Minimal Parameter Homotopies for the L^2 Optimal Model Order Reduction Problem

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Abstract—The problem of finding a reduced order model, optimal in the L^2 sense, to a given system model is a fundamental one in control system analysis and design. The problem is very difficult without the global convergence of homotopy methods, and a number of homotopy based approaches have been proposed. The issues are the number of degrees of freedom, the well posedness of the finite dimensional optimization problem, and the numerical robustness of the resulting homotopy algorithm. Homotopy algorithms based on several formulations — Hyland and Bernstein's optimal projection equations; input normal form; Ly, Bryson, and Cannon's 2×2 block parametrization; a new nonminimal parametrization — are developed and compared here. The main conclusions are that dimensionality is inversely related to numerical well conditioning, and algorithmic efficiency is inversely related to robustness of the algorithm.

Index Terms—homotopy method, input normal form, optimal projection equations, parameter optimization, reduced order model problem.

I. Introduction

The L^2 optimal model reduction problem, i.e., the problem of approximating a higher order dynamical system by a lower order one so that a quadratic model reduction criterion is minimized, is of significant importance and is under intense study. Several earlier attempts to apply homotopy methods to the L^2 optimal model order reduction problem were not entirely satisfactory. Richter and Collins [13]–[15] devised a homotopy approach which only estimated certain crucial partial derivatives and employed relatively crude curve tracking techniques. Žigić, Bernstein, Collins, Richter, and Watson [21]–[23] formulated the problem so that numerical linear algebra techniques could be used to explicitly calculate partial derivatives, and employed sophisticated homotopy curve tracking algorithms, but the number of variables made large problems intractable. We propose here several ways to reduce the dimension of the homotopy map so that large problems are computationally feasible.

The problem can be formulated as: given the asymptotically stable, controllable, observable, time invariant, continuous time system

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t), \\ y(t) &= C\,x(t), \end{aligned} \tag{1}$$

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where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{l\times n}$, the goal is to find a reduced order model

$$\begin{aligned} \dot{x}_m(t) &= A_m\, x_m(t) + B_m\, u(t), \\ y_m(t) &= C_m\, x_m(t), \end{aligned} \tag{2}$$

where $A_m \in \mathbf{R}^{n_m\times n_m}$, $B_m \in \mathbf{R}^{n_m\times m}$, $C_m \in \mathbf{R}^{l\times n_m}$, $n_m < n$ which minimizes the cost function

$$J(A_m, B_m, C_m) \equiv \lim_{t\to\infty} E\left[(y - y_m)^T R\,(y - y_m)\right], \tag{3}$$

where the input $u(t)$ is white noise with symmetric and positive definite intensity V and R is a symmetric and positive definite weighting matrix.

The optimal projection equations of Hyland and Bernstein [4], [5], described in Section V, are basis independent and correspond to the maximum number of degrees of freedom one could plausibly use. Richter and Collins [15] use this maximum number, and Žigić [21] reduced it somewhat. At the other extreme, the minimum number of degrees of freedom corresponds to the input normal form described in Section II, and developed into a homotopy algorithm in Sections III and IV. Subtle differences between the optimal projection equations and input normal form formulations are explored in Section V. Assuming a particular Jordan form for A_m leads to the minimal parameter formulation of Ly et al. [8], which is developed into a homotopy algorithm in Section VI. Section VII gives numerical results for the input normal form and Ly form homotopies on the test set of Žigić [21].

Both the input normal form and Ly parameterization use the minimum possible number of degrees of freedom, but rely on assumptions about the structure of (A_m, B_m, C_m) that do not always hold, and therefore may not exist. Even worse, they may exist but be arbitrarily badly ill conditioned, resulting in unstable numerical algorithms. Section VIII explores an alternative formulation using more than the minimal number of degrees of freedom, and compares to the minimal formulations. Comparisons between the three formulations and the optimal projection equations approach are given in Section IX. A fundamental difference between the optimal projection equations and the other formulations is that the optimal projection equations approach solves $f(x) = 0$ where f is not the gradient of the cost functional and x is not the reduced order model, while the other three formulations solve $g(y) = 0$ where g is the gradient of the cost functional and y is the reduced order model.

II. Input normal form formulations.

The following theorem is needed to present the homotopy method for the input normal form.

THEOREM 1 [6]. *Suppose $\bar{A}_m$ is asymptotically stable. Then for every minimal $(\bar{A}_m, \bar{B}_m, \bar{C}_m)$, i.e., $(\bar{A}_m, \bar{B}_m)$ is controllable and $(\bar{A}_m, \bar{C}_m)$ is observable, there exist a similarity transformation U and a positive definite matrix $\Omega = \operatorname{diag}(\omega_1, \cdots, \omega_{n_m})$ such that $A_m = U^{-1}\bar{A}_m U$, $B_m = U^{-1}\bar{B}_m$, and $C_m = \bar{C}_m U$ satisfy*

$$\begin{aligned} 0 &= A_m + A_m^T + B_m V B_m^T, \\ 0 &= A_m^T \Omega + \Omega A_m + C_m^T R C_m. \end{aligned} \tag{4}$$

In addition,

$$
\begin{aligned}
(A_m)_{ii} &= -\frac{1}{2}(B_m V B_m^T)_{ii}, \\
\omega_i &= \frac{(C_m^T R C_m)_{ii}}{(B_m V B_m^T)_{ii}}, \\
(A_m)_{ij} &= \frac{(C_m^T R C_m)_{ij} - \omega_j (B_m V B_m^T)_{ij}}{\omega_j - \omega_i}, \qquad \text{if } \omega_i \neq \omega_j.
\end{aligned} \tag{5}
$$

DEFINITION 1. The triple (A_m, B_m, C_m) satisfying (4) or (5) is said to be in *input normal form.*

Note that generically $\omega_i \neq \omega_j$ for $i \neq j$, and this is assumed henceforth. Under the assumption that a solution (A_m, B_m, C_m) in input normal form is sought, the only independent variables are B_m and C_m, and in this case the domain is

$$\{(A_m, B_m, C_m) : \ A_m \text{ is stable, } (A_m, B_m, C_m) \text{ is minimal and in input normal form}\}.$$

Assuming (A_m, B_m, C_m) is in input normal form, the cost function (3) can be written as

$$J(A_m, B_m, C_m) = \operatorname{tr}\,(\tilde{Q}\tilde{R}) \tag{6}$$

where $\tilde{Q}$ is a symmetric and positive definite matrix satisfying

$$\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{V} = 0, \tag{7}$$

and

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & A_m \end{pmatrix}, \qquad \tilde{R} = \begin{pmatrix} C^T R C & -C^T R C_m \\ -C_m^T R C & C_m^T R C_m \end{pmatrix}, \qquad \tilde{V} = \begin{pmatrix} B V B^T & B V B_m^T \\ B_m V B^T & B_m V B_m^T \end{pmatrix}. \tag{8}$$

$\tilde{Q}$ can be written as

$$\tilde{Q} = \begin{pmatrix} \tilde{Q}_1 & \tilde{Q}_{12} \\ \tilde{Q}_{12}^T & \tilde{Q}_2 \end{pmatrix}, \tag{9}$$

where $\tilde{Q}_1 \in \mathbf{R}^{n \times n}$, $\tilde{Q}_{12} \in \mathbf{R}^{n \times n_m}$, and $\tilde{Q}_2 \in \mathbf{R}^{n_m \times n_m}$.

The goal of minimizing (6) under the constraints (4) and (7) leads to the Lagrangian

$$
\begin{aligned}
L(A_m, B_m, C_m, \Omega, \tilde{Q}) = \operatorname{tr}[&\tilde{Q}\tilde{R} + (A_m + A_m^T + B_m V B_m^T) M_c \\
&+ (A_m^T \Omega + \Omega A_m + C_m^T R C_m) M_o + (\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{V})\tilde{P}],
\end{aligned}
$$

where the symmetric matrices M_o, M_c, and $\tilde{P}$ are Lagrange multipliers.

Setting $\partial L / \partial \tilde{Q} = 0$ gives

$$\tilde{A}^T \tilde{P} + \tilde{P}\tilde{A} + \tilde{R} = 0, \tag{10}$$

where $\tilde{P}$ is symmetric positive definite and can be partitioned as

$$\tilde{P} = \begin{pmatrix} \tilde{P}_1 & \tilde{P}_{12} \\ \tilde{P}_{12}^T & \tilde{P}_2 \end{pmatrix}. \tag{11}$$

$\partial L / \partial \Omega = 0$ and $\partial L / \partial A_m = 0$ yield

$$0 = 2M_c + 2\Omega M_o + 2\big(\tilde{P}_{12}^T \tilde{Q}_{12} + \tilde{P}_2 \tilde{Q}_2\big), \qquad 0 = \big(A_m M_o\big)_{ii}, \quad 1 \le i \le n_m.$$

A straightforward calculation shows

$$\begin{aligned} \frac{\partial L}{\partial B_m} &= 2\big(\tilde{P}_{12}^T B + \tilde{P}_2 B_m\big) V + 2 M_c B_m V, \\ \frac{\partial L}{\partial C_m} &= 2R\big(C_m \tilde{Q}_2 - C \tilde{Q}_{12}\big) + 2 R C_m M_o. \end{aligned} \tag{12}$$

THEOREM 2 [2]. *The matrices M_c and M_o in* (12) *satisfy*

$$\begin{aligned} M_c &= -\big(\frac{1}{2} S + \Omega M_o\big), \\ \big(M_o\big)_{ii} &= -\frac{1}{\big(A_m\big)_{ii}} \sum_{\substack{j=1 \\ j \ne i}}^{n_m} \big(A_m\big)_{ij} \big(M_o\big)_{ji}, \\ \big(M_o\big)_{ij} &= \frac{(S)_{ij} - (S)_{ji}}{2(\omega_j - \omega_i)}, \quad \text{if } \omega_j \ne \omega_i, \end{aligned} \tag{13}$$

where

$$S = 2\big(\tilde{P}_{12}^T \tilde{Q}_{12} + \tilde{P}_2 \tilde{Q}_2\big). \tag{14}$$

III. A HOMOTOPY APPROACH BASED ON THE INPUT NORMAL FORM.

A homotopy approach based on the input normal form is now described. Let A_f, B_f, C_f , R_f, and V_f denote A, B, C, R, and V in the above and define

$$\begin{aligned} A(\lambda) &= A_0 + \lambda(A_f - A_0), \qquad & R(\lambda) &= R_0 + \lambda(R_f - R_0), \\ B(\lambda) &= B_0 + \lambda(B_f - B_0), & V(\lambda) &= V_0 + \lambda(V_f - V_0). \\ C(\lambda) &= C_0 + \lambda(C_f - C_0), \end{aligned} \tag{15}$$

For brevity, $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $V(\lambda)$, and $R(\lambda)$ will be denoted by A, B, C, V, and R respectively in the following. Let

$$\begin{aligned} H_{B_m}(\theta, \lambda) &= \frac{\partial L}{\partial B_m} = 2\big(\tilde{P}_{12}^T B + \tilde{P}_2 B_m\big) V + 2 M_c B_m V, \\ H_{C_m}(\theta, \lambda) &= \frac{\partial L}{\partial C_m} = 2R\big(C_m \tilde{Q}_2 - C \tilde{Q}_{12}\big) + 2 R C_m M_o, \end{aligned}$$

where

$$\theta \equiv \begin{pmatrix} \text{Vec } (B_m) \\ \text{Vec } (C_m) \end{pmatrix}$$

denotes the independent variables B_m and C_m, M_o and M_c satisfy (13), and $\tilde{Q}$ and $\tilde{P}$ satisfy respectively (7) and (10) with partitioned forms (9) and (11). Vec(P) for a matrix $P \in \mathbf{R}^{p\times q}$ is the concatenation of its columns:

$$\text{Vec}(P) \equiv \begin{pmatrix} P_{\cdot 1} \\ P_{\cdot 2} \\ \vdots \\ P_{\cdot q} \end{pmatrix} \in \mathbf{R}^{p\times q}.$$

The homotopy map is defined as

$$\rho(\theta,\lambda) = \begin{pmatrix} \text{Vec}\left[H_{B_m}(\theta,\lambda)\right] \\ \text{Vec}\left[H_{C_m}(\theta,\lambda)\right] \end{pmatrix}, \tag{16}$$

and its Jacobian matrix is

$$D\rho(\theta,\lambda) = \big(D_\theta\rho(\theta,\lambda), D_\lambda\rho(\theta,\lambda)\big). \tag{17}$$

Define

$$\hat{H}_{B_m}\big(\tilde{P}^{(j)}, M_c^{(j)}\big) = 2\big(\tilde{P}_{12}^{T\,(j)}B + \tilde{P}_2^{(j)}B_m\big)V + 2M_c^{(j)}B_mV,$$
$$\hat{H}_{C_m}(\tilde{Q}^{(j)}, M_o^{(j)}) = 2R\big(C_m\tilde{Q}_2^{(j)} - C\tilde{Q}_{12}^{(j)}\big) + 2RC_mM_o^{(j)},$$

where the superscript (j) means $\partial/\partial\theta_j$: $Y^{(j)} \equiv \frac{\partial Y}{\partial\theta_j}$. Using the above definitions, we have for $\theta_j = \big(B_m\big)_{kl}$,

$$\begin{aligned} \frac{\partial H_{B_m}}{\partial(B_m)_{kl}} &= \hat{H}_{B_m}\big(\tilde{P}^{(j)}, M_c^{(j)}\big) + 2\big(\tilde{P}_2 + M_c\big)E^{(k,l)}V, \\ \frac{\partial H_{C_m}}{\partial(B_m)_{kl}} &= \hat{H}_{C_m}\big(\tilde{Q}^{(j)}, M_o^{(j)}\big), \end{aligned} \tag{18}$$

and for $\theta_j = \big(C_m\big)_{kl}$,

$$\begin{aligned} \frac{\partial H_{B_m}}{\partial(C_m)_{kl}} &= \hat{H}_{B_m}\big(\tilde{P}^{(j)}, M_c^{(j)}\big), \\ \frac{\partial H_{C_m}}{\partial(C_m)_{kl}} &= \hat{H}_{C_m}\big(\tilde{Q}^{(j)}, M_o^{(j)}\big) + 2RE^{(k,l)}\big(\tilde{Q}_2 + M_o\big), \end{aligned} \tag{19}$$

where $E^{(k,l)}$ is a matrix of the appropriate dimension whose only nonzero element is $e_{kl} = 1$. $\tilde{P}^{(j)}$ and $\tilde{Q}^{(j)}$ can be obtained by solving the Lyapunov equations

$$\begin{aligned} 0 &= \tilde{A}^{(j)}\tilde{Q} + \tilde{A}\tilde{Q}^{(j)} + \tilde{Q}^{(j)}\tilde{A}^T + \tilde{Q}\tilde{A}^{T\,(j)} + \tilde{V}^{(j)}, \\ 0 &= \tilde{A}^{T\,(j)}\tilde{P} + \tilde{A}^T\tilde{P}^{(j)} + \tilde{P}^{(j)}\tilde{A} + \tilde{P}\tilde{A}^{(j)} + \tilde{R}^{(j)}. \end{aligned} \tag{20}$$

Similarly for λ, using a dot to denote $\partial/\partial\lambda$,

$$\begin{aligned} \frac{\partial H_{B_m}}{\partial\lambda} &= \hat{H}_{B_m}\big(\dot{\tilde{P}}, \dot{M}_c\big) + 2\tilde{P}_{12}^T\big(\dot{B}V + B\dot{V}\big) + 2\big(\tilde{P}_2 + M_c\big)B_m\dot{V}, \\ \frac{\partial H_{C_m}}{\partial\lambda} &= \hat{H}_{C_m}\big(\dot{\tilde{Q}}, \dot{M}_o\big) + 2\dot{R}C_m\big(\tilde{Q}_2 + M_o\big) - 2\big(\dot{R}C + R\dot{C}\big)\tilde{Q}_{12}, \end{aligned} \tag{21}$$

where $\dot{\tilde{P}}$ and $\dot{\tilde{Q}}$ are obtained by solving the Lyapunov equations

$$0 = \dot{\tilde{A}}\tilde{Q} + \tilde{A}\dot{\tilde{Q}} + \dot{\tilde{Q}}\tilde{A}^T + \tilde{Q}\dot{\tilde{A}}^T + \dot{\tilde{V}},$$
$$0 = \dot{\tilde{A}}^T\tilde{P} + \tilde{A}^T\dot{\tilde{P}} + \dot{\tilde{P}}\tilde{A} + \tilde{P}\dot{\tilde{A}} + \dot{\tilde{R}}.$$

IV. Numerical algorithm for input normal form homotopy.

The initial point $(\theta, \lambda) = (\theta_0, 0) = \big((B_m)_0, (C_m)_0, 0\big)$ is chosen so that the triple $\big((A_m)_0, (B_m)_0, (C_m)_0\big)$ is in input normal form and satisfies $\rho(\theta_0, 0) = 0$.

Theorem 3 [9]. *Suppose $\bar{A}$ is asymptotically stable. Then for every minimal $(\bar{A}, \bar{B}, \bar{C})$, i.e., $(\bar{A}, \bar{B})$ is controllable and $(\bar{A}, \bar{C})$ is observable, there exist a similarity transformation T and a positive definite matrix $\Lambda =$ diag $(d_1, d_2, \cdots, d_n)$ with $d_i \geq d_{i+1}$ such that $A = T^{-1}\bar{A}T$, $B = T^{-1}\bar{B}$, and $C = \bar{C}T$ satisfy*

$$0 = A\Lambda + \Lambda A^T + BVB^T,$$
$$0 = A^T\Lambda + \Lambda A + C^TRC.$$

Definition 2. The triple (A, B, C) in the above theorem is *balanced.*

According to Moore [9], under certain conditions, the leading principal $n_m \times n_m$ block of A, the leading principal $n_m \times m$ block of B, and the leading principal $l \times n_m$ block of C in balanced form are good approximations to the reduced order model. This suggests that the initial point $(\theta_0, 0)$ be chosen as follows:

1) Transform the given triple (A_f, B_f, C_f) to balanced form (A_b, B_b, C_b).
2) Partition (A_b, B_b, C_b) as

$$A_b = {}^{n_m}\Big\{ \begin{pmatrix} \overbrace{A_{11}}^{n_m} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B_b = {}^{n_m}\Big\{ \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad C_b = \begin{pmatrix} \overbrace{C_1}^{n_m} & C_2 \end{pmatrix}.$$

3) (A_0, B_0, C_0) is chosen as

$$A_0 = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \quad B_0 = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} C_1 & 0 \end{pmatrix}.$$

4) The initial point for the reduced order model is chosen as

$$\bar{\theta}_0 = \begin{pmatrix} \text{Vec } (\bar{B}_m)_0 \\ \text{Vec } (\bar{C}_m)_0 \end{pmatrix} = \begin{pmatrix} \text{Vec } B_1 \\ \text{Vec } C_1 \end{pmatrix},$$

and $(\bar{A}_m)_0 = A_{11}$ by construction.

5) Transform the initial point $\big((\bar{A}_m)_0, (\bar{B}_m)_0, (\bar{C}_m)_0\big)$ to input normal form so that the initial reduced order model is

$$\big((A_m)_0, (B_m)_0, (C_m)_0\big) = \big(T^{-1}(\bar{A}_m)_0 T, \quad T^{-1}(\bar{B}_m)_0, \quad (\bar{C}_m)_0 T\big).$$

The initial point for the homotopy map is then $(\theta_0, 0)$, where

$$\theta_0 = \begin{pmatrix} \text{Vec } (B_m)_0 \\ \text{Vec } (C_m)_0 \end{pmatrix}.$$

(In general, the truncation to obtain the approximate reduced order model should be based on the component costs instead of on the sizes of the balanced gains d_i as done above [16]. This explains why in some cases (Examples 1 and 6) the above algorithm for choosing the initial points did not lead to a reduced order model with a minimal cost.)

Once the initial point is chosen, the rest of the computation is as follows:

1) Set $\lambda := 0$, $\theta := \theta_0$.
2) Calculate A_m from (5), $\tilde{R}$, $\tilde{V}$, and compute $\tilde{Q}$ and $\tilde{P}$ according to (7) and (10).
3) Evaluate S from (14) and M_o and M_c according to (13).
4) Evaluate the homotopy map $\rho(\theta, \lambda)$ in (16) and $D\rho(\theta, \lambda)$ in (17).
5) Predict the next point $Z^{(0)} = \left(\theta^{(0)}, \lambda^{(0)}\right)$ on the curve γ.
6) For $k := 0, 1, 2, \cdots$ until convergence do

$$Z^{(k+1)} = \left[D\rho(Z^{(k)})\right]^{\dagger} \rho(Z^{(k)}),$$

where $\left[D\rho(Z)\right]^{\dagger}$ is the Moore-Penrose inverse of $D\rho(Z)$. Let $(\theta_1, \lambda_1) = \lim_{k\to\infty} Z^{(k)}$.

7) If $\lambda_1 < 1$, then set $\theta := \theta_1$, $\lambda := \lambda_1$, and go to step 2).
8) If $\lambda_1 \geq 1$, compute the solution $\bar{\theta}$ at $\lambda = 1$. A_m is then obtained from (5).

An alternative strategy for choosing an initial point is as follows:

1) Modify A_f to $A_f' = c_1 I + c_2 A_f$, where $c_1 \leq 0$ and $c_2 \geq 0$.
1) Transform (A_f', B_f, C_f) to balanced form and choose (A_0', B_0', C_0') as before.
3) Compute the initial reduced order model $\left((A_m)_0, (B_m)_0, (C_m)_0\right)$ from the triple (A_0', B_0', C_0') as before.

When $c_1 = 0$, $c_2 = 1$, this strategy reduces to the previous one. For some problems, our numerical experiments show that HOMPACK reaches $\lambda > 1$ in fewer steps with $c_1 \neq 0$ than with $c_1 = 0$. A modification to the homotopy map $\rho(\theta, \lambda)$ in (16) is

$$\rho_1(\theta, \lambda) = \lambda\rho(\theta, \lambda) + (1 - \lambda)(\theta - \theta_0),$$

where θ_0 denotes the initial value of θ at $\lambda = 0$. For some problems this homotopy map can be more efficient than the one in (16), while in other cases it can be less efficient.

V. Comparison with optimal projection equations approach.

THEOREM 4 [4] [5]. *Suppose (A_m, B_m, C_m) is a controllable and observable solution of the problem* (1)–(3). *Then there exist positive semidefinite* pseudogramians $\hat{Q}$, $\hat{P}$ *that are a solution to* modified Lyapunov equations

$$\begin{aligned} 0 &= \tau[A\hat{Q} + \hat{Q}A^T + BVB^T], \\ 0 &= [A^T\hat{P} + \hat{P}A + C^TRC]\tau, \end{aligned} \tag{22}$$

and satisfy rank conditions

$$rank\,(\hat{Q}) = rank\,(\hat{P}) = rank\,(\hat{Q}\,\hat{P}) = n_m,$$

such that the optimal model is given by

$$\begin{aligned} A_m &= \Gamma\, A\, G^T, \\ B_m &= \Gamma\, B, \\ C_m &= C\, G^T, \end{aligned} \tag{23}$$

where G and Γ come from a (G, M, Γ)-factorization *of $\hat{Q}\hat{P}$:*

$$\begin{aligned} \hat{Q}\,\hat{P} &= G^T\, M\, \Gamma, \\ \Gamma\, G^T &= I_{n_m}, \end{aligned} \tag{24}$$

$G, \Gamma \in \mathbf{R}^{n_m \times n}$, $M \in \mathbf{R}^{n_m \times n_m}$ *is positive semisimple and* $\tau \equiv G^T\,\Gamma$.

Equations (22) are called the optimal projection equations, which after the nontrivial algebraic manipulation described in [22], can be written in a form suitable for computation as

$$\begin{aligned} U_1\, A\, W_1 \Sigma\, W_1^T + \Sigma\, W_1^T\, A^T + U_1\, B\, V\, B^T &= 0, & (n_m\ n) \\ A^T\, U_1^T\, \Sigma + U_1^T\, \Sigma\, U_1\, A\, W_1 + C^T\, R\, C\, W_1 &= 0, & (n\ n_m) \\ U_1\, W_1 - I &= 0. & (n_m^2) \end{aligned} \tag{25}$$

The unknowns are $W_1 \in \mathbf{R}^{n \times n_m}$, $U_1 \in \mathbf{R}^{n_m \times n}$ and symmetric $\Sigma \in \mathbf{R}^{n_m \times n_m}$. In terms of these new unknowns, $\hat{Q}$ and $\hat{P}$ in (24) can be written as

$$\hat{Q} = W_1\, \Sigma\, W_1^T, \qquad \hat{P} = U_1^T\, \Sigma\, U_1.$$

Hyland and Bernstein [5] stated that the optimal projection equations can have at most $\binom{n}{n_m}$ solutions. It is shown by the following 2-dimensional example that this is not true in general.

The system [7] is given by

$$A = \begin{pmatrix} -0.05 & -0.99 \\ -0.99 & -5000.0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 100 \end{pmatrix}, \quad C = (\,1 \quad 100\,). \tag{26}$$

Proposition*: For the system* (1) *defined by* (26)*, the solution set of the optimal projection equations contains three isolated solutions and a one-dimensional manifold parameterized by one element of either W_1 or U_1.*

Proof. The three isolated solutions are

$$A_m = (-0.005004234), \quad B_m = (1.000213), \quad C_m = (1.000213),$$

$$A_m = (-4998.079), \quad B_m = (100.0002), \quad C_m = (100.0002),$$

$$A_m = (-0.4659163), \quad B_m = (-1.940482), \quad C_m = (-1.940482),$$

which were obtained by both POLSYS from HOMPACK [19] and by a homotopy approach [21]–[23]. The one-dimensional manifold of solutions corresponds to

$$A_m = (-0.4851515), \quad B_m = (0.0), \quad C_m = (0.0),$$

which can be derived directly from the optimal projection equations as follows.

Let $W_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $U_1 = (x_3, x_4)$, and $\Sigma = x_5$. The optimal projection equations (25) for this problem can be written as

$$\begin{aligned}
0 &= a_{11}x_1^2x_3x_5 + a_{12}x_1x_2x_3x_5 + a_{21}x_1^2x_4x_5 + a_{22}x_1x_2x_4x_5 \\
&\quad + a_{11}x_1x_5 + a_{12}x_2x_5 + (BVB^T)_{11}x_3 + (BVB^T)_{21}x_4, \\
0 &= a_{11}x_1x_2x_3x_5 + a_{12}x_2^2x_3x_5 + a_{21}x_1x_2x_4x_5 + a_{22}x_2^2x_4x_5 \\
&\quad + a_{21}x_1x_5 + a_{22}x_2x_5 + (BVB^T)_{12}x_3 + (BVB^T)_{22}x_4, \\
0 &= a_{11}x_1x_3^2x_5 + a_{12}x_2x_3^2x_5 + a_{21}x_1x_3x_4x_5 + a_{22}x_2x_3x_4x_5 \\
&\quad + a_{11}x_3x_5 + a_{21}x_4x_5 + (C^TRC)_{11}x_1 + (C^TRC)_{12}x_2, \\
0 &= a_{11}x_1x_3x_4x_5 + a_{12}x_2x_3x_4x_5 + a_{21}x_1x_4^2x_5 + a_{22}x_2x_4^2x_5 \\
&\quad + a_{12}x_3x_5 + a_{22}x_4x_5 + (C^TRC)_{21}x_1 + (C^TRC)_{22}x_2, \\
0 &= x_1x_3 + x_2x_4 - 1.
\end{aligned} \tag{27}$$

The triple (A_m, B_m, C_m) is given by

$$\begin{aligned}
A_m &= \Gamma AG^T = (x_3 \; x_4)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
&= x_1(a_{11}x_3 + a_{21}x_4) + x_2(a_{12}x_3 + a_{22}x_4), \\
B_m &= \Gamma B = (x_3 \; x_4)\begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = b_{11}x_3 + b_{21}x_4, \\
C_m &= CG^T = (c_{11} \; c_{12})\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c_{11}x_1 + c_{12}x_2,
\end{aligned} \tag{28}$$

where $\Gamma = U_1$ and $G = W_1^T$. Substituting (26) into (27) and (28), setting $B_m = x_3 + 100x_4 = 0$ and $C_m = x_1 + 100x_2 = 0$ gives $x_1 = -100x_2$, $x_3 = -100x_4$, and $A_m = -4852x_2x_4$. Equations (27) become

$$0 = 485200x_2^2x_4x_5 - 0.49x_2x_5, \tag{29}$$

$$0 = 485200x_2x_4^2x_5 - 0.49x_4x_5, \tag{30}$$

$$0 = 4852x_2^2x_4x_5 + 4901x_2x_5, \tag{31}$$

$$0 = 4852x_2x_4^2x_5 + 4901x_4x_5, \tag{32}$$

$$0 = 10001x_2x_4 - 1. \tag{33}$$

If $x_2 = 0$ or $x_4 = 0$, equation (33) will not be satisfied. Only the situation that $x_2 \neq 0$ and $x_4 \neq 0$ is possible. Then equations (29)–(33) can be reduced to

$$\begin{aligned}
0 &= 485200x_2x_4x_5 - 0.49x_5, \\
0 &= 4852x_2x_4x_5 + 4901x_5, \\
0 &= 10001x_2x_4 - 1.
\end{aligned} \tag{34}$$

If $x_5 \neq 0$ then (34) becomes

$$\begin{aligned} 0 &= 485200x_2x_4 - 0.49, \\ 0 &= 4852x_2x_4 + 4901, \\ 0 &= 10001x_2x_4 - 1, \end{aligned} \tag{35}$$

which does not have a solution.

Thus $x_5 = 0$, and equation (34) reduces to

$$10001x_2x_4 - 1 = 0,$$

which gives $A_m = -4852/10001 = -0.4851515$ corresponding to a one-dimensional manifold parametrized by x_2 or x_4. Hence the solution $A_m = -0.4851515$, $B_m = 0$ and $C_m = 0$ (which is not controllable or observable) corresponds to a one-dimensional manifold of solutions of the optimal projection equations. Q. E. D.

The set of solutions of the input normal form equations contains the same set of isolated solutions as the optimal projection equations, and also a fourth isolated solution given by $A_m = B_m = C_m = 0$. Therefore the solution sets of the two formulations are different.

The input normal form equations can be rewritten as

$$\begin{aligned} 0 &= 2(\tilde{P}_{12}^T B + \tilde{P}_2 B_m)V + 2M_c B_m V, \\ 0 &= 2R(C_m\tilde{Q}_2 - C\tilde{Q}_{12}) + 2RC_m M_o. \end{aligned} \tag{36}$$

Setting $B_m = C_m = 0$, the equations become

$$\begin{aligned} 0 &= \tilde{P}_{12}^T BV, \\ 0 &= RC\tilde{Q}_{12}, \end{aligned} \tag{37}$$

where $\tilde{P}_{12}$ and $\tilde{Q}_{12}$ satisfy respectively

$$\begin{aligned} 0 &= A^T\tilde{P}_{12} + \tilde{P}_{12}A_m, \\ 0 &= A\tilde{Q}_{12} + \tilde{Q}_{12}A_m, \end{aligned}$$

which has a solution $\tilde{P}_{12} = \tilde{Q}_{12} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. A_m satisfies

$$A_m + A_m^T + B_m V B_m^T = A_m + A_m^T = 0$$

which gives $A_m = 0$.

It should be noted that the solutions to the optimal projection equations (22) that satisfy the rank conditions $\text{rank}(\hat{Q}) = \text{rank}(\hat{P}) = \text{rank}(\hat{Q}\,\hat{P}) = n_m$ characterize all controllable and observable extremals of the optimal model reduction problem. However, there are algebraic solutions to (22) that do not satisfy these rank conditions. The one-dimensional manifold of solutions of the previous proposition are such a set of solutions since for these solutions $\text{rank}(\hat{Q}) = \text{rank}(\hat{P}) = 0 \neq n_m = 1$. On the other hand, the input normal form equations characterize all extremals of the optimal model reduction problem for which the input normal form has the property that no two diagonal elements of Ω are equal. No restriction is placed on the controllability or observability of these extremals. Hence, the extremal sets that the optimal projection equations and the input normal form equations characterize are not identical. In addition, the optimal projection equations may also have algebraic solutions that characterize additional reduced-order models that are uncontrollable or unobservable and may or may not be related to the solutions of the input normal form equations by a similarity transformation. These differences in the solution sets were illustrated by the example of this section. However, it should be noted that if one considers their input-output properties, the two solution sets are equivalent.

VI. Homotopy algorithm based on Ly's formulation.

Ly et al. [8] introduced another canonical form also with $n_m m + n_m l$ parameters as in the input normal form formulation. The reduced order model is represented with respect to a basis such that A_m is a 2×2 block-diagonal matrix (2×2 blocks with an additional 1×1 block if n_m is odd) with 2×2 blocks in the form

$$\begin{pmatrix} 0 & 1 \\ * & * \end{pmatrix},$$

B_m is a full matrix, and

$$C_m = \left((C_m)_1 \quad (C_m)_2 \quad \cdots \quad (C_m)_r \right)$$

where

$$(C_m)_i = \begin{pmatrix} 1 & * & \cdots & * \\ 0 & * & \cdots & * \end{pmatrix}^T,$$

$$(C_m)_r = \left(1 \quad * \quad \cdots \quad * \right)^T, \quad \text{if } n_m \text{ is odd.}$$

Let $\mathcal{S}$ be the set of indices of those elements of A_m which are parameters, i.e.,

$$\mathcal{S} \equiv \{(2,1), (2,2), \ldots, (n_m, n_m)\}.$$

To find the minimum of the cost function (6), consider the Lagrangian

$$L(A_m, B_m, C_m, \tilde{Q}) = \mathrm{tr}\big[\tilde{Q}\tilde{R} + (\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{V})\tilde{P}\big], \tag{38}$$

where the symmetric matrix $\tilde{P}$ is a Lagrange multiplier, $\tilde{Q}$ satisfies (7), and $\tilde{A}$, $\tilde{R}$, and $\tilde{V}$ are defined in (8). Setting $\partial L/\partial \tilde{Q} = 0$ gives (10), and $\tilde{P}$ is symmetric positive definite and can be partitioned as in (11). A straightforward calculation shows

$$\begin{aligned}
\frac{\partial L}{\partial (A_m)_{ij}} &= 2\big(\tilde{P}_{12}^T \tilde{Q}_{12} + \tilde{P}_2 \tilde{Q}_2\big)_{ij}, \quad (i,j) \in \mathcal{S}, \\
\frac{\partial L}{\partial B_m} &= 2\big(\tilde{P}_{12}^T B + \tilde{P}_2 B_m\big) V, \\
\frac{\partial L}{\partial (C_m)_{ij}} &= 2 \frac{\partial}{\partial (C_m)_{ij}} \big[\mathrm{tr}\ (-Q_{12}^T C^T R C_m) + \mathrm{tr}\ (Q_2 C_m^T R C_m) \big] \\
&= 2R\big(C_m \tilde{Q}_2 - C\tilde{Q}_{12}\big)_{ij}, \quad i > 1.
\end{aligned} \tag{39}$$

Let A_f, B_f, C_f , R_f, and V_f denote A, B, C, R, and V in the above and define $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $R(\lambda)$, and $V(\lambda)$ as in (15) and denote them by A, B, C, V, and R respectively in the following. Let

$$
\begin{aligned}
H_{A_m}(\theta,\lambda) &= \frac{\partial L}{\partial A_m} = 2\big(\tilde{P}_{12}^T\tilde{Q}_{12} + \tilde{P}_2\tilde{Q}_2\big),\\
H_{B_m}(\theta,\lambda) &= \frac{\partial L}{\partial B_m} = 2\big(\tilde{P}_{12}^T B + \tilde{P}_2 B_m\big)V,\\
H_{C_m}(\theta,\lambda) &= \frac{\partial L}{\partial C_m} = 2R\big(C_m\tilde{Q}_2 - C\tilde{Q}_{12}\big),
\end{aligned} \tag{40}
$$

where in H_{A_m} only those elements corresponding to the parameter elements of A_m are nonzero and

$$
\theta \equiv \begin{pmatrix} (A_m)_{\mathcal{S}} \\ \text{Vec } (B_m) \\ \text{Vec } (C_m)_{\mathcal{T}}. \end{pmatrix} \tag{41}
$$

denotes the independent variables, $\tilde{Q}$ and $\tilde{P}$ satisfy respectively (7) and (10), $(A_m)_{\mathcal{S}}$ is a vector consisting of those elements in A_m with indices in the set $\mathcal{S}$, i.e.,

$$
(A_m)_{\mathcal{S}} = \big((A_m)_{21}, (A_m)_{22}, \cdots, (A_m)_{n_m n_m}\big)^T,
$$

$(C_m)_{\mathcal{T}}$. is the matrix obtained from rows $\mathcal{T} = \{2, \ldots, l\}$ of C_m.

The homotopy map is defined as

$$
\rho(\theta,\lambda) = \begin{pmatrix} \big[H_{A_m}(\theta,\lambda)\big]_{\mathcal{S}} \\ \text{Vec } \big[H_{B_m}(\theta,\lambda)\big] \\ \text{Vec } \big[H_{C_m}(\theta,\lambda)\big]_{\mathcal{T}}. \end{pmatrix}, \tag{42}
$$

and its Jacobian matrix is

$$
D\rho(\theta,\lambda) = \big(D_\theta\rho(\theta,\lambda), D_\lambda\rho(\theta,\lambda)\big).
$$

Define

$$
\begin{aligned}
\hat{H}_{A_m}\big(\tilde{P}^{(j)},\tilde{Q}^{(j)}\big) &= 2\big(\tilde{P}_{12}^{T(j)}\tilde{Q}_{12} + \tilde{P}_{12}^T\tilde{Q}_{12}^{(j)} + \tilde{P}_2^{(j)}\tilde{Q}_2 + \tilde{P}_2\tilde{Q}_2^{(j)}\big),\\
\hat{H}_{B_m}\big(\tilde{P}^{(j)}\big) &= 2\big(\tilde{P}_{12}^{T(j)}B + \tilde{P}_2^{(j)}B_m\big)V,\\
\hat{H}_{C_m}\big(\tilde{Q}^{(j)}\big) &= 2R\big(C_m\tilde{Q}_2^{(j)} - C\tilde{Q}_{12}^{(j)}\big),
\end{aligned} \tag{43}
$$

where the superscript (j) means $\partial/\partial\theta_j$. Using the above definitions, we have for $\theta_j = \big(A_m\big)_{kl}$, where $(k,l) \in \mathcal{S}$,

$$
\begin{aligned}
\frac{\partial H_{A_m}}{\partial (A_m)_{kl}} &= \hat{H}_{A_m}\big(\tilde{P}^{(j)},\tilde{Q}^{(j)}\big),\\
\frac{\partial H_{B_m}}{\partial (A_m)_{kl}} &= \hat{H}_{B_m}\big(\tilde{P}^{(j)}\big),\\
\frac{\partial H_{C_m}}{\partial (A_m)_{kl}} &= \hat{H}_{C_m}\big(\tilde{Q}^{(j)}\big),
\end{aligned} \tag{44}
$$

for $\theta_j = \left(B_m\right)_{kl}$,

$$\begin{aligned}\frac{\partial H_{A_m}}{\partial (B_m)_{kl}} &= \hat{H}_{A_m}\left(\tilde{P}^{(j)}, \tilde{Q}^{(j)}\right),\\ \frac{\partial H_{B_m}}{\partial (B_m)_{kl}} &= \hat{H}_{B_m}\left(\tilde{P}^{(j)}\right) + 2\tilde{P}_2 E^{(k,l)} V,\\ \frac{\partial H_{C_m}}{\partial (B_m)_{kl}} &= \hat{H}_{C_m}\left(\tilde{Q}^{(j)}\right),\end{aligned} \tag{45}$$

and for $\theta_j = (C_m)_{kl}$, where $k > 1$,

$$\begin{aligned}\frac{\partial H_{A_m}}{\partial (C_m)_{kl}} &= \hat{H}_{A_m}\left(\tilde{P}^{(j)}, \tilde{Q}^{(j)}\right),\\ \frac{\partial H_{B_m}}{\partial (C_m)_{kl}} &= \hat{H}_{B_m}\left(\tilde{P}^{(j)}\right),\\ \frac{\partial H_{C_m}}{\partial (C_m)_{kl}} &= \hat{H}_{C_m}\left(\tilde{Q}^{(j)}\right) + 2RE^{(k,l)}\tilde{Q}_2.\end{aligned} \tag{46}$$

$\tilde{P}^{(j)}$ and $\tilde{Q}^{(j)}$ can be obtained by solving the Lyapunov equation (20). The derivative of the homotopy map with respect to λ can be derived in a similar fashion.

The initial point $(\theta, \lambda) = (\theta_0, 0)$ is chosen so that the triple $\left((A_m)_0, (B_m)_0, (C_m)_0\right)$ is in Ly's form and satisfies $\rho(\theta_0, 0) = 0$. This can be done as follows:

1) Obtain the initial reduced order model $\left((A_m)_0, (B_m)_0, (C_m)_0\right)_b$ in balanced form in the same way as for the input normal form approach.
2) Transform the balanced $\left((A_m)_0, (B_m)_0, (C_m)_0\right)_b$ to Ly's form, and build θ_0 as described in (41).

The homotopy curve tracking computation is the same as described in Section IV.

VII. Numerical results.

In this section numerical results for both the input normal form and Ly formulations are given for eleven systems. The first nine systems have been studied and solved in [21]–[23] using the optimal projection equations approach. Comparisons are made between these two minimal formulations and the optimal projection equations in Section IX.

The cost J is computed for each model as tr $(\tilde{Q}\,\tilde{R})$, according to (6). For all examples $V = R = I$. Unless indicated otherwise, the solutions, given in input normal form, can be obtained by both formulations and are the same as those obtained by the optimal projection equations method.

Example 1 [7]. The system is given by

$$A = \begin{pmatrix} -0.05 & -0.99 \\ -0.99 & -5000.0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 100 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 100 \end{pmatrix}.$$

The homotopy algorithm converges to a solution corresponding to the model of order $n_m = 1$ given by

$$A_m = (-0.00500423), \quad B_m = (-0.100042), \quad C_m = (-10.0000),$$

which was not obtained by the optimal projection equation approach of [21]–[23]. This model yields the cost $J = 10000$.

In the first step of choosing an initial point, (A_f, B_f, C_f) is transformed to (A_b, B_b, C_b), where orthogonal decompositions of two matrices are needed. If the eigenvalues of one of the matrices are rearranged in ascending order, then a different solution is obtained, namely

$$A_m = (-4998.08), \quad B_m = (-99.9808), \quad C_m = (-100.020).$$

This model yields the (minimum) cost $J = 96.0781$.

EXAMPLE 2 [17]. The system is given by

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -10 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ 70 & 1 \end{pmatrix}, \quad C = (1 \quad -0.2).$$

A model of order $n_m = 1$ is

$$A_m = (-11.9794), \quad B_m = (-4.85914 \quad 0.589656), \quad C_m = (2.76076).$$

This model yields the cost $J = 0.598377$.

EXAMPLE 3 [7]. The system is given by

$$A = \begin{pmatrix} -0.25 & -0.4 \\ -0.4 & -0.72 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1.2 \end{pmatrix}, \quad C = (1 \quad 1.2).$$

A model of order $n_m = 1$ is

$$A_m = (-0.838521), \quad B_m = (-1.29501), \quad C_m = (1.82558).$$

This model yields the cost $J = 0.107256$.

EXAMPLE 4 [18]. The system is given by

$$A = \begin{pmatrix} -1 & 3 & 0 \\ -1 & -1 & 1 \\ 4 & -5 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} -2 \\ 2 \\ 4 \end{pmatrix}, \quad C = (1 \quad 0 \quad 0).$$

A model of order $n_m = 1$ is

$$A_m = (-0.286334), \quad B_m = (-0.756748), \quad C_m = (0.878161),$$

which is different from that obtained by the optimal projection equation method [21]-[23], and has a smaller cost $J = 1.22883$. A model of order $n_m = 2$ is

$$A_m = \begin{pmatrix} -0.215037 & 0.753968 \\ -2.51385 & -3.60074 \end{pmatrix}, \quad B_m = \begin{pmatrix} 0.655800 \\ 2.68356 \end{pmatrix}, \quad C_m^T = \begin{pmatrix} 0.888877 \\ -1.09093 \end{pmatrix}.$$

This model yields the cost $J = 0.0197781$.

EXAMPLE 5 [7]. The system is given by

$$A = \begin{pmatrix} -10 & 1 & 0 \\ -5 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad C = (1 \quad 0 \quad 0).$$

A model of order $n_m = 1$ is

$$A_m = (-0.157898), \quad B_m = (0.561956), \quad C_m = (0.318537).$$

This model yields the cost $J = 0.0107792$. A model of order $n_m = 2$ is

$$A_m = \begin{pmatrix} -0.139652 & 0.100607 \\ -0.600971 & -0.448192 \end{pmatrix}, \quad B_m = \begin{pmatrix} 0.528492 \\ 0.946775 \end{pmatrix}, \quad C_m^T = \begin{pmatrix} 0.320438 \\ -0.0961019 \end{pmatrix}.$$

This model yields the cost $J = 0.000329024$.

EXAMPLE 6 [21]. The system is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -2 & -0.02 & 1 & 0.01 \\ 0 & 0 & 0 & 1 \\ 0.1 & 0.001 & -0.1 & -0.001 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad C = (0 \quad 1 \quad 0 \quad 0).$$

A model of order $n_m = 1$ is

$$A_m = (-0.353743), \quad B_m = \begin{pmatrix} 0.184397 \\ 0.820660 \end{pmatrix}, \quad C_m = (0.805197).$$

This model yields the cost $J = 285.012$.

With the input normal form, when $n_m = 2, 3$, two of the initial ωs are approximately the same, which leads to a significant numerical error in computing M_o and the numerical failure of the homotopy algorithm. Therefore this technique for choosing initial points fails, and some modification to the algorithm is needed to avoid this kind of ill conditioning. However, it is not at all clear how to systematically avoid nearly equal ωs, and this remains an open question. It can be shown that the solutions, obtained by the optimal projection equation approach, also have close ωs, which implies that changing the strategy for choosing initial points will not suffice for this example.

The Ly formulation can obtain the solutions which are given in [21]–[23]. However, a more general strategy for choosing an initial point similar to that for getting the minimum solution in Example 1 is needed to get the same results as in [21]–[23]. If such change is not made, different solutions with a larger cost are obtained; those solutions will not be reported here.

The solutions are given in Ly's form. A model of order $n_m = 2$ is

$$A_m = \begin{pmatrix} 0.0 & 1.0 \\ -2.05093 & -0.0205208 \end{pmatrix},$$

$$B_m = \begin{pmatrix} 0.974576 & -0.4994518 \\ -0.0265942 & 0.0351745 \end{pmatrix}, \quad C_m = (1.0 \quad 0.0).$$

This model yields the cost $J = 256.432$. A model of order $n_m = 3$ is

$$A_m = \begin{pmatrix} 0.0 & 1.0 & 0 \\ -2.05137 & -0.0205971 & 0.0 \\ 0.0 & 0.0 & -0.384243 \end{pmatrix},$$

$$B_m = \begin{pmatrix} 0.976140 & -0.505800 \\ -0.0176536 & 0.00117515 \\ 0.0373661 & 0.754190 \end{pmatrix}, \quad C_m = (\,1.0 \quad 0.0 \quad 1.0\,).$$

This model yields the cost $J = 255.703$.

EXAMPLE 7 [9], [20]. The system is given by

$$A = \begin{pmatrix} 0 & 0 & 0 & -150 \\ 1 & 0 & 0 & -245 \\ 0 & 1 & 0 & -1113 \\ 0 & 0 & 1 & -19 \end{pmatrix}, \quad B = \begin{pmatrix} 4 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad C = (\,0 \quad 0 \quad 0 \quad 1\,).$$

A model of order $n_m = 1$ is

$$A_m = (\,-0.495187\,), \quad B_m = (\,0.995175\,), \quad C_m = (\,0.0148426\,).$$

This model yields the cost $J = 4.90749 \cdot 10^{-5}$. A model of order $n_m = 2$ is

$$A_m = \begin{pmatrix} -0.437964 & -0.482612 \\ 2.84007 & -3.17242 \end{pmatrix}, \quad B_m = \begin{pmatrix} 0.935911 \\ -2.51890 \end{pmatrix}, \quad C_m^T = (\,0.0149143 \quad 0.00682097\,).$$

This model yields the cost $J = 4.159 \cdot 10^{-7}$. A model of order $n_m = 3$ is

$$A_m = \begin{pmatrix} -0.437810 & -0.483078 & -0.0370108 \\ 2.82632 & -3.13536 & -0.612598 \\ -4.65184 & 13.1604 & -12.5542 \end{pmatrix},$$

$$B_m = \begin{pmatrix} 0.935746 \\ -2.50414 \\ 5.01082 \end{pmatrix}, \quad C_m = (\,0.0149143 \quad 0.00682180 \quad 0.000635413\,).$$

This model yields the cost $J = 4.59 \cdot 10^{-10}$.

EXAMPLE 8 [9]. The system is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -50 & -79 & -33 & -5 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad C = (\,50 \quad 15 \quad 1 \quad 0\,).$$

A model of order $n_m = 1$ is

$$A_m = (\,-0.576205\,), \quad B_m = (\,1.07350\,), \quad C_m = (\,0.588692\,).$$

This model yields the cost $J = 0.104740$. A model of order $n_m = 2$ is

$$A_m = \begin{pmatrix} -0.532330 & -0.598751 \\ 3.80077 & -4.81512 \end{pmatrix}, \quad B_m = \begin{pmatrix} 1.03182 \\ -3.10326 \end{pmatrix}, \quad C_m^T = \begin{pmatrix} 0.588704 \\ 0.278923 \end{pmatrix}.$$

This model yields the cost $J = 0.0269278$. A model of order $n_m = 3$ is

$$A_m = \begin{pmatrix} -0.520312 & -0.731867 & -0.162146 \\ 2.88892 & -2.23562 & -3.72129 \\ -1.08450 & 6.30540 & -0.746729 \end{pmatrix},$$

$$B_m^T = (\,1.02011 \quad -2.11453 \quad 1.22207\,), \quad C_m = (\,0.586461 \quad 0.307967 \quad 0.105043\,).$$

This model yields the cost $J = 0.00148438$.

EXAMPLE 9 [3]. The system is given by

$$A = \begin{pmatrix} -6.2036 & 15.054 & -9.8726 & -376.58 & 251.32 & -162.24 & 66.827 \\ 0.53 & -2.0176 & 1.4363 & 0 & 0 & 0 & 0 \\ 16.846 & 25.079 & -43.555 & 0 & 0 & 0 & 0 \\ 377.4 & -89.449 & -162.83 & 57.998 & -65.514 & 68.579 & 157.57 \\ 0 & 0 & 0 & 107.25 & -118.05 & 0 & 0 \\ 0.36992 & -0.14445 & -0.26303 & -0.64719 & 0.49947 & -0.21133 & 0 \\ 0 & 0 & 0 & 0 & 0 & 376.99 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 89.353 & 0 \\ 376.99 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0.21133 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

A model of order $n_m = 1$ is

$$A_m = (\,-0.199272\,), \quad B_m = (\,0.631300 \quad -0.00187918\,), \quad C_m = (\,-0.187347 \quad -354.430\,).$$

This model yields the cost $J = 27632.2$. A model of order $n_m = 2$ is

$$A_m = \begin{pmatrix} -0.199608 & -0.0763006 \\ 3.33119 & -13.2758 \end{pmatrix},$$

$$B_m = \begin{pmatrix} 0.631832 & -0.00191612 \\ -5.15182 & -0.101952 \end{pmatrix}, \quad C_m = \begin{pmatrix} -0.201050 & 0.800899 \\ -354.414 & -66.1873 \end{pmatrix}.$$

This model yields the cost $J = 23262.3$. A model of order $n_m = 3$ is

$$A_m = \begin{pmatrix} -0.198769 & 0.235666 & -0.0248136 \\ -1.08739 & -0.912444 & 9.20181 \\ -0.115288 & -9.50243 & -0.0261157 \end{pmatrix},$$

$$B_m = \begin{pmatrix} -0.630503 & 0.00216112 \\ -1.350879 & -0.00377142 \\ -0.222387 & -0.0526803 \end{pmatrix}, \quad C_m = \begin{pmatrix} 0.291338 & -0.0265117 & -4.03570 \\ 354.222 & -164.479 & 26.6355 \end{pmatrix}.$$

This model yields the cost $J = 0.673079$. A model of order $n_m = 4$ is

$$A_m = \begin{pmatrix} -0.198769 & 0.235667 & -0.0248136 & 0.000915746 \\ -1.08739 & -0.912440 & 9.20181 & -0.00904508 \\ -0.115288 & -9.50243 & -0.0261155 & 0.00159031 \\ -5.46513 & -11.6984 & -1.92997 & -37.5544 \end{pmatrix},$$

$$B_m = \begin{pmatrix} -0.630503 & 0.00216112 \\ -1.35088 & -0.00377141 \\ -0.222386 & -0.0526803 \\ -8.66651 & -0.0203036 \end{pmatrix}, \quad C_m^T = \begin{pmatrix} 0.291340 & 354.222 \\ -0.0265302 & -164.479 \\ -4.03569 & 26.6355 \\ 0.0861885 & -0.815898 \end{pmatrix}.$$

This model yields the cost $J = 3.22 \cdot 10^{-7}$.

For this example with $n_m = 3,\ 4$, the columns of the initial Jacobian matrices from input normal form formulations are so badly scaled that the numerical linear algebra in HOMPACK fails. Modifying HOMPACK to use the LINPACK subroutine DQRDC for the QR factorization of the initial Jacobian matrices enables HOMPACK to successfully overcome the ill conditioning and find a solution.

EXAMPLE 10 [1]. A is a 2×2 block diagonal matrix with each diagonal block being of the form

$$\begin{pmatrix} 0 & 1 \\ -\sigma_i^2 & -2y\sigma_i \end{pmatrix}, \quad i = 1, \ldots, n/2,$$

$$B = (0, b_1, 0, b_2, \ldots, 0, b_{n/2})^T, \quad C = (0, c_1, 0, c_2, \ldots, 0, c_{n/2}),$$

where $\sigma_i = i^2\pi^2$, $b_i = \sqrt{2}\sin(i\pi a)$, $c_i = \sqrt{2}\sin(i\pi s)$ and y, a, s are known parameters. This system was not studied in [21]–[23]. The input normal form approach can not give a solution when $n_m > 1$ because the initial ωs are generated in pairs.

Choosing $n = 16$, $n_m = 8$, $y = 0.001$, $a = 0.1$, $s = 0.2$, the reduced order model is

$$A_m = \text{ diag } (A_1, A_2, A_3, A_4),$$

$$A_1 = \begin{pmatrix} 0.0 & 1.0 \\ -24936.92 & -0.3158248 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0.0 & 1.0 \\ -97.40911 & -0.01973900 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0.0 & 1.0 \\ -7890.149 & -0.1776489 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0.0 & 1.0 \\ -1558.546 & -0.07895572 \end{pmatrix},$$

$$B_m = \begin{pmatrix} 1.118022 \\ 0.3208260 \\ 0.3632675 \\ -0.007030323 \\ 1.538809 \\ -0.1661970 \\ 1.118019 \\ -0.07921770 \end{pmatrix}, \quad C_m = \begin{pmatrix} 1.0 \\ 0.0 \\ 1.0 \\ 0.0 \\ 1.0 \\ 0.0 \\ 1.0 \\ 0.0 \end{pmatrix}^T,$$

which has cost $J = 2.59857$. Ly's formulation is very efficient for this problem.

EXAMPLE 11. The system is given by

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0.0005 & -1.000001 & 0 \\ 0.0005 & 0.0005 & -1.00001 \end{pmatrix}, \quad B = \begin{pmatrix} 1.1 \\ 1.2 \\ 1.3 \end{pmatrix}, \quad C = (2.1 \quad 2.2 \quad 2.3).$$

A model of order $n_m = 2$ with cost $J = 0.36 \cdot 10^{-14}$ is

$$A_m = \begin{pmatrix} -0.999519 & 0.00000 \\ 1.99976 & -1.00024 \end{pmatrix}, \quad B_m = \begin{pmatrix} -1.41387 \\ 1.41438 \end{pmatrix}, \quad C_m^T = \begin{pmatrix} -5.61578 \\ 0.00000 \end{pmatrix}.$$

This system was constructed to illustrate that some problems can be solved by the input normal form formulation or the over-parametrization formulation described below but not by the Ly formulation.

VIII. Over-parametrization formulation.

Both the input normal form formulation and Ly formulation can introduce ill conditioning, resulting from eliminating certain variables so that the minimal number of variables is used. To avoid such ill conditioning, one could use all the elements in A_m, B_m, and C_m as variables, i.e., not impose any restriction on the representation of (A_m, B_m, C_m).

The same Lagrangian as in (38) is used:

$$L(A_m, B_m, C_m, \tilde{Q}) = \mathrm{tr}\big[\tilde{Q}\tilde{R} + (\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T + \tilde{V})\tilde{P}\big],$$

where the symmetric matrix $\tilde{P}$ is a Lagrange multiplier. Setting $\partial L/\partial \tilde{Q} = 0$ gives (10). A straightforward calculation shows

$$\begin{aligned} \frac{\partial L}{\partial A_m} &= 2(\tilde{P}_{12}^T\tilde{Q}_{12} + \tilde{P}_2\tilde{Q}_2), \\ \frac{\partial L}{\partial B_m} &= 2(\tilde{P}_{12}^T B + \tilde{P}_2 B_m)V, \\ \frac{\partial L}{\partial C_m} &= 2R(C_m\tilde{Q}_2 - C\tilde{Q}_{12}). \end{aligned} \tag{47}$$

Let A_f, B_f, C_f , R_f, and V_f denote A, B, C, R, and V in the above and define $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $V(\lambda)$ and $R(\lambda)$ as in (15) and denote them respectively by A, B, C, V, and R. Let

$$\begin{aligned} H_{A_m}(\theta,\lambda) &= \frac{\partial L}{\partial A_m} = 2(\tilde{P}_{12}^T\tilde{Q}_{12} + \tilde{P}_2\tilde{Q}_2), \\ H_{B_m}(\theta,\lambda) &= \frac{\partial L}{\partial B_m} = 2(\tilde{P}_{12}^T B + \tilde{P}_2 B_m)V, \\ H_{C_m}(\theta,\lambda) &= \frac{\partial L}{\partial C_m} = 2R(C_m\tilde{Q}_2 - C\tilde{Q}_{12}), \end{aligned} \tag{48}$$

where

$$\theta \equiv \begin{pmatrix} \mathrm{Vec}\ (A_m) \\ \mathrm{Vec}\ (B_m) \\ \mathrm{Vec}\ (C_m) \end{pmatrix}$$

denotes the independent variables A_m, B_m, and C_m, and $\tilde{Q}$ and $\tilde{P}$ satisfy respectively (7) and (10). Define

$$\rho(\theta,\lambda) = \begin{pmatrix} \mathrm{Vec}\ [H_{A_m}(\theta,\lambda)] \\ \mathrm{Vec}\ [H_{B_m}(\theta,\lambda)] \\ \mathrm{Vec}\ [H_{C_m}(\theta,\lambda)] \end{pmatrix},$$

whose Jacobian matrix is

$$D\rho(\theta,\lambda) = \big(D_\theta\rho(\theta,\lambda),\ D_\lambda\rho(\theta,\lambda)\big).$$

Because of the over-parametrization, the Jacobian matrix of ρ is singular. The homotopy map is defined as

$$\hat{\rho}(\theta,\lambda) = \lambda\rho(\theta,\lambda) + (1-\lambda)(\theta-\theta_0), \tag{49}$$

which guarantees a well conditioned Jacobian matrix along the whole path except at the solution corresponding to $\lambda = 1$. The Jacobian matrix is given by

$$D\hat{\rho}(\theta,\lambda) = \left(\lambda D_\theta\rho(\theta,\lambda) + (1-\lambda)I, \quad \rho(\theta,\lambda) + \lambda D_\lambda\rho(\theta,\lambda) - (\theta-\theta_0)\right). \tag{50}$$

To find $D_\theta\rho(\theta,\lambda)$, define $\hat{H}_{A_m}\left(\tilde{P}^{(j)},\tilde{Q}^{(j)}\right)$, $\hat{H}_{B_m}\left(\tilde{P}^{(j)}\right)$, and $\hat{H}_{C_m}\left(\tilde{Q}^{(j)}\right)$ as in (43), where again the superscript (j) means $\partial/\partial\theta_j$. For $\theta_j = \left(A_m\right)_{kl}$,

$$\begin{aligned}
\frac{\partial H_{A_m}}{\partial(A_m)_{kl}} &= \hat{H}_{A_m}\left(\tilde{P}^{(j)},\tilde{Q}^{(j)}\right)\\
\frac{\partial H_{B_m}}{\partial(A_m)_{kl}} &= \hat{H}_{B_m}\left(\tilde{P}^{(j)}\right),\\
\frac{\partial H_{C_m}}{\partial(A_m)_{kl}} &= \hat{H}_{C_m}\left(\tilde{Q}^{(j)}\right),
\end{aligned} \tag{51}$$

for $\theta_j = \left(B_m\right)_{kl}$,

$$\begin{aligned}
\frac{\partial H_{A_m}}{\partial(B_m)_{kl}} &= \hat{H}_{A_m}\left(\tilde{P}^{(j)},\tilde{Q}^{(j)}\right)\\
\frac{\partial H_{B_m}}{\partial(B_m)_{kl}} &= \hat{H}_{B_m}\left(\tilde{P}^{(j)}\right) + 2\tilde{P}_2E^{(k,l)}V,\\
\frac{\partial H_{C_m}}{\partial(B_m)_{kl}} &= \hat{H}_{C_m}\left(\tilde{Q}^{(j)}\right),
\end{aligned} \tag{52}$$

and for $\theta_j = \left(C_m\right)_{kl}$,

$$\begin{aligned}
\frac{\partial H_{A_m}}{\partial(C_m)_{kl}} &= \hat{H}_{A_m}\left(\tilde{P}^{(j)},\tilde{Q}^{(j)}\right)\\
\frac{\partial H_{B_m}}{\partial(C_m)_{kl}} &= \hat{H}_{B_m}\left(\tilde{P}^{(j)}\right),\\
\frac{\partial H_{C_m}}{\partial(C_m)_{kl}} &= \hat{H}_{C_m}\left(\tilde{Q}^{(j)}\right) + 2RE^{(k,l)}\tilde{Q}_2.
\end{aligned} \tag{53}$$

$\tilde{P}^{(j)}$ and $\tilde{Q}^{(j)}$ can be obtained by solving the Lyapunov equations (20). The derivatives with respect to λ can be obtained in the same way as in Section III.

The initial point $(\theta,\lambda) = (\theta_0,0) = \left((A_m)_0,(B_m)_0,(C_m)_0,0\right)$ is chosen so that the triple $\left((A_m)_0,(B_m)_0,(C_m)_0\right)$ is in balanced form and satisfies $\rho(\theta_0,0) = 0$. The algorithm is similar to steps 1)–8) described in Section IV, except that the homotopy $\hat{\rho}$ from (49) is used.

For all the test problems except Example 6 with $n_m = 3$ and Example 9 with $n_m = 2, 3, 4$, the above algorithm gives satisfactory results by adjusting the curve tracking precision. For these exceptional cases, HOMPACK reaches $\lambda \geq 1$ very fast, but because of the high order singularity at the solution, the computed solution does not have acceptable accuracy. Although very sophisticated methods for dealing with singular endpoints of homotopy curves are known [10]–[12], these are difficult to implement in the present context, and the following simple algorithm was adequate.

1) Use the algorithm in Section IV to track the curve until $\lambda \geq 1$.

2) Use the last point $(\tilde{\theta}, \tilde{\lambda})$ before $\lambda \geq 1$ to redefine the homotopy map with $\theta_0 = \tilde{\theta}$ and set $\lambda = 0$.

3) Redo step 1.

4) Use Hermite polynomial interpolation to obtain the solution at $\lambda = 1$.

In Step 3 the new homotopy (49) has a zero curve that is nearly a straight line, and thus Hermite interpolation using points before $\lambda = 1$ and one point with $\lambda \geq 1$ is quite accurate. Care must be taken to use data points away from the singularity (lest they be inaccurate), but this is easily done by controlling the step size parameters in HOMPACK.

IX. Comparisons and discussions.

Table 1 gives the CPU times in seconds and the number of steps needed to obtain the results for each example (a dash indicates failure). The CPU times are for a DECstation 5000/200, using double precision, IEEE arithmetic, and the MIPS RISC f77 compiler. Table 2 gives the comparison of the optimal projection equations approach and the input normal form formulation for Examples 8 and 9. The asterisks in Table 1 denote the cases requiring Hermite polynomial interpolation to obtain the solution for the over-parametrization formulation. The asterisks in Table 2 indicate cases that required special numerical linear algebra techniques to deal with severe scaling errors.

Table 1. Algorithm measures for three formulations.

		Input normal form		Ly's form		Over-parametrization	
Ex	n_m	steps	time	steps	time	steps	time
1	1	7	0.06	8	0.08	12	0.08
2	1	25	0.13	40	0.25	31	0.20
3	1	23	0.10	25	0.14	30	0.16
4	1	16	0.14	18	0.21	18	0.19
4	2	11	0.20	16	0.37	18	0.47
5	1	14	0.12	13	0.17	14	0.15
5	2	14	0.22	12	0.27	10	0.29
6	1	205	1.8	220	2.9	19	0.29
6	2	-	-	8	0.32	35	1.3
6	3	-	-	114	6.9	125*	13.
7	1	15	0.22	15	0.33	13	0.21
7	2	12	0.30	13	0.45	10	0.38
7	3	10	0.42	12	0.70	18	1.5
8	1	14	0.20	14	0.29	17	0.25
8	2	22	0.50	35	1.1	35	1.1
8	3	16	0.65	17	0.96	14	1.3
9	1	15	0.60	2339	103.	19	0.78
9	2	127	8.0	-	-	168*	13.
9	3	9	1.3	45	6.7	21*	4.2
9	4	8	1.9	59	15.	17*	7.5
10	8	-	-	19	35.	7	49.
11	2	6	0.13	-	-	16	0.42

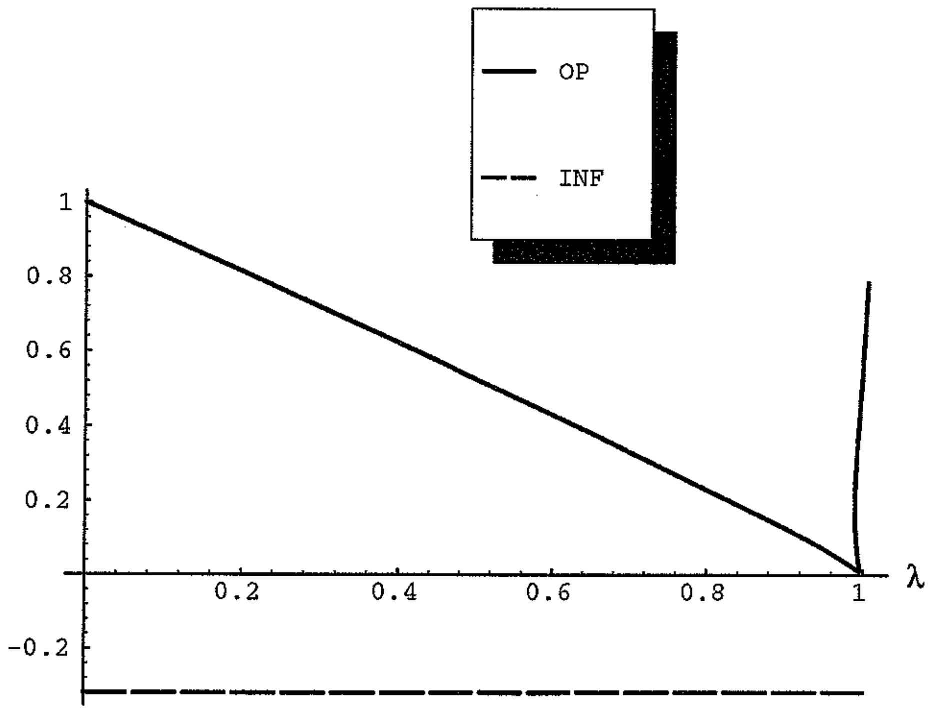

Fig. 1. x_7 (OP) and θ_1 (INF) vs. λ.

As shown by Table 1, the input normal form homotopy can be very efficient. Also there is no need to adjust any parameter to achieve this efficiency (although to obtain the minimum solution of Example 1, some adjustment of the initial point was necessary). However, note that the potential ill conditioning of the input normal form formulation can result in failure (Examples 6 and 10) or the need for extraordinarily delicate linear algebra (Example 9).

Figures 1 and 2 show the behavior of the largest variation component with respect to λ for Example 5 at $n_m = 1$ and Example 9 at $n_m = 2$ using the input normal form formulation and the optimal projection equation formulation [21]–[23]. The figures show that component of the solution vector with the largest total amount of oscillation, corresponding to the most difficult component of the homotopy path to track. Even though Fig. 1 corresponds to a good choice of the initial point for the optimal projection equations approach, it is obviously not as efficient as the input normal form formulation. Generally speaking, since the number of variables in the input normal form and Ly formulations is much smaller than that of the optimal projection equations formulation, and the strategy for choosing initial points uses balancing (hence giving an initial point closer to the final solution in most cases), the input normal form and Ly form homotopies are more efficient than the optimal projection equations homotopy.

For Example 9, when $n_m = 1$, the Ly's form homotopy is extremely inefficient, requiring c_1 and c_2 (cf. Section IV) to be adjusted to achieve a solution. All attempts to obtain a solution when $n_m = 2$ failed. The solutions of Example 9 when $n_m = 3$ and $n_m = 4$ are singular, which accounts for the large number of steps required by Ly's form.

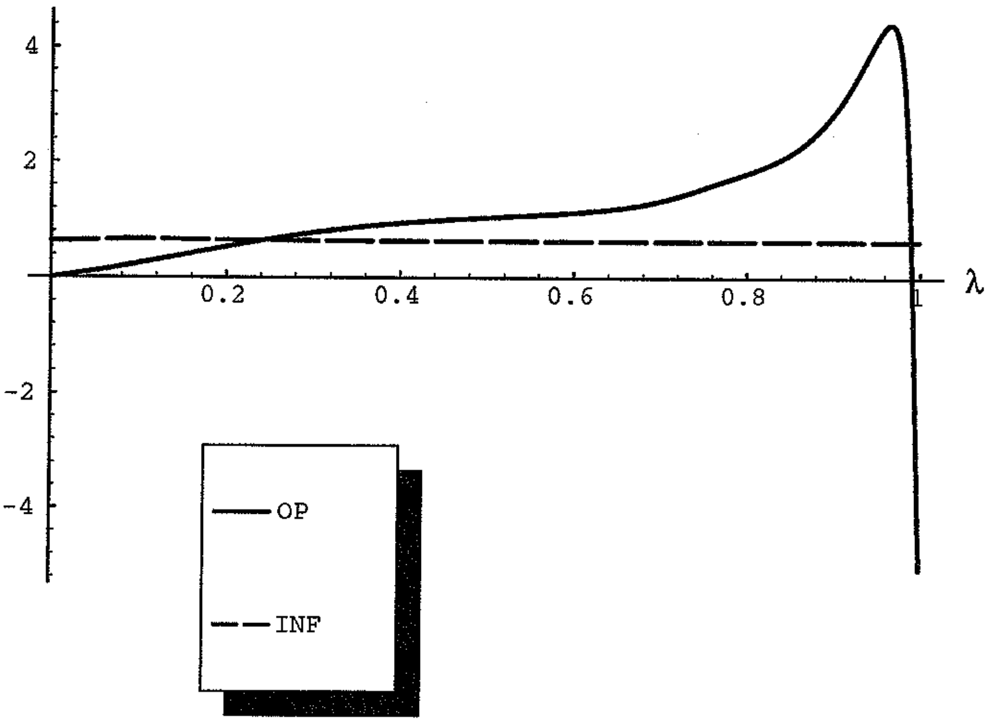

Fig. 2. x_7 (OP) and θ_2 (INF) vs. λ.

TABLE 2. COMPARISON OF METHODS.

	Example 8			
	Optimal projection		input normal form	
n_m	# steps	time (sec)	# steps	time (sec)
1	31	0.6	10	0.20
2	59	2.7	18	0.50
3	89	14.	10	0.65
	Example 9			
2	575	88	123	8.0
3	601	223	6*	1.3
4	671	518	6*	1.9

The optimal projection equations homotopy successfully solved *all* of the test problems, but Table 2, containing typical results, shows that the minimal parameter homotopies are much more efficient. However, when the input normal form and Ly's form are used, some restrictions are imposed on the structure of the triple (A_m, B_m, C_m), potentially resulting in ill conditioning. For the input normal form formulation, ill conditioning occurs if two diagonal elements of Ω in (4) are approximately the same. In other words, let Q_m and P_m be the controllability and observability

Gramians of the system represented by (A_m, B_m, C_m), and let

$$Q_m = W\Sigma W^T, \quad P_m = W^{-T}\Sigma W^{-1},$$

where Σ is diagonal and is the controllability and observability Gramian in balanced form. If two diagonal elements of Σ are approximately the same, then ill conditioning occurs. For Example 6, when $n_m = 2, 3$, both the initial point chosen using the given strategy and the solution obtained in [21]–[23] or by Ly's formulation are ill conditioned, i.e., two diagonal elements of Ω are approximately the same. Hence the input normal form method will not be able to solve this problem.

For Ly's formulation, ill conditioning occurs if the Jordan decomposition of A_m is ill conditioned. Precisely, if the two eigenvalues of A_m which are to be grouped into a 2×2 block are approximately the same, the transition matrix to 2×2 block diagonal form is ill conditioned. This can be clearly illustrated by observing that for $n_m = 2$, finding the Ly form is equivalent to finding the transition matrix $T \in \mathbf{R}^{2\times 2}$ such that

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\lambda_1\lambda_2 & \lambda_1+\lambda_2 \end{pmatrix},$$
$$C_m(1,1)t_{11} + C_m(1,2)t_{21} = 1,$$
$$C_m(1,1)t_{12} + C_m(1,2)t_{22} = 0,$$

where λ_1 and λ_2 are the eigenvalues of A_m. Trivial algebra gives

$$t_{11} = -\lambda_2 C_m(1,1)^{-1}\lambda_{12}^{-1}, \qquad t_{12} = C_m(1,1)^{-1}\lambda_{12}^{-1},$$
$$t_{21} = \lambda_1 C_m(1,2)^{-1}\lambda_{12}^{-1}, \qquad t_{22} = -C_m(1,2)^{-1}\lambda_{12}^{-1},$$
$$\operatorname{cond} T = \sigma + \sqrt{\sigma^2 - 1},$$

where

$$\lambda_{12} = \lambda_1 - \lambda_2, \qquad \tau = \frac{C_m(1,1)}{C_m(1,2)}, \qquad \sigma = \frac{1 + \tau^2 + \lambda_2^2 + \tau^2\lambda_1^2}{2|\tau\lambda_{12}|}.$$

Thus ill conditioning occurs in general when σ is large, and in particular when $\tau\lambda_{12} \approx 0$. Furthermore, note that the very existence of the Ly form is predicated on the *assumption that the Jordan form of A_m consists of 2×2 Jordan blocks*, which is a rather strong assumption.

Both the input normal form formulation and Ly's formulation can fail to exist or lead to ill conditioning and it is conceivable that both of these formulations will fail for some problems. This failure of existence in general is related to the insistence on using the minimal number of parameters $n_m m + n_m l$. The over-parametrization formulation solves the ill conditioning issue, but introduces a very high order singularity at the solution. It is doubtful whether either the Hermite interpolation used here or the techniques of [10]–[12] can handle a large problem with a singularity of order 100. A pragmatic suggestion is to try in order the input normal form, Ly's form, and the over-parametrization form, switching if ill conditioning or failure occurs. The ideal paradigm would be to have a family of minimal formulations, almost all of which exist for any given problem. The homotopy algorithm would then dynamically adjust the formulation, finding a well conditioned one and tracking its zero curve simultaneously. Such a paradigm remains an open question.

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