An Assessment of Quasi-Newton Sparse Update Techniques for Nonlinear Structural Analysis

by

M. P. Kamat, L. T. Watson and D. J. Vanden Brink

Technical Report Number CS80013-R

AN ASSESSMENT OF QUASI-NEWTON SPARSE UPDATE TECHNIQUES FOR NONLINEAR STRUCTURAL ANALYSIS

M. P. Kamat*, L. T. Watson+ and D. J. Vanden Brink*
Virginia Polytechnic Institute and State University
Blacksburg, Virginia 24061, U.S.A.

SUMMARY

In this paper an attempt is made to evaluate the performance of a few algorithms for unconstrained minimization of nonlinear functions that exploit sparsity of the Hessians of such functions. The evaluation is centered around large scale, geometrically nonlinear problems of structural analysis in general. In particular, the snap-through response of finite element models of a shallow elastic arch under a concentrated load at the crown is considered. The sensitivity of these algorithms to varying degrees of refinement of these finite element models as well as to the sparsity pattern of the Hessian of the potential surface in question are examined. The paper concludes by making recommendations on the choice of an algorithm based on the scale of the problem and the degree and type of nonlinearity.

I. INTRODUCTION

Nonlinear analysis of structures has been of increasing interest to engineers by virtue of their interest in minimizing human and property damage resulting from the catastropic failure of such structures under crash, blast or seismic conditions. Complexities of the structural configuration and its equally complex large deflection response in the presence of material inelasticity make finite element modeling of such

---

*Associate Professor and Graduate Student respectively in the Department of Engineering Science and Mechanics.

+Associate Professor, Department of Computer Science.

structures a very natural and plausible recourse.

Two distinct solution approaches exist for the prediction of the nonlinear response of finite element models of structures: (i) the vector approach and (ii) the scalar approach. In the former, the mathematical model is derived on the basis of the principle of virtual work and reduces to a system of nonlinear second-order differential equations in time. In the latter approach, a scalar or a potential function associated with the energy of the model is introduced, minimization of which yields the desired stable equilibrium configuration.

In both approaches a temporal finite difference scheme is utilized to effectively eliminate time as a variable. As a result, in the vector approach the equations of motion are reduced to a system of nonlinear algebraic equations in the unknown nodal parameters of the finite element model. In the scalar approach, which is of relevance to this paper, the problem is reduced to a well-known problem in mathematical programming namely the unconstrained minimization of a nonlinear function of several variables.

II. THE BFGS ALGORITHM

The scalar approach has been used for nonlinear structural analysis by several investigators [1]-[5]. Quite recently, this approach was successfully used for analyzing the dynamic response of a finite element model of a twin-engine, low-wing airplane section subjected to a vertical impact velocity crash condition [6]. Of interest was the prediction of the time of occurrence of the initial peak and the magnitude of the acceleration of the occupant inside the substructure. The finite element model of Fig. 1 used for this analysis involved a total of 336 degrees of freedom, 105 nodes, 209 elements (including 96 membranes, 77 frame elements and 36 stringer elements). For unconstrained minimization of the potential function the formulation used the well known Broyden-Fletcher-Goldfarb-Shanno (BFGS) variable metric algorithm [7] which dispenses with the exact line search while using an inverse Hessian update formula which, in the case of a quadratic functional, guarantees convergence of the approximating matrix to the inverse of the actual Hessian.

Let $f(\underline{x})$ be the potential function, where $\underline{x}$ is the vector of generalized displacements. The BFGS algorithm starts with an approximation $\underline{x}_0$ to $\underline{x}$ and an approximation $\underline{H}_0$ to the inverse Hessian matrix of $f(\underline{x})$ at $\underline{x}_0$. BFGS has the important property that successive inverse Hessian approximations are symmetric and positive definite if $\underline{H}_0$ has these properties. At the ith iteration, a search direction $\underline{p}_i$ is defined by

$$\underline{p}_i = - \underline{H}_i \underline{g}_i \qquad (1)$$

where $\underline{H}_i$ is the current inverse Hessian approximation and $\underline{g}_i$ is the gradient of $f(\underline{x})$ at $\underline{x}_i$. The next iterate is then

$$\underline{x}_{i+1} = \underline{x}_i + t_i \underline{p}_i \qquad (2)$$

where t_i is chosen to decrease or minimize a certain norm of the function or its gradient. The rationale for choosing t_i is somewhat technical and the choice of t_i is a subject of active research. A good scheme, with rigorous mathematical justification, is described in [7]. Next the inverse Hessian approximation is updated by

$$\underline{H}_{i+1} = \left(I - \frac{sy^t}{y^t s}\right) \underline{H}_i \left(I - \frac{ys^t}{y^t s}\right) + \frac{ss^t}{y^t s} \qquad (3)$$

where $s = \underline{x}_{i+1} - \underline{x}_i$ and $y = - t_i \underline{g}_i$. (4)

The iterative scheme is begun with the vector of the unknown generalized displacements being the null vector and the inverse Hessian approximation being the identity matrix. Although overall a very powerful algorithm, in spite of its sluggishness in the first load or time step, its storage requirements (upper or lower half of the symmetrix matrix requiring n(n+1)/2 storage locations) does not make it very cost-effective with incremental techniques like the pseudo force technique often referred to as a one step Newton-Raphson technique. For instance, for comparable models of the aircraft substructure shown in Fig. 1 a time and cost comparisons of the analyses on a CYBER 175 machine using the BFGS algorithm and the pseudo force technique revealed that the BFGS algorithm was superior to the pseudo force technique in terms of CPU time by a factor of 2 or better while costing 22% more for every 0.01 second of response [6]. This is presumably because of higher in-core storage requirements of the BFGS algorithm. It needs to be pointed out however, that the pseudo force technique utilized a model with an optimized bandwidth with a Cholesky decomposition scheme for solutions of the incrementally linearized equations. The performance of the BFGS algorithm on the other hand is insensitive to bandwidth optimization and cannot therefore exploit this property of most finite element models.

III. ALGORITHMS FOR NONLINEAR SYSTEMS THAT EXPLOIT SPARSITY

Past experiments using minimization algorithms for structural analysis reveal that at least for small scale problems the energy minimization technique is better suited than most other incremental techniques for solving highly nonlinear problems [5]. Like the second-order minimization algorithms most incremental techniques use some variants of the Newton-Ralphson technique, but whereas the minimization algorithms attempt to optimize their move in the Quasi-Newton direction most incremental techniques are usually locked into taking a full step in the predicted Newton or Quasi-Newton direction. As will be apparent from this paper a full step in the Newton direction is not always the best strategy.

Extension of the minimization algorithms to large scale problems centers on reducing the storage requirements of the second order Quasi-Newton methods (BFGS, DFP [8], etc.) or improving the efficiency of the first order conjugate gradient techniques. In the past few years the mathematicians and computer scientists have been attacking the problem areas which inhibit the extension of the minimization algorithms to large scale problems. Two alternatives are presently available: (a) the preconditioned conjugate gradient technique or (b) the variable metric methods that exploit sparsity and utilize singular perturbation theory or scaling to eliminate ill-conditioning [9]. In the present paper we look at some of the variable metric methods that exploit sparsity.

a. Schubert's Algorithm

The first attempts at exploiting sparsity in the matrix updating process appear to have been those of Schubert [10] who proposed a modification of the Broyden's method [11] for solving sparse nonlinear systems of equations

$$\underline{F}(\underline{x}) = \underline{0}. \tag{5}$$

The iterates are computed by

$$\underline{B}_i \underline{p}_i = - \underline{F}(\underline{x}_i), \tag{6}$$

$$\underline{x}_{i+1} = \underline{x}_i + t_i \underline{p}_i, \tag{7}$$

where $\underline{B}_i$ is the current approximation to the Jacobian matrix of $\underline{F}(\underline{x})$. $\underline{B}_i$ is updated row by row according to

$$\underline{B}_{i+1}^{(k)} = \underline{B}_i^{(k)} + \frac{[F_k(\underline{x}_{i+1})-(1-t_i)F_k(\underline{x}_i)]\hat{\underline{p}}_i}{t_i \hat{\underline{p}}_i^{\,t} \hat{\underline{p}}_i} \tag{8}$$

where $\underline{B}_i^{(k)}$ is the kth row of $\underline{B}_i$ and $\hat{\underline{p}}_i$ is obtained from $\underline{p}_i$ by setting to zero those coordinates corresponding to known zeros in $\underline{B}_i^{(k)}$. Note that $\hat{\underline{p}}_i$ is dependent on k also.

However, the method has the drawback that it cannot retain symmetry of the resulting matrix even when starting with a symmetric, positive definite one. Not only does this place slightly increased demands on storage, but it also requires special sparse linear equation solvers that can accomodate matrices that are not symmetric and positive definite. Our experiments with Schubert's sparse update algorithm indicate that the technique is not suitable for nonlinear problems of structural analysis wherein the Hessians are symmetric, banded and mostly positive definite.

b. Curtis, Powell, Reid Strategy (CPR) and Powell, Toint Strategy (PT)

Because of sparsity the full Newton method wherein the Hessian matrix is evaluated at each iteration does appear to be a viable alternate especially if sparsity and or symmetry can be exploited not only in the solution of the resulting linear systems of equations but in the estimation of the relatively few nonzero entries in the Hessian matrix. Such a technique was proposed by Curtis, Powell and Reid [12] and will be referred to as the CPR strategy. The method divides the columns of the Hessian into groups, so that in each group the row numbers of the unknown elements of the column vectors are all different. After the formation of the first group other groups are formed successively by applying the same strategy to columns not included in the previous groups. The number of such groups for banded matrices of typical finite element models is usually a very small number by comparison with the number of degrees of freedom of the model. After an initial calculation of the gradient vector only as many more gradient evaluations as the number of groups are needed to evaluate all the nonzero elements of the Hessian using one-sided finite difference approximation. Thus using forward differences

$$B_{k\ell} = \frac{\partial g_k}{\partial x_\ell} \simeq \frac{g_k(\underline{x}+h_\ell \underline{e}_\ell)-g_k(\underline{x})}{h_\ell} \tag{9}$$

where $\underline{e}_\ell$ is the ℓ-th coordinate vector and h_ℓ is a suitable step size. Each step size may be adjusted such that the greatest ratio of roundoff to truncation error for any column of the Hessian falls within a specified range. However, such an adjustment of the step sizes would necessitate a significantly large number of gradient evaluations thereby rendering the CPR strategy perhaps altogether ineffective. Hence to economize on the number of gradient evaluations the step sizes are not allowed to leave the range

$$[\max(\varepsilon|x_\ell|,\ \eta h_{u\ell}),\ h_{u\ell}] \tag{10}$$

where ε is the greatest relative roundoff in a single operation, η is the relative machine precision and $h_{u\ell}$ is an upper bound on h_ℓ. Unfortunately, the CPR strategy does not account for symmetry of the Hessian matrix in the formation of the groups. Powell and Toint in their recent paper [13] have proposed two new strategies which not only account for sparsity but also symmetry in the formation of the groups thereby reducing the number of gradient evaluations for estimating the Hessian even further. One of these strategies known as the substitution method is extremely well suited for banded matrices.

The substitution method described by Powell and Toint in [13] is obtained by applying the CPR strategy to the lower triangular part, $\underline{L}$ of the symmetric Hessian, $\underline{B}$. Corresponding to m free elements in $\underline{L}$, m equations (9) for the gradient components in the last row of $\underline{L}$ are selected. Because the last rows of $\underline{L}$ and $\underline{B}$ have the same sparsity pattern the last row of $\underline{B}$ is defined. The elements B_{kn}, $k = n-m,\ldots n-1$ of the last column of $\underline{B}$ are defined by symmetry. Now the unknown elements in the next to last (n-1)th rows of $\underline{L}$ and $\underline{B}$ have the same sparsity pattern. Therefore, these elements can be calculated from the selected equations (9) for the gradient components in the (n-1)th row of $\underline{L}$. The elements of $\underline{B}$ in the (n-1)th column are then found by symmetry. A continuation of this process then enables the calculation of all the elements of $\underline{B}$.

Most nonlinear finite element codes which employ the full or the modified Newton-Ralphson technique obtain the effective stiffness matrix (sum of $[K^0]$,$[K^1]$ and sometimes even $[K^2]$) by an assembly of the stiffness matrices of all the elements of the structure. In fact the rather high CPU time requirement of the pseudo force technique by comparison with that of BFGS (see footnote for Tables 1 and 2) may in part be due to this expensive assembly process even after accounting for symmetry of the stiffness matrices. Therefore, we suspect that Powell-Toint's or for that matter even the CPR strategy of estimating the sparse Hessians using one-sided finite difference approximation will be extremely cost-effective by comparison with the conventional process of assembly of the individual element stiffness matrices especially in structures involving a large number of higher order 3-D elements with material inelasticity requiring very costly numerical integrations for evaluating the entries of element stiffness matrices.

c. Toint's Algorithm for Sparse Systems

Toint has recently proposed an algorithm [14], [15] which finds updating formulas for symmetric matrices that preserve known sparsity conditions. The update is obtained by calculating the smallest correction matrix in the Frobenius norm subject to some linear constraints which include the sparsity conditions. Precisely, let A be a given symmetric matrix with sparsity conditions

$$A_{ij} = 0, \quad (i,j) \in I \tag{11}$$

where I is some set of indices. The updating problem is to find a correction matrix E such that

$$A^* = A + E, \tag{12}$$

$$A^{*t} = A^*, \tag{13}$$

$$A^*x = y \text{ for given vectors } x,y, \tag{14}$$

$$A^*_{ij} = 0, \quad (i,j) \in I. \tag{15}$$

Define the vectors x(i) by

$$x(i)_j = \begin{cases} x_j, & (i,j) \notin I \\ 0, & (i,j) \in I \end{cases} \tag{16}$$

and the matrix Q by

$$Q_{ij} = x(i)_j \, x(j)_i + ||x(i)||^2 \delta_{ij}. \tag{17}$$

Let λ be the solution of the linear system (which has the same sparsity pattern as A)

$$Q\lambda = y - Ax \tag{18}$$

Then Toint proves that the correction matrix E is given by

$$E_{ij} = \begin{cases} 0, & (i,j) \in I \\ \lambda_i x_j + \lambda_j x_i, & (i,j) \notin I. \end{cases} \tag{19}$$

In our notation, $A = \underline{B}_i$, the current approximation to the Hessian matrix of $f(\underline{x})$ at $\underline{x}_i$, $A^* = \underline{B}_{i+1}$, $x = t_i\underline{p}_i = \underline{x}_{i+1} - \underline{x}_i$, $y = \underline{g}_{i+1} - \underline{g}_i = \nabla f(\underline{x}_{i+1}) - \nabla f(\underline{x}_i)$. More details of this algorithm may be found in reference [14].

To solve the minimization problem starting with an initial guess $\underline{x}_0$ for the vector of unknowns and an initial guess for the Hessian $\underline{B}_0$, a direction of travel $\underline{p}_i$ is generated by a constrained minimization of a local quadratic approximation of the potential function $f(\underline{x})$. The function is then minimized along this direction by a cubic line search to obtain a new starting point $\underline{x}_{i+1}$. Next, the Hessian is updated at this new point and the iteration is repeated. Presummably to minimize storage requirements Toint proposes the solution of the constrained minimization of the quadratic function by a Levenberg-Marquardt procedure [15] which is similar to a conjugate gradient scheme. The updating procedure also involves the solution of a sparse linear system of equations with the same sparsity pattern as the Hessian. Again, presumably for the same reasons as before, Toint proposes the solution of this system also by a conjugate gradient scheme.

d. Proposed Quasi-Newton Algorithm using Toint's Sparse Updates

Our rather limited experiments indicate that the performance of Toint's algorithm as described in the previous section is rather disappointing for large nonlinear systems. This may be brought on by three sources: (i) the constrained minimization of the local quadratic approximation to the potential function, (ii) the use of the conjugate gradient scheme for the solution of sparse linear system of equations and (iii) the use of a rather expensive cubic line search. To overcome these apparent limitations the following remedies which have proven to be extremely effective are proposed.

It is proposed to use a Quasi-Newton algorithm similar to Broyden's algorithm [11] which begins with an initial guess $\underline{x}_0$ for the vector $\underline{x}$ of unknown generalized displacements and the initial Hessian approximation $\underline{B}_0$ calculated by the CPR (or PT) strategy. At the ith iteration, a new direction of travel $\underline{p}_i$ is then obtained by the solution of the sparse linear system of equations

$$\underline{p}_i = -\underline{B}_i^{-1}\underline{g}_i \tag{20}$$

where $\underline{B}_i$ is the ith approximation to the Hessian matrix and $\underline{g}_i$ the gradient vector at $\underline{x}_i$. Incidently, this is equivalent to obtaining $\underline{p}_i$ by an unconstrained minimization of the quadratic approximation of the function at $\underline{x}_i$. $\underline{p}_i$ in Eq. (20) is obtained by a triple factorization of the matrix $\underline{B}_i$ using a highly optimized scheme outlined by Bathe and Wilson [16] for sparse linear systems of equations.

A simple modification of Newton's method then gives

$$\underline{x}_{i+1} = \underline{x}_i + t_i\underline{p}_i \tag{21}$$

where t_i, whose derivation may be found in reference [11], is a scalar chosen to reduce the norm of the gradient vector and prevent divergence. The Hessian $\underline{B}_{i+1}$ at $\underline{x}_{i+1}$ is obtained by the application of Toint's update scheme. The ensuing sparse linear system of equations is solved again by using the same triple factorization scheme of Bathe and Wilson.

Because of poor approximations to the Hessian a value of t_i in Eq. (21) may be nonexistent or intolerably small in which case a Hessian is calculated at the point $\underline{x}_i$ by the CPR (or PT) strategy and the process continued from there on. It is clear that because of triple factorization the storage requirements for the Hessian are twice those of Toint's algorithm.

IV. NUMERICAL EXPERIMENTS WITH ALGORITHMS THAT EXPLOIT SPARSITY

For evaluating the performances of these various algorithms consider the problem of the snap-through buckling of a shallow elastic arch under a concentrated load at the crown as shown in Fig. 2. This problem is regarded as being a highly nonlinear problem. Figure 2 also illustrates the snap-through response predictions using energy minimization and pseudo force technique for the 29 degree of freedom model. As a point of reference the square symbols present a first order self-correcting solution from [17].

a. Finite Element Models of the Arch

The arch is modeled as an assemblage of straight frame elements. The deformation model for the assemblage is synthesized from deformation states of each element of the structure. These states are expressed in terms of generalized displacements of the nodes of the structure at which the elements interface. The axial and transverse displacement fields within each element are a linear and a cubic function respectively of the local spatial coordinates. The fields maintain interelement continuity of the essential derivatives thereby providing a Galerkin model of the system. The deformation model uses the co-rotational or the rigid-convected formulation thereby permitting arbitrarily large rotations of the element. The co-rotational formulation decomposes the total displacements into a rigid body motion and a strain producing component. To allow large rigid body rotations the transformations between the local and global co-ordinate system are accomplished using Euler angles which are independent by virtue of the fact that the rotations are performed in a prescribed order.

The load response curve of the arch is not a single-valued function of the load but is a composite of stable and unstable branches. Using straight-forward load incrementation with the potential energy of the system as the function to be minimized it is possible to locate only the stable equilibrium configurations. Using displacement incrementation, however, with the function to be minimized being the strain energy of the structure, the entire load response curve can be easily obtained. The gradient of the strain energy is evaluated analytically. Details of this evaluation may be found in reference [18].

To evaluate sensitivity of the minimization algorithms to the scale of the problem consider three different models with the degrees of freedom, N, taking on the values of 29, 89 and 179 respectively.

b. Performance of the Various Minimization Algorithms

The numerical results presented in Table 1 pertain to the following different variations of the Toint and our proposed Quasi-Newton algorithm using Toint's sparse updates.

(i) TCNA (Toint's Constrained Newton Algorithm) designates Toint's algorithm wherein the Hessian is evaluated by the CPR (PT) strategy at each iteration.

(ii) TCUNA (Toint's Constrained Updated Newton Algorithm) designates Toint's algorithm wherein the Hessian is evaluated every pth iteration (p = 4) but updated on all other iterations using Toint's sparse updating formula.

(iii) NA (Newton Algorithm) designates Newton's method, Eqs. (20), (21), wherein the Hessian is evaluated by the CPR (PT) strategy and factored at each iteration by the triple factorization scheme of Bathe and Wilson [16].

(iv) TUNA (Toint's Updated Newton Algorithm) designates a Quasi-

Newton algorithm with initial approximation to the Hessian evaluated by the CPR (PT) strategy and updated on all other iterations using Toint's sparse updating formula. In places where Broyden's line search [11] failure occurs Hessian evaluation by the CPR (PT) strategy is activated.

(v) MNA (Modified Newton Algorithm) designates NA algorithm wherein the Hessian is calculated by the CPR (PT) strategy only initially and held constant on all other iterations.

In Tables 1 and 2, NZUH denotes the nonzero entries in the upper half of the Hessian matrix. The CPU time for each case is normalized with respect to the CPU time required for the same problem using the BFGS algorithm. For the model with N = 29 additional normalization with respect to the pseudo force technique [19] is also presented since this was the only model for which the CPU time was readily available from previous studies. The Hessian is divided into 9 groups by the CPR strategy and 6 groups by the PT strategy for all the three models with the regular sparsity pattern as illustrated in Fig. 3 for the 29 degree-of-freedom model. This implied that only 9 gradient evaluations were required by the CPR strategy and 6 by the PT strategy to evaluate all the nonzero entries in the upper half of the Hessian matrix using a forward difference approximation. The entire response involved a total of 12 displacement steps with a displacement step of one inch.

c. Sensitivity of the Various Algorithms to the Sparsity Pattern

To evaluate the sensitivity to changes in sparsity pattern of some of the more promising algorithms considered previously the sparsity pattern of Fig. 3 was disturbed by a rather arbitrary disturbance of the node numbering scheme of the finite element models. Figure 4 illustrates a typical disturbed sparsity pattern for a model with N =

29. This disturbed sparsity pattern required that the Hessian be divided into 18 groups requiring 18 gradient evaluations by the CPR strategy and 12 groups requiring 12 gradient evaluations by the PT strategy for evaluating all the non-zero entries in the upper half of the Hessian matrix. The results for this sparsity pattern are presented in Table 2.

V. CONCLUSIONS

The performance of the Toint's Algorithms (TCNA, TCUNA) is rather disappointing and it appears that they may not be suited well for highly nonlinear large scale problems. Toint's original sparse updating algorithm (SUA in reference [15]) was tried and found to be considerably worse than both TCNA and TCUNA. It needs to be pointed out, however, that the code used for the evaluation of Toint's algorithm was experimental and the conclusions may, therefore, be rather premature. Results indicate that, especially for the larger models, the modified Newton method (MUA) requires small displacement steps to be able to yield a complete response in a reasonable amount of CPU time and even so it is no match for any of the other techniques considered. Judging from the trends it would appear that for extremely large scale geometrically nonlinear problems Newton's method (NA) using Hessian evaluations by the CPR (or PT) strategy will be superior to the Quasi-Newton algorithm using Toint's updates with occasional Hessian evaluation by the CPR (or PT) strategy (TUNA) unless the sparsity is extremely poor. Note that generally Hessian evaluation by the PT strategy is superior to the CPR strategy, but PT may be worse as Table 2 shows. CPR and PT produce different Hessian approximations, hence different search directions, and which direction is best depends on the problem and point $\underline{x}_i$. For problems involving three dimensional isoparametric elements with material inelasticity wherein the Hessian evaluation is likely to be extremely expensive in spite of using the Powell-Toint strategies, the TUNA algorithm may be expected to have a significant advantage. In conclusion as Toint correctly points out in his paper [15] "a more sophisticated procedure using sparse update seems still to be desirable in order to use all the information in an optimal way".

VI. ACKNOWLEDGEMENT

The authors wish to express their thanks for all the help they received towards the preparation of this manuscript to Dr. L. K. Schubert and Professor Ph. L. Toint.

VI. REFERENCES

1. BOGNER, F. K., MALLET, R. H., MINICH, M. D. and SCHMIT, L. A. - Development and Evaluation of Energy Search Methods of Nonlinear Structural Analysis. AFFDL-TR-65-113, WPAFB, Dayton, Ohio, 1966.
2. MALLET, R. H. and BERKE, L. - Automated Method for Large Deflection and Instability Analysis of Three Dimensional Truss and Frame Assemblies. AFFDL-TR-66-102, WPAFB, Dayton, Ohio, 1966.
3. YOUNG, J. W. - CRASH: A Computer Simulation of Nonlinear Transient Response of Structures. DOT-HS-091-1-125-13, March 1972.
4. BERGAN, P. G., SOREIDE, T. - A Comparative Study of Different Numerical Solutions Techniques as Applied to a Nonlinear Structural Problem. Comp. Meth. Appl. Mech. Eng., Vol. 2, pp. 185-201, 1973.
5. KAMAT, M. P. and HAYDUK, R. J. - Energy Minimization versus Pseudo Force Technique for Nonlinear Structural Analysis. J. Comp. Struct. (In Print).

6. HAYDUK, R. J., THOMSON, R. G., WITLIN, G. and KAMAT, M. P. - Nonlinear Structural Crash Dynamics Analyses. SAE Business Aircraft Meeting, SAE Paper No. 790588, 1979.

7. DENNIS, J E., Jr. and MORÉ, J. J. - Quasi-Newton Methods, Motivation and Theory, SIAM Review, Vol. 19, No. 1, pp. 46-79, 1977.

8. FLETCHER, R. and POWELL, M. J. D. - A Rapidly Convergent Descent Method for Minimization. Comp. J., Vol. 6, pp. 163-168, 1963.

9. SHANNO, D. F. and PHUA, K. H. - Matrix Conditioning and Nonlinear Optimization, Math. Prog., Vol. 14, pp. 149-160, 1978.

10. SCHUBERT, L. K. - Modification of a Quasi-Newton Method for Nonlinear Equations with a Sparse Jacobian. Math. Comp., Vol. 24, pp. 27-30, 1970.

11. BROYDEN, C. G. - A Class of Methods for Solving Nonlinear Simultaneous Equations. Math. Comp., Vol. 10, pp. 577-593, 1965.

12. CURTIS, A. R., POWELL, M. J. D. and REID, J. K. - On the Estimation of Sparse Jacobian Matrices. J. Inst. Math. Appl., Vol. 13, pp. 117-119, 1974.

13. POWELL, M .J. D. and TOINT, Ph. L. - On the Estimation of Sparse Hessian Matrices. SIAM J. Numer. Anal., Vol. 11, No. 6, 1979.

14. TOINT, Ph. L. - On Sparse and Symmetric Matrix Updating Subject to a Linear Equation. Math. Comp., Vol. 31, No. 140, pp. 954-961, 1977.

15. TOINT, Ph. L. - Some Numerical Results Using a Sparse Matrix Updating Formula in Unconstrained Optimization. Math. Comp., Vol. 32, No. 143, pp. 839-851, 1978.

16. BATHE, K. J. and WILSON, E. L. - Numerical Methods in Finite Element Analysis. Prentice-Hall Inc., 1976.

17. STRICKLIN, J. A. and HAISLER, W. E. - Formulations and Solution Procedures for Nonlinear Structural Analysis, Comp. Struct., Vol. 7, pp. 125-136, 1977.

18. KAMAT, M. P. - Nonlinear Transient Analysis by Energy Minimization - A Theoretical Basis for the ACTION Computer Code, NASA CR-3287, 1980.

19. PIFKO, A. B., LEVINE, H. S. and ARMEN, H. Jr. - PLANS - A Finite Element Program for Nonlinear Analysis of Structures, Vol. I - Theoretical Manual. NASA-CR-2568, 1975.

Algorithm	Model 1		Model 2		Model 3	
	N=29, NZUH=134		N=89, NZUH=434		N=179, NZUH=884	
	CPR	PT	CPR	PT	CPR	PT
1. TCNA	1.213	-	1.243	-	5.843†	-
2. TCUNA	1.228	-	1.729	-	4.378†	-
3. NA	1.007	0.583	0.532	0.414	0.158	0.114
4. TUNA	0.593	0.577	0.469	0.400	0.200	0.147
5. MUA	1.022	-	10.433†*	-	-	-

TABLE 1. NORMALIZED†† CPU TIMES WITH REGULAR SPARSITY**

---

†Extrapolated value. Not carried to conclusion because of excessive time requirements.

*Because of convergence difficulties the size of the displacement step was chosen to be 0.5".

††Running time have been normalized with respect to the running time for the model using the BFGS algorithm.

**To obtain approximate CPU times for the N = 29 model normalized with respect to the Pseudo Force Technique, divide the corresponding entries in this column by a factor of 5.

Algorithm	Model 1 N=29, NZUH=188		Model 2 N=89, NZUH=596		Model 3 N=179, NZUH=1208	
	CPR	PT	CPR	PT	CPR	PT
1. NA	1.532	1.158	0.894	0.648	0.264	0.179
2. TUNA	0.742	0.698	0.582	0.504	0.286	0.315

TABLE 2. NORMALIZED CPU TIMES†† WITH DISTURBED SPARSITY

††Running times have been normalized with respect to the running time for the model using the BFGS algorithm.

Figure Legend

Note:

A figure should be included in the body of the text immediately after its very first citation.

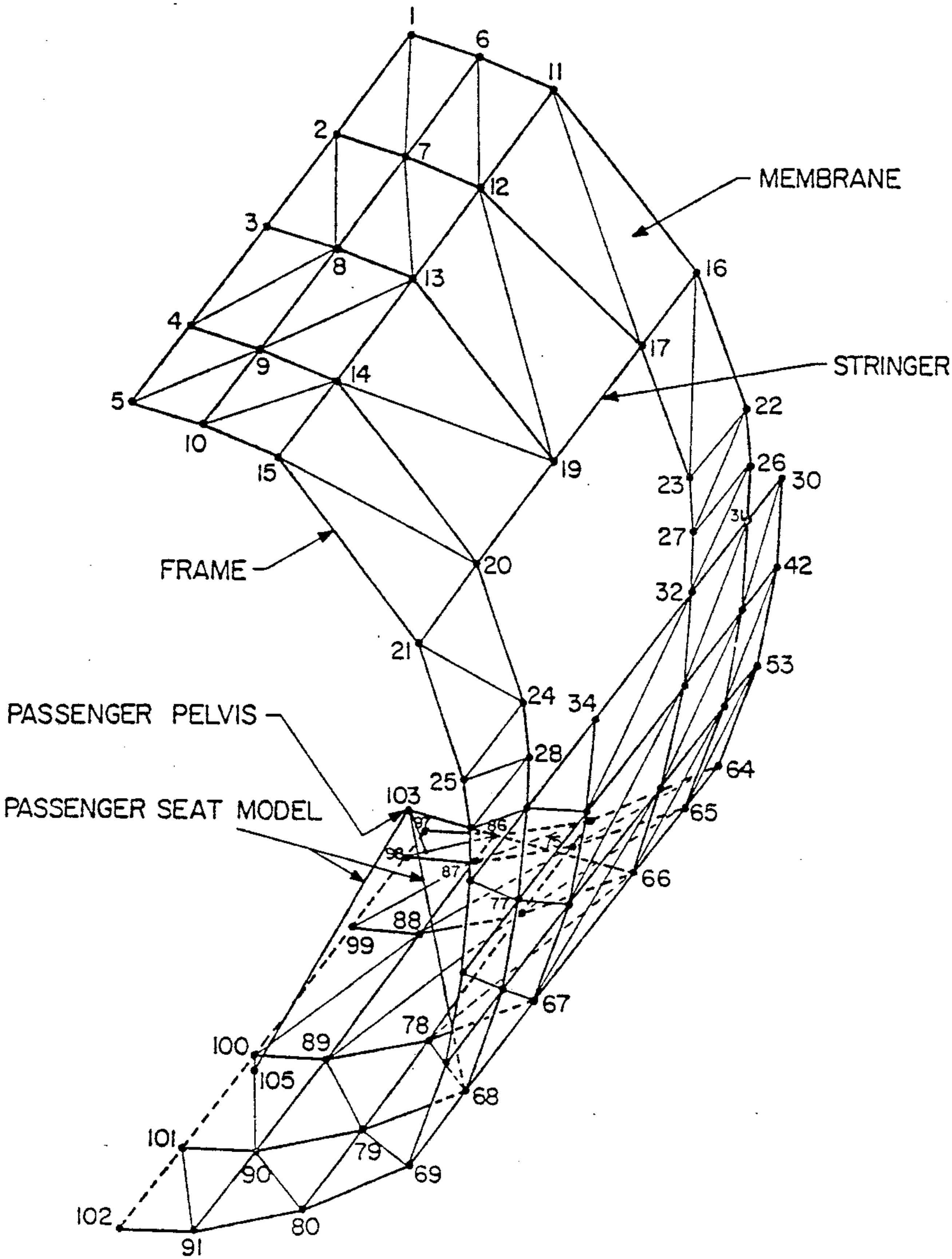
MEMBRANE
STRINGER
FRAME
PASSENGER PELVIS
PASSENGER SEAT MODEL

Figure 1.

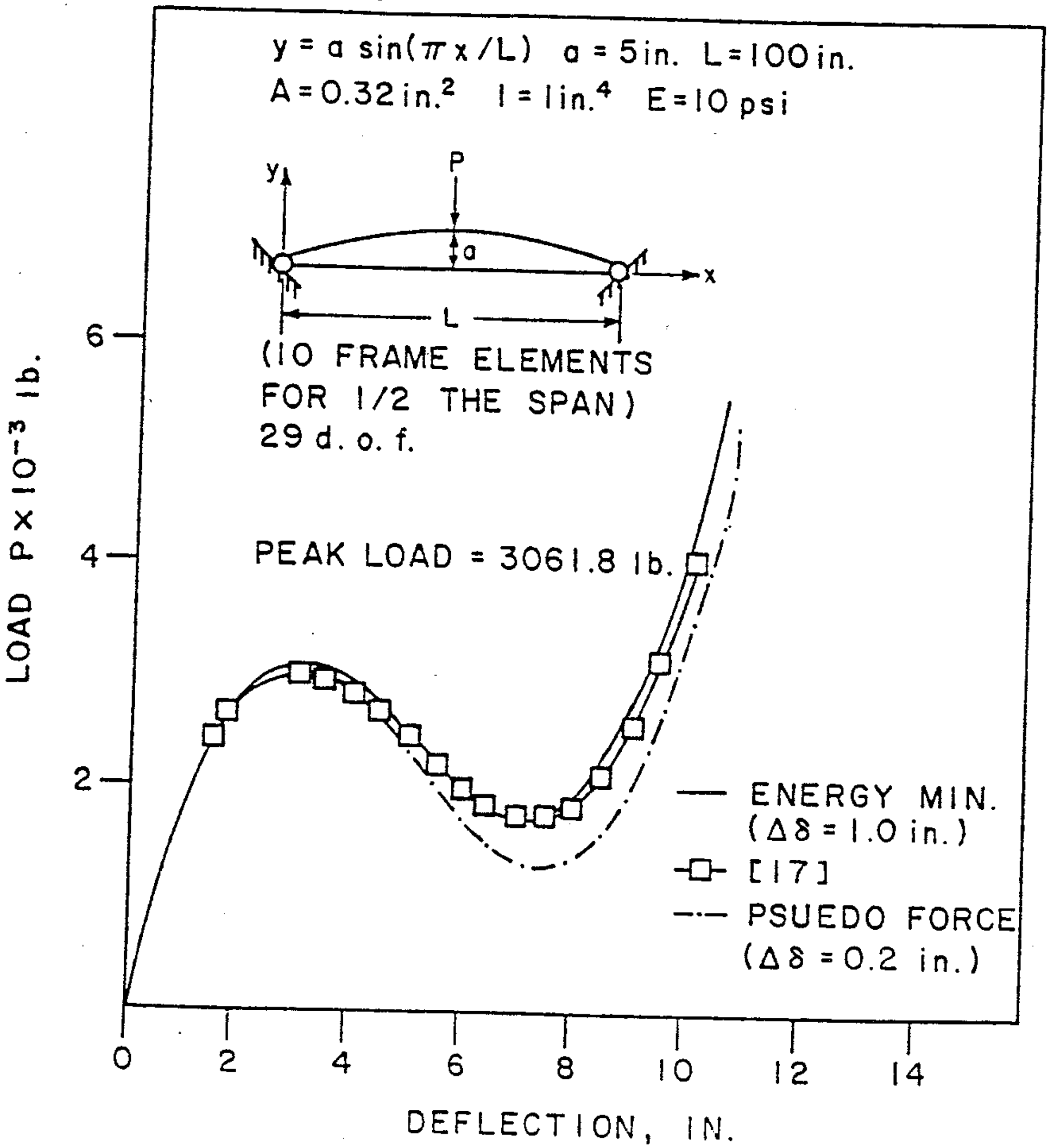
y = a sin(πx/L) a = 5in. L=100in.
A=0.32in.2 I=1in.4 E=10 psi
y
P
a
x
L
(10 FRAME ELEMENTS
FOR 1/2 THE SPAN)
29 d.o.f.
PEAK LOAD = 3061.8 lb.
LOAD P×10-3 lb.
6
4
2
0
2
4
6
8
10
12
14
DEFLECTION, IN.
ENERGY MIN.
(Δδ = 1.0 in.)
[17]
PSUEDO FORCE
(Δδ = 0.2 in.)

Figure 2.

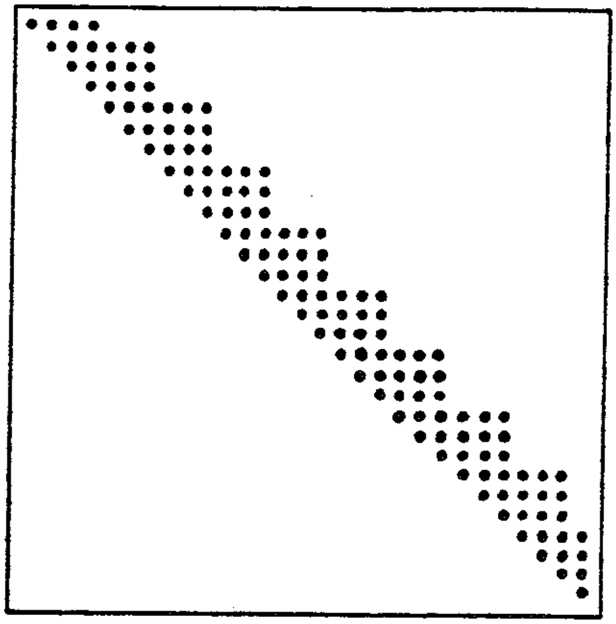

Figure 3.

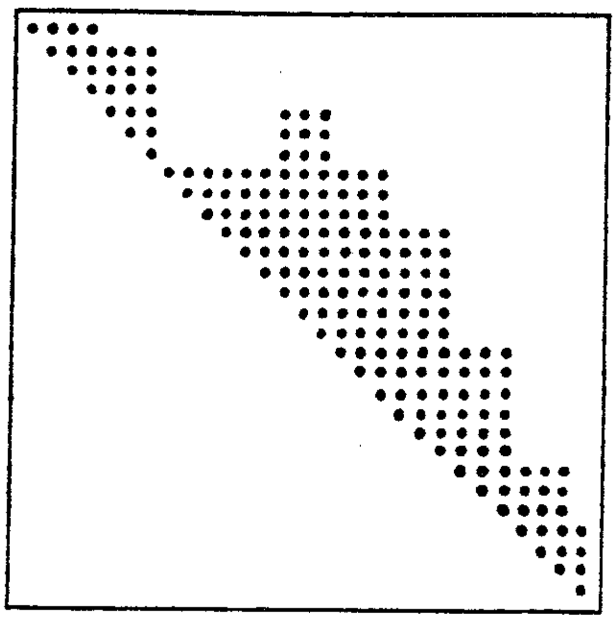

Figure 4.