u_5 + \frac{1}{1680}u_0^2u_6 + \frac{11}{15120}u_2u_6 + \frac{1}{3780}u_1u_7 + \frac{1}{332640}u_{10}
\end{aligned}
$$

$$
\begin{aligned}
a_7(x;x) = {} & \frac{1}{5040}u_0^7 + \frac{1}{720}u_2u_0^5 + \frac{1}{288}u_1^2u_0^4 + \frac{1}{240}u_2^2u_0^3 + \frac{1}{180}u_1u_3u_0^3 + \frac{11}{720}u_1^2u_2u_0^2 + \frac{1}{560}u_1u_5u_0^2 \\
& + \frac{1}{288}u_1^4u_0 + \frac{61}{15120}u_2^3u_0 + \frac{43}{2520}u_1u_2u_3u_0 + \frac{5}{1008}u_1^2u_4u_0 + \frac{1}{332640}u_{10}u_0 + \frac{23}{10080}u_3^2u_0^2 \\
& + \frac{19}{5040}u_2u_4u_0^2 + \frac{1}{5040}u_6u_0^3 + \frac{83}{10080}u_1^2u_2^2 + \frac{1}{252}u_1^3u_3 + \frac{31}{10080}u_2u_3^2 + \frac{1}{280}u_1u_3u_4 + \frac{1}{1440}u_0^4u_4 \\
& + \frac{5}{2016}u_2^2u_4 + \frac{23}{30240}u_0u_4^2 + \frac{1}{420}u_1u_2u_5 + \frac{19}{15120}u_0u_3u_5 + \frac{71}{665280}u_5^2 + \frac{1}{2016}u_1^2u_6 \\
& + \frac{11}{15120}u_0u_2u_6 + \frac{61}{332640}u_4u_6 + \frac{1}{3780}u_0u_1u_7 + \frac{19}{166320}u_3u_7 + \frac{1}{30240}u_0^2u_8 + \frac{17}{332640}u_2u_8 \\
& + \frac{1}{66528}u_1u_9 + \frac{1}{8648640}u_{12}
\end{aligned}
$$

$$
\begin{aligned}
a_8(x;x) = {} & \frac{1}{40320}u_0^8 + \frac{1}{960}u_2^2u_0^4 + \frac{1}{720}u_1u_3u_0^4 + \frac{1}{576}u_1^4u_0^2 + \frac{1}{252}u_1^3u_3u_0 + \frac{1}{280}u_1u_3u_4u_0 + \frac{1}{420}u_1u_2u_5u_0 \\
& + \frac{31}{10080}u_2u_3^2u_0 + \frac{5}{2016}u_2^2u_4u_0 + \frac{1}{2016}u_1^2u_6u_0 + \frac{1}{8648640}u_{12}u_0 + \frac{23}{60480}u_4^2u_0^2 + \frac{19}{30240}u_3u_5u_0^2 \\
& + \frac{11}{30240}u_2u_6u_0^2 + \frac{1}{7560}u_1u_7u_0^2 + \frac{11}{2160}u_1^2u_2u_0^3 + \frac{1}{90720}u_8u_0^3 + \frac{1}{7200}u_4u_0^5 + \frac{1}{1440}u_0^5u_1^2 \\
& + \frac{1}{4320}u_0^6u_2 + \frac{17}{8640}u_1^4u_2 + \frac{83}{10080}u_0u_1^2u_2^2 + \frac{61}{30240}u_0^2u_2^3 + \frac{1261}{1814400}u_2^4 + \frac{43}{5040}u_0^2u_1u_2u_3 \\
& + \frac{227}{37800}u_1u_2^2u_3 + \frac{23}{30240}u_0^3u_3^2 + \frac{659}{302400}u_1^2u_3^2 + \frac{5}{2016}u_0^2u_1^2u_4 + \frac{19}{15120}u_0^3u_2u_4 + \frac{527}{151200}u_1^2u_2u_4 \\
& + \frac{7939}{9979200}u_3^2u_4 + \frac{6353}{9979200}u_2u_4^2 + \frac{1}{1680}u_0^3u_1u_5 + \frac{17}{30240}u_1^3u_5 + \frac{13}{12320}u_2u_3u_5 + \frac{3067}{4989300}u_1u_4u_5 \\
& + \frac{71}{665280}u_0u_5^2 + \frac{1}{20160}u_0^4u_6 + \frac{3001}{9979200}u_2^2u_6 + \frac{13}{29700}u_1u_3u_6 + \frac{61}{332640}u_0u_4u_6 \\
& + \frac{3433}{259459200}u_6^2 + \frac{109}{498960}u_1u_2u_7 + \frac{19}{166320}u_0u_3u_7 + \frac{1501}{64864800}u_5u_7 + \frac{71}{1995840}u_1^2u_8 \\
& + \frac{17}{332640}u_0u_2u_8 + \frac{2003}{129729600}u_4u_8 + \frac{1}{66528}u_0u_1u_9 + \frac{5}{648648}u_3u_9 + \frac{1}{665280}u_0^2u_{10} \\
& + \frac{73}{25945920}u_2u_{10} + \frac{1}{1441440}u_1u_{11} + \frac{1}{259459200}u_{14}
\end{aligned}
$$

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