Generalized Linear Product Homotopy Algorithms and the Computation of Reachable Surfaces

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Abstract

In this paper, we apply a homotopy algorithm to the problem of finding points in a moving body that lie on specific algebraic surfaces for a given set of spatial configurations of the body. This problem is a generalization of Burmester's determination of points in a body that lie on a circle for five planar positions. We focus on seven surfaces that we term "reachable" because they correspond to serial chains with two degree-of-freedom positioning structures combined with a three degree-of-freedom spherical wrist. A homotopy algorithm based on generalized linear products is used to provide a convenient estimate of the number of solutions of these polynomial systems. A parallelized version of this algorithm was then used to numerically determine all of the solutions.

1 Introduction

The problem that we consider originates with the determination by Burmester (1886)[4] of those points in a body that lie on a circle for a given set of five planar positions. He used these so-called *Burmester points* to design a linkage to guide a body through the given positions. His result was a graphical solution to a set of five quadratic equations in five unknown parameters, see Suh and Radcliffe (1978)[27], Sandor and Erdman (1984)[22], or McCarthy (2000)[17].

Chen and Roth (1967)[7] generalized this problem by seeking points and lines in a moving body that take positions on surfaces associated with articulated serial chains, in order to design robot manipulators. A subset of these serial chains consists of two joints that support a spherical wrist, and we consider the surfaces that are reachable by the wrist center of these chains. Considering the various ways of assembling these articulated chains, we obtain seven reachable algebraic surfaces. The equations of these surfaces can be evaluated on the displacement positions of a generic point in order to define a set of polynomial equations. The solution of these equations define the surface and the dimensions of the associated chains that guide the end-effector through the given displacements.

To illustrate this problem, consider the set of points, $\mathbf{P}^i = (X_i, Y_i, Z_i)^T$, $i = 1, \ldots, n$, that are the images of a point $\mathbf{p} = (x, y, z)^T$ in a moving body defined by a set of spatial displacements $T_i = [A_i, \mathbf{d}_i]$ $i = 1, \ldots, n$, which means $\mathbf{P}^i = [A_i]\mathbf{p} + \mathbf{d}_i$ —note $[A_i]$ is a 3×3 rotation matrix and $\mathbf{d}_i$ is a 3×1 translation vector (Bottema and Roth 1979[3], McCarthy 1990[16]). We now ask if there is a point $\mathbf{p}$ in the moving body that has the property that the image points $\mathbf{P}^i$ lie on a sphere, such that

$$(X_i - u)^2 + (Y_i - v)^2 + (Z_i - w)^2 - R^2 = 0, \quad i = 1, \ldots, n, \tag{1}$$

where R is the radius of the sphere and $\mathbf{B} = (u, v, w)$ is its center. This sphere is defined by the seven parameters $\mathbf{p} = (x, y, z)$, $\mathbf{B} = (u, v, w)$ and R. Thus $n = 7$ spatial displacements yield seven quadratic polynomials (1) that determine these parameters. The system of polynomials has a total degree of $2^7 = 128$, but it is known to have only 20 solutions, Chen and Roth 1967[7], Liao and McCarthy 2001)[15].

In what follows, we study the cases of the plane, sphere, circular cylinder, circular hyperboloid, elliptic cylinder, circular torus and general torus. These are the surfaces reachable by the PPS, TS, CS, RPS, PRS, right RRS, and RRS serial chains. It is interesting how quickly the complexity of the problem increases with the number of dimensional parameters and the degree of the surface. The total degree of the polynomial systems that we consider range from 32 for the simplest to over 4 million for the most complex.

We show that these polynomial systems have a generalized linear product structure (Morgan et al. 1995[19]) that yields a bound on the number of solutions that ranges

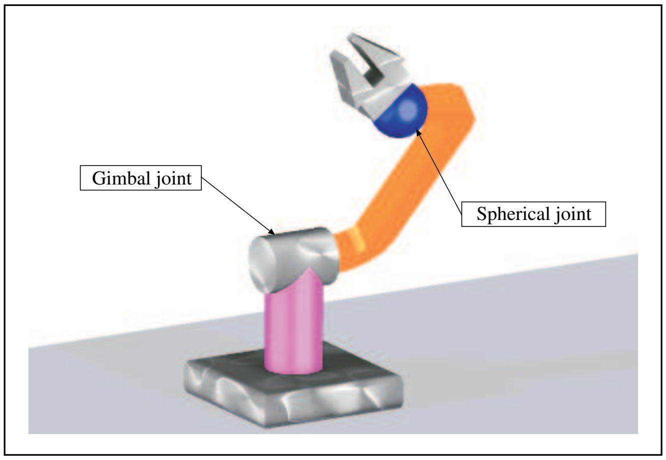


Figure 1: The TS serial constraints the wrist center to the surface of a sphere

from 10 to over 800,000. In addition, this generalized linear product structure provides a convenient start system for a homotopy algorithm POLSYS_GLP developed for this application to numerically determine all of the solutions of these polynomial systems (Verschelde and Haegemans 1993[29] and Wise et al. $\tilde{2}$000[35]).

Our results are summarized in Table 3 which compares the total degree of each polynomial system, the bound obtained using the generalized linear product structure of these polynomials, and the number of solutions obtained using the homotopy algorithm POLSYS_GLP.

2 Homotopy Algorithms

Our concern is finding all of the solutions of a set of n polynomial equations in n unknowns that arise in finding surfaces reachable by articulated chains. For the cases of the plane and sphere, the systems of polynomials can be solved by direct elimination of the unknown parameters to obtain a univariate polynomial. Numerical solution of this polynomial, combined with back-substitution yields the desired solutions. However, the remaining surfaces yield systems of polynomials that are simply too complicated to solve by direct parameter elimination, therefore we use a numerical method called a homotopy algorithm.

Consider the array of polynomials $P(\mathbf{z})$ obtained from (1),

$$P(\mathbf{z}) = \begin{Bmatrix} S_1(\mathbf{z}) \\ S_2(\mathbf{z}) \\ \vdots \\ S_7(\mathbf{z}) \end{Bmatrix} = 0, \tag{2}$$

where $\mathbf{z} = (\mathbf{p}, \mathbf{B}, R)$ is the vector of parameters that define the sphere. If we start with a polynomial system $Q(\mathbf{z}) = 0$ that has the same structure as $P(\mathbf{z}) = 0$ but with a known set of solutions, then we can continuously transform $Q(\mathbf{z})$ into $P(\mathbf{z})$ and track its roots in order to find the solutions of $P(\mathbf{z}) = 0$. This continuous transformation of $Q(\mathbf{z})$ into $P(\mathbf{z})$ is called a *homotopy* map.

A numerical homotopy technique was used by Tsai and Morgan (1985)[28] to solve the inverse kinematics equations of a general 6R robot manipulator. Wampler et al. (1990)[31] and Sommese et al. (2002)[23] describe the use of numerical homotopy for applications in the kinematics of linkages and robots. Our focus on the design of serial chain robots follows Lee and Mavroidis (2002)[14], who used numerical homotopy to solve the design equations for an RRR manipulator.

For our purposes, we use the convex combination homotopy map

$$H(\lambda, \mathbf{z}) = (1 - \lambda)Q(\mathbf{z}) + \lambda P(\mathbf{z}), \tag{3}$$

where $\lambda \in [0, 1)$ is the real-valued homotopy parameter. The coefficients of our polynomial system $P(\mathbf{z}) = 0$ are real, however, its roots $\mathbf{z}$ need not be. Therefore, the homotopy $H(\lambda, \mathbf{z})$ must be viewed as an array of n complex functions in n complex variables $\mathbf{z}$ together with a single real variable λ.

For each root of the start system $Q(\mathbf{z}) = 0$, denoted $\mathbf{z} = \mathbf{a}_j$, $j = 1, \ldots, N$, the homotopy equation $H(\lambda, \mathbf{z}) = 0$ has an associated zero curve γ_a, which is the connected component of $H^{-1}(0)$ containing the start point $(0, \mathbf{a}_j)$. The zero curve leads either to a point $(1, \mathbf{z}_a)$ where $P(\mathbf{z}_a) = 0$, or diverges to a root "at infinity."

Each zero curve can be parameterized by its arc length s, so γ_a has the form $(\lambda(s), \mathbf{z}(s))$. Tracking this curve involves numerical computation of points $\mathbf{y}_i \approx (\lambda(s_i), \mathbf{z}(s_i))$, where $\{s_i\}$ is an increasing sequence of arc lengths. This can be done using a predictor-corrector strategy described in Watson et al. (1997)[34] and Wise et al. (2000)[35].

Along the zero curve γ_a, we have $H(\lambda(s), \mathbf{z}(s)) = 0$, therefore we can compute

$$\frac{d}{ds} H(\lambda, \mathbf{z}) = \begin{bmatrix} H_\lambda & H_{\mathbf{z}} \end{bmatrix} \begin{Bmatrix} d\lambda/ds \\ d\mathbf{z}/ds \end{Bmatrix} = 0, \tag{4}$$

where $[J_H] = [H_\lambda, H_{\mathbf{z}}]$ is the $n \times (n+1)$ matrix of partial derivatives of the homotopy $H(\lambda, \mathbf{z})$. Notice that the vector $\mathbf{v} = (d\lambda/ds, d\mathbf{z}/ds)^T$ tangent to the zero curve γ_a is in the null space of the Jacobian matrix $[J_H]$. This null space has dimension one by the theory of polynomial homotopy maps, (Wampler et al. 1990[31] and Wise et al. 2000[35]).

The unit tangent vector $\mathbf{v}_i$, in the direction of increasing arc length, at a point $\mathbf{y}_i$ on γ_a is used to predict a value for the next point $\mathbf{y}^0_{i+1}$, that is

$$\mathbf{y}^0_{i+1} = \mathbf{y}_i + (s_{i+1} - s_i)\mathbf{v}_i, \tag{5}$$

where $s_{i+1} - s_i$ is a chosen arc length step. The predicted value of $\mathbf{y}^0_{i+1}$ is corrected using the Taylor series expansion of the homotopy given by

$$H(\mathbf{y}^0_{i+1}) + [J_H(\mathbf{y}^0_{i+1})](\mathbf{y}^1_{i+1} - \mathbf{y}^0_{i+1}) \approx 0, \tag{6}$$

which yields the correction formula

$$\mathbf{y}^1_{i+1} = \mathbf{y}^0_{i+1} - [J_H(\mathbf{y}^0_{i+1})]^\dagger H(\mathbf{y}^0_{i+1}). \tag{7}$$

The dagger denotes the Moore-Penrose pseudoinverse of the $n \times (n+1)$ Jacobian matrix. Geometrically, iteration of the correction formula moves $\mathbf{y}^k_{i+1}$ toward the zero curve γ_a along a normal direction, and is termed the "normal flow algorithm."

The predictor can be improved by interpolation at previous computed points along the zero curve, and a projective transformation can be used to bound the arc length of all of the paths so that none diverge to infinity. Finally, an "end-game" strategy can improve the calculation of $\mathbf{y}$ near $\lambda = 1$. See Wise et al. (2000)[35] for details.

Fundamental to this approach to solving the equations $P(\mathbf{z}) = 0$ is the determination of a start system $Q(\mathbf{z}) = 0$ with a known set of solutions. A general purpose homotopy algorithm must systematically construct a start system with known roots that is appropriate for the given set of polynomials. In the next section, we show how to construct a start system using a generalized linear product representation of the system of polynomials.

3 Linear Product Decomposition

The fundamental theorem of algebra states that the number of roots of a polynomial is equal to or less than its degree, which is the integer value of its highest power—equality is obtained if roots are counted with the appropriate multiplicity. This has been generalized to Bezout's theorem which states that the number of roots of a system of polynomials is less than or equal to the product of the degrees of the individual polynomials, called the *total degree* of the system. This fact leads to a relatively simple start system $Q(\mathbf{z}) = 0$, where d_i, $i = 1, \ldots, n$ is the degree of the ith polynomial in the target system $P(\mathbf{z}) = 0$, given by

$$Q(\mathbf{z}) = \left\{ \begin{array}{c} a_1 z_1^{d_1} - b_1 \\ a_2 z_2^{d_2} - b_2 \\ \vdots \\ a_n z_n^{d_n} - b_n \end{array} \right\} = 0. \tag{8}$$

The coefficients a_i and b_i are randomly selected complex numbers. The solutions to this start system are easy to determine and provide the starting coordinates for tracing the $d = d_1 d_2 \ldots d_n$ zero curves to the solutions of $P(\mathbf{z}) = 0$.

In the problems that we consider in this paper, the total degree over-estimates the number of roots in the target polynomial $P(\mathbf{z})$ by a significant amount. For example in order to solve our example problem (1) the polynomial homotopy algorithm with the start system (8) would track 128 paths to find 20 roots, which means over 80% of the computation is spent tracing paths that are extraneous.

The problem of extraneous paths arises from the fact that the polynomials we wish to solve are not general, but instead have internal structure that reduces the number of solutions. Morgan et al. (1995)[19] show that a "generic" system of polynomials that includes every monomial of a particular system of polynomials will have as many or more solutions as any version obtained by specifying values for the coefficients. This leads to the construction of the *linear product decomposition* of a system of polynomials. Associated with a linear product decomposition is a start system that is easy to construct and solve called the *generalized linear product*.

In order to illustrate the linear product decomposition, we analyze the example (1) in more detail. Write these polynomials in vector form to obtain

$$(\mathbf{P}^i - \mathbf{B}) \cdot (\mathbf{P}^i - \mathbf{B}) = R^2, \quad i = 1, \ldots, 7, \tag{9}$$

where the dot denotes the vector dot product. Now subtract the first equation from the rest in order to eliminate R^2. This reduces the problem to six equations in the unknowns $\mathbf{z} = (x, y, z, u, v, w)$, given by

$$S(\mathbf{z}) = \left\{ \begin{array}{c} (\mathbf{P}^2 \cdot \mathbf{P}^2 - \mathbf{P}^1 \cdot \mathbf{P}^1) - 2\mathbf{B} \cdot (\mathbf{P}^2 - \mathbf{P}^1) \\ \vdots \\ (\mathbf{P}^7 \cdot \mathbf{P}^7 - \mathbf{P}^1 \cdot \mathbf{P}^1) - 2\mathbf{B} \cdot (\mathbf{P}^7 - \mathbf{P}^1) \end{array} \right\} = 0. \tag{10}$$

We now consider these equations as linear combinations of monomials formed by the unknown parameters.

Let $\langle x, y, 1 \rangle$ represent the set of linear combinations of parameters x, y and 1, which means a typical term is $\alpha x + \beta y + \gamma \in \langle x, y, 1 \rangle$, where α, β and γ are arbitrary constants. Using this notation, we define the product of $\langle x, y, 1\rangle\langle u, v, 1\rangle$ as the set of linear combinations of the product of the elements of the two sets, that is

$$\langle x, y, 1\rangle\langle u, v, 1\rangle = \langle xu, xv, yu, yv, x, y, u, v, 1\rangle. \tag{11}$$

This product commutes, which means $\langle x\rangle\langle y\rangle = \langle y\rangle\langle x\rangle$, and it distributes over unions, such that $\langle x\rangle\langle y\rangle \cup \langle x\rangle\langle z\rangle = \langle x\rangle(\langle y\rangle \cup \langle z\rangle) = \langle x\rangle\langle y, z\rangle$. Furthermore, we represent repeated factors using exponents, so $\langle x, y, 1\rangle\langle x, y, 1\rangle = \langle x, y, 1\rangle^2$.

Recall that $\mathbf{P}^i = [A_i]\mathbf{p} + \mathbf{d}_i$ where $[A_i]$ and $\mathbf{d}_i$ are known, so it is easy to see that

$$2\mathbf{B} \cdot (\mathbf{P}^{j+1} - \mathbf{P}^1) \in \langle u, v, w\rangle\langle x, y, z, 1\rangle. \tag{12}$$

It is also possible to compute

$$\mathbf{P}^{j+1} \cdot \mathbf{P}^{j+1} - \mathbf{P}^1 \cdot \mathbf{P}^1 = 2\mathbf{d}_{j+1} \cdot [A_{j+1}]\mathbf{p} - 2\mathbf{d}_1 \cdot [A_1]\mathbf{p} + \mathbf{d}_{j+1}^2 - \mathbf{d}_1^2 \in \langle x, y, z, 1\rangle. \tag{13}$$

Each of the equations in (10) has the same monomial structure given by

$$\langle x, y, z, 1\rangle \cup \langle u, v, w\rangle\langle x, y, z, 1\rangle \subset \langle x, y, z, 1\rangle\langle u, v, w, 1\rangle. \tag{14}$$

¿From this we see that a generic set of polynomials that contains our system as a special case can be constructed as a product of linear factors, as

$$Q(\mathbf{z}) = \left\{ \begin{matrix} (a_1x + b_1y + c_1z + d_1)(e_1u + f_1v + g_1w + h_1) \\ \vdots \\ (a_6x + b_6y + c_6z + d_6)(e_6u + f_6v + g_6w + h_6) \end{matrix} \right\} = 0, \tag{15}$$

where the coefficients are known complex constants. This structure is called the *linear product decomposition* of the target system.

Solutions to a linear product decomposition of a set of polynomials are easily determined by assembling all combinations of factors, one from each equation, that can be set to zero and solved for the unknown parameters (Wampler 1994[32]). In our example, select three factors $a_ix + b_iy + c_iz + d_i = 0$ from the six equations, and combine with the three factors $e_iu + f_iv + g_iw + h_i = 0$ in the remaining equations. A solution of this set of six linear equations is a root of (15). Thus, we find that this system has $\binom{6}{3} = 20$ solutions, which matches the known result for (10).

For the problems we consider the linear product decomposition provides a bound on the number of solutions that is significantly less than the total degree.

4 Generalized Linear Product

The "generalized linear product" is a start system constructed from the linear product decomposition of a polynomial system. It is an extended version of the "partitioned linear product" used to construct m-homogeneous start systems, (Wise et al. 2000[35]).

We begin with a linear product decomposition for each of the polynomials P_i, $i = 1, \ldots, n$ in the unknowns z_i, $i = 1, \ldots, n$. Augment each factor in this decomposition with a constant term, if it is not already present. This means that a factor of the form $\langle z_1, z_2, z_3\rangle$ is replaced by $\langle z_1, z_2, z_3, 1\rangle$. Now for notational convenience we introduce the "mask" $S_{ij} = (s_{ij1}, \ldots, s_{ijn})$ constructed from n 1s and 0s in order to identify the unknowns in $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ that appear in a specific linear factor. This allows us to write a general linear product decomposition as

$$P_i \in \prod_{j=1}^{m_i} \langle s_{ij1}z_1, \ldots, s_{ijn}z_n, 1\rangle^{d_{ij}}, \tag{16}$$

where m_i is the number of different factors in polynomial P_i. Notice that $d_i = \sum_{j=1}^{m_i} d_{ij}$ is the degree of P_i. This decomposition is specified by identifying the masks S_{ij} and the associated degrees d_{ij}.

We now construct the start system by introducing the polynomial

$$G_{ij} = \left(\sum_{k=1}^{n} c_{ijk}s_{ijk}z_k\right)^{d_{ij}} - 1, \tag{17}$$

for each factor in the augmented linear product decomposition. The coefficients c_{ijk} are randomly specified complex numbers. Thus, the generalized linear product start system is given by

$$Q(\mathbf{z}) = \left\{ \begin{array}{c} \prod_{j=1}^{m_1} G_{1j} \\ \vdots \\ \prod_{j=1}^{m_n} G_{nj} \end{array} \right\}. \tag{18}$$

In order to determine the roots of this start system, we follow Wise et al. (2000)[35] and introduce the *factor lexicographic vector* $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_n)$ which is the lexicographically ordered combinations of factors taken one from each polynomial in the system. Notice that Φ ranges from $(1, 1, \ldots, 1) \leq \Phi \leq (m_1, m_2, \ldots, m_n)$. Next, we introduce the *degree lexicographic vector* $\Delta = (\Delta_1, \Delta_2, \ldots, \Delta_n)$ which is the lexicographically ordered combinations of the count of the roots of unity associated with the degree of the factor. The set Δ ranges from $(0, 0, \ldots, 0) \le \Delta \le (d_{1\Phi_1} - 1, d_{2\Phi_2} - 1, \ldots, d_{n\Phi_n} - 1)$, where $1 \le d_{i\Phi_i}$ by definition of our linear product decomposition.

Given a combination of factors Φ, we have one or more arrays Δ depending on the degrees of the specific factors identified by Φ. These two vectors specify the linear system of equations

$$[A_\Phi]\mathbf{z} = \left\{ \begin{array}{c} \sum_{k=1}^{m_1} c_{1\Phi_1 k} s_{1\Phi_1 k} z_k \\ \sum_{k=1}^{m_2} c_{2\Phi_2 k} s_{2\Phi_2 k} z_k \\ \vdots \\ \sum_{k=1}^{m_n} c_{n\Phi_n k} s_{n\Phi_n k} z_k \end{array} \right\} = \left\{ \begin{array}{c} e^{i\frac{\Delta_1}{d_{1\Phi_1}}} \\ e^{i\frac{\Delta_2}{d_{2\Phi_2}}} \\ \vdots \\ e^{i\frac{\Delta_n}{d_{n\Phi_n}}} \end{array} \right\} = b_\Delta. \tag{19}$$

If $[A_\Phi]$ is non-singular then the solution of this equation contributes a root to the start system for every root of unity in the array Δ. Wise et al. (2000)[35] provide an efficient algorithm for computing the solutions to linear systems that are organized in this way, which was implemented in the polynomial homotopy software POLSYS_PLP. We use the same algorithm to determine the roots of our generalized linear product start systems. For this reason, we term our algorithm POLSYS_GLP.

5 Verifying the Linear Product Decomposition

In order to execute POLSYS_GLP, the user provides both the target polynomials and their associated linear product decompositions, which are used to construct the start system. If there is an error and the polynomial does not actually lie in the span of the specified generic linear products, then the homotopy is meaningless. Therefore, it is imperative to verify the linear product decomposition as follows.

For each polynomial P_i, we check that each monomial ${z_1}^{\alpha_1} {z_2}^{\alpha_2} \cdots {z_n}^{\alpha_n}$ is contained in the associated linear product decomposition $\prod_{j=1}^{m_i} \langle s_{ij1} z_1, \ldots, s_{ijn} z_n, 1 \rangle^{d_{ij}}$. Our approach is to create a "set structure table" that has the linear terms of $\langle s_{ij1} z_1, \ldots, s_{ijn} z_n, 1 \rangle$ as its column headings, and the factors of the expanded monomial ${z_1}^{\alpha_1} {z_2}^{\alpha_2} \cdots {z_n}^{\alpha_n}$ as its rows. This set structure table has as many columns as the total degree d_i of P_i, and as many rows as the total degree of the monomial, which must be less than or equal to d_i.

The defining characteristic of a linear product decomposition is that each factor of the expanded monomial arises from a different linear term in the decomposition. This means that each row of the set structure table must be assignable to a separate column. If this assignment does not exist then the linear decomposition is invalid.

We begin with the first row and search the columns left to right to find a linear term (column) that contains the associated monomial factor (row). This column number is saved in a list that denotes the linear terms that have been taken. The row is incremented and the search applied to the columns that have not been taken. When the final row is assigned to an available linear term the verification for the monomial is complete.

If a row is found to have a factor that cannot be assigned to a linear term, then the assignment of the factor in the previous row is advanced to the next linear term (column) in which it is contained. This step continues until either all of the factors are assigned to separate columns, or there is no available assignment for the factor in the first row. If this occurs then the monomial is not contained in the span of the linear product decomposition.

6 Parallelizing the Path Tracking Step

Each solution of the GLP start system (19) defines a starting point to begin tracing an individual zero curve. The zero curve for every root must be traced to determine whether it leads to a root of the target system or to a point at infinity. Because these calculations are independent, they can be distributed among different processors in a parallel computing cluster. See Allison et al. (1989ab)[1, 2], and Chakraborty et al. (1991, 1993)[5, 6].

We use MPI-2 (Message Passing Interface) described by Gropp et al. (2002)[11] to distribute an identical set of POLSYS_GLP routines among each of $n-1$ slave processing nodes, numbered $r = 1, \ldots, n-1$. The number r is called the *rank* of the processor. The processor of rank 0 is the master node. Each of the slave nodes executes a loop consisting of a request to the master node for a path index. This index identifies the root that begins a particular zero curve. The slave node traces the zero curve, reports the results and requests another path index. The master node receives the requests by the slave nodes, identifies the rank of the requesting node, distributes the next path index, and sends a stop code when all the paths are traced.

Recall that the start system is constructed using random values for the coefficients in the polynomials G_{ij}. We generate these coefficients separately and provided them to the slave routines via a data file. In this way each slave node has the same start system with the same array of roots. This reduces the need for inter-processor communication. The result is a convenient parallel computation of the homotopy zero curves leading from the roots of the start system to the roots of the target polynomial system.

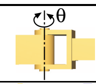
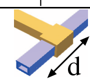
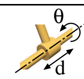
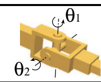
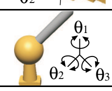
Joint	Diagram	Symbol	DOF
Revolute		R	1
Prismatic		P	1
Cylindric		C	2
Universal		T	2
Spherical		S	3

Table 1: The five basic joints.

7 Reachable Surfaces

We now consider the problem of finding surfaces that contain a set of points generated by a displaced rigid body. Our focus is on the surfaces reachable by the wrist center of an articulated serial chain. In general, each joint of an articulated serial chain is designed to allow either pure rotation about, or a linear slide along, the joint axis, and is termed a revolute or prismatic joint, denoted R and P, respectively. See Craig (1989)[8] for an introduction to the kinematics of articulated serial chains.

Revolute and prismatic joints can be combined to define other specialized joints. In particular, the sequence of two revolute joints that have axes that intersect at right angles is called a gimbal, or universal joint, denoted by a T. Similarly, the sequence of a revolute and a prismatic joint constructed so their axes are parallel is called a cylindric (C) joint. Finally, a three revolute chain with concurrent joint axes form a spherical (S), or ball, joint. See Table 1.

A spherical wrist is an S-joint that allows full orientation of the gripper about its wrist center, **P**, therefore our reachable surfaces by **P** under the control of two other joints in the articulated chain. The combinations available for revolute and prismatic joints yields four basic chains: PPS, RPS, PRS, and RRS. The reachable surfaces defined by these chains are the plane, the circular hyperboloid, the elliptic cylinder and the general torus.

We can obtain additional reachable surfaces by specializing the dimensional parameters that characterize the first two joints. In particular, the RR chain has two defining parameters the distance, ρ, between the joint axes along their common normal line, and the angle α between them measured around this common normal. For $\alpha = \frac{\pi}{2}$ we have the chain "right" RRS that traces a circular torus. For the case

Case	Chain	angle	length	Surface
1	PPS	-	-	plane
2	TS	$\frac{\pi}{2}$	0	sphere
3	CS	0	-	circular cylinder
4	RPS	α	-	circular hyperboloid
5	PRS	α	-	elliptic cylinder
6	right RRS	$\frac{\pi}{2}$	ρ	circular torus
7	RRS	α	ρ	general torus

Table 2: The basic serial chains and their associated reachable surfaces.

$\alpha = 0$, the "parallel" RRS traces a plane and is equivalent to the PPS chain. If the parameter $\rho = 0$, then the surface is part of a sphere, and fills the sphere for $\alpha = \frac{\pi}{2}$ which characterizes a TS chain.

For RP and PR chains, only the angle α is important because this joint ensures that all points to travel on lines parallel to its direction. We can identify the special cases of the RPS and PRS for which this angle is $\alpha = 0$, which in both cases become the CS chain that traces a circular cylinder. If this angle is $\alpha = \frac{\pi}{2}$, called a "right" RPS, then the surface is again a plane equivalent to that traced by the PPS chain.

Finally, all PP chains are essentially the same as long the directions of the two joints are not parallel, so that some component of movement perpendicular to the first prismatic joint is available by sliding along the second joint.

The result is a set of seven algebraic surfaces that are reachable by the wrist centers of a set of articulated chains. See Table 2. In what follows, we formulate sets of polynomial equations that define these surfaces for a given set of spatial displacements. We then provide a linear product decomposition and the results of our polynomial homotopy solution.

8 The Plane

The PPS serial chain has the property that the wrist center $\mathbf{P}$ is constrained to lie on a plane (Figure 2). We now seek points in a moving body that can be used for this wrist center, such that they lie on a plane for each of a set of spatial positions defined for the body. If the positions be defined by $[T_i] = [A_i, \mathbf{d}_i]$, $i = 1, \ldots, n$, then $\mathbf{P}^i = [A_i]\mathbf{p} + \mathbf{d}_i$ are the positions of the wrist center. The goal is to find both a plane and the point $\mathbf{p} = (x, y, z)$, such that the $\mathbf{P}^i$ all lie on the plane.

A point $\mathbf{P} = (X, Y, Z)$ lies on a plane with the surface normal $\mathbf{G} = (a, b, c)$ if its

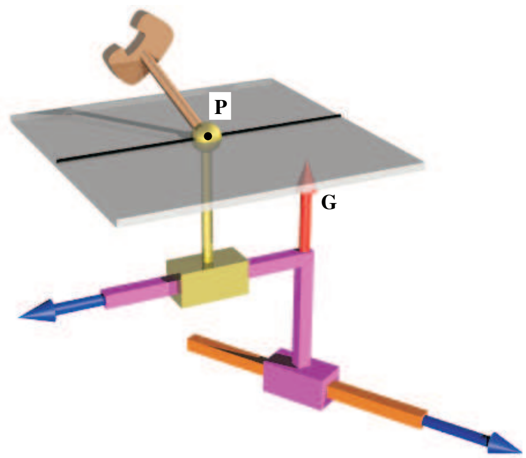


Figure 2: A plane as traced by a point at the wrist center of a PPS serial chain.

coordinates satisfy the equation

$$aX + bY + cZ - d = \mathbf{G} \cdot \mathbf{P} - d = 0. \tag{20}$$

The parameter d is the product of the magnitude $|G|$ and the signed normal distance to the plane.

There are seven parameters in this problem, the three coordinates of $\mathbf{G}$, the three coordinates of $\mathbf{p}$, and d. Notice, however, the components of $\mathbf{G}$ are not independent because the depends on the direction of $\mathbf{G}$, not its magnitude. A convenient way to constrain this magnitude is to choose a vector $\mathbf{m}$ and scalar e, and require that $\mathbf{m} \cdot \mathbf{G} = e$.

Thus, to determine the plane we use this constraint equation together with the equation of the plane (20) evaluated for six spatial displacements, $[T_i], i = 1, \dots, 6$,

$$\mathbf{G} \cdot \mathbf{P}^i - d = 0, \quad i = 1, \dots, 6. \tag{21}$$

Subtract the first of these equations from the remaining to eliminate d, The result is the polynomial system

$$P(\mathbf{z}) = \left\{ \begin{matrix} \mathbf{G} \cdot (\mathbf{P}^2 - \mathbf{P}^1) \\ \vdots \\ \mathbf{G} \cdot (\mathbf{P}^6 - \mathbf{P}^1) \\ \mathbf{m} \cdot \mathbf{G} - e \end{matrix} \right\} = 0. \tag{22}$$

This is a set of five quadratic equations and one linear equation in the six unknowns $\mathbf{z} = (a, b, c, x, y, z)$. The total degree of this system is $2^5 = 32$.

It is easy to see that this polynomial system has the linear product decomposition (22) as

$$P(\mathbf{z}) \in \left\{ \begin{array}{c} \langle a, b, c\rangle\langle x, y, z, 1\rangle|_1 \\ \vdots \\ \langle a, b, c\rangle\langle x, y, z, 1\rangle|_5 \\ \langle a, b, c, 1\rangle \end{array} \right\}. \tag{23}$$

The root count for this linear product decomposition (LPD) is given by the combinations of linear factors that can be set to zero and solved for the unknown parameters. In this case, we have $\binom{5}{2} = 10$ roots, which means that may be as many as 10 points in the moving body that lie on a plane for six specified positions of the end-effector.

This system of polynomials (22) is small enough that direct elimination of the parameters can be used to obtain a univariate polynomial, which is found to be of degree 10. Thus, in this case the LPD bound is exact. It is interesting to note that our numerical calculations have not yielded more than four real solutions.

Once the plane P and point $\mathbf{p}$ are defined, then it is possible to determine a PPS chain, a parallel RRS or a right RPS chain that guides this point through the specified positions.

9 The Sphere

We now return to our opening example in which a point $\mathbf{P} = (X, Y, Z)$ constrained to lie on a sphere of radius R around the point $\mathbf{B} = (u, v, w)$, Figure 3. This means its coordinates satisfy the equation

$$(X - u)^2 + (Y - v)^2 + (Z - w)^2 - R^2 = (\mathbf{P} - \mathbf{B})^2 - R^2 = 0. \tag{24}$$

We now consider $\mathbf{P}^i$ to be the image of a point $\mathbf{p} = (x, y, z)$ in a moving frame M that takes positions in space defined by the displacements $[T_i] = [A_i, \mathbf{d}_i]$, $i = 1, \dots, n$. See Innocenti (1995)[12], Liao and McCarthy (2001)[15] and Raghavan (2002)[21].

This problem has seven parameters, the three components each of $\mathbf{p}$ and $\mathbf{B}$ and the radius R. Therefore we can evaluate (24) on $n = 7$ displacements,

$$(\mathbf{P}^i - \mathbf{B})^2 - R^2 = 0, \quad i = 1, \dots, 7. \tag{25}$$

Subtract the first equation from the remainder in order to eliminate R, and obtain the equations $S(\mathbf{z})$ (10) where $\mathbf{z} = (x, y, z, u, v, w)$.

We have already seen that this system has the linear product decomposition

$$S(\mathbf{z}) \in \left\{ \begin{array}{c} \langle x, y, z, 1\rangle\langle u, v, w, 1\rangle|_1 \\ \vdots \\ \langle x, y, z, 1\rangle\langle u, v, w, 1\rangle|_6 \end{array} \right\}, \tag{26}$$

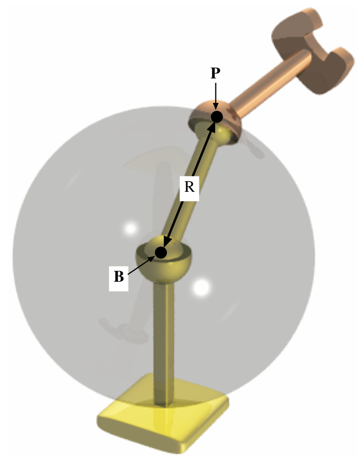


Figure 3: A sphere traced by a point at the wrist center of a TS serial chain.

from which we can compute the LPD bound $\binom{6}{3} = 20$. Parameter elimination yields a univariate polynomial of degree 20, which means that this bound is exact. Innocenti (1995)[12] presents an example that results in 20 real roots.

Thus, given seven arbitrary spatial positions there can be as many as 20 points in the moving body that have positions lying on a sphere. For each real point, it is possible to determine an associated TS chain.

10 The Circular Cylinder

In order to define the equation of a circular cylinder, let the line $L(t) = \mathbf{B} + t\mathbf{G}$ be its axis. A general point $\mathbf{P}$ on the cylinder lies on a circle about the point $\mathbf{Q}$ closest to it on the axis $L(t)$. See Figure 4.

Introduce the unit vectors $\mathbf{u}$ and $\mathbf{v}$ along $\mathbf{G}$ and the radius R of the cylinder, respectively, so we have

$$\mathbf{P} - \mathbf{B} = d\mathbf{u} + R\mathbf{v}, \tag{27}$$

where d is the distance from $\mathbf{B}$ to $\mathbf{Q}$. Compute the cross product of this equation with $\mathbf{G}$, in order to cancel d before squaring both sides. The result is

$$((\mathbf{P} - \mathbf{B}) \times \mathbf{G})^2 = R^2\mathbf{G}^2. \tag{28}$$

In this calculation we use the fact that $(\mathbf{v} \times \mathbf{G})^2 = \mathbf{G}^2$.

Another version of the equation of the cylinder is obtained by substituting $d = (\mathbf{P} - \mathbf{B}) \cdot \mathbf{u}$ into (27) and squaring both sides to obtain

$$(\mathbf{P} - \mathbf{B})^2 - ((\mathbf{P} - \mathbf{B}) \cdot \mathbf{G})^2 \frac{1}{\mathbf{G} \cdot \mathbf{G}} - R^2 = 0. \tag{29}$$

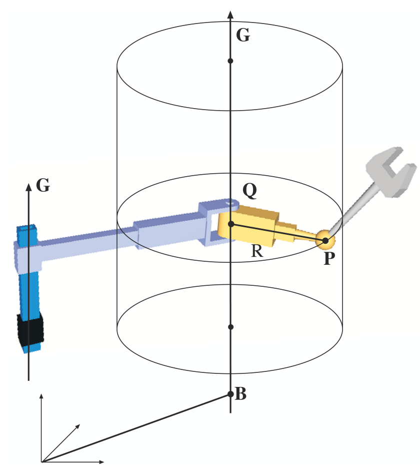


Figure 4: The circular cylinder reachable by a CS serial chain.

Notice that we allow $\mathbf{G}$ to have an arbitrary magnitude. This form of the cylinder is related to the equation of the circular hyperboloid, which is discussed in the next section.

Equation (28) has 10 parameters, the radius R and three each in the vectors $\mathbf{P} = (X, Y, Z)$, $\mathbf{B} = (u, v, w)$ and $\mathbf{G} = (a, b, c)$. However, because only the direction of $\mathbf{G}$ is important to the definition of the cylinder, its three components are not independent. We set the magnitude of $\mathbf{G}$ as we did above for the equation of the plane. Choose an arbitrary vector $\mathbf{m}$ and scalar e and require the components of G satisfy the constraint,

$$\mathbf{G} \cdot \mathbf{m} - e = 0. \tag{30}$$

The components of the point $\mathbf{B}$ are also not independent, but for a different reason. It is because any point on the line $L(t)$ can be selected as the reference point $\mathbf{B}$. We identify this point by requiring $\mathbf{B}$ to lie on a specific plane $U : (\mathbf{n}, f)$, that is

$$\mathbf{B} \cdot \mathbf{n} - f = 0, \tag{31}$$

where $\mathbf{n}$ and f are chosen arbitrarily to avoid the possibility that the line $L(t)$ may lie entirely on U.

Eight polynomials are obtained by specifying eight spatial displacements $[T_i] = [A_i, \mathbf{d}_i]$, $i = 1, \ldots, 8$, that is $\mathbf{P}^i = [A_i]\mathbf{p} + \mathbf{d}_i$, $i = 1, \ldots, 8$ and evaluating (29) for $\mathbf{p} = (x, y, z)$. The result is

$$((\mathbf{P}^i - \mathbf{B}) \times \mathbf{G})^2 - R^2\mathbf{G}^2 = 0, \quad i = 1, \ldots, 8. \tag{32}$$

Subtract the first equation from the remaining seven to eliminate R and obtain

$$((\mathbf{P}^{j+1} - \mathbf{B}) \times \mathbf{G})^2 - ((\mathbf{P}^1 - \mathbf{B}) \times \mathbf{G})^2 = 0, \quad j = 1, \ldots, 7. \tag{33}$$

These equations can be simplified and assembled with the two constraint equations to define the system of polynomials

$$C(\mathbf{z}) = \left\{ \begin{matrix} (\mathbf{P}^2 \times \mathbf{G})^2 - (\mathbf{P}^1 \times \mathbf{G})^2 - 2((\mathbf{P}^2 - \mathbf{P}^1) \times \mathbf{G}) \cdot (\mathbf{B} \times \mathbf{G}) \\ \vdots \\ (\mathbf{P}^8 \times \mathbf{G})^2 - (\mathbf{P}^1 \times \mathbf{G})^2 - 2((\mathbf{P}^8 - \mathbf{P}^1) \times \mathbf{G}) \cdot (\mathbf{B} \times \mathbf{G}) \\ \mathbf{G} \cdot \mathbf{m} - e \\ \mathbf{B} \cdot \mathbf{n} - f \end{matrix} \right\} = 0. \qquad (34)$$

This is a set of seven polynomials of degree four and two of degree one. The total degree is $4^7 = 16,384$. See Neilsen and Roth (1995)[20] and Su et al. (2003)[25] for additional details about this problem.

We now consider the monomial structure of polynomial system (34). Each polynomial P_i is a linear combination of monomials in the set generated by

$$(\langle x, y, z, 1\rangle\langle a, b, c\rangle)^2 \cup \langle x, y, z, 1\rangle\langle a, b, c\rangle\langle u, v, w\rangle\langle a, b, c\rangle. \qquad (35)$$

This can be manipulated to show the system of polynomials (34) is a special case of the linear product decomposition,

$$C(\mathbf{z}) \in \left\{ \begin{matrix} \langle a, b, c\rangle^2\langle x, y, z, 1\rangle\langle x, y, z, u, v, w, 1\rangle|_1 \\ \vdots \\ \langle a, b, c\rangle^2\langle x, y, z, 1\rangle\langle x, y, z, u, v, w, 1\rangle|_7 \\ \langle u, v, w, 1\rangle \\ \langle a, b, c, 1\rangle \end{matrix} \right\} = 0. \qquad (36)$$

In order to determine the number of roots, we notice that the components of $\mathbf{G} = (a, b, c)$ are determined by its linear constraint combined with two terms taken from $\langle a, b, c\rangle$ in the seven polynomials. Furthermore, because this term is squared, the number of choices is increased by a factor of $2^2 = 4$. Next we choose from zero to three of the terms $\langle x, y, z, 1\rangle$ from the remaining five polynomials to define $\mathbf{p} = (x, y, z)$. The remaining factors and the last linear equation define the parameters $\mathbf{B} = (u, v, w)$. This yields the LPD bound of

$$B_{LPD} = 2^2 \binom{7}{2} \sum_{i=0}^{3} \binom{5}{i} = 2,184, \qquad (37)$$

which is significantly less than the total degree.

We use our POLSYS_GLP homotopy algorithm to determine the roots for this system of polynomials for a random set of test cases and obtain the exact root count for this problem as 804. Thus, for eight arbitrary spatial positions we can find as many as 804 points in the moving body each of which has all eight positions on a circular cylinder. For each of these points, we can determine an associated CS chain.

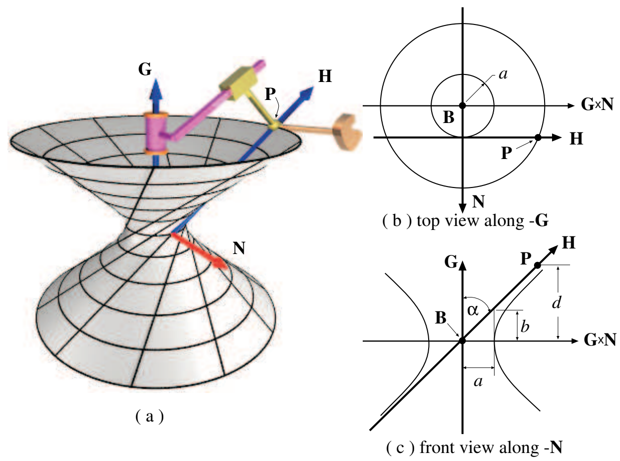


Figure 5: The circular hyperboloid traced by the wrist center of an RPS serial chain.

11 The Circular Hyperboloid

A circular hyperboloid is generated by rotating one line around another so that every point on the moving line traces a circle around the fixed line, G, which is the axis of the hyperboloid, Figure 5. Of all of these circles there is one with the smallest radius, R, and its center $\mathbf{B} = (u, v, w)$ is the center of the hyperboloid. Let $\mathbf{G} = (a, b, c)$ be the direction of the axis $L(t) = \mathbf{B} + t\mathbf{G}$. The unit vector $\mathbf{N}$ perpendicular to $\mathbf{G}$ though $\mathbf{B}$ is the common normal between the axis G and one of the generated lines, H. The generator is located at the distance R along $\mathbf{N}$, and lies at an angle α around $\mathbf{N}$ relative to the axis G.

The distance d measured along the axis G from $\mathbf{B}$ to a point $\mathbf{P}$ on the generator is given by

$$d = \frac{(\mathbf{P} - \mathbf{B}) \cdot \mathbf{G}}{\sqrt{\mathbf{G} \cdot \mathbf{G}}}. \tag{38}$$

Notice that we are not assuming that $\mathbf{G}$ is a unit vector. The magnitude of $\mathbf{P} - \mathbf{B}$ is now computed to be

$$(\mathbf{P} - \mathbf{B})^2 = R^2 + d^2 + (d \tan \alpha)^2. \tag{39}$$

Substitute d into this equation to obtain the equation of a circular hyperboloid

$$(\mathbf{P} - \mathbf{B})^2 - ((\mathbf{P} - \mathbf{B}) \cdot \mathbf{G})^2 \left(\frac{1 + \tan^2 \alpha}{\mathbf{G} \cdot \mathbf{G}}\right) - R^2 = 0. \tag{40}$$

When $\alpha = 0$, this becomes the equation of a cylinder presented in the previous section.

Figure 5(a) shows the RPS chain associated with the circular hyperboloid. The R-joint axis is G, and its P-joint axis in the direction α measured around the common normal. The point $\mathbf{P}$ is the center of the S-joint, and lies at the distance R in the direction $\mathbf{N}$ of the common normal.

Expand equation (40) and collect terms to obtain

$$k_0\mathbf{P}\cdot\mathbf{P} + 2\mathbf{K}\cdot\mathbf{P} - (\mathbf{P}\cdot\mathbf{G})^2 - \zeta = 0, \tag{41}$$

where we have introduce the parameters k_0, $\mathbf{K} = (k_1, k_2, k_3)$ and ζ defined by

$$k_0 = \frac{\mathbf{G}\cdot\mathbf{G}}{1+\tan^2\alpha},\ \mathbf{K} = (\mathbf{B}\cdot\mathbf{G})\mathbf{G} - k_0\mathbf{B},\ \zeta = (\mathbf{B}\cdot\mathbf{G})^2 - k_0\mathbf{B}\cdot\mathbf{B} + k_0R^2. \tag{42}$$

Given values for ζ, k_0, $\mathbf{K}$, and $\mathbf{G}$, we can compute $\mathbf{B}$ by solving the linear equations

$$\begin{Bmatrix} k_1\\ k_2\\ k_3\end{Bmatrix} = \begin{bmatrix} a^2-k_0 & ab & ac\\ ab & b^2-k_0 & ac\\ ac & bc & c^2-k_0\end{bmatrix}\begin{Bmatrix} u\\ v\\ w\end{Bmatrix}. \tag{43}$$

Then the length and twist parameters, R and α, are obtained from the formulas

$$\alpha = \arccos\left(\sqrt{\frac{k_0}{\mathbf{G}\cdot\mathbf{G}}}\right),\ R = \sqrt{\frac{\zeta - (\mathbf{B}\cdot\mathbf{G})^2 + k_0\mathbf{B}\cdot\mathbf{B}}{k_0}}. \tag{44}$$

Thus, the 11 dimensional parameters ζ, k_0, $\mathbf{K}$, $\mathbf{G}$, and $\mathbf{P}$ define a circular hyperboloid.

As we have seen previously, it is the direction of $\mathbf{G}$ and not its magnitude that is required, so this magnitude can be set using an arbitrary vector $\mathbf{m}$ and scalar e in the constraint equation,

$$\mathbf{G}\cdot\mathbf{m} - e = 0. \tag{45}$$

Thus, the remaining ten dimensional parameters can be determined by evaluating (41) on the displaced positions $\mathbf{P}^i = [A_i]\mathbf{p}+\mathbf{d}_i$ $i = 1,\ldots,10$, of the point $\mathbf{p} = (x, y, z)$. The result is

$$k_0\mathbf{P}^i\cdot\mathbf{P}^i + 2\mathbf{K}\cdot\mathbf{P}^i - (\mathbf{P}^i\cdot\mathbf{G})^2 - \zeta = 0, \quad i = 1,\ldots,10. \tag{46}$$

Subtract the first of these equations from the remaining in order to eliminate ζ and obtain

$$k_0(\mathbf{P}^{j+1}\cdot\mathbf{P}^{j+1} - \mathbf{P}^1\cdot\mathbf{P}^1) + 2\mathbf{K}\cdot(\mathbf{P}^{j+1} - \mathbf{P}^1) - (\mathbf{P}^{j+1}\cdot\mathbf{G})^2 + (\mathbf{P}^1\cdot\mathbf{G})^2 = 0, \quad j = 1,\ldots,9. \tag{47}$$

The result is that the right circular hyperboloid is defined by the system of polynomial equations

$$H(\mathbf{z}) = \left\{\begin{matrix} k_0(\mathbf{P}^2\cdot\mathbf{P}^2 - \mathbf{P}^1\cdot\mathbf{P}^1) + 2\mathbf{K}\cdot(\mathbf{P}^2 - \mathbf{P}^1) - (\mathbf{P}^2\cdot\mathbf{G})^2 + (\mathbf{P}^1\cdot\mathbf{G})^2\\ \vdots\\ k_0(\mathbf{P}^{10}\cdot\mathbf{P}^{10} - \mathbf{P}^1\cdot\mathbf{P}^1) + 2\mathbf{K}\cdot(\mathbf{P}^{10} - \mathbf{P}^1) - (\mathbf{P}^{10}\cdot\mathbf{G})^2 + (\mathbf{P}^1\cdot\mathbf{G})^2\\ \mathbf{G}\cdot\mathbf{m} - e\end{matrix}\right\} = 0. \tag{48}$$

This is a system of nine fourth degree polynomials and one linear equation which has a total degree of $4^9 = 262,144$. See Neilsen and Roth (1995)[20] and Kim and Tsai (2002)[13] for other formulations of this problem.

A better bound on the number of solutions can be obtained by considering the monomial structure of these equations. Recall that the term $\mathbf{P}^{j+1} \cdot \mathbf{P}^{j+1} - \mathbf{P}^1 \cdot \mathbf{P}^1$ is linear in x, y, and z, because the quadratic terms cancel, see (13). This means the polynomials (47) have the monomial structure

$$\langle k_0 \rangle \langle x, y, z, 1 \rangle \cup \langle k_1, k_2, k_3 \rangle \langle x, y, z, 1 \rangle \cup (\langle x, y, z, 1 \rangle \langle a, b, c \rangle)^2. \quad (49)$$

This simplifies to yield the linear product decomposition for the system (48) as (48) as

$$H(\mathbf{z}) \in \left\{ \begin{matrix} \langle a, b, c \rangle^2 \langle x, y, z, 1 \rangle \langle x, y, z, k_0, k_1, k_2, k_3, 1 \rangle |_1 \\ \vdots \\ \langle a, b, c \rangle^2 \langle x, y, z, 1 \rangle \langle x, y, z, k_0, k_1, k_2, k_3, 1 \rangle |_9 \\ \langle a, b, c, 1 \rangle \end{matrix} \right\}. \quad (50)$$

This structure allows us to count the number of roots from the number of admissible sets of linear equations that yield solutions for the unknown parameters. In this case, we obtain the LPD bound

$$B_{LPD} = 2^2 \binom{9}{2} \sum_{j=0}^{3} \binom{7}{j} = 9,216. \quad (51)$$

Our POLSYS_GLP algorithm yielded a generic root count of 1,024, see Su and McCarthy (2003)[26]. This calculation took approximately 24 hours on a single 2.4GHz PC (384 paths/processor-hour). The parallel version of POLSYS_GLP was run on 8 64-bit processors of UCI's Beowulf cluster, and required 30 minutes (2304 paths/processor-hour). This particular problem has a structure that is convenient for polyhedral homotopy algorithms, which yield the same solutions in minutes on a single processor by tracking only 1024 paths (Gao et al. 1999[9], Gao et al. 2003[10]).

Thus, for ten spatial positions, we can find as many as 1024 points that have all 10 positions on a circular hyperboloid. For each of these points we can find an associated RPS chain.

12 The Elliptic Cylinder

An elliptic cylinder is generated by a circle that has its center swept along a line $L(t) = \mathbf{B} + t\mathbf{S}_1$ such that the vector through the center normal to the plane of the circle maintains a constant direction $\mathbf{S}_2$ at an angle α relative to the direction $\mathbf{S}_1$ of

$L(t)$, see Figure 6. The major axis of the elliptic cross-section is the radius R of the circle and the minor axis is $R\cos\alpha$. This surface is generated by the wrist center of a PRS chain that has its P-joint aligned with the axis $L(t)$ and its R-joint positioned so its axis is along $\mathbf{S}_2$.

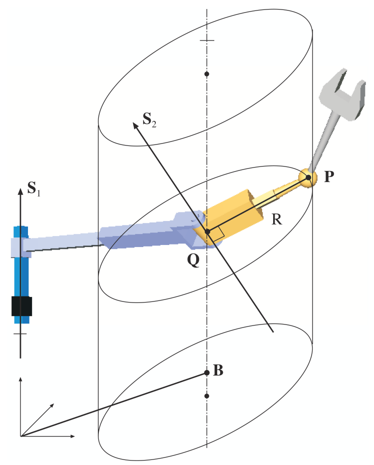


Figure 6: The elliptic cylinder reachable by a PRS serial chain.

Consider a general point on the cylinder $\mathbf{P}$, and let $\mathbf{Q}$ be the center of the circle. The point $\mathbf{Q}$ moves along the axis $L(t)$ which has the Plucker coordinates $\mathsf{S}_1 = (\mathbf{S}_1, \mathbf{B}\times\mathbf{S}_1)$. The distance from the reference point $\mathbf{B}$ to $\mathbf{Q}$ is denoted d. These definitions allow us to express the location of $\mathbf{P}$ relative to $\mathbf{B}$ as

$$\mathbf{P} - \mathbf{B} = d\mathbf{S}_1 + R\mathbf{u}, \tag{52}$$

where $\mathbf{u}$ is a unit vector in the direction $\mathbf{S}_1 \times \mathbf{S}_2$. Compute the cross product with $\mathbf{S}_1$ to eliminate d, and the cross product with $\mathbf{S}_2$ to obtain

$$\mathbf{S}_2 \times ((\mathbf{P} - \mathbf{B}) \times \mathbf{S}_1) = R(\mathbf{S}_2 \cdot \mathbf{S}_1)\mathbf{u}. \tag{53}$$

The magnitude of this vector identity yields our equation of the elliptic cylinder

$$\big(\mathbf{S}_2 \times \big((\mathbf{P} - \mathbf{B}) \times \mathbf{S}_1\big)\big)^2 = R^2(\mathbf{S}_1 \cdot \mathbf{S}_2)^2. \tag{54}$$

This equation has 13 dimensional parameters: the radius R, three each for the directions $\mathbf{S}_1$, $\mathbf{S}_2$, and the points $\mathbf{P}$ and $\mathbf{B}$. Notice that if $\mathbf{S}_1 = \mathbf{S}_2 = \mathbf{G}$ this simplifies to the equation of a circular cylinder.

There are actually only 10 independent parameters in (54), because magnitude of the directions $\mathbf{S}_1$ and $\mathbf{S}_2$ can be set arbitrarily, and the point $\mathbf{B}$ can be any point on the line S_1. We set these values using three additional linear constraints. For,the directions of $\mathbf{S}_1$ and $\mathbf{S}_2$, we introduce two arbitrary planes $V_k : (\mathbf{m}_k, e_k), k = 1, 2$ and require

$$\mathbf{m}_k \cdot \mathbf{S}_k - e_k = 0, \quad k = 1, 2. \tag{55}$$

The point $\mathbf{B}$ is specified using the intersection of $\mathbf{S}_1$ with the arbitrary plane $U : (\mathbf{n}, f)$, so that

$$\mathbf{n} \cdot \mathbf{B} - f = 0. \tag{56}$$

Recall that $\mathbf{n}$ is the unit normal to the plane and f the directed distance from the origin to the plane.

Now consider the images of a point $\mathbf{p} = (x, y, z)$ generated by 10 spatial displacements, that is $\mathbf{P}^i = [A_i]\mathbf{p} + \mathbf{d}_i$, $i = 1, \ldots, 10$. Evaluate equation (54) on these 10 points to obtain

$$\big(\mathbf{S}_2 \times \big((\mathbf{P}^i - \mathbf{B}) \times \mathbf{S}_1\big)\big)^2 - R^2(\mathbf{S}_1 \cdot \mathbf{S}_2)^2 = 0, \quad i = 1, \ldots, 10. \tag{57}$$

Subtract the first of these equations from the remaining to obtain

$$(\mathbf{S}_2 \times ((\mathbf{P}^{j+1} - \mathbf{B}) \times \mathbf{S}_1))^2 - (\mathbf{S}_2 \times ((\mathbf{P}^1 - \mathbf{B}) \times \mathbf{S}_1))^2 = 0, \; j = 1, \ldots, 9, \tag{58}$$

Thus, the elliptic cylinder is obtained as the solution to the system of polynomials

$$E(\mathbf{z}) = \begin{Bmatrix} (\mathbf{S}_2 \times ((\mathbf{P}^2 - \mathbf{B}) \times \mathbf{S}_1))^2 - (\mathbf{S}_2 \times ((\mathbf{P}^1 - \mathbf{B}) \times \mathbf{S}_1))^2 \\ \vdots \\ (\mathbf{S}_2 \times ((\mathbf{P}^{10} - \mathbf{B}) \times \mathbf{S}_1))^2 - (\mathbf{S}_2 \times ((\mathbf{P}^1 - \mathbf{B}) \times \mathbf{S}_1))^2 \\ \mathbf{m}_1 \cdot \mathbf{S}_1 - e_1 \\ \mathbf{m}_2 \cdot \mathbf{S}_2 - e_2 \\ \mathbf{n} \cdot \mathbf{B} - f \end{Bmatrix} = 0. \tag{59}$$

The result is nine polynomials of degree six, and three linear equations. The total degree of this polynomial system is $6^9 = 10,077,696$.

The total degree of this system can be reduced by expanding the triple product in (59) and introducing new variables. that is

$$\begin{aligned} \mathbf{S}_2 \times ((\mathbf{P} - \mathbf{B}) \times \mathbf{S}_1) &= (\mathbf{S}_1 \cdot \mathbf{S}_2)(\mathbf{P} - \mathbf{B}) - ((\mathbf{P} - \mathbf{B}) \cdot \mathbf{S}_2)\mathbf{S}_1 \\ &= (\mathbf{S}_1 \cdot \mathbf{S}_2)(\mathbf{P} - (\mathbf{P} \cdot \mathbf{K})\mathbf{S}_1 + \mathbf{Q}), \end{aligned} \tag{60}$$

where

$$\mathbf{K} = \frac{\mathbf{S}_2}{\mathbf{S}_1 \cdot \mathbf{S}_2}, \quad \text{and} \quad \mathbf{Q} = (\mathbf{B} \cdot \mathbf{K})\mathbf{S}_1 - \mathbf{B}. \tag{61}$$

Add to this the constraints

$$\mathbf{S}_1 \cdot \mathbf{S}_1 = 1, \; \mathbf{K} \cdot \mathbf{S}_1 = 1, \; \text{and } \mathbf{Q} \cdot \mathbf{K} = 0. \tag{62}$$

These definitions reduce the degree of the polynomials from six to four, so we have

$$\begin{aligned}&(\mathbf{P} - (\mathbf{P}\cdot\mathbf{K})\mathbf{S}_1 + \mathbf{Q})^2 = \\ &\mathbf{P}^2 + (\mathbf{P}\cdot\mathbf{K})^2 + \mathbf{Q}^2 - 2(\mathbf{P}\cdot\mathbf{S}_1)(\mathbf{P}\cdot\mathbf{K}) + 2\mathbf{P}\cdot\mathbf{Q} - 2(\mathbf{P}\cdot\mathbf{K})(\mathbf{Q}\cdot\mathbf{S}_1).\end{aligned} \tag{63}$$

The result is a new version of the polynomial system

$$E'(\mathbf{z}) = \left\{\begin{array}{c} (\mathbf{P}^2 - (\mathbf{P}^2\cdot\mathbf{K})\mathbf{S}_1 + \mathbf{Q})^2 - (\mathbf{P}^1 - (\mathbf{P}^1\cdot\mathbf{K})\mathbf{S}_1 + \mathbf{Q})^2 \\ \vdots \\ (\mathbf{P}^{10} - (\mathbf{P}^{10}\cdot\mathbf{K})\mathbf{S}_1 + \mathbf{Q})^2 - (\mathbf{P}^1 - (\mathbf{P}^1\cdot\mathbf{K})\mathbf{S}_1 + \mathbf{Q})^2 \\ \mathbf{S}_1\cdot\mathbf{S}_1 - 1 \\ \mathbf{K}\cdot\mathbf{S}_1 - 1 \\ \mathbf{Q}\cdot\mathbf{K} \end{array}\right\} = 0, \tag{64}$$

which has the total degree $2^3 4^9 = 2,097,152$.

As we have done previously, we examine the monomial structure of this system of polynomials. Let $\mathbf{S}_1 = (a,b,c)$, $\mathbf{K} = (k_1,k_2,k_3)$, and $\mathbf{Q} = (q_1,q_2,q_3)$, and recall that the quadratic terms in $\mathbf{P}^{j+1}\cdot\mathbf{P}^{j+1} - \mathbf{P}^1\cdot\mathbf{P}^1$ cancel, as does the term $\mathbf{Q}^2$. Thus, the polynomial (58) has the monomial structure

$$\begin{aligned}&\langle x,y,z,1\rangle \cup \langle x,y,z,1\rangle^2\langle k_1,k_2,k_3\rangle^2 \cup \langle x,y,z,1\rangle^2\langle k_1,k_2,k_3\rangle\langle a,b,c\rangle \\ &\quad\cup \langle x,y,z,1\rangle\langle q_1,q_2,q_3\rangle \cup \langle x,y,z,1\rangle\langle k_1,k_2,k_3\rangle\langle a,b,c\rangle\langle q_1,q_2,q_3\rangle.\end{aligned} \tag{65}$$

This leads to the linear product decomposition of (64) given by

$$E'(\mathbf{z}) \in \left\{\begin{array}{c} \langle x,y,z,1\rangle\langle x,y,z,q_1,q_2,q_3,1\rangle\langle k_1,k_2,k_3,1\rangle\langle k_1,k_2,k_3,a,b,c,1\rangle|_1 \\ \vdots \\ \langle x,y,z,1\rangle\langle x,y,z,q_1,q_2,q_3,1\rangle\langle k_1,k_2,k_3,1\rangle\langle k_1,k_2,k_3,a,b,c,1\rangle|_9 \\ \langle a,b,c,1\rangle^2 \\ \langle k_1,k_2,k_3,1\rangle\langle a,b,c,1\rangle \\ \langle k_1,k_2,k_3,1\rangle\langle q_1,q_2,q_3,1\rangle \end{array}\right\}. \tag{66}$$

The LPD bound for this system is 247,968, which is large.

This system was solved using our parallelized POLSYS_GLP on 128 nodes of the Blue Horizon supercomputer at the San Diego Supercomputer Center. The result was 18,120 solutions in almost 33 minutes. Because each node of Blue Horizon has eight processors, this corresponds to 563 cpu hours, or approximately 440 paths/processor-hour.

13 The Circular Torus

A circular torus is generated by sweeping a circle around an axis so its center traces a second circle. Let the axis be $L(t) = \mathbf{B}+t\mathbf{G}$, with Plucker coordinates $\mathsf{G} = (\mathbf{G}, \mathbf{B}\times\mathbf{G})$. See Figure 7. Introduce a unit vector $\mathbf{v}$ perpendicular to this axis so the center of the generating circle is given by $\mathbf{Q} - \mathbf{B} = \rho\mathbf{v}$. Now define $\mathbf{u}$ to be the unit vector in the direction $\mathbf{G}$, then a point $\mathbf{P}$ on the torus is defined by the vector equation,

$$\mathbf{P} - \mathbf{B} = \rho\mathbf{v} + R(\cos\phi\mathbf{v} + \sin\phi\mathbf{u}), \tag{67}$$

where ϕ is the angle measured from $\mathbf{v}$ to the radius vector of the generating circle.

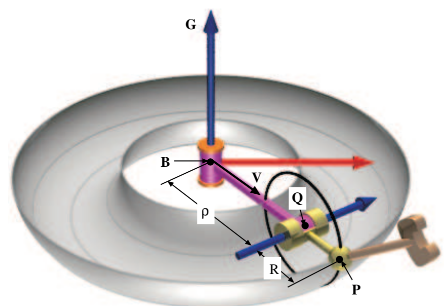


Figure 7: The circular torus traced by the wrist center of a "right" RRS serial chain.

An algebraic equation of the torus is obtained from (67) by first computing the magnitude

$$(\mathbf{P} - \mathbf{B})^2 = \rho^2 + R^2 + 2\rho R\cos\phi. \tag{68}$$

Next compute the dot product with $\mathbf{u}$, to obtain

$$(\mathbf{P} - \mathbf{B})\cdot\mathbf{u} = R\sin\phi. \tag{69}$$

Finally, eliminate $\cos\phi$ and $\sin\phi$ from these equations, and the result is

$$\mathbf{G}^2((\mathbf{P} - \mathbf{B})^2 - \rho^2 - R^2)^2 + 4\rho^2((\mathbf{P} - \mathbf{B})\cdot\mathbf{G})^2 = 4\rho^2\mathbf{G}^2R^2. \tag{70}$$

This is the equation of a circular torus. It has 11 parameters, the scalars ρ and R, and the three vectors $\mathbf{G}$, $\mathbf{P}$ and $\mathbf{B}$.

In contrast to what we have done previously, here we set the magnitude of $\mathbf{G}$ to a constant, in order to simplify the polynomial (70),

$$\mathbf{G}\cdot\mathbf{G} = 1. \tag{71}$$

Unfortunately, this doubles the number of solutions since $-\mathbf{G}$ and $\mathbf{G}$ define the same torus, however, it reduces this polynomial from degree six to degree four.

Let $[T_i] = [A_i, \mathbf{d}_i]$, $i = 1, \ldots, 10$ be a specified set of displacements, so we have the 10 positions $\mathbf{P}^i = [A_i]]\mathbf{p} + \mathbf{d}_i$ of a point $\mathbf{p} = (x, y, z)$ that is fixed in the moving frame M. Evaluating (70) on these points, we obtain the polynomial system

$$T(\mathbf{z}) = \left\{ \begin{array}{c} ((\mathbf{P}^1 - \mathbf{B})^2 - \rho^2 - R^2)^2 + 4\rho^2((\mathbf{P}^i - \mathbf{B}) \cdot \mathbf{G})^2 - 4\rho^2 R^2 \\ \vdots \\ ((\mathbf{P}^1 0 - \mathbf{B})^2 - \rho^2 - R^2)^2 + 4\rho^2((\mathbf{P}^i - \mathbf{B}) \cdot \mathbf{G})^2 - 4\rho^2 R^2 \\ \mathbf{G} \cdot \mathbf{G} - 1 \end{array} \right\} = 0. \qquad (72)$$

The total degree of this system is $2(4^{10}) = 2,097,152$.

In order to simplify this system of polynomials we introduce the parameters

$$\mathbf{H} = 2\rho\mathbf{G} \quad \text{and} \quad k_1 = \mathbf{B}^2 - \rho^2 - R^2, \qquad (73)$$

which yields the identity

$$4\rho^2 R^2 = \mathbf{H}^2(\mathbf{B}^2 - \frac{\mathbf{H}^2}{4} - k_1). \qquad (74)$$

Substitute these relations into (72) which eliminates R^2 and we obtain the system of 10 polynomials

$$T'(\mathbf{z}) = \left\{ \begin{array}{c} ((\mathbf{P}^1)^2 - 2\mathbf{P}^1 \cdot \mathbf{B} + k_1)^2 + ((\mathbf{P}^1 - \mathbf{B}) \cdot \mathbf{H})^2 - \mathbf{H}^2(\mathbf{B}^2 - \frac{\mathbf{H}^2}{4} - k_1) \\ \vdots \\ ((\mathbf{P}^{10})^2 - 2\mathbf{P}^{10} \cdot \mathbf{B} + k_1)^2 + ((\mathbf{P}^{10} - \mathbf{B}) \cdot \mathbf{H})^2 - \mathbf{H}^2(\mathbf{B}^2 - \frac{\mathbf{H}^2}{4} - k_1) \end{array} \right\} = 0. \qquad (75)$$

It is difficult to find a simplified formulation for these equations, even if we subtract the first equation from the remaining in order to cancel terms.

Expanding the polynomials in this system and examining each of the terms, we can identify the linear product decomposition

$$T'(\mathbf{z}) \in \left\{ \begin{array}{c} \langle x, y, z, h_1, h_2, h_3, 1\rangle^2 \langle x, y, z, h_1, h_2, h_3, u, v, w, k_1, 1\rangle^2|_1 \\ \vdots \\ \langle x, y, z, h_1, h_2, h_3, 1\rangle^2 \langle x, y, z, h_1, h_2, h_3, u, v, w, k_1, 1\rangle^2|_{10} \end{array} \right\}. \qquad (76)$$

This allows us to compute the LPD bound on the number of roots as

$$B_{LPD} = 2^{10} \sum_{j=0}^{6} \binom{10}{j} = 868,352. \qquad (77)$$

The computation of these homotopy paths took 72 minutes on 128 nodes of the Blue Horizon supercomputer. This means the over 800,000 paths were tracked on 1024 processors at a rate of approximately 707 paths per hour.

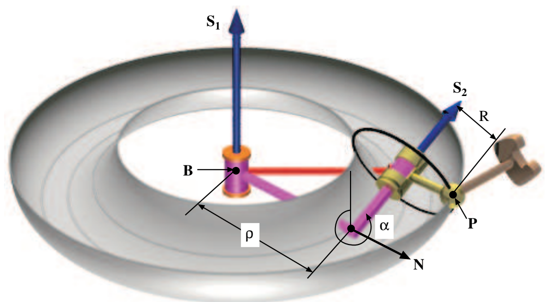


Figure 8: The general torus reachable by the wrist center of an RRS serial chain.

14 The General Torus

A general torus is defined by sweeping a circle that has a general orientation in space around an arbitrary axis. See Figure 8. Let $\mathsf{S}_1 = (\mathbf{S}_1, \mathbf{B} \times \mathbf{S}_1)$ be the Plucker coordinates of the line that forms the axis of the torus, and $\mathsf{S}_2 = (\mathbf{S}_2, \mathbf{Q} \times \mathbf{S}_2)$ be the through the center of the sweeping circle, perpendicular to its plane. These two lines define a common normal N and we choose its intersection with S_1 and S_2 to be the reference points $\mathbf{B}$ and $\mathbf{Q}$, respectively. The normal angle and distance between these lines around and along their common normal are denoted α and ρ. Finally, we identify the center of the sweeping circle as lying a distance d along $\mathbf{S}_2$ measured from $\mathbf{Q}$.

In this derivation, we constrain $\mathbf{S}_1$ and $\mathbf{S}_2$ to be unit vectors, in order to reduce the degree of the resulting equation. This allows us to define the unit vector in the common normal direction as $\mathbf{n} = (\mathbf{S}_1 \times \mathbf{S}_2)/\sin\alpha$, so we obtain a general point $\mathbf{P}$ on the torus from the vector equation,

$$\mathbf{P} - \mathbf{B} = \rho\mathbf{n} + d\mathbf{S}_2 + R(\cos\phi\mathbf{n} + \sin\phi(\mathbf{S}_2 \times \mathbf{n})). \quad (78)$$

The algebraic equation for the torus is obtained by first computing

$$(\mathbf{P} - \mathbf{B})^2 = \rho^2 + d^2 + R^2 + 2\rho R\cos\phi, \quad (79)$$

and

$$(\mathbf{P} - \mathbf{B}) \cdot (\mathbf{S}_2 \times \mathbf{n}) = R\sin\phi. \quad (80)$$

Notice that $\mathbf{S}_2 \times \mathbf{n}$ is

$$\mathbf{S}_2 \times \frac{\mathbf{S}_1 \times \mathbf{S}_2}{\sin\alpha} = \frac{1}{\sin\alpha}(\mathbf{S}_1 - \cos\alpha\mathbf{S}_2). \quad (81)$$

Now, eliminate ϕ between these two equations to obtain

$$((\mathbf{P} - \mathbf{B})^2 - \rho^2 - d^2 - R^2)^2 + \frac{4\rho^2}{\sin^2\alpha}((\mathbf{P} - \mathbf{B}) \cdot \mathbf{S}_1 - d\cos\alpha)^2 - 4\rho^2R^2 = 0. \quad (82)$$

This equation has four scalar parameters ρ, α, d and R, and three vector parameters $\mathbf{P}$, $\mathbf{B}$, and $\mathbf{S}_1$which combine with the constraint, $|\mathbf{S}_1| = 1$, to yield 12 independent parameters.

In order to simplify the use of equation (82), we introduce the new parameters

$$\begin{aligned} k_1 =& \mathbf{B}\cdot\mathbf{B} - \rho^2 - R^2 - d^2, \\ k_2 =& (\mathbf{B}\cdot\mathbf{S}_1 + d\cos\alpha)\frac{2\rho}{\sin\alpha}, \\ k_3 =& 4\rho^2R^2, \\ \mathbf{H} =& \frac{2\rho}{\sin\alpha}\mathbf{S}_1, \end{aligned} \tag{83}$$

This allow us to write (82) in the form

$$(\mathbf{P}\cdot\mathbf{P} - 2\mathbf{P}\cdot\mathbf{B} + k_1)^2 + (\mathbf{P}\cdot\mathbf{H} - k_2)^2 - k_3 = 0. \tag{84}$$

This is a quartic polynomial in the 12 unknowns, consisting of $k_1, i = 1, 2, 3$ and the components $\mathbf{P}$, $\mathbf{B}$, and $\mathbf{H}$.

Given a set of displacements $[T_i] = [A_i, \mathbf{d}_1]$, $i = 1, \ldots, 12$, we evaluate (84) on the points $\mathbf{P}^i = [A_i]\mathbf{p} + \mathbf{d}_i$, $i = 1, \ldots, 12$. Subtract the first of these equations from the remaining to cancel k_3 and obtain

$$G(\mathbf{z}) = \left\{\begin{array}{r} (\mathbf{P}^2\cdot\mathbf{P}^2 - 2\mathbf{P}^2\cdot\mathbf{B} + k_1)^2 - (\mathbf{P}^1\cdot\mathbf{P}^1 - 2\mathbf{P}^1\cdot\mathbf{B} + k_1)^2 \\ +(\mathbf{P}^2\cdot\mathbf{H} - k_2)^2 - (\mathbf{P}^1\cdot\mathbf{H} - k_2)^2 \\ \vdots \\ (\mathbf{P}^{12}\cdot\mathbf{P}^{12} - 2\mathbf{P}^{12}\cdot\mathbf{B} + k_1)^2 - (\mathbf{P}^1\cdot\mathbf{P}^1 - 2\mathbf{P}^1\cdot\mathbf{B} + k_1)^2 \\ +(\mathbf{P}^{12}\cdot\mathbf{H} - k_2)^2 - (\mathbf{P}^1\cdot\mathbf{H} - k_2)^2 \end{array}\right\} = 0. \tag{85}$$

The total degree of this system of polynomials is $4^{11} = 4,194,304$.

We can refine the estimate of the number of roots of this polynomial system by using the linear product decomposition. Expanding these polynomials, we obtain the terms

$$\begin{aligned} \mathbf{P}^{j+1^4} - \mathbf{P}^{1^4} \in& \langle x, y, z, 1\rangle^3, \\ (2\mathbf{P}^{j+1}\cdot\mathbf{B})^2 - (2\mathbf{P}^1\cdot\mathbf{B})^2 \in& \langle x, y, z, 1\rangle^2\langle u, v, w\rangle^2, \\ -4\mathbf{P}^{j+1^2}(\mathbf{P}^{j+1}\cdot\mathbf{B}) + 4\mathbf{P}^{1^2}(\mathbf{P}^1\cdot\mathbf{B}) \in& \langle x, y, z, 1\rangle^3\langle u, v, w\rangle, \\ 2k_1(\mathbf{P}^{j+1^2} - \mathbf{P}^{1^2} - 2\mathbf{P}^{j+1}\cdot\mathbf{B} + 2\mathbf{P}^1\cdot\mathbf{B}) \in& \langle x, y, z, 1\rangle\langle u, v, w, 1\rangle\langle k_1\rangle, \\ (\mathbf{P}^{j+1}\cdot\mathbf{H}))^2 - (\mathbf{P}^1\cdot\mathbf{H})^2 \in& \langle x, y, z, 1\rangle^2\langle h_1, h_2, h_3\rangle^2 \\ -2k_2(\mathbf{P}^{j+1}\cdot\mathbf{H} - \mathbf{P}^1\cdot\mathbf{H}) \in& \langle x, y, z, 1\rangle\langle h_1, h_2, h_3\rangle\langle k_2\rangle \end{aligned} \tag{86}$$

Case	Surface	Total degree	LPD bound	Number of roots
1	plane	32	10	10
2	sphere	64	20	20
3	circular cylinder	16,384	2,184	804
4	circular hyperboloid	262,144	9,216	1,024
5	elliptic cylinder	2,097,152	247,968	18,120
6	circular torus	2,097,152	868,352	94,622
7	general torus	4,194,304	448,702	42,615

Table 3: Summary of the total degree, LPD bound, and number of solutions of the polynomial equations that define each reachable surface.

Notice that the quartic terms in the first expression cancel. We combine these monomials into the linear product decomposition,

$$G(\mathbf{z}) \in \left\{ \begin{array}{c} \langle x, y, z, 1\rangle^2 \langle u, v, w, h_1, h_2, h_3, 1\rangle \langle x, y, z, u, v, w, h_1, h_2, h_3, k_1, k_2, 1\rangle|_1, \\ \vdots \\ \langle x, y, z, 1\rangle^2 \langle u, v, w, h_1, h_2, h_3, 1\rangle \langle x, y, z, u, v, w, h_1, h_2, h_3, k_1, k_2, 1\rangle|_{11}, \end{array} \right\}. \tag{87}$$

This allows us to compute the LPD bound of 448,702.

Our parallel POLSYS_GLP algorithm computed 42,615 solutions in 42 minutes using 128 nodes of Blue Horizon. This is approximately 626 paths/processor-hour. Each real solution can be used to design an RRS chain to reach the specified displacements. The distribution and utility of these solutions requires further study.

15 Conclusion

In this paper, we seek points in a moving body that lie on seven algebraic surfaces that are reachable by an articulated chain with a spherical wrist, see Table 2. The algebraic equations of these *reachable surfaces* are evaluated for a specified set of spatial displacements, in order to define a system of polynomial equations that are solved to determine the surface.

The complexity of this problem increases with degree of the surface and the number of parameters that define it, and for all but the simplest cases we use a numerical homotopy algorithm to find all of the roots. Vector operations in the derivation of these equations yield a general linear product structure that allows us to show the number of roots is (often) less than the total degree of the system. This linear product bound defines the number of paths that we must track using our homotopy algorithm

POLSYS_GLP to find these roots. Table 3 summarizes the results of our analysis.

Except for the plane and sphere, this is the first computation of the solutions for these polynomial systems. The three most challenging cases were the elliptic cylinder, right circular torus and the general torus, which correspond to the PRS, the right RRS, and general RRS chains. In these cases, our algorithm required the Blue Horizon supercomputer in order to compute tens of thousands of solutions. More research is required to increase the efficiency of the calculation and to evaluate the utility of each solution.

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