Flows of Mellin transforms with periodic integrator

Titus Hilberdink
Department of Mathematics, University of Reading, Whiteknights,
PO Box 220, Reading RG6 6AX, UK; t.w.hilberdink@reading.ac.uk

Abstract

We study Mellin transforms $\hat{N}(s) = \int_{1-}^{\infty} x^{-s} dN(x)$ for which $N(x) - x$ is periodic with period 1 in order to investigate 'flows' of such functions to Riemann's $\zeta(s)$ and the possibility of proving the Riemann Hypothesis with such an approach. We show that, excepting the trivial case where $N(x) = x$, the supremum of the real parts of the zeros of any such function is at least $\frac{1}{2}$.

We investigate a particular flow of such functions $\{\hat{N}_\lambda\}_{\lambda \geq 1}$ which converges locally uniformly to $\zeta(s)$ as $\lambda \to 1$, and show that they exhibit features similar to $\zeta(s)$. For example, $\hat{N}_\lambda(s)$ has roughly $\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi}$ zeros in the critical strip up to height T and an infinite number of negative zeros, roughly at the points $\lambda - 1 - 2n$ ($n \in \mathbb{N}$).

2010 AMS Mathematics Subject Classification: 11M41, 30C15.
Keywords and phrases: Zeros of Mellin transforms, Lindelöf function.

Introduction

One idea of approaching the Riemann Hypothesis (RH) is to construct a sequence or a flow of holomorphic functions converging to $\zeta(s)$, uniformly on compact subsets of $\mathbb{C} \setminus \{1\}$ in such a way that all the functions in the sequence have no zeros in[1] $H_{\frac{1}{2}}$. Then by Hurwitz's Theorem on the zeros of the limit function, RH would follow. Less stringently, we would only require that there are no zeros in half-planes converging to $H_{\frac{1}{2}}$. To make it worthwhile, it should be easier to locate the zeros of the sequence than of $\zeta(s)$ itself.

The problem with such an approach is of course how to choose your sequence or flow (if indeed this is possible). We shall restrict ourselves to Mellin transforms; i.e.

$$\hat{N}_\lambda(s) = \int_0^\infty x^{-s} \, dN_\lambda(x),$$

where λ ranges over some interval, say $\lambda \in [0, 1]$ with $N_\lambda(x) \to [x]$ as $\lambda \to 1$. Thus $\hat{N}_\lambda(s) \to \zeta(s)$.

For instance, one can imagine starting from very 'smooth' generalised primes and integers and 'flowing' to the actual primes and integers as time progresses. For example, we could start from $N_0(x) = x$ ($x \geq 1$) and zero otherwise and 'flow' to the function $N_1(x) = [x]$. Then $\hat{N}_0(s) = \frac{s}{s-1}$ 'flows' to $\hat{N}_1(s) = \zeta(s)$.

There are many 'natural' properties that a typical integrator $N(x)$ (or its Mellin transform) in such a flow could be assumed to have, by analogy with $[x]$ and its Mellin transform $\zeta(s)$. One property we shall assume at the outset is that $N(x) = 0$ for $x < 1$ and $N(1) = 1$. Thus N has a jump at 1 and so $\hat{N}(s) = 1 + \int_1^\infty x^{-s} dN(x)$, ensuring that $\hat{N}(s)$ is bounded away from zero in half-planes far enough to the right. In this paper we shall further assume that for $x \geq 1$, $N(x) - x$ is periodic with period 1. (This is true for the cases $N(x) = x$ and $N(x) = [x]$

[1]For $\theta \in \mathbb{R}$, we denote by H_θ the half plane $\{s \in \mathbb{C} : \Re s > \theta\}$.

mentioned above). A further property that could be considered is that $N(x)$ forms part of a generalised prime system; i.e. $N(x) = \exp_* \Pi(x)$ for some increasing function $\Pi(x)$, or in terms of Mellin transforms; $\log \hat{N}(s) = \hat{\Pi}(s)$. However, we shall not assume this here.

On the above assumptions $\hat{N}(s)$ has an analytic continuation to $H_0 \setminus \{1\}$ with a simple pole at $s = 1$. In fact, using the Fourier development of $N(x) - x$, we shall show (Theorem 1) that there is an analytic continuation to the rest of the complex plane as well, and furthermore $\hat{N}(s)$ satisfies a 'functional relationship' akin to the functional equation for $\zeta(s)$. As a corollary (Corollary 2) it follows that the associated Lindelöf function (see below for the definition) is at least $\frac{1}{2} - \sigma$ for $\sigma < \frac{1}{2}$, except in the case when $N(x) = x$. Denoting by Θ the supremum of the real parts of the zeros of $\hat{N}$, this further implies that $\Theta \geq \frac{1}{2}$.

In particular, this shows it is impossible to have a flow of such Mellin transforms from $\frac{s}{s-1}$ to $\zeta(s)$ in which the zeros gradually move to the right (unless RH is false).

In the final section, we discuss the zeros of a particular flow of such Mellin transforms $\{\hat{N}_\lambda\}_{\lambda \geq 1}$ whose integrator N_λ has Fourier coefficients proportional to $n^{-\lambda}$.

1. Some preliminaries and notation

Let S denote the space of functions $f : \mathbb{R} \to \mathbb{C}$ which are zero on $(-\infty, 1)$, right-continuous, and of local bounded variation. (See e.g. [2], pp.50-70.) For $\alpha \in \mathbb{R}$, let $S_\alpha = \{f \in S : f(1) = \alpha\}$.

Let $f \in S$. If $f(x) = O(x^A)$ for some A, then we define the *Mellin transform* by

$$\hat{f}(s) = \int_{1-}^{\infty} x^{-s}\, df(x).$$

This is well-defined for $\sigma = \Re s > \alpha$, where α is the infimum of A for which $f(x) = O(x^A)$. Indeed, in this half-plane, $\hat{f}$ is holomorphic. Integrating by parts gives

$$\hat{f}(s) = s \int_{1}^{\infty} \frac{f(x)}{x^{s+1}}\, dx.$$

A function F holomorphic in a vertical strip (except possibly at a finite number of isolated singularities) is said to be of *finite order* if

$$F(\sigma + it) = O(|t|^A) \qquad (|t| \geq t_0, \text{ some } t_0),$$

for each σ in the interval of the strip. As such, we may define the *Lindelöf* function $\mu(\sigma)$ to be the infimum of those A for which the above holds. It is well-known that μ is a convex function. In our case (with $F = \hat{N}$ and $N \in S_1$), μ will be decreasing and eventually zero since

$$|\hat{N}(s) - 1| \leq \int_{1}^{\infty} x^{-\sigma}\, d|N|(x) \to 0$$

as $\sigma \to \infty$.

Knowledge of the positivity of μ can be used for locating zeros because of the following result: *if f is of finite order in H_β and has at most finitely many zeros here and $\mu(\sigma) = 0$ for σ sufficiently large, then $\mu(\sigma) = 0$ for $\sigma > \beta$.* (This was shown to hold for Beurling zeta functions in [4], but actually the proof readily extends to general functions.) Thus, for example, if $\mu(\sigma) > 0$ for $\sigma < \frac{1}{2}$, then $f(s)$ has infinitely many zeros in each half-plane $H_{\frac{1}{2}-\delta}$ for every $\delta > 0$.

2. Main results and proofs

Suppose $N \in S_1$ and $N(x) = x - R(x)$ where $R(x)$ has period 1. Extend R to the whole real line by periodicity. Thus R is right continuous, locally of bounded variation, and $R(1) = 0$. Since R is of bounded variation, it possesses a Fourier series

$$a_0 + \sum_{n=1}^{\infty} b_n \cos 2\pi nx + \sum_{n=1}^{\infty} c_n \sin 2\pi nx$$

which converges to $\frac{1}{2}(R(x+0) + R(x-0))$, and the series is boundedly convergent (see [5], p.408). Also $b_n, c_n = O(\frac{1}{n})$.

Theorem 1
Suppose that $N(x) = x - R(x) \in S_1$ where R is periodic with period 1. Then $\hat{N}(s)$ has an analytic continuation to $\mathbb{C} \setminus \{1\}$ with a simple pole at $s = 1$ with residue 1. Furthermore $\hat{N}(s)$ is of finite order and for $\sigma < 0$ satisfies the relation

$$\hat{N}(s) = \frac{s}{s-1} + \int_0^1 x^{-s}\, dR(x) + (2\pi)^s \Gamma(1-s) \Big(\cos \frac{\pi s}{2} \sum_{n=1}^{\infty} b_n n^s - \sin \frac{\pi s}{2} \sum_{n=1}^{\infty} c_n n^s \Big). \qquad (2.1)$$

The proof of Theorem 1 shows that the Lindelöf function of $\hat{N}$ satisfies $\mu(\sigma) \leq \frac{1}{2} - \sigma$ for $\sigma \leq 0$, while of course $\mu(\sigma) = 0$ for $\sigma \geq 1$. By convexity one obtains upper bounds for all σ. We can get equality if we know that b_n and c_n are not identically zero. (Equivalently, since R is right-continuous, if R is not constant; i.e. non-zero.)

Corollary 2
Under the assumptions of Theorem 1, if $R \not\equiv 0$ then $\mu(\sigma) = \frac{1}{2} - \sigma$ for $\sigma \leq 0$ and $\mu(\sigma) \geq \mu_0(\sigma)$ for all σ, where

$$\mu_0(\sigma) = \begin{cases} 0 & \text{if } \sigma \geq \frac{1}{2} \\ \frac{1}{2} - \sigma & \text{if } \sigma < \frac{1}{2} \end{cases}.$$

It follows that $\hat{N}$ has infinitely many zeros in $H_{\frac{1}{2}-\delta}$ for any $\delta > 0$.

In particular, if we let Θ denote the supremum of the real parts of the zeros of $\hat{N}$, then $\Theta \geq \frac{1}{2}$.

Proof of Theorem 1. We have for $\sigma > 1$,

$$\hat{N}(s) = 1 + s \int_1^{\infty} \frac{N(x)}{x^{s+1}}\, dx = \frac{s}{s-1} - s \int_1^{\infty} \frac{R(x)}{x^{s+1}}\, dx.$$

The integral on the right converges for $\sigma > 0$, and so $\hat{N}(s)$ has an analytic continuation to H_0 except for a simple pole at $s = 1$ with residue 1. We can extend further to the left by noting that $a_0 = \int_0^1 R(x)dx$ so that $\int_0^X (R(x) - a_0)dx = O(1)$. Hence for $\sigma > 0$,

$$\hat{N}(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{a_0}{x^{s+1}}\, dx - s \int_1^{\infty} \frac{R(x) - a_0}{x^{s+1}}\, dx = \frac{s}{s-1} - a_0 - s \int_1^{\infty} \frac{R(x) - a_0}{x^{s+1}}\, dx.$$

The final integral converges and is holomorphic for $\sigma > -1$ and so this extends $\hat{N}(s)$ holomorphically to H_{-1}. Thus $\hat{N}(0) = -a_0$. Note that $\hat{N}(s)$ has finite order for $\sigma > -1$ since in this range, writing $V(x) = \int_1^x (R(y) - a_0)dy = O(1)$, we have

$$s \int_1^{\infty} \frac{R(x) - a_0}{x^{s+1}}\, dx = s(s+1) \int_1^{\infty} \frac{V(x)}{x^{s+2}}\, dx = O(|t|^2).$$

Also $s\int_0^1 \frac{R(x)-a_0}{x^{s+1}}\,dx$ converges for $\sigma<0$ and equals $s\int_0^1 \frac{R(x)}{x^{s+1}}\,dx+a_0=\int_0^1 x^{-s}\,dR(x)+a_0$. Thus,

$$\hat{N}(s)=\frac{s}{s-1}+\int_0^1 x^{-s}\,dR(x)-s\int_0^\infty \frac{R(x)-a_0}{x^{s+1}}\,dx \qquad \text{for } -1<\sigma<0. \tag{2.2}$$

Now we insert the Fourier series for $R(x)-a_0$. If we ignore all problems of convergence for the moment, the final integral of (2.2) becomes

$$\begin{aligned}
s\int_0^\infty \frac{R(x)-a_0}{x^{s+1}}\,dx &= s\int_0^\infty \frac{1}{x^{s+1}}\left(\sum_{n=1}^\infty b_n\cos 2\pi nx+\sum_{n=1}^\infty c_n\sin 2\pi nx\right)dx\\
&= s\sum_{n=1}^\infty\left(b_n\int_0^\infty \frac{\cos 2\pi nx}{x^{s+1}}\,dx+c_n\int_0^\infty \frac{\sin 2\pi nx}{x^{s+1}}\right)dx\\
&= s\sum_{n=1}^\infty (2\pi n)^s\left(b_n\Gamma(-s)\cos\frac{\pi s}{2}-c_n\Gamma(-s)\sin\frac{\pi s}{2}\right)\\
&= -\Gamma(1-s)(2\pi)^s\left(\cos\frac{\pi s}{2}\sum_{n=1}^\infty b_n n^s-\sin\frac{\pi s}{2}\sum_{n=1}^\infty c_n n^s\right),
\end{aligned} \tag{2.3}$$

and the result follows formally. However, the term-by-term integration is permissible since the Fourier series is boundedly convergent and b_n and c_n are both $O(1/n)$ (the argument is identical to the special case $c_n=\frac{1}{n}$ as in [6], p.15).

Thus (2.3) holds for $-1<\sigma<0$. But the RHS of (2.3) is holomorphic for $\sigma<0$. Hence this provides the analytic continuation of $\hat{N}(s)$ to $\mathbb{C}\setminus\{1\}$ and (2.3) holds for $\sigma\le -1$ also.

That $\hat{N}(s)$ is of finite order follows directly from (2.3). For $|\Gamma(1-s)(2\pi)^s\cos\frac{\pi s}{2}|=O(|t|^{1/2-\sigma})$ and similarly for the term involving sin, while $|\sum b_n n^s|\le\sum|b_n|n^\sigma=O(1)$ for $\sigma<0$ and also for $\sum c_n n^s$. Since $|\int_0^1 x^{-s}dR(x)|\le\int_0^1 1d|R|(x)=O(1)$, (2.3) gives, for $\sigma<0$,

$$|\hat{N}(\sigma+it)|=O(1)+O(|t|^{1/2-\sigma}).$$

□

Proof of Corollary 2. Consider the final term in (2.1) which can be written

$$\Gamma(1-s)(2\pi)^s\cos\frac{\pi s}{2}\sum_{n=1}^\infty n^s\left(b_n-c_n\tan\frac{\pi s}{2}\right) \tag{2.4}$$

and use the asymptotic bounds

$$\begin{aligned}
&|\Gamma(1-s)|=|\Gamma(1-\sigma-it)|\sim\sqrt{2\pi}|t|^{1/2-\sigma}e^{-\frac{\pi}{2}|t|},\\
&\left|\cos\frac{\pi s}{2}\right|\sim\frac{1}{2}e^{\frac{\pi}{2}|t|}, \qquad \text{and} \qquad \tan\frac{\pi s}{2}=\tan\left(\frac{\pi\sigma}{2}+i\frac{\pi t}{2}\right)=\operatorname{sgn}(t)i+O(e^{-\pi|t|}).
\end{aligned}$$

Thus the term in (2.4) is, in modulus, asymptotic to

$$\sqrt{\frac{\pi}{2}}\,|t|^{1/2-\sigma}\left(\left|\sum_{n=1}^\infty(b_n\pm ic_n)n^s\right|+O(e^{-\pi|t|})\right).$$

Since the coefficients b_n and c_n are not identically zero and, furthermore, are real, there is a least integer n_0 for which $b_{n_0} \pm ic_{n_0} \neq 0$. It follows that for σ sufficiently large and negative,

$$\left| \sum_{n=1}^{\infty} (b_n \pm ic_n) n^s \right| \geq \frac{1}{2} n_0^{\sigma} |b_{n_0} + ic_{n_0}|.$$

This implies that $\mu(\sigma) = \frac{1}{2} - \sigma$ for σ sufficiently large and negative. By convexity, $\mu(\sigma) \geq \mu_0(\sigma)$ for all σ. But for $\sigma \leq 0$, we already know that $\mu(\sigma) \leq \frac{1}{2} - \sigma$, so we have equality here.

□

Remarks

(a) Theorem 1 and Corollary 2 extend immediately to the case where $N(x) - cx$ is periodic for some constant c.

(b) Similar results can be obtained more generally if $R(x) = N(x) - x$ is almost-periodic under some extra assumptions. For example, suppose that

$$R(x) = a_0 + \sum_{n=1}^{\infty} b_n \cos 2\pi\lambda_n x + \sum_{n=1}^{\infty} c_n \sin 2\pi\lambda_n x,$$

and that the series is boundedly convergent with b_n and c_n both $O(1/n)$. Here suppose $\lambda_n > 0$ increases strictly and without bound. If we assume that $\sum \frac{\lambda_n^{\sigma}}{n}$ converges for every $\sigma < 0$, then the same method as in Theorem 1 shows that $\hat{N}$ has an analytic continuation to $\mathbb{C} \setminus \{1\}$, is of finite order and satisfies

$$\hat{N}(s) = \frac{s}{s-1} + \int_0^1 x^{-s}\, dR(x) + (2\pi)^s \Gamma(1-s) \left(\cos \frac{\pi s}{2} \sum_{n=1}^{\infty} b_n \lambda_n^s - \sin \frac{\pi s}{2} \sum_{n=1}^{\infty} c_n \lambda_n^s \right),$$

for $\sigma < 0$. Corollary 2 also holds in this case if the b_n and c_n are not identically zero (i.e. $R(x)$ not constant).

(c) The inequality $\mu \geq \mu_0$ seems quite robust. It holds for the Beurling zeta function associated to discrete g-prime systems (see [3]) but also for those Mellin transforms contained in (a) and (b) above. What is a natural setting for which this inequality is true?

3. A particular flow of Mellin transforms to $\zeta(s)$

As Corollary 2 shows, it is impossible to construct a flow of Mellin transforms with 'periodic' integrator converging to $\zeta(s)$ such that the supremum of the real parts of the zeros converges to $\frac{1}{2}$ from below. Nevertheless, it might still be of interest to investigate a particular flow of such systems with $N(x) - x$ periodic.

Here we consider a particular flow of Mellin transforms $\{\hat{N}_\lambda(s)\}_{\lambda \geq 1}$ converging uniformly to $\zeta(s)$ as $\lambda \to 1$, and for which $N_\lambda(x) - x$ has period 1 with Fourier coefficients proportional to $\frac{1}{n^\lambda}$. We shall see that for $\lambda > 1$, $\hat{N}_\lambda(s)$ shares a number of characteristics of $\hat{N}_1(s) = \zeta(s)$. Thus $\hat{N}_\lambda(s)$ has roughly $\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi}$ zeros in H_0 up to height T and an infinite number of negative zeros, roughly at the points $\lambda - 1 - 2n$ ($n \in \mathbb{N}$).

The Hurwitz zeta function $\zeta(s, a)$, defined for $\Re s > 1$ and $0 < a \leq 1$ by the series $\sum_{n=0}^{\infty} (n + a)^{-s}$ has (as a function of s) an analytic continuation to $\mathbb{C} \setminus \{1\}$ and a simple pole at $s = 1$

with residue 1 (see for example [1], Chapter 12). Its analytic continuation is given by $\zeta(s,a) = \Gamma(1-s)I(s,a)$, where $I(s,a)$ is the entire function

$$I(s,a) = \frac{1}{2\pi i}\int_C \frac{z^{s-1}e^{az}}{1-e^z}\,dz,$$

where C is the contour which starts at $-\infty$, goes along the negative real axis (on the lower side) to $-c$ where $0 < c < 2\pi$, encircles the origin back to $-c$ and returns to $-\infty$ on the upper side of the negative real axis. Note that $\zeta(s,1) = \zeta(s)$. The definition actually makes sense whenever $\Re a > 0$ (any s). As a function of a (for any given s), $I(s,a)$ is holomorphic for $\Re a > 0$.

Definition: Let $N_\lambda(x) = x - R_\lambda(x)$ for $x \geq 1$ and zero otherwise and $\lambda \geq 1$, where $R_\lambda(x)$ is periodic with period 1 and be defined for $0 \leq x < 1$ by

$$R_\lambda(x) = \rho_\lambda(\zeta(1-\lambda, 1-x) - \zeta(1-\lambda)) = \frac{\rho_\lambda\Gamma(\lambda)}{2\pi i}\int_C \frac{z^{-\lambda}(e^{-xz}-1)}{e^{-z}-1}\,dz. \tag{3.1}$$

Here ρ_λ is a continuous function of λ (to be determined) and we set $\rho_1 = 1$ so that $R_1(x) = \{x\}$.

Some properties

(a) For $\lambda = m \in \mathbb{N}$, R_m is a polynomial in $[0,1)$ since $\zeta(-n,a) = -\frac{B_{n+1}(a)}{n+1}$ where $B_n(\cdot)$ is the n^{th} Bernoulli polynomial; i.e.

$$R_m(x) = \frac{\rho_m}{m}(B_m(1) - B_m(1-x)) = \frac{(-1)^{m-1}\rho_m}{m}(B_m(x) - B_m(0)). \qquad (0 \leq x < 1)$$

(b) For $\lambda > 1$ R_λ is continuous, while R_1 is right continuous but has jump continuities at the integers. On the interval $[0,1)$, R_λ is holomorphic since the function

$$R^*_\lambda(z) = \rho_\lambda(\zeta(1-\lambda, 1-z) - \zeta(1-\lambda)),$$

which agrees with R_λ on [0,1), is holomorphic for $\Re z < 1$. Hence we have an expansion

$$R_\lambda(x) = \sum_{n=1}^{\infty} a_n(\lambda)x^n \qquad (0 \leq x < 1)$$

for some coefficients $a_n(\lambda)$. Expanding the integrand in (3.1) gives a formula for the coefficients.

$$\begin{aligned} R_\lambda(x) &= \frac{\rho_\lambda\Gamma(\lambda)}{2\pi i}\int_C \frac{z^{-\lambda}}{e^{-z}-1}\sum_{n=1}^{\infty}(-1)^n\frac{x^n z^n}{n!}\,dz = \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left(\frac{\rho_\lambda\Gamma(\lambda)}{2\pi i}\int_C \frac{z^{n-\lambda}}{e^{-z}-1}\,dz\right)x^n \\ &= \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{\rho_\lambda\Gamma(\lambda)\zeta(n-\lambda+1)}{\Gamma(\lambda-n)}x^n. \end{aligned}$$

Hence

$$a_n(\lambda) = (-1)^n\rho_\lambda\binom{\lambda-1}{n}\zeta(n+1-\lambda). \tag{3.2}$$

For $\lambda > 1$ the expansion is also valid for $x = 1$, since $a_n(\lambda) = O(n^{-\lambda})$. For $\lambda = m \in \mathbb{N}$ and $n = m$, (3.2) should be interpreted as $\lim_{\lambda\to m} a_m(\lambda) = (-1)^{m-1}\rho_m/m$. Of course in this case the expansion is finite and is a polynomial of degree m.

(c) Fourier expansion: We have

$$R_\lambda(x) = -\frac{2\rho_\lambda\Gamma(\lambda)}{(2\pi)^\lambda}\left(\cos\frac{\pi\lambda}{2}\sum_{n=1}^\infty \frac{1-\cos 2\pi nx}{n^\lambda} + \sin\frac{\pi\lambda}{2}\sum_{n=1}^\infty \frac{\sin 2\pi nx}{n^\lambda}\right)$$

which holds for all $x \in \mathbb{R}$ if $\lambda > 1$ and for $x \in \mathbb{R}\setminus\mathbb{Z}$ if $\lambda = 1$ ([1], p.257).

By Theorem 1, $\hat{N}_\lambda$ extends analytically to the complex plane except for a simple pole at 1 and (after some calculation)

$$\hat{N}_\lambda(s) = \frac{s}{s-1} + \int_0^1 x^{-s}\, dR_\lambda(x) + 2\rho_\lambda(2\pi)^{s-\lambda}\Gamma(\lambda)\Gamma(1-s)\cos\frac{\pi(s-\lambda)}{2}\zeta(\lambda-s). \tag{3.3}$$

Using the functional equation for $\zeta(\lambda - s)$ this becomes

$$\hat{N}_\lambda(s) = \frac{s}{s-1} + \int_0^1 x^{-s}\, dR_\lambda(x) + \rho_\lambda\frac{\Gamma(\lambda)\Gamma(1-s)}{\Gamma(\lambda-s)}\zeta(s-\lambda+1). \tag{3.4}$$

For $\lambda > 1$ we have for $\sigma < 1$,

$$\int_0^1 x^{-s}\, dR_\lambda(x) = \int_0^1 x^{-s}R'_\lambda(x)\, dx = \int_0^1 \sum_{n=1}^\infty na_n(\lambda)x^{n-s-1}\, dx = \sum_{n=1}^\infty \frac{na_n(\lambda)}{n-s}. \tag{3.5}$$

This series converges for all $s \notin \mathbb{N}$ and provides the meromorphic continuation of the LHS to $\mathbb{C}$ with (at most) simple poles at the positive integers. Thus (3.3)-(3.5) hold for all s.

Theorem 3
With N_λ as defined above, we have $\hat{N}_\lambda(s) \to \zeta(s)$ as $\lambda \to 1$ uniformly on compact subsets of $\mathbb{C}\setminus\{1\}$.

Proof. This basically follows from the fact that $R_\lambda \to R_1$ uniformly on $[0, a]$ for every $a < 1$, but we need to be a little careful near 1 since R_1 is not continuous here. First consider $\sigma > 0$. Let K be a compact subset of $H_0 \setminus \{1\}$. We have for $s \in K$

$$|\hat{N}_\lambda(s) - \hat{N}_1(s)| = \left|s\int_1^\infty \frac{R_\lambda(x) - R_1(x)}{x^{s+1}}\, dx\right| \le A\int_1^\infty \frac{|R_\lambda(x) - R_1(x)|}{x^{\sigma_0+1}}\, dx$$

for some constants $A, \sigma_0 > 0$. Let $\eta > 0$. Then for all $\varepsilon > 0$, there exists λ_0 such that for $1 < \lambda < \lambda_0$, $|R_1(x) - R_\lambda(x)| < \varepsilon$ for $n \le x \le n+1-\eta$ (any $n \in \mathbb{Z}$). Hence

$$|\hat{N}_\lambda(s) - \hat{N}_1(s)| \le A\varepsilon\int_1^\infty \frac{1}{x^{\sigma_0+1}}\, dx + A\sum_{n=1}^\infty \int_{n+1-\eta}^{n+1} \frac{C}{x^{\sigma_0+1}}\, dx \le A_1\varepsilon + AC\eta\sum_{n=1}^\infty \frac{1}{n^{\sigma_0+1}},$$

which can be made as small as we please. Hence $\hat{N}_\lambda(s) \to \hat{N}_1(s)$ uniformly on compact subsets of $H_0 \setminus \{1\}$.

In fact the same argument works for compact subsets of $H_{-1} \setminus \{1\}$ if we use the expression

$$\hat{N}_\lambda(s) = \frac{s}{s-1} - a_0 + s(s+1)\int_1^\infty \frac{V_\lambda(x)}{x^{s+2}}\, dx,$$

where $V_\lambda(x) = \int_1^x (R_\lambda(\cdot) - a_0)$, and noting that $V_\lambda \to V_1$ uniformly.

For $\sigma < 0$ we can use (3.4). The final term tends locally uniformly to $\zeta(s)$, while

$$\int_0^1 x^{-s}\, dR_\lambda(x) = s\int_0^1 \frac{R_\lambda(x)}{x^{s+1}}\, dx \to s\int_0^1 \frac{R_1(x)}{x^{s+1}}\, dx = -\frac{s}{s-1},$$

the convergence again being uniform. The result now follows.

□

Zeros

Since $\hat{N}_\lambda(s) \to \zeta(s)$ locally uniformly, the Riemann Hypothesis will follow if we can show that for all λ close to 1 (with some particular choice of ρ_λ), $\hat{N}_\lambda(s)$ has no zeros with $\sigma > \frac{1}{2}$. Slightly less restrictively, RH is true if the following conjecture is true:

Conjecture: Given $\theta > \frac{1}{2}$, there exists $\lambda_\theta > 1$ such that for $1 < \lambda < \lambda_\theta$ and some suitable choice of ρ_λ, $\hat{N}_\lambda$ has no zeros in H_θ.

It may even be the case that this conjecture is equivalent to RH. The hope is of course that it is easier to show that for $\lambda > 1$, $\hat{N}_\lambda$ has no zeros in H_θ than it is for $\lambda = 1$.

Now we show that for $\lambda > \frac{3}{2}$, $\hat{N}_\lambda$ has only *finitely* many zeros in $H_{\frac{1}{2}+\delta}$ (any $\delta > 0$). As λ gets closer to 1 however, we can only be certain of having finitely many zeros in half-planes further to the right, since we do not have the strong bounds on ζ in vertical strips. If we assume the Lindelöf Hypothesis (LH), then $\hat{N}_\lambda$ has only finitely many zeros in $H_{\frac{1}{2}+\delta}$ for *every* $\lambda > 1$.

Theorem 4

(i) Let $\lambda \geq \frac{3}{2}$. Then for every $\delta > 0$, $\hat{N}_\lambda(s)$ has at most finitely many zeros in $H_{\frac{1}{2}+\delta}$ and in every strip where $\sigma \in [-A, \frac{1}{2} - \delta]$ (any A).

(ii) Let $1 < \lambda < \frac{3}{2}$. Then for every $\delta > 0$, $\hat{N}_\lambda(s)$ has at most finitely many zeros in $H_{2-\lambda+\delta}$ ($H_{\frac{1}{2}+\delta}$ on LH) and in every strip where $\sigma \in [-A, \lambda - 1 - \delta]$ ($\sigma \in [-A, \frac{1}{2} - \delta]$ on RH).

Proof. For $\lambda > 1$, $\int_0^1 x^{-s} dR_\lambda(x) = \sum_{n=1}^\infty \frac{na_n(\lambda)}{n-s} \to 0$ as $|t| \to \infty$ for every σ. Hence from (3.4),

$$\hat{N}_\lambda(\sigma + it) = 1 + o(1) + \rho_\lambda \frac{\Gamma(\lambda)\Gamma(1-\sigma-it)}{\Gamma(\lambda-\sigma-it)}\zeta(\sigma - \lambda + 1 + it).$$

The term on the right is, in modulus, asymptotic to

$$|\rho_\lambda|\Gamma(\lambda)\frac{|\zeta(\sigma-\lambda+1+it)|}{|t|^{\lambda-1}} = O(|t|^{\mu(\sigma-\lambda+1)-\lambda+1+\varepsilon}), \tag{3.6}$$

for every $\varepsilon > 0$, where $\mu(\cdot)$ is the Lindelöf function for ζ. Note that the implied constant is independent of σ for $a \leq \sigma \leq b$, any a, b.

Let $\lambda > \frac{3}{2}$. Consider $\sigma \leq \lambda - 1$ and $\sigma > \lambda - 1$ separately. If $\sigma \leq \lambda - 1$, then $\mu(\sigma - \lambda + 1) = \lambda - \sigma - \frac{1}{2}$, and the exponent of $|t|$ in (3.6) is $\frac{1}{2} - \sigma + \varepsilon$. This is negative (for sufficiently small ε) if $\sigma > \frac{1}{2}$. If $\sigma > \lambda - 1$, $\mu(\sigma - \lambda + 1) < \frac{1}{2}$, so the exponent is also negative for ε small enough. Since the bound is uniform in σ, and there are no zeros in H_A for A sufficiently large, this implies that for $\lambda \geq \frac{3}{2}$, $\hat{N}_\lambda$ has only finitely many zeros in $H_{\frac{1}{2}+\delta}$ for each $\delta > 0$.

If $\sigma < \frac{1}{2}$, then $\sigma < \lambda - 1$ and the expression in (3.6) is at least[2]

$$c|t|^{\frac{1}{2}-\sigma},$$

[2]Assuming $\rho_\lambda \neq 0$. If $\rho_\lambda = 0$, the result is trivially true.

for some $c > 0$, depending continuously on λ and σ. Hence for $-A \le \sigma \le \frac{1}{2} - \delta$, this is at least $c_1|t|^\delta$ (some constant $c_1 > 0$) which tends to infinity. Thus there are no zeros with $|t|$ sufficiently large in such a strip, proving assertion (i).

Now consider $1 < \lambda < \frac{3}{2}$. If $\sigma \ge \lambda$, then $\mu(\sigma - \lambda + 1) = 0$ and the exponent in (3.6) is negative. For $\lambda - 1 \le \sigma < \lambda$, $\mu(\sigma - \lambda + 1) \le \frac{\lambda - \sigma}{2}$ (using $\mu(\alpha) \le \frac{1-\alpha}{2}$ for $0 \le \alpha \le 1$) and the exponent in (3.6) is $1 - \frac{\lambda+\sigma}{2} + \varepsilon$. This is negative for $\sigma > 2 - \lambda$, and the result follows.

If L.H. holds, then $\mu(\sigma - \lambda + 1) = 0$ for $\sigma > \lambda - \frac{1}{2}$ and $\mu(\sigma - \lambda + 1) = \lambda - \sigma - \frac{1}{2}$ for $\sigma \le \lambda - \frac{1}{2}$. Hence the exponent in (3.6) is now

$$\begin{array}{ll} 1 - \lambda + \varepsilon & \text{if } \sigma > \lambda - \frac{1}{2} \\ \frac{1}{2} - \sigma + \varepsilon & \text{if } \sigma \le \lambda - \frac{1}{2} \end{array}.$$

Both are negative if $\sigma > \frac{1}{2}$ for sufficiently small ε.

As in part(i), if $\sigma < \lambda - 1$, then $\sigma - \lambda + 1 < 0$ and the expression in (3.6) is at least $c|t|^{\frac{1}{2} - \sigma} \to \infty$. For $\sigma \ge \lambda - 1$ we cannot deduce anything about (3.6) for large $|t|$ unless we know that ζ has no zeros in certain strips inside the critical strip. On R.H., the above argument applies for $\sigma - \lambda + 1 < \frac{1}{2}$, and (ii) follows.

□

Remark. For $\lambda > \frac{3}{2}$, the zeros in any right half-plane (apart from at most a finite number of exceptions) actually lie in a region

$$\left\{\sigma + it : -\frac{A}{\log|t|} \le \sigma - \frac{1}{2} \le \frac{B}{\log|t|}, |t| \ge 2\right\},$$

for some constants A, B. For $\hat{N}_\lambda(s) = 0$ if and only if

$$\frac{s}{s-1} + \int_0^1 x^{-s}\, dR_\lambda(x) = -2\rho_\lambda(2\pi)^{s-\lambda}\Gamma(\lambda)\Gamma(1-s)\cos\frac{\pi(s-\lambda)}{2}\zeta(\lambda - s). \tag{3.7}$$

Take σ such that $|\sigma - \frac{1}{2}| \le \lambda - \frac{3}{2} - \delta$ for some $\delta > 0$, and $|t| \ge 2$. The LHS of (3.7) is $1 + o(1)$, while the RHS is, in modulus,

$$\sim \frac{|\rho_\lambda|\Gamma(\lambda)}{(2\pi)^{\lambda - \sigma - \frac{1}{2}}}|t|^{\frac{1}{2} - \sigma}|\zeta(\lambda - \sigma - it)|.$$

Since $\lambda - \sigma \ge 1 + \delta$, this is $\asymp |t|^{\frac{1}{2} - \sigma}$, uniformly in σ. In particular, for $\frac{1}{2} - \sigma > A/\log|t|$ and A sufficiently large, the LHS of (3.7) is less than the RHS in modulus, and hence there are no zeros for $|t|$ sufficiently large in this range. Similarly, for $\sigma - \frac{1}{2} > B/\log|t|$ and B sufficiently large, the LHS is greater than the RHS in modulus.

We can be more precise. Let $\sigma = \frac{1}{2} + \frac{\theta_t}{\log|t|}$ where $\theta_t = O(1)$. Then for a zero $\sigma + it$ with large $|t|$, we need

$$\frac{|\rho_\lambda|\Gamma(\lambda)}{(2\pi)^{\lambda-1}}e^{-\theta_t}|\zeta(\lambda - \sigma - it)| \sim 1.$$

Since $|\zeta(\lambda - \sigma - it)| \sim |\zeta(\lambda - \frac{1}{2} - it)|$, this requires

$$\theta_t = \log\left(\frac{|\rho_\lambda|\Gamma(\lambda)}{(2\pi)^{\lambda-1}}\left|\zeta\left(\lambda - \frac{1}{2} - it\right)\right|\right) + o(1).$$

As such and taking $t \geq 2$, the RHS of (3.7) is, using Stirling's formula, asymptotically

$$-\frac{\rho_\lambda \Gamma(\lambda)}{(2\pi)^{\lambda-1}} e^{\theta_t + \frac{i\pi}{2}(\lambda - \frac{1}{2})} e^{-i(t\log t - t - t\log 2\pi)} \zeta\left(\lambda - \frac{1}{2} - it\right) = -\frac{\rho_\lambda \zeta(\lambda - \frac{1}{2} - it)}{|\rho_\lambda \zeta(\lambda - \frac{1}{2} - it)|} e^{-i(t\log t - t - t\log 2\pi - \frac{\pi}{2}(\lambda - \frac{1}{2}))}.$$

At a zero, we want this to be asymptotic to the LHS of (3.7); i.e. to 1. Thus we want $t\log t - t - t\log 2\pi = 2\pi k + O(1)$ for $k \in \mathbb{Z}$; i.e.

$$f(t) := \frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi} = k + O(1).$$

Since $f(t)$ is continuous we should expect a zero $\sigma_k + it_k$ for each k sufficiently large. The number of such zeros with $t_k \leq T$ is therefore roughly $f(T)$; i.e. we should expect, for $\lambda > \frac{3}{2}$,

$$\frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(1)$$

zeros up to height T.

Theorem 5
Let $\lambda > 1$. Then $\hat{N}_\lambda$ has

$$\frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

zeros in the rectangular strip $\{\sigma + it : 0 \leq \sigma \leq 1, 0 \leq t \leq T\}$.

Proof. Choose σ_0 sufficiently large so that $|\Re \hat{N}_\lambda(\sigma_0 + it)| \geq c > 0$ for all t.

Denote by $n(T)$ the number of zeros in the rectangular strip

$$\{\sigma + it : 0 \leq \sigma \leq \sigma_0, 1 \leq t \leq T\}.$$

This differs from the required number by $O(1)$. Let γ denote the (anti-clockwise) boundary path of this strip. We may assume without loss of generality that there are no zeros of $\hat{N}_\lambda$ on γ. Then

$$n(T) = \frac{1}{2\pi}\Delta_\gamma \arg \hat{N}_\lambda,$$

where $\Delta_\gamma \arg \hat{N}_\lambda$ is the continuous variation of the argument of $\hat{N}_\lambda$ around γ.

On the right-hand vertical, $\hat{N}_\lambda(\sigma_0 + it) \to 1$ as $t \to \infty$. Hence the variation of the argument along this vertical line segment is $O(1)$. For the top horizontal, we use Lemma 9.4 of [6] (with '2' replaced by 'σ_0'). Since $\hat{N}_\lambda$ has finite order, this Lemma implies that the variation along here is at most $O(\log T)$. The variation along the bottom horizontal is trivially $O(1)$. Finally on the left vertical, we have

$$\begin{aligned}\hat{N}_\lambda(it) &= 2\rho_\lambda \Gamma(\lambda)(2\pi)^{it-\lambda}\Gamma(1-it)\cos\frac{\pi(it-\lambda)}{2}\zeta(\lambda - it) + 1 + o(1) \\ &\sim \frac{\rho_\lambda \Gamma(\lambda)}{(2\pi)^{\lambda - \frac{1}{2}}} t^{\frac{1}{2}} e^{-i(t\log t - t - t\log 2\pi)} e^{\frac{i\pi}{2}(\lambda - \frac{1}{2})}\zeta(\lambda - it).\end{aligned}$$

Since $\zeta(\lambda - it)$ is bounded and bounded away from zero, $\arg \hat{N}_\lambda(it) = -(t\log t - t - t\log 2\pi) + O(1)$, and the variation of the argument along the (downward) left hand vertical is $T\log T - T - T\log 2\pi + O(1)$.

□

Remark. It seems plausible that the $O(\log T)$-term can be replaced by $O((\log T)^\kappa)$, with κ decreasing steadily from 1 to 0 as λ varies from 1 to $\frac{3}{2}$.

Zeros on the negative real axis: For $\lambda = 1$, $\hat{N}_\lambda(s) = \zeta(s)$ has zeros on the negative real axis at $-2k$ for each positive integer k — the so-called trivial zeros. Very similar behaviour occurs for $\lambda > 1$.

We require the following elementary result.

Lemma 6
Suppose f is holomorphic and real valued on $[0, \infty)$. Suppose further that, as $x \to \infty$,

$$f(x) = \cos\frac{\pi x}{2} + o(1) \quad \text{and} \quad f'(x) = -\frac{\pi}{2}\sin\frac{\pi x}{2} + o(1).$$

Then for every sufficiently large integer n, the interval $(2n, 2n+2)$ contains exactly one zero, say x_n, and $x_n = 2n + 1 + o(1)$.

Proof. For $n \in \mathbb{N}$, $f(2n) - (-1)^n \to 0$, so for n sufficiently large, the sign of $f(2n)$ is $(-1)^n$. Hence there is at least one zero in each interval $(2n, 2n+2)$ (for n large). In fact the zero(s) must be close to $2n+1$ since for $|h| \leq 1$,

$$f(2n+h) - (-1)^n \cos\frac{\pi h}{2} \to 0,$$

uniformly in h, and $\cos\frac{\pi h}{2}$ is bounded away from zero if $|h| < 1$.

Now for $x = 2n + y$, $f'(x) = (-1)^{n-1}\frac{\pi}{2}\sin\frac{\pi y}{2} + o(1)$, so for $x \in [2n+h, 2n+2-h]$ (any fixed $h > 0$), $(-1)^{n-1}f'(x) > 0$ for n large enough; i.e. f is monotonic in this interval. Thus can be at most one zero, say x_n. This must satisfy $x_n = 2n + 1 + o(1)$.

□

Theorem 7
For every sufficiently large positive integer n, $\hat{N}_\lambda(\lambda - x)$ has exactly one zero x_n in each interval $(2n, 2n+2)$ $(n \in \mathbb{N})$. Furthermore $x_n = 2n + 1 + o(1)$ as $n \to \infty$.

Proof. Apply Lemma 6 with

$$f(x) = \frac{(2\pi)^x \hat{N}_\lambda(\lambda - x)}{2\rho_\lambda \Gamma(\lambda)\Gamma(x+1-\lambda)} = \zeta(x)\cos\frac{\pi x}{2} + \frac{(2\pi)^x}{2\rho_\lambda \Gamma(\lambda)\Gamma(x+1-\lambda)}\Big(\frac{x}{x+1} + \sum_{m=1}^{\infty}\frac{m a_\lambda(m)}{m+x}\Big)$$

(using (3.5)). The final term and its derivarive tend to 0 with x, while $\zeta(x) \to 1$, $\zeta'(x) \to 0$, so f satisfies the conditions of Lemma 6 and the result follows.

□

Acknowledgement. It is a pleasure for me to thank Eric Saias for some stimulating discussions on flows of Mellin transforms to the Riemann zeta function.

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976.

[2] P. T. Bateman and H. G. Diamond, *Analytic Number Theory*, World Scientific Publishing, 2004.

[3] T. W. Hilberdink, A lower bound for the Lindelöf function associated to generalised integers, *J. Number Theory* **122** (2007) 336-341.

[4] T. W. Hilberdink and M. L. Lapidus, Beurling zeta functions, Generalised Primes, and Fractal Membranes, *Acta Appl Math* **94** (2006) 21-48.

[5] E. C. Titchmarsh, *The Theory of Functions*, Second edition, Oxford University Press, 1986.

[6] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, Second edition, Oxford University Press, 1986.