Some results on the eigenfunctions of the quantum trigonometric Calogero-Sutherland model related to the Lie algebra E_6

J. Fernandez Nuñez[†], W. García Fuertes[†], A.M. Perelomov[‡]*
[†] Dep. de Física, Facultad de Ciencias, Universidad de Oviedo, E-33007 Oviedo, Spain
[‡] Dep. de Física Teórica, Facultad de Ciencias, Universidad de Zaragoza, E-50009 Zaragoza, Spain

Abstract

The quantum trigonometric Calogero-Sutherland models related to Lie algebras admit a parametrization in which the dynamical variables are the characters of the fundamental representations of the algebra. We develop here this approach for the case of the exceptional Lie algebra E_6.

PACS numbers: 02.30.Ik, 03.65.Ge, 02.20.Sv

I Introduction

The so-called Calogero-Sutherland or Calogero-Moser models were introduced by Calogero [1], who studied, from the quantum standpoint, the dynamics on the infinite line of a set of pairwise interacting particles through rational plus quadratic potentials, and found that the problem was exactly solvable. Soon afterwards, Sutherland [2] arrived to similar results for the quantum problem on the circle, this time with trigonometric interaction, and Moser [3] showed that the classical version of both models enjoyed integrability in the Liouville sense. The identification of the general scope of these discoveries came with the works of Olshanetsky and Perelomov [4]-[6], who realized that it was possible to associate models of this kind to all the root systems of the simple Lie algebras, and that all these models were integrable, both in the classical and in the quantum framework [7, 8]. Nowadays, there is a widespread interest in this type of integrable systems, and many mathematical and physical applications for them have been found, see for instance [9].

The Calogero-Sutherland Hamiltonian associated to the root system of a simple Lie algebra L can be written as a second-order differential operator whose variables are the characters of the fundamental representations of the algebra. As it was shown in the papers [10, 11, 12], and later in [13, 14, 15, 16, 17, 18], this approach gives the possibility of developping some systematic procedures to solve the Schrödinger equation and determine important properties of the eigenfunctions, such as recurrence relations or generating functions for some subsets of them. For the moment, the approach has been used only for classical algebras of A_n and D_n type, and recently [18] for the exceptional algebra E_6 for a special value of the coupling constant for which the eigenfunctions are proportional to the characters of the irreducible representations of the algebra. The aim of this paper is to show how to generalize the treatment of [18] to arbitrary values of the coupling constant and to extend some of the particular results found there to the general case.

* On leave of absence from the Institute for Theoretical and Experimental Physics, 117259, Moscow, Russia. Current E-mail address: perelomo@dftuz.unizar.es

II The Calogero–Sutherland model for E_6 in z–variables

The Hamiltonian operator for the trigonometric Calogero–Sutherland model related to the root system of a simple Lie algebra has the generic form

$$H = \frac{1}{2}(p,p) + \sum_{\alpha\in R^+} \kappa_\alpha(\kappa_\alpha - 1)\sin^{-2}(\alpha,q),$$

where R^+ is the set of positive roots, q and p are vectors of dimension r = rank of the algebra, $(\ ,\ )$ is the usual euclidean scalar product in $\mathbb{R}^r$, and the coupling constants κ_α are such that $\kappa_\alpha = \kappa_\beta$ if $||\alpha|| = ||\beta||$. In particular, because E_6 is simply–laced (for all details about the structure of E_6 needed to follow the main text, see Appendix A), the Calogero–Sutherland model associated to E_6 depends only on one coupling constant κ. To write H in a more explicit way, it is convenient to use the orthonormal basis $\{e_i;\ i = 1,\ldots,6\}$ which is related to the generating system ϵ of the Appendix A through $e_i = \epsilon_i - \frac{1}{6}\sum_{j=1}^{6}\epsilon_j$. The expression of q and p in this basis is simply $q = \sum_{i=1}^{6} q_i e_i$, $p = \sum_{i=1}^{6} p_i e_i$, while the simple roots are given by:

$$\begin{aligned}
\alpha_1 &= e_1 - e_2\\
\alpha_2 &= -\frac{1}{2}\left(1+\frac{\sqrt{3}}{3}\right)\sum_{j=1}^{3} e_j + \frac{1}{2}\left(-1+\frac{\sqrt{3}}{3}\right)\sum_{j=4}^{6} e_j\\
\alpha_k &= e_{k-1} - e_k, \qquad k = 3,4,5,6.
\end{aligned}$$

The q coordinates are assumed to take values in the interval $[0,\pi]$, and therefore the Hamiltonian can be interpreted as describing the dynamics of a system of six particles moving on the circle, but notice that there is not translational invariance. We recapitulate some important facts about this model which follow from the general structure of the quantum Calogero–Sutherland models related to Lie algebras [8]. The ground state energy and (non-normalized) wave function are

$$\begin{aligned}
E_0(\kappa) &= 2\kappa^2(\rho,\rho) = 156\kappa^2\\
\Psi_0(q) &= \prod_{\alpha\in R^+}\sin^{\kappa}(\alpha,q),
\end{aligned}$$

with ρ being the Weyl vector, while the excited states depend on a six-tuple of quantum numbers $m = (m_1,m_2,m_3,m_4,m_5,m_6)$, and satisfy the Schrödinger equation

$$\begin{aligned}
H\Psi_m &= E_m(\kappa)\Psi_m\\
E_m(\kappa) &= 2(\lambda+\kappa\rho,\lambda+\kappa\rho),
\end{aligned} \tag{1}$$

where λ is the highest weight of the irreducible representation of E_6 labelled by m, i.e. $\lambda = \sum_{i=1}^{6} m_i\lambda_i$. By substitution in (1) of

$$\Psi_m(q) = \Psi_0(q)\Phi_m(q), \tag{2}$$

we are led to the eigenvalue problem

$$\Delta\Phi_m = \varepsilon_m(\kappa)\Phi_m \tag{3}$$

with

$$\Delta = \frac{1}{2}\Delta_q + \kappa\sum_{\alpha\in R^+}\operatorname{ctg}(\alpha,q)(\alpha,\nabla_q), \tag{4}$$

and

$$\varepsilon_m(\kappa) = E_m(\kappa) - E_0(\kappa) = 2(\lambda;\lambda + 2\kappa\rho): \tag{5}$$

Taking into account that $A^{-1}_{jk} = (\lambda_j;\lambda_k)$, it is possible to give a more explicit expression for $\varepsilon_m(\kappa)$:

$$\varepsilon_m(\kappa) = 2\sum_{j,k=1}^{6} A^{-1}_{jk} m_j m_k + 4\kappa \sum_{j,k=1}^{6} A^{-1}_{jk} m_j: \tag{6}$$

Now the main problem is to solve (3). As it has been shown for other algebras [10, 11, 12, 16], the best way to do this is to use a set of independent variables which are invariant under the Weyl symmetry of the Hamiltonian, namely the characters $z_k, k = 1,\dots,6$, of the six fundamental representations of the Lie algebra E_6. We can infer from (4) the structure of Δ when written in the z-variables:

$$\Delta = \sum_{j,k=1}^{6} a_{jk}(z)\partial_{z_j}\partial_{z_k} + \sum_{j=1}^{6}\left[b_j^{(0)}(z) + \kappa b_j^{(1)}(z)\right]\partial_{z_j}: \tag{7}$$

As a matter of fact, the eigenfunctions of $\Delta^{(0)}$ and $\Delta^{(1)}$ are (proportional to) the monomial symmetric functions $M_\lambda = \sum_{s\in W} \exp[i(s\lambda;q)]$ (W is the Weyl group) and the characters χ_λ of the irreducible representations of the algebra E_6, respectively [8]. Thus, knowing the characters $z_i = \chi_{\lambda_i}$ of the fundamental representations and the products $z_i z_j$ through the Clebsch-Gordan series for the algebra, we are able to find the Hamiltonian $\Delta^{(1)}$, that is, we obtain the coefficients $a_{jk}(z)$ entering in the expression of all the Hamiltonians Δ and also the coefficients $b_j^{(0)}(z) + \kappa b_j^{(1)}(z)$. In the previous paper [18] we computed the needed Clebsch-Gordan series and showed these coefficients.

On the other hand, knowing enough monomial symmetric functions in terms of the fundamental characters, $M_\lambda(z)$, we can complete the form of Δ, for we know that

$$\Delta^{(0)} M_\lambda = \varepsilon_m(0) M_\lambda = 2(\lambda;\lambda) M_\lambda; \tag{8}$$

a system of linear equations which can be solved for the coefficients $b_j^{(0)}(z)$, $j = 1,\dots,6$. To this end, remind that the characters can be expanded as sums of monomial functions ([19]),

$$\chi_\lambda = M_\lambda + a_1 M_{\lambda_1} + a_2 M_{\lambda_2} + \cdots;$$

where the set of λ_k entering in the expansion is easy to determine: they are the dominant weights such that $\lambda_i = \lambda - \sum_{j=1}^{6} n_j \alpha_j$ with $n_j \geq 0$ and $(\lambda_i;\alpha_k) \geq 0$ for the six simple roots α_k. The coefficients a_k, on the other hand, represent the multiplicities of the weights λ_i in the representation with highest weight λ. Here it will suffice to deal with the following expansions:

$$\begin{aligned}
\chi^{(27)}_{100000} &= z_1 = M^{(27)}_{100000};\\
\chi^{(78)}_{010000} &= z_2 = M^{(72)}_{010000} + aM^{(1)}_{000000};\\
\chi^{(351)}_{001000} &= z_3 = M^{(216)}_{001000} + bM^{(27)}_{000001};\\
\chi^{(2925)}_{000100} &= z_4 = M^{(720)}_{000100} + cM^{(270)}_{100001} + dM^{(72)}_{010000} + eM^{(1)}_{000000};\\
\chi^{(351)}_{000010} &= z_5 = M^{(216)}_{000010} + bM^{(27)}_{100000};\\
\chi^{(27)}_{000001} &= z_6 = M^{(27)}_{000001};\\
\chi^{(650)}_{100001} &= z_1 z_6 - z_2 - 1 = M^{(270)}_{100001} + fM^{(72)}_{010000} + gM^{(1)}_{000000};
\end{aligned} \tag{9}$$

where the form of χ_{100001} comes from the list in [18] and the numbers appearing in parentheses as superscripts are either the dimensions of the representations or the dimensions of the linear spaces generated by the orbits of the Weyl group corresponding to the monomial functions. The former can be computed from the Weyl dimension formula, while the latter follow easily from the fact that the Weyl group of the subalgebra of E_6 obtained by removing from the Dynkin diagram the dots corresponding to the weight defining the monomial function acts trivially on such a weight: for instance, removing the dot associated to λ_1 we obtain the Dynkin diagram of D_5, and hence

$$\dim M_{100000} = \frac{|W_{E_6}|}{|W_{D_5}|} = \frac{2^7 \cdot 3^4 \cdot 5}{\frac{1}{2} 10!!} = 27;$$

and so on.

While the dimensions shown in (9) suffice for fixing $a = 6$, $b = 5$, for the computation of remaining multiplicities we need to use the Freudenthal formula [20]

$$n_\mu = \frac{\sum_{\alpha>0} \sum_{k=1}^{\infty} 2 n_{\mu+k\alpha} (\mu + k\alpha, \alpha)}{(\Lambda + \mu + 2\rho, \Lambda - \mu)}: \qquad (10)$$

Here n_μ stands for the multiplicity of the weight μ in the representation of highest weight Λ, the first sum extends over positive roots, and ρ is the Weyl vector. The application of the Freudenthal formula is quite easy for the representations at stake. Let us see, for instance, how to compute c in (9). In this case $\Lambda = \lambda_4$ and $c = n_\mu = n_{\lambda_1+\lambda_6}$. The scalar product of the vector $\nu = \Lambda - \mu = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$ with μ is $(\mu, \nu) = 0$ and, due to the fact that the length of $\mu + \alpha$, with α a positive root, is $|\mu + \alpha| = (|\mu|^2 + 2 + 2(\mu, \alpha))^{\frac{1}{2}}$, the only roots entering in (10) are the positive roots α such that $(\mu, \alpha) = 0$, because otherwise $|\mu + \alpha| > |\mu + \nu| = |\Lambda|$ and $\mu + \alpha$ would lie outside of the weight diagram for the representation R_Λ. Looking at the table of positive roots in Appendix A, we check that there are 12 of them with $(\mu, \alpha) = 0$. For all of these, $\mu + \alpha$ lies on the orbit of Λ, and thus $n_{\mu+\alpha} = 1$. This gives

$$c = n_{\lambda_1+\lambda_6} = 12 \frac{2(\mu, \alpha) + 2(\alpha, \alpha)}{(\lambda_4 + \lambda_1 + \lambda_6 + 2\sum_{i=1}^{6} \lambda_i; \alpha_1 + \alpha_3 + 2\alpha_4 + \alpha_5)} = 12 \frac{2 \cdot 0 + 2 \cdot 2}{2 + 2 \cdot 5} = 4:$$

To compute d we proceed much in the same way. Now $\mu = \lambda_2$ and $\nu = \Lambda - \mu = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$. It follows that $(\mu, \alpha) \leq 1$, and thus only positive roots with $(\mu, \alpha) = 0$ or $(\mu, \alpha) = 1$ enter in the Freudenthal formula. There are 20 positive roots with $(\mu, \alpha) = 1$, and for them $n_{\mu+\alpha} = 1$ because they are in the orbit of Λ. The number of positive roots with $(\mu, \alpha) = 0$ is 15, and for them $\mu + \alpha$ lies in the orbit of $\lambda_1 + \lambda_6$, so that their multiplicities are $n_{\mu+\alpha} = 4$. This gives $d = n_{\lambda_2} = 15$. Once we know c and d, we compute e by balancing dimensions in (9), and obtain $e = 45$. A similar use of the Freudenthal formula gives $f = 5$, and therefore $g = 20$.

With all the coefficients in (9) being fixed, we can now solve for the monomial functions corresponding to the fundamental weights. We find

$$\begin{aligned}
M_{100000} &= z_1; \\
M_{010000} &= z_2 - 6; \\
M_{001000} &= z_3 - 5z_6; \\
M_{000100} &= z_4 - 4z_1 z_6 + 9z_2 + 9; \\
M_{000010} &= z_5 - 5z_1; \\
M_{000001} &= z_6:
\end{aligned}$$

The remaining step is to substitute these monomials in (8) and to solve the linear system for the coefficients $b_j^{(0)}(z)$; the outcome is

$$b_1^{(0)} = \tfrac{8}{3} z_1; \qquad b_2^{(0)} = 4z_2 - 6; \qquad b_3^{(0)} = \tfrac{20}{3} z_3 - 20z_6;$$
$$b_4^{(0)} = 12z_4 - 16z_1 z_6 - 24z_2 + 36; \qquad b_5^{(0)} = \tfrac{20}{3} z_5 - 20z_1; \qquad b_6^{(0)} = z_6:$$

W ith this and the form of $^{(1)}$ given in [18], we can now write the full set of coe cients in (7):

$$a_{11}(z) = \frac{8}{3}z_1^2 \quad 4z_3 \quad 20z_6;$$
$$a_{12}(z) = 2z_1z_2 \quad 26z_1 \quad 10z_5;$$
$$a_{13}(z) = \frac{10}{3}z_1z_3 + 18 \quad 12z_2 \quad 6z_4 \quad 18z_1z_6;$$
$$a_{14}(z) = 4z_1z_4 + 18z_1 \quad 10z_1z_2 \quad 18z_5 \quad 8z_2z_5 \quad 8z_3z_6 + 8z_6^2;$$
$$a_{15}(z) = \frac{8}{3}z_1z_5 \quad 10z_3 \quad 26z_6 \quad 10z_2z_6;$$
$$a_{16}(z) = \frac{4}{3}z_1z_6 \quad 36 \quad 12z_2;$$
$$a_{22}(z) = 2z_2^2 \quad 18 \quad 6z_2 \quad 2z_4 \quad 8z_1z_6;$$
$$a_{23}(z) = 4z_2z_3 \quad 24z_1^2 + 14z_3 \quad 8z_1z_5 \quad 2z_6 \quad 10z_2z_6;$$
$$a_{24}(z) = 6z_2z_4 \quad 18z_2 \quad 12z_2^2 \quad 10z_1z_3 + 24z_4 \quad 6z_3z_5 + 26z_1z_6 \quad 8z_1z_2z_6 \quad 10z_5z_6;$$
$$a_{25}(z) = 4z_2z_5 \quad 2z_1 \quad 10z_1z_2 + 14z_5 \quad 8z_3z_6 \quad 24z_6^2;$$
$$a_{26}(z) = 2z_2z_6 \quad 10z_3 \quad 26z_6;$$
$$a_{33}(z) = \frac{10}{3}z_3^2 + 14z_1 \quad 12z_1z_2 \quad 2z_1z_4 + 16z_5 \quad 4z_2z_5 \quad 8z_1^2z_6 + 4z_3z_6 \quad 6z_6^2;$$
$$a_{34}(z) = 8z_3z_4 + 10z_1^2 \quad 10z_1^2z_2 + 18z_3 \quad 2z_2z_3 \quad 6z_1z_2z_5 \quad 10z_5^2 \quad 18z_6 + 8z_2z_6 \quad 10z_2^2z_6$$
$$8z_1z_3z_6 + 20z_4z_6 + 8z_1z_6^2;$$
$$a_{35}(z) = \frac{16}{3}z_3z_5 \quad 36 + 24z_2 \quad 12z_2^2 \quad 10z_1z_3 + 24z_4 \quad 16z_1z_6 \quad 8z_1z_2z_6 \quad 10z_5z_6;$$
$$a_{36}(z) = \frac{8}{3}z_3z_6 \quad 26z_1 \quad 10z_1z_2 \quad 10z_5;$$
$$a_{44}(z) = 6z_4^2 \quad 4z_1^3 \quad 6z_2^3 + 18z_1z_3 \quad 6z_1z_2z_3 \quad 18z_4 + 18z_2z_4 + 8z_1^2z_5 \quad 18z_3z_5 \quad 2z_2z_3z_5 \quad 4z_1z_5^2$$
$$18z_1z_6 + 14z_1z_2z_6 \quad 4z_1z_2^2z_6 \quad 4z_3^2z_6 + 8z_1z_4z_6 + 18z_5z_6 \quad 6z_2z_5z_6 + 8z_3z_6^2 \quad 4z_6^3;$$
$$a_{45}(z) = 8z_4z_5 \quad 18z_1 + 8z_1z_2 \quad 10z_1z_2^2 \quad 10z_3^2 + 20z_1z_4 + 18z_5 \quad 2z_2z_5 + 8z_1^2z_6 \quad 6z_2z_3z_6$$
$$8z_1z_5z_6 + 10z_6^2 \quad 10z_2z_6^2;$$
$$a_{46}(z) = 4z_4z_6 + 8z_1^2 \quad 18z_3 \quad 8z_2z_3 \quad 8z_1z_5 + 18z_6 \quad 10z_2z_6;$$
$$a_{55}(z) = \frac{10}{3}z_5^2 \quad 6z_1^2 + 16z_3 \quad 4z_2z_3 + 4z_1z_5 + 14z_6 \quad 12z_2z_6 \quad 2z_4z_6 \quad 8z_1z_6^2;$$
$$a_{56}(z) = \frac{10}{3}z_5z_6 + 18 \quad 12z_2 \quad 6z_4 \quad 18z_1z_6;$$
$$a_{66}(z) = \frac{4}{3}z_6^2 \quad 10z_1 \quad 2z_5;$$
$$b_1(z) = b_1^{(0)}(z) + \quad b_1^{(1)}(z) = \quad 32 \quad + \frac{8}{3} \quad z_1;$$
$$b_2(z) = b_2^{(0)}(z) + \quad b_2^{(1)}(z) = (44 \quad + 4)z_2 + 24(\quad 1);$$
$$b_3(z) = b_3^{(0)}(z) + \quad b_3^{(1)}(z) = \quad 60 \quad + \frac{20}{3} \quad z_3 + 20(\quad 1)z_6;$$
$$b_4(z) = b_4^{(0)}(z) + \quad b_4^{(1)}(z) = (84 \quad + 12)z_4 + (\quad 1)(16z_1z_6 + 24z_2 \quad 36);$$
$$b_5(z) = b_5^{(0)}(z) + \quad b_5^{(1)}(z) = \quad 60 \quad + \frac{20}{3} \quad z_5 + 20(\quad 1)z_1;$$
$$b_6(z) = b_6^{(0)}(z) + \quad b_6^{(1)}(z) = \quad 32 \quad + \frac{8}{3} \quad z_6:$$

III Computation of polynomials and deformed Clebsch-Gordan series

The eigenfunctions $\Psi_m(q)$ are polynomials when expressed in z variables, $\Psi_m(q) = P_m(z)$. The Schrodinger equation can then be solved by applying a systematic procedure, which is suitable to be implemented in a computer program able to carry out symbolic calculations. We propose two alternative methods to find the Schrodinger eigenfunctions:

1. Given a weight $n_1\lambda_1 + n_2\lambda_2 + n_3\lambda_3 + n_4\lambda_4 + n_5\lambda_5 + n_6\lambda_6$, let us denote $z^n = z_1^{n_1} z_2^{n_2} z_3^{n_3} z_4^{n_4} z_5^{n_5} z_6^{n_6}$. Thus, Δ acting on z^n gives
$$\Delta z^n = \sum_{\mu \in \Phi} k_{\mu,n}(\;) z^{n-\mu} \; ; \qquad (11)$$
where Φ includes only integral linear combinations of the simple roots with non-negative coefficients and, of course, in the exponent of (11) we express μ in the basis of fundamental weights. In particular, $k_{0,n}(\;) = \varepsilon_n(\;)$. The polynomials $P_m(z)$ can be written as
$$P_m(z) = \sum_{\mu \in Q^+(m)} c_\mu(\;) z^{m-\mu} \; ; \quad c_0 = 1;$$
where again the μ in $Q^+(m)$ are integral linear combinations of the simple roots with non-negative coefficients such that they do not give rise to negative powers of the z's. By substituting in the Schrodinger equation we find the iterative formula
$$c_\mu(\;) = \frac{1}{\varepsilon_m(\;) - \varepsilon_{m-\mu}(\;)} \sum_{\nu \in \Phi, \nu \neq 0} k_{\nu, m-(\mu-\nu)}(\;) c_{\mu-\nu}(\;):$$
To use this formula in practice, one should take into account the heights of the μ's, because each coefficient c_μ can depend only on some of the c_ν such that $ht(\nu) < ht(\mu)$.

2. The product $z_1^{m_1} z_2^{m_2} z_3^{m_3} z_4^{m_4} z_5^{m_5} z_6^{m_6}$ can be expanded on the basis of the orthogonal Polynomials $P_m(z)$ as
$$z_1^{m_1} z_2^{m_2} z_3^{m_3} z_4^{m_4} z_5^{m_5} z_6^{m_6} = P_m(z) + \sum_{\mu \in S_m} n_\mu(\;) P_{m-\mu}(z):$$
In each particular case, it is not difficult to elaborate a list with all the elements in S_m (they are the same integral dominant weights which appear in the corresponding Clebsch-Gordan series, see [18]). Furthermore, the operator $\Delta - \varepsilon_n(\;)$ annihilates the character P_n. Taking this into account, we can obtain the eigenfunctions using the formula
$$P_m = \Big\{ \prod_{\mu \in S_m} (\Delta - \varepsilon_{m-\mu}(\;)) \Big\} z^m :$$

Through any of these methods, it is possible to compute the characters rather quickly. As an illustration, we offer a list of polynomials and monomial functions in Appendix B. For a similar list of characters, see [18].

Once we have a method for the computation of the polynomials, we can extend it to produce an algorithm for calculating deformed Clebsch-Gordan series for the product of them. Suppose that we want to obtain the series for $P_m \cdot P_n$. We list the possible dominant weights entering in the series arranged by heights
$$P_m \cdot P_n = P_{m+n} + n_{\mu_1}(\;) P_{\mu_1} + n_{\mu_2}(\;) P_{\mu_2} + \cdots$$
The coefficient $n_{\mu_1}(\;)$ is simply the difference between the coefficients of z^{μ_1} in $P_m \cdot P_n$ and in P_{m+n}. Then, $n_{\mu_2}(\;)$ is the difference between the coefficient of z^{μ_2} in $P_m \cdot P_n$ and the sum of the corresponding coefficients in P_{m+n} and P_{μ_1}, and so on. As an example, we present a list with all the cuadratic deformed Clebsch-Gordan series in Appendix C.

IV Some recurrence relations

The approach we are describing is also useful to nd the form of the recurrence relations for products $z_j P_m(z)$. Considered in full generality, these recurrence relations are extremely complicated, but for some special cases they can be written in explicit form. Let us consider, for instance, the recurrence relation for $z_1 P_{n\ 1}$ with arbitrary n. If we express the weights of the representation $R_{\ 1}$ (which are all the combinations $"_i\ \ "$, $"_i\ \ "_j$) in the basis of fundamental weights, we see that there are only three whose coe cients for $\ _i$, $i \neq 1$, are all non-negative, namely $\ _1$; $\ _1 + \ _3$ and $\ _1 + \ _6$. Hence, the form of the series should be

$$z_1 P_{n00000} = P_{(n+1)00000} + a_n(\ )P_{(n\ 1)01000} + b_n(\ )P_{(n\ 1)00001}\ ; \tag{12}$$

where we have to x $a_n(\ )$ and $b_n(\ )$. Now, solving the Schrodinger equation by means of the rst of the two methods described in Sect. 3, one nds

$$\begin{aligned}
P_{n00000} &= z_1^n + \frac{(1\ \ n)n}{n+\ \ \ 1} z_1^{n\ 2} z_3 + \frac{(1\ \ n)n\ (n+5\ \ \ 2)}{(n+\ \ \ 1)(n+\ \ \ 2)(n+4\ \ \ 1)} z_1^{n\ 2} z_6 + \ldots; \\
P_{(n\ 1)01000} &= z_1^{n\ 1} z_3 + \frac{10\ ^3\ \ 5(1+3n)\ ^2\ \ 2(\ 1\ \ 9n+5n^2)\ \ \ n(2\ \ 3n+n^2)}{(n+\ \ \ 2)(n+\ \ \ 1)(n+7\ )} z_1^{n\ 1} z_6 + \ldots; \\
P_{(n\ 1)00001} &= z_1^{n\ 1} z_6 + \ldots:
\end{aligned}$$

Substituting in (12), we can solve for $a_n(\ )$ and $b_n(\ )$ with the results

$$\begin{aligned}
a_n(\ ) &= \frac{n(n+2\ \ \ 1)}{(n+\ \ )(n+\ \ \ 1)}; \\
b_n(\ ) &= \frac{n(n+3\ )(n+5\ \ \ 1)(n+8\ \ \ 1)}{(n+\ \ \ 1)(n+4\ \ \ 1)(n+4\ )(n+7\ )}:
\end{aligned}$$

We list below the series of the form $z_1 P_{n\ k}$ obtained through the same procedure:

$$\begin{aligned}
z_1 P_{0n0000} &= P_{1n0000} + c_n(\ )P_{0(n\ 1)0010} + d_n(\ )P_{1(n\ 1)0000}\ ; \\
z_1 P_{00n000} &= P_{10n000} + e_n(\ )P_{00(n\ 1)100} + f_n(\ )P_{10(n\ 1)001} + g_n(\ )P_{10(n\ 1)001}\ ; \\
z_1 P_{000n00} &= P_{100n00} + h_n(\ )P_{010(n\ 1)10} + i_n(\ )P_{001(n\ 1)01} + j_n(\ )P_{110(n\ 1)00} + k_n(\ )P_{000(n\ 1)10}\ ; \\
z_1 P_{0000n0} &= P_{1000n0} + l_n(\ )P_{0100(n\ 1)1} + p_n(\ )P_{0010(n\ 1)0} + q_n(\ )P_{0000(n\ 1)1}\ ; \\
z_1 P_{00000n} &= P_{10000n} + r_n(\ )P_{01000(n\ 1)} + s_n(\ )P_{00000(n\ 1)}\ ;
\end{aligned}$$

where

$$\begin{aligned}
c_n(\ ) &= \frac{n(\ 1+n+5\ )}{(\ 1+n+\ \ )(n+4\ )}; \\
d_n(\ ) &= \frac{2n(n+2\ )(\ 1+n+6\ )(\ 1+n+8\ )(\ 1+2n+12\ )}{(\ 1+n+\ \ )(\ 1+n+3\ )(n+7\ )(\ 1+2n+11\ )(2n+11\ )}; \\
e_n(\ ) &= \frac{n(\ 1+n+3\ )}{(\ 1+n+\ \ )(n+2\ )}; \\
f_n(\ ) &= \frac{2n(n+2\ )(\ 1+n+4\ )(\ 1+n+6\ )(\ 1+2n+8\ )}{(\ 1+n+\ \ )(\ 1+n+3\ )(n+5\ )(\ 1+2n+7\ )(2n+7\ )}; \\
g_n(\ ) &= \frac{n(n+\ \ )(n+3\ )(\ 1+n+5\ )(\ 1+n+6\ )(\ 1+2n+11\ )}{(\ 1+n+\ \ )(\ 1+n+2\ )(n+4\ )(n+5\ )^2(\ 1+2n+7\ )};
\end{aligned}$$

$$h_n(\ ) = \frac{n(\ 1+n+4\ )}{(\ 1+n+\ \ )(n+3\ )};$$

$$i_n(\ ) = \frac{n(n+\ \ )(\ 1+n+3\ )(\ 1+n+4\ )(\ 1+2n+7\ )}{(\ 1+n+\ \ )(\ 1+n+2\ )(n+3\ )^2(\ 1+2n+5\ )};$$

$$j_n(\ ) = \frac{6n(n+\ \ )(n+2\ )(\ 1+n+4\ )(\ 1+n+5\ )(\ 1+2n+8\ )(\ 1+3n+11\ )}{(\ 1+n+\ \ )(\ 1+n+2\ )(n+4\ )(\ 1+2n+5\ )(2n+7\ )(\ 1+3n+10\ )(3n+10\ )};$$

$$k_n(\ ) = \frac{3n(n+\ \ )(n+2\ )(\ 1+n+4\ )(\ 1+n+5\ )(2n+5\ )(\ 1+2n+8\ )}{(\ 1+n+\ \ )(\ 1+n+2\ )(n+4\ )^2(\ 1+2n+5\ )(\ 1+2n+6\ )(2n+7\ )}$$

$$\frac{(\ 1+2n+9\ )(\ 1+3n+12\ )}{(\ 1+3n+10\ )(3n+11)};$$

$$l_n(\ ) = \frac{n(\ 1+n+5\ )}{(\ 1+n+\ \ )(n+4\ )};$$

$$p_n(\ ) = \frac{n(n+\ \ )(\ 1+n+4\ )(\ 1+n+5\ )(\ 1+2n+9\ )}{(\ 1+n+\ \ )(\ 1+n+2\ )(n+4\ )^2(\ 1+2n+7\ )};$$

$$q_n(\ ) = \frac{2n(n+2\ )(n+3\ )(\ 1+n+6\ )(\ 1+n+8\ )(\ 1+2n+12\ )}{(\ 1+n+\ \ )(\ 1+n+3\ )(n+5\ )(n+7\ )(\ 1+2n+7\ )(2n+11\ )}$$

$$r_n(\ ) = \frac{n(\ 1+n+6\ )}{(\ 1+n+\ \ )(n+5\ )};$$

$$s_n(\ ) = \frac{n(n+3\ )(\ 1+n+9\ )(\ 1+n+12\ )}{(\ 1+n+\ \ )(\ 1+n+4\ )(n+8\ )(n+11\ )}:$$

Note that the series $z_6 P_{n\ j}$ immediately follow by duality.

V Conclusions

In this paper, we have shown how to solve the Schrodinger equation for the trigonometric Calogero–Sutherland model related to the Lie algebra E_6 and we have explored some properties of the energy eigenfunctions. The main point is that the use of a Weyl-invariant set of variables, the characters of the fundamental representations, leads to a formulation of the Schrodinger equation by means of a second order di erential operator which is simple enough to make feasible a recursive method for the treatment of the spectral problem. The eigenfunctions provide a complete system of orthogonal polynomials in six variables, and these polynomials obey recurrence relations which are deformations of the Clebsch-Gordan series of the algebra. The structure of some of these recurrence relations has been xed.

Acknowledgements

This work has been partially supported by the Spanish Ministerio de Educacion y Ciencia under grants BFM 2003-02532 (J.F.N.) and BFM 2003-00936 / FISI (W.G.F and A.M.P.). A.M.P. would like to thank the Departament of Theoretical Physics of University of Zaragoza for hospitality.

Appendix A: Summary of results on the Lie algebra E_6

In this Section, we review some standard facts about the root and weight systems of the Lie algebra E_6, with the aim of xing the notation and help the reader to follow the rest of the paper. More extensive and sound treatments of these topics can be found in many excellent textbooks, see for instance [20], [21].

The complex Lie algebra E_6, the lowest-dimensional one in the E-family of exceptional Lie algebras in the Cartan-Killing classification, has dimension 78 and rank 6, as the name suggests. From the geometrical point of view, it admits (with some subtleties, see [22]) an interpretation which extends the standard-one for the classical algebras: in the same way that these correspond to the isometries of projective spaces over the first three normed division algebras — $SO(n+1) \simeq \mathrm{Isom}(\mathbb{R}P^n)$, $SU(n+1) \simeq \mathrm{Isom}(\mathbb{C}P^n)$, $Sp(n+1) \simeq \mathrm{Isom}(\mathbb{H}P^n)$— F_4, E_6, E_7 and E_8 are the Lie algebras of the projective planes over extensions of the octonions, giving rise to the so-called "magic square": $F_4 \simeq \mathrm{Isom}(\mathbb{O}P^2)$, $E_6 \simeq \mathrm{Isom}[(\mathbb{C}\otimes\mathbb{O})P^2]$, $E_7 \simeq \mathrm{Isom}[(\mathbb{H}\otimes\mathbb{O})P^2]$, $E_8 \simeq \mathrm{Isom}[(\mathbb{O}\otimes\mathbb{O})P^2]$. In Physics, the most remarkable role played by E_6 is in the heterotic ten-dimensional $E_8 \otimes E_8$ superstring theory when the extra six dimensions are compactified to a manifold of $SU(3)$ holonomy. In such a case, one of the E_8 breaks to an E_6 which gives the Grand Unification group of four-dimensional physics [23]. The Dynkin diagram of E_6, see Figure 1,

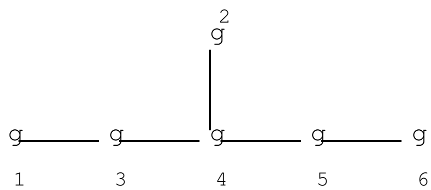

Figure 1. The Dynkin diagram for the Lie algebra E_6.

encodes the euclidean relations among the simple roots, which are

$$
\begin{aligned}
(\alpha_i;\alpha_i) &= 2; & i&=1;2;3;4;5;6;\\
(\alpha_4;\alpha_i) &= -1; & i&=2;3;5;\\
(\alpha_1;\alpha_3) &= (\alpha_5;\alpha_6) = -1;\\
(\alpha_i;\alpha_j) &= 0; & &\text{in all other cases.}
\end{aligned}
$$

Therefore, the Cartan matrix reads

$$
A = \begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0\\
0 & 2 & 0 & -1 & 0 & 0\\
-1 & 0 & 2 & -1 & 0 & 0\\
0 & -1 & -1 & 2 & -1 & 0\\
0 & 0 & 0 & -1 & 2 & -1\\
0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}.
$$

We will use a realization of the simple roots in terms of a generating system $\{\varepsilon_1;\varepsilon_2;\varepsilon_3;\varepsilon_4;\varepsilon_5;\varepsilon_6;\varepsilon\}$ of $\mathbb{R}^7$ (endowed with the standard Euclidean metric) satisfying $\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6 = 0$, $(\varepsilon_i;\varepsilon_j) = -\frac{1}{6}+\delta_{ij}$, $(\varepsilon;\varepsilon) = \frac{1}{2}$ and $(\varepsilon;\varepsilon_j) = 0$ [20]. With reference to this system, we have

$$
\begin{aligned}
\alpha_1 &= \varepsilon_1-\varepsilon_2; & \alpha_2 &= \varepsilon_4+\varepsilon_5+\varepsilon_6+\varepsilon;\\
\alpha_3 &= \varepsilon_2-\varepsilon_3; & \alpha_4 &= \varepsilon_3-\varepsilon_4;\\
\alpha_5 &= \varepsilon_4-\varepsilon_5; & \alpha_6 &= \varepsilon_5-\varepsilon_6.
\end{aligned} \tag{13}
$$

The positive roots, which are given by all linear combinations of the forms

$$
\varepsilon_i-\varepsilon_j;\quad \varepsilon_i+\varepsilon_j+\varepsilon_k+\varepsilon;\quad 2\varepsilon;\quad i\neq j\neq k; \tag{14}
$$

can be classified by heights as indicated in the Table 1. The fundamental weights λ_k follow from the

Height	Positive roots
1	${}_1;\ {}_2;\ {}_3;\ {}_4;\ {}_5;\ {}_6$
2	${}_1 + {}_3;\ {}_3 + {}_4;\ {}_4 + {}_5;\ {}_5 + {}_6;\ {}_2 + {}_4$
3	${}_1 + {}_3 + {}_4;\ {}_3 + {}_4 + {}_5;\ {}_4 + {}_5 + {}_6;\ {}_2 + {}_3 + {}_4;$ ${}_2 + {}_4 + {}_5$
4	${}_1 + {}_3 + {}_4 + {}_5;\ {}_3 + {}_4 + {}_5 + {}_6;\ {}_1 + {}_2 + {}_3 + {}_4;$ ${}_2 + {}_3 + {}_4 + {}_5;\ {}_2 + {}_4 + {}_5 + {}_6$
5	${}_1 + {}_3 + {}_4 + {}_5 + {}_6;\ {}_1 + {}_2 + {}_3 + {}_4 + {}_5;\ {}_2 + {}_3 + 2\,{}_4 + {}_5;$ ${}_2 + {}_3 + {}_4 + {}_5 + {}_6$
6	${}_1 + {}_2 + {}_3 + 2\,{}_4 + {}_5;\ {}_1 + {}_2 + {}_3 + {}_4 + {}_5 + {}_6;$ ${}_2 + {}_3 + 2\,{}_4 + {}_5 + {}_6$
7	${}_1 + {}_2 + 2\,{}_3 + 2\,{}_4 + {}_5;\ {}_2 + {}_3 + 2\,{}_4 + 2\,{}_5 + {}_6;$ ${}_1 + {}_2 + {}_3 + 2\,{}_4 + {}_5 + {}_6$
8	${}_1 + {}_2 + 2\,{}_3 + 2\,{}_4 + {}_5 + {}_6;\ {}_1 + {}_2 + {}_3 + 2\,{}_4 + 2\,{}_5 + {}_6$
9	${}_1 + {}_2 + 2\,{}_3 + 2\,{}_4 + 2\,{}_5 + {}_6$
10	${}_1 + {}_2 + 2\,{}_3 + 3\,{}_4 + 2\,{}_5 + {}_6$
11	${}_1 + 2\,{}_2 + 2\,{}_3 + 3\,{}_4 + 2\,{}_5 + {}_6$

Table 1: Heights of positive roots.

equation ${}_i = \sum_{j=1}^{4} A_{ji}\ {}_j$. They are

$$\begin{aligned} {}_1 &= "_1 + "; \\ {}_2 &= 2"; \\ {}_3 &= "_1 + "_2 + 2"; \\ {}_4 &= "_1 + "_2 + "_3 + 3"; \\ {}_5 &= "_1 + "_2 + "_3 + "_4 + 2"; \\ {}_6 &= "_1 + "_2 + "_3 + "_4 + "_5 + ": \end{aligned}$$

The geometry of the weight system is summarized by the relations

$$({}_i; {}_j) = A^{-1}_{ij};$$

with (A^{-1}_{ij}) being the inverse Cartan matrix. The Weyl vector is

$$= \frac{1}{2}\sum_{2R^+} = \sum_{i=1}^{6} {}_i = 8\,{}_1 + 11\,{}_2 + 15\,{}_3 + 21\,{}_4 + 15\,{}_5 + 8\,{}_6;$$

with R^+ being the set of positive roots of the algebra. The Weyl formula for dimensions applied to the irreducible representation associated to the integral dominant weight $= m_1\,{}_1 + m_2\,{}_2 + m_3\,{}_3 + m_4\,{}_4 + m_5\,{}_5 + m_6\,{}_6$ gives

$$\dim R = \prod_{2R^+} \frac{(; +)}{(;)} = \frac{P}{2^5\ 3^5\ 4^5\ 5^4\ 6^3\ 7^3\ 8^2\ 9\ 10\ 11}$$

where P is a product extended to the set of positive roots in which the root $= \sum_{i=1}^{6} c_i\ {}_i$ contributes with a factor $ht( ) + \sum_{i=1}^{6} c_i m_i$ where $ht( )$ is the height of . In particular, for the fundamental representations, one nds:

$$\dim R_1 = 27 \qquad \dim R_2 = 78$$
$$\dim R_3 = 351 \qquad \dim R_4 = 2925$$
$$\dim R_5 = 351 \qquad \dim R_6 = 27:$$

Note that, these dimensions re ect the fact, coming from the Z_2 symmetry (duality) of the Dynkin diagram, that the representations R_{l_1} and R_{l_6} are complex conjugates. The same is true for R_{l_3} and R_{l_5}, while R_{l_2} (the adjoint representation) and R_{l_4} are real.

Appendix B: Some polynomials and monomial functions

We list here the polynomials up to degree two, and the monomial functions up to degree three. Some of them are omitted for they can be obtained by duality.

Polynomials

$$P_{200000} = z_1^2 \quad \frac{2z_3}{1+\ } \quad \frac{10\ z_6}{(1+\ )(1+4\ )};$$

$$P_{110000} = z_1 z_2 \quad \frac{5z_5}{1+4\ } + \frac{(6\ \ 95\ +24\ ^2)z_1}{(1+4\ )(2+11\ )};$$

$$P_{020000} = z_2^2 \quad \frac{2z_4}{1+\ } \quad \frac{8\ z_1 z_6}{(1+\ )(1+3\ )} + \frac{6(\ 1+\ )(1\ \ +6\ ^2)z_2}{(1+\ )(1+3\ )(3+11\ )} + \frac{18(\ 1+\ )(2+13\ \ 7\ ^2+6\ ^3)}{(1+\ )(1+3\ )(2+11\ )(3+11\ )};$$

$$P_{101000} = z_1 z_3 \quad \frac{3z_4}{1+2\ } + \frac{(\ 2\ \ 35\ +10\ ^2)z_1 z_6}{(1+2\ )(2+7\ )} \quad \frac{6(\ 1+\ )(2+15\ )z_2}{(1+2\ )(1+4\ )(2+7\ )} + \frac{9(\ 2+17\ )}{(1+2\ )(1+4\ )(2+7\ )};$$

$$P_{011000} = z_2 z_3 \quad \frac{4z_1 z_5}{1+3\ } + \frac{5(\ 1+\ )(\ 2+3\ )z_2 z_6}{(1+3\ )(2+7\ )} \quad \frac{4(\ 1+7\ )z_1^2}{(1+3\ )(1+5\ )}$$
$$+ \frac{6(\ 4+51\ +311\ ^2+41\ ^3+105\ ^4)z_3}{(1+3\ )(1+5\ )(2+7\ )(3+11\ )} + \frac{6(\ 16+17\ +673\ ^2\ \ 245\ ^3+75\ ^4)z_6}{(1+3\ )(1+5\ )(2+7\ )(3+11\ )};$$

$$P_{002000} = z_3^2 \quad \frac{2z_1 z_4}{1+\ } \quad \frac{2(\ 1+\ )z_2 z_5}{(1+\ )(1+2\ )} \quad \frac{8\ z_1^2 z_6}{(1+\ )(1+3\ )} + \frac{2(\ 3+7\ +49\ ^2+37\ ^3+30\ ^4)z_3 z_6}{(1+\ )(1+2\ )(1+3\ )(3+7\ )}$$
$$\frac{2(\ 1+\ )\ (17+69\ )z_1 z_2}{(1+\ )(1+2\ )(1+3\ )(3+7\ )} + \frac{(\ 1+\ )(42+291\ +478\ ^2+59\ ^3+150\ ^4)z_6^2}{(1+\ )(1+2\ )(1+3\ )(2+7\ )(3+7\ )}$$
$$+ \frac{12(6+27\ +41\ ^2+103\ ^3+3\ ^4)z_5}{(1+\ )(1+2\ )(1+3\ )(2+7\ )(3+7\ )} \quad \frac{6(\ 2\ \ 7\ +58\ ^2\ \ 433\ ^3+24\ ^4)z_1}{(1+\ )(1+2\ )(1+3\ )(2+7\ )(3+7\ )};$$

$$P_{100100} = z_1 z_4 \quad \frac{4z_2 z_5}{1+3\ } + \frac{4(\ 1+\ )z_1^2 z_6}{1+5\ } \quad \frac{2(\ 1+\ )(5+21\ )z_3 z_6}{(1+3\ )^2(1+5\ )} + \frac{(1\ \ )(27+292\ +723\ ^2\ \ 270\ ^3)z_1 z_2}{(1+3\ )^2(1+5\ )(3+10\ )}$$
$$+ \frac{2(7+56\ \ 15\ ^2)z_6^2}{(1+3\ )^2(1+5\ )} \quad \frac{6(34+321\ +712\ ^2+55\ ^3+750\ ^4)z_5}{(1+3\ )^2(1+5\ )(2+7\ )(3+10\ )}$$
$$+ \frac{3(\ 1+\ )(42+703\ +1634\ ^2\ \ 2937\ ^3\ \ 270\ ^4)z_1}{(1+3\ )^2(1+5\ )(2+7\ )(3+10\ )};$$

$$P_{010100} = z_2 z_4 \quad \frac{3z_3 z_5}{1+2\ } + \frac{4(\ 1+\ )(\ 1+2\ )z_1 z_2 z_6}{(1+2\ )(2+5\ )} + \frac{6(\ 1+\ )(\ 1+2\ )(\ 2+5\ )z_2^2}{(1+2\ )(2+5\ )(3+10\ )}$$
$$\frac{5(\ \ 1)(2+11\ )z_1 z_3}{(1+2\ )(1+3\ )(2+5\ )} \quad \frac{5(\ \ 1)(2+11\ )z_5 z_6}{(1+2\ )(1+3\ )(2+5\ )} + \frac{6(\ \ 1)(56+548\ +1465\ ^2+1000\ ^3+300\ ^4)z_4}{(1+2\ )(1+3\ )(2+5\ )(3+10\ )(4+11\ )}$$
$$+ \frac{(\ 456\ \ 2930\ +2063\ ^2+23981\ ^3\ \ 7718\ ^4+1440\ ^5)z_1 z_6}{(1+2\ )(1+3\ )(2+5\ )(3+10\ )(4+11\ )}$$
$$+ \frac{3(272+244\ \ 11336\ ^2\ \ 28933\ ^3+8109\ ^4\ \ 18036\ ^5+540\ ^6)z_2}{(1+2\ )(1+3\ )(2+5\ )(2+7\ )(3+10\ )(4+11\ )}$$

$$+\frac{18(1\quad)(112+1200\quad+2570\ ^{2}\quad 1215\ ^{3}+1788\ ^{4}+180\ ^{5})}{(1+2\ )(1+3\ )(2+5\ )(2+7\ )(3+10\ )(4+11\ )};$$

$$P_{001100}=z_3z_4\quad\frac{3z_1z_2z_5}{1+2}\quad\frac{5(\ 1+\ )z_2^2z_6}{(1+2\ )(1+3\ )}+\frac{4(\ 1+\ )(\ 1+2\ )z_1z_3z_6}{(1+2\ )(2+5\ )}\quad\frac{5(\ 1+\ )z_5^2}{(1+2\ )(1+3\ )}$$

$$+\frac{(\ 42+25\ +444\ ^{2}+263\ ^{3}+150\ ^{4})z_4z_6}{(1+2\ )(1+3\ )(2+5\ )(3+7\ )}\quad\frac{5(\ 1+\ )(2+11\ )z_1^2z_2}{(1+2\ )(1+3\ )(2+5\ )}$$

$$+\frac{4(\ 15\quad 32\ +276\ ^{2}+626\ ^{3}\quad 105\ ^{4}+90\ ^{5})z_1z_6^2}{(1+2\ )(1+3\ )^2(2+5\ )(3+7\ )}+\frac{2(\ 1+\ )(84+574\ +904\ ^{2}+69\ ^{3}+315\ ^{4})z_2z_3}{(1+2\ )(1+3\ )(2+5\ )^2(3+7\ )}$$

$$\frac{4(1\quad)(30+109\quad 87\ ^{2}+158\ ^{3}+1680\ ^{4})z_1z_5}{(1+2\ )(1+3\ )^2(2+5\ )^2(3+7\ )}+\frac{(\quad 1)(\ 36\quad 1080\quad 8095\ ^{2}\quad 12988\ ^{3}+5847\ ^{4})z_1^2}{(1+2\ )(1+3\ )^2(2+5\ )^2(3+7\ )}$$

$$+\frac{2(\ 1+\ )(12+734\ +5474\ ^{2}+9705\ ^{3}\quad 1620\ ^{4}+675\ ^{5})z_2z_6}{(1+2\ )(1+3\ )^2(2+5\ )^2(3+7\ )}$$

$$+\frac{3(1\quad)(1+6\ )(44+492\quad 41\ ^{2}+252\ ^{3}+45\ ^{4})z_3}{(1+2\ )(1+3\ )^2(2+5\ )^2(3+7\ )}$$

$$\frac{3(\ 120\quad 3020\quad 14654\ ^{2}+9383\ ^{3}+99779\ ^{4}\quad 34713\ ^{5}+12555\ ^{6}+1350\ ^{7})z_6}{(1+2\ )(1+3\ )^2(2+5\ )^2(2+7\ )(3+7\ )};$$

$$P_{000200}=z_4^2\quad\frac{2z_2z_3z_5}{1+}\quad\frac{2(\ 1+\ )z_3^2z_6}{(1+\ )(1+2\ )}\quad\frac{2(\ 1+\ )z_1z_5^2}{(1+\ )(1+2\ )}\quad\frac{2(\ 1+\ )z_1z_2^2z_6}{(1+\ )(1+2\ )}\quad\frac{2(\ 1+\ )(\ 1+2\ )z_2^3}{(1+\ )(1+2\ )(1+3\ )}$$

$$+\frac{4(\ 3+5\ +6\ ^{2}+4\ ^{3})z_1z_4z_6}{(1+\ )(1+2\ )(3+5\ )}\quad\frac{2(\ 1+\ )(\ 3+2\ +28\ ^{2})z_2z_5z_6}{(1+\ )(1+2\ )^2(3+5\ )}\quad\frac{2(\ 1+\ )(\ 3+2\ +28\ ^{2})z_1z_2z_3}{(1+\ )(1+2\ )^2(3+5\ )}$$

$$+\frac{6(\ 1+\ )(\ 15+2\ +335\ ^{2}+754\ ^{3}+436\ ^{4}+120\ ^{5})z_2z_4}{5(1+\ )(1+2\ )^3(1+3\ )(3+5\ )}+\frac{16(\ 1+\ )(3+10\ +3\ ^{2}+2\ ^{3})z_1^2z_6^2}{(1+\ )(1+2\ )(2+5\ )(3+5\ )}$$

$$\frac{4(\ 18\quad 65\quad 82\ ^{2}\quad 109\ ^{3}+22\ ^{4})z_3z_6^2}{(1+\ )(1+2\ )^2(2+5\ )(3+5\ )}\quad\frac{4(\ 18\quad 65\quad 82\ ^{2}\quad 109\ ^{3}+22\ ^{4})z_1^2z_5}{(1+\ )(1+2\ )^2(2+5\ )(3+5\ )}$$

$$\frac{4(150+1507\ +6668\ ^{2}+17329\ ^{3}+27482\ ^{4}+23584\ ^{5}+9800\ ^{6}+4200\ ^{7})z_3z_5}{5(1+\ )(1+2\ )^4(1+3\ )(2+5\ )(3+5\ )}$$

$$\frac{2(\ 1+\ )(6+39\quad 118\ ^{2}\quad 453\ ^{3}+70\ ^{4})z_6^3}{(1+\ )(1+2\ )^2(1+3\ )(2+5\ )(3+5\ )}$$

$$+\frac{2(\ 1+\ )(\ 30\quad 21\ +3383\ ^{2}+22456\ ^{3}+52408\ ^{4}+39680\ ^{5}\quad 3216\ ^{6}+2880\ ^{7})z_1z_2z_6}{5(1+\ )(1+2\ )^4(1+3\ )(2+5\ )(3+5\ )}$$

$$\frac{4(\ 1+\ )(\ 6\quad 37\ +225\ ^{2}+1328\ ^{3}+1224\ ^{4}\quad 616\ ^{5}+600\ ^{6})z_5z_6}{(1+\ )(1+2\ )^4(1+3\ )(2+5\ )(3+5\ )}$$

$$\frac{2(\ 1+\ )(6+39\quad 118\ ^{2}\quad 453\ ^{3}+70\ ^{4})z_1^3}{(1+\ )(1+2\ )^2(1+3\ )(2+5\ )(3+5\ )}$$

$$+\frac{9(\ 1+\ )(\ 60\quad 784\quad 4813\ ^{2}\quad 15896\ ^{3}\quad 24883\ ^{4}\quad 9500\ ^{5}+9296\ ^{6}+80\ ^{7}+1200\ ^{8})z_2^2}{5(1+\ )(1+2\ )^4(1+3\ )(2+5\ )^2(3+5\ )}$$

$$\frac{4(\ 1+\ )(\ 6\quad 37\ +225\ ^{2}+1328\ ^{3}+1224\ ^{4}\quad 616\ ^{5}+600\ ^{6})z_1z_3}{(1+\ )(1+2\ )^4(1+3\ )(2+5\ )(3+5\ )}\quad\frac{A\,z_1z_6}{a}\quad\frac{B\,z_2}{a}\quad\frac{C\,z_4}{a}+\frac{D}{a};$$

$$P_{001010}=z_3z_5\quad\frac{4z_1z_2z_6}{1+3}\quad\frac{9(\ 1+\ )z_2^2}{(1+3\ )(1+4\ )}+\frac{5(\ 1+\ )(\ 2+3\ )z_1z_3}{(1+3\ )(2+7\ )}+\frac{5(\ 1+\ )(\ 2+3\ )z_5z_6}{(1+3\ )(2+7\ )}$$

$$+\frac{24(\ 1+\ +21\ ^{2}+9\ ^{3})z_4}{(1+3\ )^2(1+4\ )(2+7\ )}+\frac{(\ 44+140\ +3413\ ^{2}+7150\ ^{3}\quad 5079\ ^{4}+900\ ^{5})z_1z_6}{(1+3\ )^2(1+4\ )(2+7\ )^2}$$

$$\frac{36\ (16+26\quad 231\ ^{2}+9\ ^{3})z_2}{(1+3\ )^2(1+4\ )(2+7\ )^2}\quad\frac{108\ (6+103\ +311\ ^{2}\quad 123\ ^{3}+63\ ^{4})}{(1+3\ )^2(1+4\ )(1+5\ )(2+7\ )^2};$$

$$P_{100001}=z_1z_6\quad\frac{6z_2}{1+5}\quad\frac{9(\ 1+7\ )}{(1+5\ )(1+8\ )};$$

where the coe cients $A;B;C;D$ and a are

$$A=8\ (\ 12\quad 6872\quad 74937\ ^{2}\quad 237510\ ^{3}\quad 15495\ ^{4}+979026\ ^{5}+989844\ ^{6}\quad 199504\ ^{7}+142260\ ^{8}+10800\ ^{9});$$

$$B=18(1\quad)(180+2196\ +12403\ ^{2}+34729\ ^{3}+9833\ ^{4}\quad 153277\ ^{5}\quad 225096\ ^{6}\quad 37608\ ^{7}\quad 36240\ ^{8}\quad 3600\ ^{9});$$

$C = 6(420 + 6424 \quad + 50807 \ ^{2} + 228922 \ ^{3} + 594476 \ ^{4} + 938974 \ ^{5} + 1027217 \ ^{6} + 835680 \ ^{7} + 400680 \ ^{8} + 132000 \ ^{9}$
$+ 18000 \ ^{10});$

$D = 27(120 + 1772 \quad + 7970 \ ^{2} + 5421 \ ^{3} + 21440 \ ^{4} + 503710 \ ^{5} + 1712910 \ ^{6} + 1652129 \ ^{7} + 44920 \ ^{8} + 259768 \ ^{9}$
$+ 19840 \ ^{10} + 3600 \ ^{11});$

$a = 5(1 + \quad)(1 + 2 \ )^{4}(1 + 3 \ )(2 + 5 \ )^{2}(3 + 5 \ )(3 + 7 \ ):$

Monomial functions

$M_{200000} = z_1^2 \quad 2z_3;$

$M_{110000} = z_1 z_2 \quad 5z_5 + 3z_1;$

$M_{020000} = z_2^2 \quad 2z_4 \quad 2z_2 \quad 6;$

$M_{101000} = z_1 z_3 \quad 3z_4 \quad z_1 z_6 + 6z_2 \quad 9;$

$M_{011000} = z_2 z_3 \quad 4z_1 z_5 + 5z_2 z_6 + 4z_1^2 \quad 4z_3 \quad 16z_6;$

$M_{002000} = z_3^2 \quad 2z_1 z_4 + 2z_2 z_5 \quad 2z_3 z_6 \quad 7z_6^2 + 12z_5 + 2z_1;$

$M_{100100} = z_1 z_4 \quad 4z_2 z_5 \quad 4z_1^2 z_6 + 10z_3 z_6 + 9z_1 z_2 + 14z_6^2 \quad 34z_5 \quad 21z_1;$

$M_{010100} = z_2 z_4 \quad 3z_3 z_5 + 2z_1 z_2 z_6 \quad 2z_2^2 + 5z_1 z_3 + 5z_5 z_6 \quad 14z_4 \quad 19z_1 z_6 + 17z_2 + 42;$

$M_{001100} = z_3 z_4 \quad 3z_1 z_2 z_5 + 5z_2^2 z_6 + 2z_1 z_3 z_6 + 5z_5^2 \quad 7z_4 z_6 + 5z_1^2 z_2 \quad 10z_1 z_6^2 \quad 14z_2 z_3 + 10z_1 z_5 \quad 3z_1^2 \quad 2z_2 z_6$
$+ 11z_3 + 15z_6;$

$M_{000200} = z_4^2 \quad 2z_2 z_3 z_5 + 2z_3^2 z_6 + 2z_1 z_5^2 + 2z_1 z_2^2 z_6 \quad 2z_2^3 \quad 4z_1 z_4 z_6 \quad 2z_2 z_5 z_6 \quad 2z_1 z_2 z_3 + 6z_2 z_4 \quad 8z_1^2 z_6^2 + 12z_3 z_6^2$
$+ 12z_1^2 z_5 \quad 20z_3 z_5 + 2z_6^3 + 2z_1 z_2 z_6 \quad 4z_5 z_6 + 2z_1^3 + 9z_2^2 \quad 4z_1 z_3 \quad 14z_4 \quad 18z_2 + 9;$

$M_{001010} = z_3 z_5 \quad 4z_1 z_2 z_6 + 9z_2^2 + 5z_1 z_3 + 5z_5 z_6 \quad 12z_4 \quad 11z_1 z_6;$

$M_{100001} = z_1 z_6 \quad 6z_2 + 9;$

$M_{300000} = z_1^3 \quad 3z_1 z_3 + 3z_4;$

$M_{210000} = z_1^2 z_2 \quad 2z_2 z_3 \quad z_1 z_5 \quad z_1^2 + 5z_2 z_6 \quad 5z_3 \quad 9z_6;$

$M_{120000} = z_1 z_2^2 \quad 2z_1 z_4 \quad z_2 z_5 + 4z_3 z_6 \quad 4z_6^2 \quad 8z_1 z_2 + 9z_5 + 7z_1;$

$M_{030000} = z_2^3 \quad 3z_2 z_4 + 3z_3 z_5 \quad 3z_1 z_2 z_6 + 3z_4 + 3z_1 z_6 \quad 9;$

$M_{201000} = z_1^2 z_3 \quad 2z_3^2 \quad z_1 z_4 \quad z_1^2 z_6 + 4z_2 z_5 \quad 2z_3 z_6 \quad 4z_6^2 + z_1 z_2 \quad z_5 + 8z_1;$

$M_{111000} = z_1 z_2 z_3 \quad 3z_2 z_4 \quad 4z_1^2 z_5 + 6z_3 z_5 + 7z_1 z_2 z_6 \quad 10z_5 z_6 \quad 12z_2^2 + 4z_1^3 \quad 19z_1 z_3 + 33z_4 + 7z_1 z_6 + 15z_2 \quad 9;$

$M_{021000} = z_2^2 z_3 \quad 2z_3 z_4 \quad z_1 z_2 z_5 + 5z_5^2 + 4z_1 z_3 z_6 \quad 8z_4 z_6 \quad 6z_1^2 z_2 \quad 12z_1 z_6^2 + 3z_2 z_3 + 11z_1 z_5 + 14z_1^2 + 18z_2 z_6$
$18z_3 \quad 14z_6;$

$M_{102000} = z_1 z_3^2 \quad 2z_1^2 z_4 \quad z_3 z_4 + 5z_1 z_2 z_5 \quad 5z_5^2 \quad 5z_2^2 z_6 \quad 5z_1 z_3 z_6 + 10z_4 z_6 + 5z_2 z_3 + 4z_1 z_6^2 \quad 3z_1 z_5 + 6z_2 z_6$
$10z_3 \quad z_6;$

$M_{012000} = z_2 z_3^2 \quad 2z_1 z_2 z_4 \quad z_1 z_3 z_5 + 2z_2^2 z_5 + 3z_4 z_5 + 4z_1^2 z_2 z_6 \quad 9z_2 z_3 z_6 \quad 6z_1^2 z_3 \quad 3z_1 z_5 z_6 \quad 4z_1 z_2^2 + 11z_3^2 + 8z_1 z_4$
$+ 2z_1^2 z_6 \quad 2z_2 z_6^2 + 6z_2 z_5 + 11z_3 z_6 \quad 12z_1 z_2 \quad 6z_6^2 + 19z_5 + 18z_1;$

$M_{003000} = z_3^3 \quad 3z_1 z_3 z_4 + 3z_4^2 + 3z_1^2 z_2 z_5 \quad 3z_2 z_3 z_5 \quad 3z_1 z_2^2 z_6 \quad 3z_1^2 z_3 z_6 + 3z_3^2 z_6 \quad 3z_1 z_5^2 + 6z_1 z_4 z_6 + 3z_1^2 z_6^2 + 3z_2 z_5 z_6$
$+ 3z_2^3 + 3z_1 z_2 z_3 \quad 9z_2 z_4 \quad 3z_1^2 z_5 \quad 8z_6^3 \quad 3z_1 z_2 z_6 + 21z_5 z_6 \quad 21z_4;$

$M_{200100} = z_1^2 z_4 \quad 2z_3 z_4 \quad z_1 z_2 z_5 \quad 4z_1^3 z_6 + 5z_5^2 + 5z_2^2 z_6 + 12z_1 z_3 z_6 \quad 19z_4 z_6 + 4z_1^2 z_2 \quad 11z_2 z_3 \quad 8z_1 z_6^2 + 8z_1^2 + 4z_2 z_6$
$6z_3 \quad 7z_6;$

$M_{110100} = z_1 z_2 z_4 \quad 3z_1 z_3 z_5 \quad 4z_2^2 z_5 + 6z_4 z_5 + 2z_1^2 z_2 z_6 + 7z_2 z_3 z_6 + 5z_1^2 z_3 \quad 7z_1 z_2^2 \quad 3z_1 z_5 z_6 \quad 15z_3^2 + 2z_1 z_4 \quad 3z_1^2 z_6$
$z_2 z_6^2 + 15z_2 z_5 \quad 6z_3 z_6 + 24z_1 z_2 + 9z_6^2 \quad 16z_5 \quad 28z_1;$

$M_{020100} = z_2^2 z_4 \quad 2z_4^2 \quad z_2 z_3 z_5 + 4z_3^2 z_6 + 4z_1 z_5^2 \quad 6z_1 z_4 z_6 \quad 3z_2 z_5 z_6 \quad 3z_1 z_2 z_3 \quad 8z_1^2 z_6^2 + 4z_3 z_6^2 + 6z_2 z_4 + 4z_1^2 z_5$
$8z_3 z_5 + 8z_6^3 + 27z_1 z_2 z_6 + 8z_1^3 \quad 22z_5 z_6 \quad 19z_2^2 \quad 22z_1 z_3 + 20z_4 \quad 14z_1 z_6 \quad 4z_2 + 42;$

$M_{101100} = z_1 z_3 z_4 \quad 3z_4^2 \quad 3z_1^2 z_2 z_5 + 4z_2 z_3 z_5 + 2z_1^2 z_3 z_6 + 7z_1 z_2^2 z_6 \quad 4z_3^2 z_6 + 7z_1 z_5^2 \quad 9z_1 z_4 z_6 \quad 10z_1^2 z_6^2 \quad 20z_2 z_5 z_6$
$12z_2^3 + 5z_1^3 z_2 \quad 20z_1 z_2 z_3 + 12z_3 z_6^2 + 45z_2 z_4 + 2z_1^2 z_5 + 40z_6^3 + 12z_3 z_5 + 24z_1 z_2 z_6 + 3z_1^3 \quad 92z_5 z_6 \quad 21z_1 z_3 \quad 18z_2^2$
$+ 96z_4 \quad 7z_1 z_6 + 33z_2 \quad 9;$

$M_{011100} = z_2 z_3 z_4 \quad 3z_3^2 z_5 \quad 3z_1 z_2^2 z_5 + 5z_2^3 z_6 + 4z_1 z_4 z_5 + 8z_1 z_2 z_3 z_6 + 7z_2 z_5^2 \quad 22z_2 z_4 z_6 \quad 5z_1 z_3^2 \quad 4z_1^2 z_5 z_6 \quad 5z_1^2 z_2^2$

$+ z_3 z_5 z_6 \quad 10z_1 z_2 z_6^2 + 8z_5 z_6^2 + 8z_1^2 z_4 \quad 3z_2^2 z_3 + 8z_3 z_4 + 19z_1 z_2 z_5 \quad 2z_2^2 z_6 + 4z_1^3 z_6 \quad 23z_5^2 + z_1 z_3 z_6 \quad 4z_1^2 z_2 + 8z_4 z_6$
$+ 2z_1 z_6^2 + 10z_2 z_3 \quad 18z_1 z_5 + 17z_1^2 + 3z_2 z_6 \quad 14z_3 \quad 6z_6 ;$

$M_{002100} = z_3^2 z_4 \quad 2z_1 z_4^2 \quad z_1 z_2 z_3 z_5 + 5z_2 z_4 z_5 + 4z_1^2 z_5^2 + 4z_1^2 z_2^2 z_6 \quad 7z_3 z_5^2 \quad 6z_1^2 z_4 z_6 \quad 7z_2^2 z_3 z_6 \quad 3z_1^2 z_2 z_3 + 10z_3 z_4 z_6$
$\quad 4z_1 z_2^3 \quad 9z_1 z_2 z_5 z_6 + 6z_2 z_3^2 \quad 8z_1^3 z_6^2 + 5z_5^2 z_6 + 5z_2^2 z_6^2 + 9z_1 z_2 z_4 + 24z_1 z_3 z_6^2 + 13z_2^2 z_5 + 4z_1^3 z_5 \quad 8z_1 z_3 z_5 \quad 8z_4 z_6^2$
$\quad 4z_1 z_6^3 \quad 18z_4 z_5 + 7z_1^2 z_2 z_6 \quad 23z_2 z_3 z_6 + 4z_1 z_5 z_6 + 8z_1^4 \quad 29z_1^2 z_3 \quad 20z_2 z_6^2 + 4z_1 z_2^2 + 16z_3^2 + z_1 z_4$
$+ z_1^2 z_6 + 10z_2 z_5 + 24z_3 z_6 \quad z_1 z_2 + 19z_6^2 \quad 16z_5 \quad 27z_1 ;$

$M_{100200} = z_1 z_4^2 \quad 2z_1 z_2 z_3 z_5 \quad z_2 z_4 z_5 + 2z_1^2 z_5^2 + 2z_1 z_3^2 z_6 + 2z_1^2 z_2^2 z_6 + 3z_3 z_5^2 \quad 4z_1^2 z_4 z_6 + 3z_2^2 z_3 z_6 \quad 2z_1^2 z_2 z_3 \quad 5z_3 z_4 z_6$
$\quad 7z_1 z_2^3 \quad 7z_1 z_2 z_5 z_6 \quad 3z_2 z_3^2 \quad 8z_1^3 z_6^2 + 21z_1 z_2 z_4 + 14z_1 z_3 z_6^2 + 7z_2^2 z_5 + 12z_1^3 z_5 \quad 25z_1 z_3 z_5 \quad z_4 z_6^2 + 8z_1 z_6^3 + 3z_4 z_5$
$+ 13z_1^2 z_2 z_6 \quad 19z_2 z_3 z_6 \quad 17z_1 z_5 z_6 + 2z_1^4 \quad 16z_1^2 z_3 \quad 8z_2 z_6^2 + 10z_1 z_2^2 + 27z_3^2 \quad 11z_1 z_4 \quad z_1^2 z_6 + 7z_2 z_5$
$+ 11z_3 z_6 \quad 31z_1 z_2 \quad 8z_6^2 + 37z_5 + 22z_1 ;$

$M_{010200} = z_2 z_4^2 \quad 2z_2^2 z_3 z_5 \quad z_3 z_4 z_5 + 5z_2 z_3^2 z_6 + 5z_1 z_2 z_5^2 \quad 5z_3^3 \quad 5z_5^3 + 2z_1 z_3^3 z_6 \quad 9z_1 z_2 z_4 z_6 \quad 5z_2^2 z_5 z_6$
$\quad 5z_1 z_3 z_5 z_6 + 15z_4 z_5 z_6 \quad 5z_1 z_2^2 z_3 \quad 2z_2^4 + 15z_1 z_3 z_4 \quad 6z_1^2 z_2 z_6^2 + 7z_2 z_3 z_6^2 + 14z_2^2 z_4 \quad 16z_4^2 + 11z_1 z_5 z_6^2$
$+ 7z_1^2 z_2 z_5 \quad 10z_2 z_3 z_5 + 11z_1^2 z_3 z_6 + 4z_1 z_2^2 z_6 \quad 12z_1 z_5^2 \quad 12z_3^2 z_6 \quad 17z_1 z_4 z_6 \quad 5z_1^2 z_6^2 \quad 10z_1^3 z_2 \quad 10z_2 z_6^3$
$+ 29z_2 z_5 z_6 + 29z_1 z_2 z_3 + 3z_3 z_6^2 + 10z_6^3 \quad 21z_2 z_4 + 3z_1^2 z_5 \quad 34z_3 z_5 + 7z_1 z_2 z_6 \quad 14z_5 z_6 + 10z_1^3 \quad 14z_1 z_3$
$\quad 6z_2^2 \quad 6z_4 \quad 34z_1 z_6 + 38z_2 + 42 ;$

$M_{001200} = z_3 z_4^2 \quad 2z_2 z_3^2 z_5 + 2z_3^3 z_6 \quad z_1 z_2 z_4 z_5 + 5z_1 z_3 z_5^2 + 4z_2^2 z_5^2 \quad 7z_4 z_5^2 + 5z_1 z_2^2 z_3 z_6 \quad 7z_2^2 z_4 z_6 \quad 9z_1 z_3 z_4 z_6$
$+ 14z_4^2 z_6 \quad 5z_1^2 z_2 z_5 z_6 \quad 6z_1^2 z_3 z_6^2 \quad 5z_1^2 z_2^3 \quad 6z_2 z_3 z_5 z_6 \quad 5z_1 z_2 z_3^2 + 15z_1^2 z_2 z_4 + 6z_3^2 z_6^2 + 7z_1^2 z_3 z_5 + 2z_2^3 z_3$
$+ z_2 z_3 z_4 \quad 4z_3^2 z_5 + 24z_1 z_4 z_6^2 + 8z_1 z_2^2 z_5 \quad 12z_2 z_5 z_6^2 \quad 33z_1 z_4 z_5 + 2z_1^2 z_6^3 + 11z_1^3 z_2 z_6 + 8z_3 z_6^3 + 8z_6^4 \quad 28z_1 z_2 z_3 z_6$
$+ 16z_2 z_5^2 + 7z_1^2 z_2^2 + 2z_2 z_4 z_6 \quad z_1^2 z_5 z_6 \quad 6z_3 z_5 z_6 + 11z_2^2 z_3 \quad 10z_1^3 z_3 + 38z_1 z_3^2 \quad 29z_1^2 z_4 \quad 22z_3 z_4 \quad z_1^3 z_6$
$\quad z_1 z_2 z_5 \quad 20z_5 z_6^2 \quad 9z_1 z_3 z_6 \quad 20z_1^2 z_2 \quad 5z_2^2 z_6 + 22z_5^2 \quad 3z_4 z_6 + 14z_2 z_3 \quad 24z_1 z_6^2$
$+ 29z_1 z_5 + 10z_2 z_6 + 27z_3 + 17z_6 ;$

$M_{000300} = z_4^3 \quad 3z_2 z_3 z_4 z_5 + 3z_3^2 z_5^2 + 3z_1 z_2^2 z_5^2 + 3z_2^2 z_3^2 z_6 \quad 3z_3^2 z_4 z_6 \quad 3z_1 z_2^2 z_4 z_6 \quad 3z_1 z_4 z_5^2 \quad 3z_2 z_3^3 \quad 3z_2^3 z_5 z_6 + 6z_1 z_4^2 z_6$
$\quad 9z_1 z_2 z_3 z_5 z_6 \quad 3z_2 z_5^3 + 12z_2 z_4 z_5 z_6 \quad 3z_1 z_2^3 z_3 + 3z_3 z_5^2 z_6 + 9z_1^2 z_4 z_6^2 + 12z_1 z_2 z_3 z_4 + 3z_2^2 z_3 z_6^2 + 6z_2^3 z_4 \quad 18z_2 z_4^2$
$+ 3z_1^2 z_2^2 z_5 \quad 12z_3 z_4 z_6^2 + 3z_1 z_3^2 z_5 \quad 12z_1^2 z_4 z_5 + 3z_2^2 z_6^3 \quad 4z_1^3 z_6^3 + 9z_3 z_4 z_5 + 3z_1^2 z_3^2 + 3z_5^2 z_6^2 \quad 12z_1 z_2 z_4 z_6$
$+ 12z_1 z_3 z_6^3 \quad 24z_4 z_6^3 \quad 3z_5^3 + 12z_1^3 z_5 z_6 \quad 33z_1 z_3 z_5 z_6 \quad 3z_3^3 + 3z_1^3 z_2^2 \quad 24z_1^3 z_4 + 54z_4 z_5 z_6 + 54z_1 z_3 z_4 \quad 45z_4^2$
$\quad 9z_2 z_3 z_5 \quad 9z_1 z_2^2 z_6 \quad 6z_1 z_5 z_6^2 \quad 6z_2 z_6^3 \quad 6z_1^2 z_3 z_6 \quad 6z_1^3 z_2 + 27z_1 z_4 z_6 + 9z_2 z_5 z_6 + 12z_2^3 + 3z_1^2 z_6^2$
$+ 9z_1 z_2 z_3 + 9z_1^2 z_5 \quad 36z_2 z_4 + 9z_3 z_6^2 + 9z_3 z_5 \quad 18z_1 z_2 z_6 + 3z_6^3 + 3z_1^3 + 9z_4 + 9z_1 z_6 \quad 9 ;$

$M_{101010} = z_1 z_3 z_5 \quad 3z_4 z_5 \quad 4z_1^2 z_2 z_6 + 5z_1^2 z_3 + 6z_2 z_3 z_6 + 9z_1 z_2^2 + 7z_1 z_5 z_6 \quad 10z_3^2 \quad 13z_1 z_4 \quad 13z_1^2 z_6 \quad 10z_2 z_6^2 \quad 7z_2 z_5$
$+ 16z_3 z_6 + 50z_6^2 + 11z_1 z_2 \quad 57z_5 \quad 59z_1 ;$

$M_{011010} = z_2 z_3 z_5 \quad 4z_3^2 z_6 \quad 4z_1 z_5^2 \quad 4z_1 z_2^2 z_6 + 9z_2^3 + 12z_1 z_4 z_6 + 11z_2 z_5 z_6 + 11z_1 z_2 z_3 \quad 45z_2 z_4 + 16z_1^2 z_6^2 \quad 28z_3 z_6^2$
$\quad 28z_1^2 z_5 + 45z_3 z_5 \quad 16z_6^3 \quad 17z_1 z_2 z_6 + 45z_5 z_6 \quad 16z_1^3 + 45z_1 z_3 \quad 27z_4 \quad 3z_1 z_6 + 18z_2 \quad 27 ;$

$M_{002010} = z_3^2 z_5 \quad 2z_1 z_4 z_5 + 2z_2 z_5^2 \quad z_1 z_2 z_3 z_6 + 3z_2 z_4 z_6 + 4z_1^2 z_5 z_6 + 5z_1^2 z_2^2 \quad 9z_3 z_5 z_6 \quad 3z_1 z_2 z_6^2 \quad 8z_1^2 z_4$
$\quad 9z_2^2 z_3 \quad 2z_5 z_6^2 + 16z_3 z_4 \quad 6z_1 z_2 z_5 + 7z_5^2 \quad 12z_1^3 z_6 + 8z_2^2 z_6 + 37z_1 z_3 z_6 \quad 9z_4 z_6 + 2z_1^2 z_2 + z_1 z_6^2$
$\quad 27z_2 z_3 + 6z_1 z_5 + 13z_1^2 \quad 11z_2 z_6 \quad 14z_3 + 13z_6 ;$

$M_{001110} = z_3 z_4 z_5 \quad 3z_1 z_2 z_5^2 \quad 3z_2 z_3^2 z_6 + 5z_3^3 + 4z_1 z_2 z_4 z_6 + 8z_1 z_3 z_5 z_6 + 5z_5^3 + 7z_2^2 z_5 z_6 + 7z_1 z_2^2 z_3 \quad 22z_4 z_5 z_6$
$\quad 4z_1^2 z_2 z_6^2 \quad 18z_2^2 z_4 + z_2 z_3 z_6^2 \quad 22z_1 z_3 z_4 + 36z_4^2 + z_1^2 z_2 z_5 \quad 10z_1 z_5 z_6^2 + 8z_2 z_6^3 \quad 8z_2 z_3 z_5 \quad 10z_1^2 z_3 z_6 + 8z_3^2 z_6$
$+ 8z_1 z_5^2 + 9z_2^3 + 52z_1 z_4 z_6 + 16z_1^2 z_6^2 \quad 40z_2 z_5 z_6 + 8z_1^3 z_2 \quad 40z_1 z_2 z_3 \quad z_3 z_6^2 \quad z_1^2 z_5 + 9z_2 z_4 + 24z_3 z_5$
$\quad 23z_1 z_2 z + 18z_2^2 + 27z_1 z_3 + 27z_5 z_6 \quad 63z_4 \quad 9z_1 z_6 \quad 36z_2 \quad 27 ;$

$M_{200001} = z_1^2 z_6 \quad 2z_3 z_6 \quad z_1 z_2 + 5z_5 \quad 4z_1 ;$

$M_{110001} = z_1 z_2 z_6 \quad 6z_2^2 \quad 5z_1 z_3 \quad 5z_5 z_6 + 21z_4 + 24z_1 z_6 \quad 30z_2 \quad 9 ;$

$M_{101001} = z_1 z_3 z_6 \quad 3z_4 z_6 \quad z_1 z_6^2 \quad 5z_1^2 z_2 + 8z_2 z_3 + 9z_1 z_5 + z_1^2 \quad 9z_2 z_6 + 3z_3 + 2z_6 ;$

$M_{100101} = z_1 z_4 z_6 \quad 4z_2 z_5 z_6 \quad 4z_1 z_2 z_3 \quad 4z_1^2 z_6^2 + 9z_2 z_4 + 10z_3 z_6^2 + 10z_1^2 z_5 \quad 9z_3 z_5 \quad z_1 z_2 z_6 + 14z_6^3 \quad 39z_5 z_6$
$+ 14z_1^3 \quad 39z_1 z_3 + 27z_4 + 24z_1 z_6 \quad 81:$

Appendix C: Deformed quadratic Clebsch-Gordan series

$$P_{100000}\quad P_{100000} = P_{200000} + \frac{2}{1+\;}P_{001000} + \frac{10(1+3\;)}{(1+4\;)(1+7\;)}P_{000001}\;;$$

$$P_{100000}\quad P_{010000} = P_{110000} + \frac{5}{1+4\;}P_{000010} + \frac{32(1+2\;)(1+12\;)}{(1+7\;)(1+11\;)(2+11\;)}P_{100000}\;;$$

$$P_{010000}\quad P_{010000} = P_{020000} + \frac{2}{1+\;}P_{000100} + \frac{8(1+2\;)}{(1+3\;)(1+5\;)}P_{100001} + \frac{12(5+84\;+255\;^{2}+160\;^{3})}{(1+5\;)^{2}(1+11\;)(3+11\;)}P_{010000}$$

$$+\frac{144(1+2\;)(1+3\;)(1+5\;)(1+12\;)}{(1+7\;)(1+8\;)(1+11\;)^{2}(2+11\;)}P_{000000}\;;$$

$$P_{100000}\quad P_{001000} = P_{101000} + \frac{3}{1+2\;}P_{000100} + \frac{16(1+2\;)(1+8\;)}{(1+5\;)(1+7\;)(2+7\;)}P_{100001} + \frac{15(1+\;)(1+3\;)(1+11\;)}{(1+4\;)(1+5\;)^{2}(1+7\;)}P_{010000}\;;$$

$$P_{010000}\quad P_{001000} = P_{011000} + \frac{4}{1+3\;}P_{100010} + \frac{20(1+\;)(1+8\;)}{(1+4\;)(1+7\;)(2+7\;)}P_{010001} + \frac{16(1+2\;)}{(1+5\;)(1+7\;)}P_{200000}$$

$$+\frac{20(1+2\;)(1+9\;)(3+46\;+71\;^{2}\;\;8\;^{3})}{(1+\;)(1+4\;)^{2}(1+7\;)(1+11\;)(3+11\;)}P_{001000} + \frac{80(1+\;)(1+2\;)(1+3\;)(1+12\;)}{(1+4\;)(1+5\;)(1+7\;)^{2}(2+11\;)}P_{000001}\;;$$

$$P_{001000}\quad P_{001000} = P_{002000} + \frac{2}{1+\;}P_{100100} + \frac{6(1+\;)}{(1+2\;)(1+3\;)}P_{010010} + \frac{8(1+2\;)}{(1+3\;)(1+5\;)}P_{200001}$$

$$+\frac{4(9+113\;+305\;^{2}+231\;^{3}\;\;18\;^{4})}{(1+\;)(1+3\;)^{2}(1+7\;)(3+7\;)}P_{001001} + \frac{120(1+\;)(1+2\;)(2+33\;+56\;^{2})}{(1+4\;)(2+5\;)(1+7\;)(2+7\;)(3+10\;)}P_{110000}$$

$$+\frac{80(1+\;)(1+2\;)(1+3\;)(1+8\;)}{(1+4\;)(1+5\;)(1+7\;)^{2}(2+7\;)}P_{000002} + \frac{80(1+8\;)(3+56\;+176\;^{2}+108\;^{3}\;\;63\;^{4})}{(1+4\;)^{2}(1+7\;)^{2}(2+7\;)(3+11\;)}P_{000010}$$

$$+\frac{160(1+\;)^{2}(1+2\;)(1+3\;)(1+9\;)(1+12\;)}{(1+4\;)^{2}(1+5\;)(1+7\;)^{3}(2+11\;)}P_{100000}\;;$$

$$P_{100000}\quad P_{000100} = P_{100100} + \frac{4}{1+3\;}P_{010010} + \frac{6(1+\;)(1+7\;)}{(1+3\;)^{2}(1+5\;)}P_{001001}$$

$$+\frac{30(1+\;)(1+2\;)(1+8\;)(2+11\;)}{(1+4\;)(1+5\;)^{2}(2+7\;)(3+10\;)}P_{110000} + \frac{30(1+\;)(1+2\;)(2+5\;)(1+8\;)(1+9\;)}{(1+4\;)^{2}(1+5\;)^{2}(2+7\;)(3+11\;)}P_{000010}\;;$$

$$P_{010000}\quad P_{000100} = P_{010100} + \frac{3}{1+2\;}P_{001010} + \frac{12(1+\;)(1+6\;)}{(1+3\;)(1+5\;)(2+5\;)}P_{110001} + \frac{30(1+\;)(1+2\;)(2+11\;)}{(1+4\;)(1+5\;)^{2}(3+10\;)}P_{020000}$$

$$+\frac{20(1+\;)(1+2\;)(1+8\;)}{(1+3\;)(1+4\;)(1+5\;)(2+7\;)}P_{101000} + \frac{20(1+\;)(1+2\;)(1+8\;)}{(1+3\;)(1+4\;)(1+5\;)(2+7\;)}P_{000011}$$

$$+\frac{72(1+7\;)(1+22\;+115\;^{2}+87\;^{3}\;\;45\;^{4})}{(1+3\;)^{2}(1+5\;)^{2}(1+11\;)(4+11\;)}P_{000100} + \frac{144(1+\;)^{2}(1+2\;)(2+5\;)(1+8\;)^{2}}{(1+3\;)(1+5\;)^{3}(2+7\;)^{2}(3+11\;)}P_{100001}$$

$$+\frac{60(1+\;)(1+2\;)^{2}(1+3\;)(2+5\;)(1+8\;)(1+9\;)(1+11\;)}{(1+4\;)^{2}(1+5\;)^{4}(1+7\;)(2+7\;)(3+11\;)}P_{010000}\;;$$

$$P_{001000}\quad P_{000100} = P_{001100} + \frac{3}{1+2\;}P_{110010} + \frac{10(1+\;)}{(1+3\;)(1+4\;)}P_{020001} + \frac{12(1+\;)(1+6\;)}{(1+3\;)(1+5\;)(2+5\;)}P_{101001}$$

$$+\frac{10(1+\;)}{(1+3\;)(1+4\;)}P_{000020} + \frac{72(1+6\;)(1+11\;+13\;^{2}\;\;5\;^{3})}{(1+2\;)(1+5\;)(2+5\;)(1+7\;)(3+7\;)}P_{000101}$$

$$+\frac{20(1+\;)(1+2\;)(1+8\;)}{(1+3\;)(1+4\;)(1+5\;)(2+7\;)}P_{210000} + \frac{48(1+\;)^{2}(1+2\;)(1+8\;)}{(1+3\;)^{2}(1+5\;)^{2}(2+7\;)}P_{100002}$$

$$+\frac{6(36+1134\;+12624\;^{2}+65771\;^{3}+172189\;^{4}+224179\;^{5}+127295\;^{6}+17700\;^{7})}{(1+2\;)(1+3\;)(1+5\;)^{2}(2+5\;)^{2}(1+7\;)(3+8\;)}P_{011000}$$

$$+\frac{24(1+\;)(42+1024\;+7069\;^{2}+17092\;^{3}+13653\;^{4}+720\;^{5})}{(1+3\;)^{2}(1+5\;)^{2}(3+7\;)(3+8\;)(4+11\;)}P_{100010}$$

$$+\frac{480(1+\;)^{2}(1+2\;)^{2}(1+6\;)(1+8\;)}{(1+4\;)(1+5\;)^{3}(1+7\;)(2+7\;)(3+11\;)}P_{200000}$$

$$+\frac{60(1+\;)(1+2\;)(1+8\;)(32+842\;+6313\;^{2}+16912\;^{3}+16251\;^{4}+3330\;^{5})}{(1+3\;)(1+4\;)(1+5\;)^{2}(2+5\;)(1+7\;)(2+7\;)(3+10\;)(4+11\;)}P_{010001}$$

$$+\frac{60(1+\ )(1+2\ )(2+5\ )(1+8\ )(1+9\ )(3+40\ +39\ ^2\ \ 18\ ^3)}{(1+3\ )^2(1+4\ )^2(1+5\ )^2(1+7\ )(2+7\ )(3+11\ )}P_{001000}$$

$$+\frac{480(1+\ )^2(1+2\ )^2(1+3\ )(2+5\ )(1+8\ )(1+9\ )(1+12\ )}{(1+4\ )^2(1+5\ )^3(1+7\ )^2(2+7\ )(2+11\ )(3+11\ )}P_{000001};$$

$$P_{000100}\quad P_{000100}=P_{000200}+\frac{2}{1+\ }P_{011010}+\frac{6(1+\ )}{(1+2\ )(1+3\ )}P_{002001}+\frac{6(1+\ )}{(1+2\ )(1+3\ )}P_{100020}$$

$$+\frac{6(1+\ )}{(1+2\ )(1+3\ )}P_{120001}+\frac{20(1+\ )(1+2\ )}{(1+3\ )(1+4\ )(1+5\ )}P_{030000}+\frac{24(1+7\ \ \ 2\ ^2)}{(1+2\ )(1+5\ )(3+5\ )}P_{100101}$$

$$+\frac{36(1+\ )^2(1+13\ +16\ ^2)}{(1+2\ )^2(1+3\ )(1+5\ )(3+7\ )}P_{010011}+\frac{36(1+\ )^2(1+13\ +16\ ^2)}{(1+2\ )^2(1+3\ )(1+5\ )(3+7\ )}P_{111000}$$

$$+\frac{36(1+\ )(40+1064\ +7172\ ^2+16301\ ^3+13138\ ^4+940\ ^5\ \ 1800\ ^6)}{5(2+\ )(1+2\ )^3(1+5\ )^2(2+5\ )(4+9\ )}P_{010100}$$

$$+\frac{48(1+\ )^2(1+2\ )(1+6\ )}{(1+3\ )^2(1+5\ )^2(2+5\ )}P_{200002}+\frac{144(1+\ )(1+6\ )(1+11\ +13\ ^2\ \ 5\ ^3)}{(1+3\ )^2(1+5\ )^2(2+5\ )(3+7\ )}P_{001002}$$

$$+\frac{144(1+\ )(1+6\ )(1+11\ +13\ ^2\ \ 5\ ^3)}{(1+3\ )^2(1+5\ )^2(2+5\ )(3+7\ )}P_{200010}$$

$$+\frac{2(195+5540\ +49198\ ^2+163456\ ^3+239715\ ^4+157964\ ^5+57452\ ^6+26320\ ^7)}{(1+2\ )^4(1+5\ )^2(3+7\ )(5+11\ )}P_{001010}$$

$$+E\,P_{000003}+F\,P_{110001}+G\,P_{000011}+E\,P_{300000}$$

$$+\frac{180(1+\ )^2(1+2\ )(90+2499\ +31155\ ^2+193684\ ^3+611355\ ^4+972155\ ^5+708750\ ^6+171000\ ^7)}{(1+3\ )(1+4\ )(1+5\ )^4(2+5\ )^2(3+8\ )(3+10\ )(5+11\ )}P_{020000}$$

$$+G\,P_{101000}+H\,P_{000100}+IP_{100001}$$

$$+\frac{2160(1+\ )^2(1+2\ )^2(2+5\ )(1+8\ )(1+9\ )(1+22\ +115\ ^2+87\ ^3\ \ 45\ ^4)}{(1+3\ )(1+4\ )^2(1+5\ )^6(2+7\ )(3+11\ )(4+11\ )}P_{010000}$$

$$+\frac{4320(1+\ )^2(1+2\ )^3(1+3\ )^2(2+5\ )(1+9\ )(1+12\ )}{(1+4\ )^2(1+5\ )^3(1+7\ )^2(2+7\ )(1+11\ )(2+11\ )(3+11\ )}P_{000000};$$

$$P_{100000}\quad P_{000010}=P_{100010}+\frac{5}{1+4\ }P_{010001}+\frac{10(1+\ )(1+9\ )}{(1+4\ )^2(1+7\ )}P_{001000}+\frac{32(1+2\ )(1+3\ )(1+12\ )}{(1+5\ )(1+7\ )^2(2+11\ )}P_{000001};$$

$$P_{001000}\quad P_{000010}=P_{001010}+\frac{4}{1+3\ }P_{110001}+\frac{15(1+\ )}{(1+4\ )(1+5\ )}P_{020000}+\frac{20(1+\ )(1+8\ )}{(1+4\ )(1+7\ )(2+7\ )}P_{101000}$$

$$+\frac{20(1+\ )(1+8\ )}{(1+4\ )(1+7\ )(2+7\ )}P_{000011}+\frac{6(3+40\ +39\ ^2\ \ 18\ ^3)}{(1+2\ )(1+3\ )^2(1+7\ )}P_{000100}$$

$$+\frac{8(1+2\ )(48+1342\ +11893\ ^2+41323\ ^3+59235\ ^4+31311\ ^5)}{(1+3\ )(1+5\ )(1+7\ )^2(2+7\ )^2(3+11\ )}P_{100001}$$

$$+\frac{60(1+2\ )(1+3\ )(1+9\ )(3+46\ +71\ ^2\ \ 8\ ^3)}{(1+4\ )^2(1+5\ )^2(1+7\ )^2(3+11\ )}P_{010000}+\frac{432(1+\ )(1+2\ )(1+3\ )^2(1+12\ )}{(1+5\ )(1+7\ )^2(1+8\ )(1+11\ )(2+11\ )}P_{000000};$$

$$P_{100000}\quad P_{000001}=P_{100001}+\frac{6}{1+5\ }P_{010000}+\frac{27(1+3\ )}{(1+8\ )(1+11\ )}P_{000000};$$

where the coe cients $E;F;G;H$ and I are such that

$$E\,(1+3\ )(1+4\ )(1+5\ )^2(1+7\ )(2+7\ )=160(1+\ )^2(1+2\ )^2(1+8\ );$$

$$F\,(1+2\ )(1+3\ )(2+3\ )(1+5\ )^3(2+5\ )^2(3+5\ )(3+7\ )^2(5+11\ )=144(1+\ )^2(270+10965$$
$$+166113\ ^2+1237287\ ^3+5078136\ ^4+12177475\ ^5+17282049\ ^6+13976605\ ^7+5700600\ ^8+818500\ ^9);$$

$$G=\frac{720(1+\ )^2(1+2\ )(1+8\ )(8+208\ +1312\ ^2+1877\ ^3+360\ ^4\ \ 300\ ^5)}{(2+\ )(1+3\ )(1+4\ )(1+5\ )^3(2+5\ )(2+7\ )(3+8\ )(4+11\ )};$$

$$H\,\frac{(1+3\ )^2(1+5\ )^4(2+5\ )^2(1+7\ )(3+7\ )(3+8\ )(4+11\ )}{24(1+\ )}=444+21566\ +436658\ ^2+4716853\ ^3$$
$$+29111132\ ^4+102644506\ ^5+195972356\ ^6+176806835\ ^7+45083850\ ^8\ \ 6894000\ ^9+10935000\ ^{10};$$

$$I\,(1+3\ )^2(1+5\ )^5(2+7\ )^2(3+8\ )(3+11\ )(4+11\ )=864(1+\ )^2(1+2\ )(1+8\ )(16+626\ +8775\ ^2+55745\ ^3$$
$$+172984\ ^4+268299\ ^5+193845\ ^6+48150\ ^7):$$

References

[1] Calogero F., J. Math. Phys. 12, 419–436, 1971.

[2] Sutherland B., Phys. Rev. A 4, 2019–2021, 1972.

[3] Moser J., Adv. Math. 16, 197–220, 1975.

[4] Olshanetsky M.A. and Perelomov A.M., Invent. Math. 37, 93–108, 1976.

[5] Olshanetsky M.A. and Perelomov A.M., Lett. Math. Phys. 2, 7–13, 1977.

[6] Olshanetsky M.A. and Perelomov A.M., Funct. Anal. Appl. 12, 121–128, 1978.

[7] Olshanetsky M.A. and Perelomov A.M., Phys. Rep. 71, 314–400, 1981.

[8] Olshanetsky M.A. and Perelomov A.M., Phys. Rep. 94, 313–404, 1983.

[9] van Diejen J.F and Vinet L. (Eds.), Calogero–Moser–Sutherland Models, Springer, 2000.

[10] Perelomov A.M., J. Phys. A 31, L31–L37, 1998.

[11] Perelomov A.M., Ragoucy E. and Zaugg Ph., J. Phys. A 31, L559–L565, 1998.

[12] Perelomov A.M., J. Phys. A 32, 8563–8576, 1999.

[13] Perelomov A.M., in Proc. Clausthal Conference 1999, Lie Theory and Its Applications in Physics. III, pp. 139–154, World Scientific, Singapore, 2000; math-ph/0111021.

[14] García Fuertes W., Lorente M., Perelomov A.M., J. Phys. A 34, 10963–10973, 2001.

[15] García Fuertes W., Perelomov A.M., Theor. Math. Phys., 131, 609–611, 2002; math-ph/0201026.

[16] Fernandez Núñez J., García Fuertes W., Perelomov A.M., Phys. Lett. A 307, 233–238, 2003.

[17] Fernandez Núñez J., García Fuertes W., Perelomov A.M., J. Math. Phys. 44, 4957–4974, 2003.

[18] Fernandez Núñez J., García Fuertes W., Perelomov A.M., J. Nonlinear Math. Phys. to appear 2005; math-ph/0406067.

[19] Ram R., Proc. 4th. Conference "Formal Power Series and Algebraic Combinatorics", Publ. LACIM 11, 327–342, 1992.

[20] Onishchik A.L. and Vinberg E.B., Lie Groups and Algebraic Groups, Springer, 1990.

[21] Cornwell J.F., Group Theory in Physics, vol II,, Academic Press, 1984; Bourbaki N. Groupes et Algebres de Lie, Hermann, 1975.

[22] Baez J.C., Bull. Amer. Math. Soc. 39, 145–205, 2002.

[23] Ramond P., Plenary Talk in the XXIV International Colloquium on Group Theoretical Methods in Physics, Paris, 2002, hep-th/0301050.