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Decomposition of complete bipartite graphs into generalized prisms

Sylwia Cichacz*
Faculty of Applied Mathematics, AGH University of Science and Technology
Al. Mickiewicza 30, 30-059 Kraków, Poland
Dalibor Fronček
Department of Mathematics and Statistics, University of Minnesota Duluth
1117 University Dr, Duluth, MN 55812-3000 U.S.A.
Petr Kovář
Department of Applied Mathematics, VŠB – Technical University of Ostrava
17. listopadu 15, 708 33 Ostrava-Poruba, Czech Republic

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Abstract

R. Häggkvist proved that every 3-regular bipartite graph of order $2n$ with no component isomorphic to the Heawood graph decomposes the complete bipartite graph $K_{6n,6n}$. In [2] the first two authors established a necessary and sufficient condition for the existence of a factorization of the complete bipartite graph $K_{n,n}$ into certain families of 3-regular graphs of order $2n$. In this paper we tackle the problem of decompositions of $K_{n,n}$ into certain 3-regular graphs called generalized prisms. We will show that certain families of 3-regular graphs of order $2n$ decompose the complete bipartite graph $K_{\frac{3n}{2},\frac{3n}{2}}$.

*Corresponding author, email cichacz@agh.edu.pl

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. We use standard terminology and notation of graph theory.

Graph decompositions have been widely studied in many different settings. We say that a graph B has a G*-decomposition* if there are subgraphs $G_1, G_2, \ldots, G_s$ of B, all isomorphic to G, such that each edge of B belongs to exactly one G_i. If each G_i for $i \in \{1, \ldots, s\}$ contains all vertices of B, then we say that B has a G*-factorization*.

Recall that a *prism* is a graph of the form $C_m \times P_2$. As in [2] we generalize prisms and let the $(0, j)$*-prism* (pronounced "oh-jay prism") of order $2n$ for j even be the graph with two vertex disjoint cycles $R_n^i = v_0^i, v_1^i, \ldots, v_{n-1}^i, v_0^i$ for $i \in \{1, 2\}$ of length n called *rims* and edges $v_1^1v_1^2, v_3^1v_3^2, v_5^1v_5^2, \ldots$ and $v_0^1v_j^2$, $v_2^1v_{2+j}^2, v_4^1v_{4+j}^2, \ldots$ called *spokes* of *type* 0 and *type* j, respectively (see Fig. 1). It is easy to observe that an $(0, j)$-prism is a 3-regular graph and is isomorphic to an $(0, -j)$-prism, $(j, 0)$-prism and $(-j, 0)$-prism. We can therefore always assume that $j \leq \frac{n}{2}$. In our terminology the usual prism is an $(0, 0)$-prism.

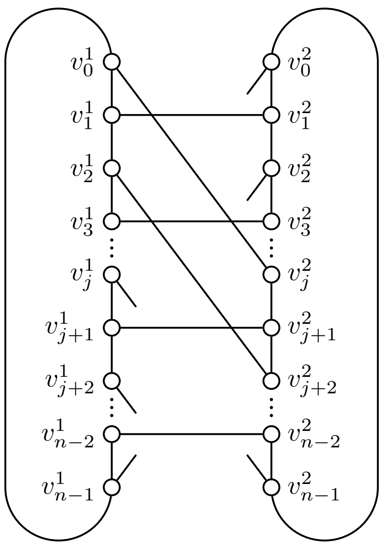

Figure 1: $(0, j)$-prism.

For many years, one of the most popular problems in graph decompositions has been the problem of decompositions and factorizations of complete

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and complete bipartite graphs into 2-regular graphs, that is, into cycles and unions of cycles. Investigation of analogous problems for 3-regular graphs is a natural next step in this field of research.

The problem of factorization of $K_{n,n}$ into $(0, j)$-prisms was solved in [2]. In this paper we approach the decomposition problem of $K_{n,n}$ into $(0, j)$-prisms. As in [5] we denote by $G[H]$ the composition of graphs G and H which is obtained by replacing every vertex of G by a copy of H and every edge of G by the complete bipartite graph $K_{|H|,|H|}$. We say that $G[H]$ arose from G by *blowing up* by H and recall that $\overline{K}_m$ is the complement of K_m, i.e., the graph consisting of m independent vertices.

A labeling of a graph G is a function from $V(G)$ into a group Γ. A. Rosa [8] introduced several types of graph labelings as tools for decompositions of complete graphs. In this paper we will use a decomposition method based on certain vertex labeling.

Definition 1 *Let Z_a be a cyclic group of order a and let G be a bipartite graph with k edges. Let $V(G) = V_0 \cup V_1$, $V_0 \cap V_1 = \emptyset$ and $|V_0| \leqslant |V_1| \leqslant k$. Let λ be an injection such that $\lambda : V_i \to \{(u, v)_i : u \in Z_a,\ v \in Z_b,\ ab = k\}$ for $i \in \{0, 1\}$. We define the* dimension *of an edge x_0y_1 with $\lambda(x_0) = (u, v)_0$ and $\lambda(y_1) = (t, z)_1$ as $\dim(x_0y_1) = ((t-u) \bmod a, (z-v) \bmod b)$ for $x_0 \in V_0$ and $y_1 \in V_1$.*

Problems of decomposition of graphs into k-regular graphs were studied widely. R. Häggkvist [6] proved that every 3-regular bipartite graph of order $2n$ with no component isomorphic to the Heawood graph decomposes the complete bipartite graph $K_{6n,6n}$. In [2] it was proved that $K_{n,n}$ can be factorized into $(0, j)$-prisms of order $2n$ if and only if $n \equiv 0 \pmod 6$. It is natural to consider also a more general problem. In this paper we decompose complete bipartite graphs $K_{k,k}$ into (non-spanning) $(0, j)$-prisms on $2n$ vertices. It is obvious that for $n \equiv 0 \pmod 6$ we can decompose every graph $K_{mn,mn}$ by first decomposing it into m^2 copies of $K_{n,n}$ and then factorizing each copy into the $(0, j)$-prisms. Hence, for $(0, j)$-prisms our construction for $n \not\equiv 0 \pmod 6$ gives stronger results than Häggkvist's theorem. On the other hand, we notice that the obvious necessary conditions allow wider classes of complete bipartite graphs than just $K_{mn,mn}$ for consideration. For if we want to decompose $K_{k,k}$ into $(0, j)$-prisms of order $2n$, then it follows that $k^2 \equiv 0 \pmod{3n}$, because the number of edges of the $(0, j)$-prism is $3n$. Moreover, since an $(0, j)$-prism has to be bipartite in order to decompose $K_{k,k}$, it follows that n must be even and the $(0, j)$-prism has an even number of edges.

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Therefore, k must be even, which implies that $k \equiv 0 \pmod 6$. However, these conditions may be in some cases satisfied even when $k \neq mn$. For instance, if $n \equiv 0 \pmod 4$, then $K_{\frac{3n}{2},\frac{3n}{2}}$ satisfies the necessary conditions. In this paper we will deal with two cases of decomposition of $K_{\frac{3n}{2},\frac{3n}{2}}$ into $(0, j)$-prisms.

2 Decomposition for $n \equiv 0 \pmod 8$

The underlying idea of the proof of the main result of this section is the following. First we decompose $K_{\frac{n}{2},\frac{n}{2}}$ into C_n, then we blow up $K_{\frac{n}{2},\frac{n}{2}}$ into $K_{\frac{3n}{2},\frac{3n}{2}} = K_{\frac{n}{2},\frac{n}{2}}[\overline{K}_3]$ and each C_n into $C_n[\overline{K}_3]$. Then we "glue together" certain pairs of $C_n[\overline{K}_3]$ and decompose the resulting graphs into six copies of $(0, j)$-prisms.

The decomposition of K_{k_1,k_2} into cycles was completely solved by J.C. Bermond, C. Huang [1], and D. Sotteau [7].

Theorem 2 [1, 7] *K_{k_1,k_2} can be decomposed into C_n if and only if n, k_1, k_2 are all even, n divides k_1k_2 and both $k_1, k_2 \geqslant \frac{n}{2}$.*

In order to prove the main result of this section, we first need the following three lemmas.

Lemma 3 [3] *Let G be an $(0, 0)$-prism of order $2n$, where n is even. Then K_{k_1,k_2} can be decomposed into G if $9n$ divides k_1k_2, both $k_1, k_2 \geqslant \frac{3n}{2}$ and 6 divides both k_1 and k_2.*

Lemma 4 [3] *Let $n \equiv 0 \pmod 4$ and G be an $(0, 2)$-prism of order $2n$. Then K_{k_1,k_2} can be decomposed into G if $18n$ divides k_1k_2 and both $k_1, k_2 \geqslant \frac{3n}{2}$ and $k_1, k_2 \equiv 0 \pmod{12}$.*

In the proof of Theorem 6 we want to find a pair of $(0, 0)$-prisms with the property that we can remove every other spoke in each of them and mutually swap the two sets of spokes between the two prisms so that they become type j spokes and hence we obtain two $(0, j)$-prisms. Therefore we need to make sure that we can decompose $K_{\frac{3n}{2},\frac{3n}{2}}$ into unions of these pairs of $(0, 0)$-prisms.

Lemma 5 [3] *Let $m \equiv 0 \pmod 4$, $c \equiv 0 \pmod 2$, $m/\gcd(c, m)$ be even and H be a 4-regular bipartite graph with bipartition $X = \{x_0, x_1, \ldots, x_{m-1}\}$, $Y =$*

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$\{y_0, y_1, \dots, y_{m-1}\}$ and edges $x_iy_i, x_iy_{i+1}, x_iy_{i+c}, x_iy_{i+c+1}$ for some positive $c \leqslant (m-2)/2$, where the addition in subscripts is taken modulo m. Then $K_{m,m}$ can be decomposed into H.

The main result of this section is restricted to the case when $n \equiv 0 \pmod 8$, and $j/\gcd(j,n)$ is odd. Recall that j is always even.

In the following section we prove another result for $n \equiv j \equiv 0 \pmod 4$ without the restriction that $j/\gcd(j,n)$ is odd.

Theorem 6 *Let $n \equiv 0 \pmod 8$, and $n/\gcd(j,n)$ be even. If G is an $(0,j)$-prism of order $2n$, then G decomposes $K_{\frac{3n}{2},\frac{3n}{2}}$.*

Proof. For $j = 0$ or $j = 2$ we are done by Lemma 3 or 4, respectively. From now on assume that $j \geqslant 4$ and let $p = \frac{j}{2}$. By the definition of an $(0,j)$-prism that j is always even.

Notice that if G is an $(0,j)$-prism of order $2n$ where n is even, then we can label vertices of G in such a way that $R_n^1 = (0,0)_1, (0,1)_0, (1,0)_1, (1,1)_0, (2,0)_1, (2,1)_0, (3,0)_1, \dots, (\frac{n}{2}-1,1)_0, (0,0)_1$, $R_n^2 = (0,2)_0, (1,2)_1, (1,2)_0, (2,2)_1, (2,2)_0, (3,2)_1, \dots, (\frac{n}{2}-1,2)_0, (0,2)_1, (0,2)_0$ and $(i+1,0)_1(i,2)_0, (i,1)_1(i+p,2)_0 \in E(G)$, where $i \in Z_{\frac{n}{2}}$ (see Fig. 2). Observe that such a labeling implies that spokes have dimension either $(1,1)$ or $(-p,1)$. Notice that we can also label the end-vertices of the spokes as $(i,1)_0(i,2)_1, (i,2)_0(i+p+1,0)_1$, where $i \in Z_{\frac{n}{2}}$ and then they have dimension either $(0,1)$ or $(p+1,1)$.

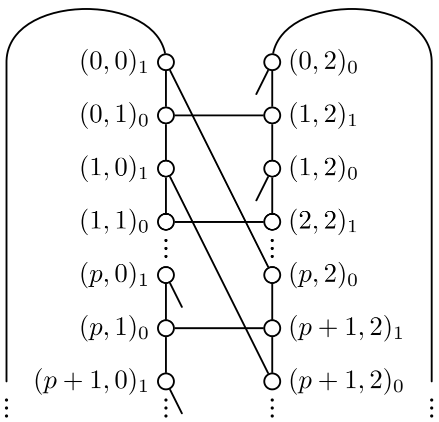

Figure 2: Labeling of $(0,j)$-prism.

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The graph $K_{\frac{n}{2},\frac{n}{2}}$ can be decomposed into C_n by Theorem 2. For $(0,0)$-prisms the method is based on decomposition of $C_n[\overline{K}_3]$ into three prisms, but for general $(0,2p)$-prisms we need to pair up two $(0,0)$-prisms and swap half of their spokes (of type 0) so that they will be of type $j = 2p$ in the other prism.

So we need two cycles C_n^1 and C_n^2, which together give us an appropriate collection of $(0,2p)$-prisms. We denote the union $C_n^1 \cup C_n^2$ of the appropriate cycles by H. Obviously, H is a bipartite 4-regular graph of order n. Let the partite sets be $X = \{x_0, x_1, \ldots, x_{\frac{n}{2}-1}\}$ and $Y = \{y_0, y_1, \ldots, y_{\frac{n}{2}-1}\}$. If the neighbors of vertex $x_i \in X$ in H are $y_i, y_{i+1}, y_{i+d}, y_{i+d+1}$ for some even $d \geqslant 2$, H consists of cycles $B_n^1 = y_0, x_0, y_1, x_1, \ldots, y_j, x_j, y_{j+1}, x_{j+1}, \ldots, y_{\frac{n}{2}-1}, x_{\frac{n}{2}-1}, y_0$ and $B_n^2 = y_d, x_0, y_{d+1}, x_1, \ldots, y_{d+j}, x_j, y_{d+j+1}, x_{j+1}, \ldots, y_{d+\frac{n}{2}-1}, x_{\frac{n}{2}-1}, y_d$. Now if $n/2\mathrm{gcd}(d, n/2)$ is even, then $K_{\frac{n}{2},\frac{n}{2}}$ can be decomposed into H by Lemma 5.

We now construct cycles C_n^1 and C_n^2 and present their mapping onto B_n^1 and B_n^2 for $d = p$. We denote the vertices of C_n^1 by $0_1, 0_0, 1_1, 1_0, \ldots, (\frac{n}{2}-1)_1, (\frac{n}{2}-1)_0$. By blowing up C_n^1 by $\overline{K}_3$ we obtain $C_n^1[\overline{K}_3]$. Then we can decompose $C_n^1[\overline{K}_3]$ into three $(0,0)$-prisms with the spokes of dimension either $(1,1)$ or $(0,1)$. We consider the following cases:
Case 1. p is odd.
We use $C_n^1[\overline{K}_3]$ and $C_n^2[\overline{K}_3]$ in such a way that we swap spokes of type 0 and dimension $(0,1)$ between $C_n^1[\overline{K}_3]$ and $C_n^2[\overline{K}_3]$ and obtain spokes of dimension $(-p,1)$ that will be of type $j = 2p$ in their new prisms.

It implies that we need edges $0_0p_1, 1_0(p+1)_1, 2_0(p+2)_1, \ldots$ and $0_0x_1, 1_0(x+1)_1, 2_0(x+2)_1, \ldots$ in the cycle C_n^2. Because these two matchings need to form the cycle C_n^2 of length n, we must have $px - x - p^2 + p + x \equiv 0 \pmod{\frac{n}{2}}$.

Because $x \equiv (p-1) \pmod{\frac{n}{2}}$ is a solution, we can set $x = p-1$ and get $C_n^2 = p_0, 1_1, (p+1)_0, 2_1 \ldots, (p-1)_0, 0_1$. Now we define a mapping β taking C_n^1 and C_n^2 onto B_n^1 and B_n^2 as $\beta(k_1) = x_{k-1}$ and $\beta(k_0) = y_k$. It can be checked that $\beta(C_n^1) = B_n^1$ and $\beta(C_n^2) = B_n^2$ and $d = p$. We assumed that $n/\mathrm{gcd}(j,n)$ is even and $j = 2p$. Using Lemma 5 for $c = p = j/2$ and $m = n/2$ we can see that H decomposes $K_{\frac{n}{2},\frac{n}{2}}$.

Now we blow up graph H by $\overline{K}_3$ to obtain from each vertex j_i three vertices $(j,0)_i, (j,1)_i, (j,2)_i$. Using this labeling we will show that we can decompose the graph $H[\overline{K}_3]$ into six copies of $(0,2p)$-prisms.

Let G be an $(0,2p)$-prism of order $2n$. Notice that we can find two edge-disjoint copies G_0, G_3 of G in $H[\overline{K}_3]$ in such a way that for G_0 we define

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the rims $R_n^{10} = (0,0)_1,(0,1)_0,(1,0)_1,(1,1)_0,(2,0)_1,\ (2,1)_0,(3,0)_1,\dots,(\frac{n}{2}-1,1)_0,(0,0)_1$, $R_n^{20} = (0,2)_1,(0,2)_0,(1,2)_1,(1,2)_0,(2,2)_1,(2,2)_0,(3,2)_1,\dots,(\frac{n}{2}-1,2)_0,(0,2)_1$ and the spokes $(i+p,1)_0(i,2)_1,(i+1,0)_1(i+1,2)_01 \in E(G_0)$, where $i \in Z_{\frac{n}{2}}$. Whereas for G_3 we define the rims $R_n^{13} = (p-1,0)_0,(0,1)_1,(p,0)_0,(1,1)_1,(p+1,0)_0,(2,1)_1,(p+2,0)_0,\dots,(p-2,0)_0,(\frac{n}{2}-1,1)_1,(p-1,0)_0$, $R_n^{23} = (p-1,2)_0,(0,2)_1,(p,2)_0,(1,2)_1,(p+1,2)_0,(2,2)_1,(p+2,2)_0,\dots,(p-2,2)_0,(\frac{n}{2}-1,2)_1(p-1,2)_0$ and the spokes $(i,1)_0(i,2)_1,\ (i+p-1,2)_0(i,0)_1 \in E(G_3)$. We obtain six edge-disjoint copies $G_0,G_1,\dots,G_5$ in $H[\overline{K}_3]$, where $G_m = \phi_m(G_0)$ and $\phi_m((a,b)_i) = (a,b+m)_i$, and $G_{3+m} = \phi_m(G_3)$ and $\phi_{3+m}((a,b)_i) = (a,b+m)_i$ for $m \in \{0,1,2\}$ and $i \in \{0,1\}$.

Case 2. p is even.

In that case we want to "glue together" two $C_n^1[\overline{K}_3]$ and $C_n^2[\overline{K}_3]$ in such a way that we can swap spokes of type 0 and dimension $(1,1)$ obtaining spokes of dimension $(p+1,1)$, which are now of type $j=2p$. It follows that $C_n^2 = (p+1)_1,1_0,(p+2)_1,2_0\dots,p_1,0_0,(p+1)_1$. Here we have $\beta(k_0)=x_k$ and $\beta(k_1)=y_k$. Define a graph H as a union $C_n^1 \cup C_n^2$. Similarly as in Case 1, H decomposes $K_{\frac{n}{2},\frac{n}{2}}$ by Lemma 5. By blowing up H by $\overline{K}_3$ we obtain from each vertex j_i three vertices $(j,0)_i,(j,1)_i,(j,2)_i$. As in Case 1 we show that we can decompose the graph $H[\overline{K}_3]$ into six copies of $(0,2p)$-prisms. Let G be an $(0,2p)$-prism of order $2n$. We can find two edge-disjoint copies G_0,G_3 of G in $H[\overline{K}_3]$ in such a way that G_0 has the rims $R_n^{10} = (0,0)_1,(0,1)_0,(1,0)_1,(1,1)_0,(2,0)_1,\ (2,1)_0,(3,0)_1,\dots,(\frac{n}{2}-1,1)_0,(0,0)_1$, $R_n^{20} = (0,2)_1,(0,2)_0,(1,2)_1,(1,2)_0,(2,2)_1,(2,2)_0,(3,2)_1,\dots,(\frac{n}{2}-1,2)_0,(0,2)_1$ and the spokes $(i,1)_0(i+p,2)_1,(i,2)_0(i+1,0)_1 \in E(G_0)$, where $i \in Z_{\frac{n}{2}}$. Whereas G_3 has the rims $R_n^{13} = (p,0)_1,(0,1)_0,(p+1,0)_1,(1,1)_0,(p+2,0)_1,(2,1)_0,(p+3,0)_1,\dots,(p-1,0)_1,(\frac{n}{2}-1,1)_0,(p,0)_1$, $R_n^{23} = (p,2)_1,(0,2)_0,(p+1,2)_1,(1,2)_0,(p+2,2)_1,(2,2)_0,(p+3,2)_1,\dots,(p-1,2)_1,(\frac{n}{2}-1,2)_0(p,2)_1$ and the spokes $(i,1)_0(i+p,2)_1,(i,2)_0(i+1,0)_1 \in E(G_3)$, We obtain six edge-disjoint copies $G_0,G_1,\dots,G_5$ in $H[\overline{K}_3]$, where $G_m = \phi_m(G_0)$ and $\phi_m((a,b)_i) = (a,b+m)_i$ and $G_{3+m} = \phi_m(G_3)$ and $\phi_{3+m}((a,b)_i) = (a,b+m)_i$ for $m \in \{0,1,2\}$ and $i \in \{0,1\}$. ■

3 Decomposition for $n, j \equiv 0 \pmod 4$

The technique used in the previous section cannot be used for the case when $n \equiv 4 \pmod 8$, because we glued together *two* $(0,0)$-prisms and swapped spokes, while for $n \equiv 4 \pmod 8$ we want to decompose $K_{\frac{3n}{2},\frac{3n}{2}}$ into an *odd*

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number $3n/4$ of $(0, j)$-prisms of order $2n$. The result in this section differs from the previous one in the following way. We drop the restriction that $n/\gcd(j, n)$ is even and require only $n \equiv 0 \pmod 4$ instead of $n \equiv 0 \pmod 8$ but on the other hand we assume that $j \equiv 0 \pmod 4$ instead of $j \equiv 0 \pmod 2$.

Theorem 7 *If $n, j \equiv 0 \pmod 4$, then $K_{\frac{3n}{2},\frac{3n}{2}}$ can be decomposed into $(0, j)$-prisms of order $2n$.*

Proof. Notice that we want to decompose $K_{\frac{3n}{2},\frac{3n}{2}}$ into $3n/4$ copies of $(0, j)$-prisms of order $2n$. The main idea of the proof is to "glue together" three prisms into a graph H and then prove that $K_{\frac{3n}{2},\frac{3n}{2}}$ can be decomposed into H.

Let $p = \frac{j}{2} + 1$. Notice that p is odd for $j \equiv 0 \pmod 4$. We denote the vertices of $K_{\frac{3n}{2},\frac{3n}{2}}$ by $(a, b)_i$ where $a \in Z_{\frac{n}{2}}$, $b \in Z_3$, and $i \in \{0, 1\}$. Notice that we can find one copy G of an $(0, j)$-prism in $K_{\frac{3n}{2},\frac{3n}{2}}$ in such a way that $C_n^1 = (0, 0)_0, (1, 1)_1, (1, 0)_0, (2, 1)_1, (2, 0)_0, (3, 1)_1, \dots, (\frac{n}{2} - 1, 0)_0, (0, 1)_1, (0, 0)_0$, $C_n^2 = (0, 0)_1, (0, 1)_0, (1, 0)_1, (1, 1)_0, (2, 0)_1, (2, 1)_0, (3, 0)_1, \dots, (\frac{n}{2} - 1, 1)_0, (0, 0)_1$ and $(i, 0)_0(i, 0)_1, (i, 1)_0(i + p, 1)_1 \in E(G)$, where $i \in Z_{\frac{n}{2}}$ (see Fig. 3).

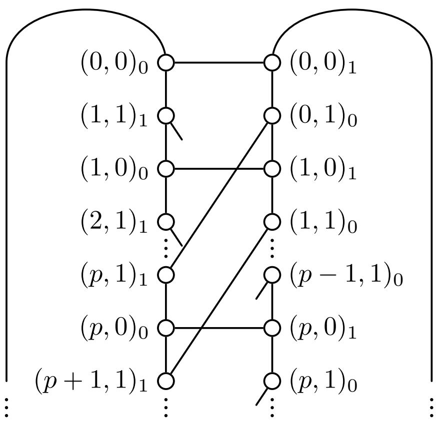

Figure 3: Labeling of $(0, j)$-prism

Claim 1. We can obtain three edge-disjoint copies G_0, G_1, G_2 of G in $K_{\frac{3n}{2},\frac{3n}{2}}$ by setting $G_m = \phi_m(G)$ and $\phi_m((a, b)_0) = (a, b + m)_0$, $\phi_m((c, d)_1) = (c, d + m)_1$ for $m \in \{0, 1, 2\}$.

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We first denote by $K_{3,3}^{a,c}$ the complete bipartite graph with partite sets $\{(a,0)_0, (a,1)_0, (a,2)_0\}$ and $\{(c,0)_1, (c,1)_1, (c,2)_1\}$. We observe that in any $K_{3,3}^{a,c}$ for any fixed $m \in \{0,1,2\}$ there are precisely three edges of dimension $(c-a, m)$ and no two of them share a vertex. Now for some fixed a, c let the copy G_0 contain an edge $(a,r)_0(c,s)_1$ of dimension $(c-a, s-r)$. By applying ϕ_1 and ϕ_2, we obtain edges $(a, r+1)_0(c, s+1)_1$ and $(a, r+2)_0(c, s+2)_1$, respectively, both of dimension $(c-a, s-r)$. These three edges are obviously independent, and the claim is proved.

We recall that $n, j \equiv 0 \pmod 4$ and $j \le \frac{n}{2}$. Also $p = \frac{j}{2} + 1$ and hence $p-1$ is even. Denote by g the greatest common divisor of $\frac{n}{2}$ and $p-1$, observe that g must be even, say $g = 2q$, and write $\frac{n}{2} = gk$. Then g in the additive group $Z_{\frac{n}{2}}$ generates a subgroup $\langle g \rangle$ of order k. It is well known that the order of the subgroup $\langle p-1 \rangle$ of $Z_{\frac{n}{2}}$ generated by $p-1$ is the same as the order of the subgroup $\langle g \rangle$ generated by g, which is k.

Now let $H = G_0 \cup G_1 \cup G_2$ and $H_i = \delta_i(H)$, where we define

$$\delta_i((a,b)_0) = (a,b)_0,$$

$$\delta_i((c,d)_1) = (c + (p-1)i, d)_1$$

for $i \in \{0, 1, \ldots, k-1\}$. Recall that the entries are elements of $Z_{\frac{n}{2}}$ and Z_3, respectively. We want to show that all these copies are edge-disjoint.

Claim 2. The copies $H_0, H_1, \ldots, H_{k-1}$ of H defined above are mutually edge-disjoint.

First we observe that H_0 contains only edges of dimensions $(0,0), (0,1)$, $(0,2), (1,1)$, $(1,2)$ and $(p,0)$. When we apply δ_i for $i \in \{0,1,\ldots,k-1\}$ to any edge of type $(0,m)$ for $m \in \{0,1,2\}$, we obtain edges of dimensions $(0,m), (p-1,m), (2(p-1),m), \ldots, ((k-1)(p-1),m)$ and notice that the first entries form a subgroup $\langle p-1 \rangle$ of $Z_{\frac{n}{2}}$.

When we apply δ_i for $i \in \{0,1,\ldots,k-1\}$ to edges $(1,m)$ for $m \in \{1,2\}$, we obtain edges of dimensions $(1,m), (p,m), (2p-1,m), \ldots, (i(p-1)+1,m), \ldots, ((k-1)(p-1)+1,m)$. The first entries of these dimensions form the coset $1+\langle p-1 \rangle$ of $Z_{\frac{n}{2}}$. Then we apply δ_i for $i \in \{0,1,\ldots,k-1\}$ to an edge of dimension $(p,0)$ and obtain edges of dimensions $(p,0), (2p-1,0), (3p-2,0), \ldots, (i(p-1)+p,0), \ldots, ((k-1)(p-1)+p,0)$, where in fact the last edge has dimension $(1,0)$. Therefore, the first entries of these dimensions again form the coset $1+\langle p-1 \rangle$. It should be now obvious that the claim is proved because no two copies of H contain edges of the same dimension.

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Now we denote by F_0 the union $H_0 \cup H_1 \cup \ldots \cup H_{k-1}$ and recall that $g = 2q$ for some $q \geq 1$. If $g = 2$, then $k = \frac{n}{4}$ and the union of the cosets $\langle p-1 \rangle$ and $1 + \langle p-1 \rangle$ gives the whole group $Z_{\frac{n}{2}}$. Because δ_i fixes all vertices $(a,b)_0$, we can see that for every fixed vertex $(a,b)_0$ we have used in F_0 each edge $(a,b)_0(a+t,b+m)_1$ of every possible dimension (t,m) for $t \in Z_{\frac{n}{2}}, m \in Z_3$ exactly once. Therefore, F_0 contains precisely the edges of $K_{\frac{3n}{2},\frac{3n}{2}}$ and the proof is complete.

If $g > 2$, we construct for $i \in \{0, 1, \ldots, q-1\}$ graphs $F_i = \psi_i(F_0)$ by setting

$$\psi_i((a,b)_0) = (a,b)_0$$

and

$$\psi_i((c,d)_1) = (c+2i,d)_1.$$

We observe that F_i then contains edges whose first entries are elements of the cosets $2i + \langle p-1 \rangle$ and $2i + 1 + \langle p-1 \rangle$ and the union D of the graphs $F_0, F_1, \ldots, F_{q-1}$ contains as first entries all elements of the group $Z_{\frac{n}{2}}$. Consequently, we observe that again for every fixed vertex $(a,b)_0$ we have used in D each edge $(a,b)_0(a+t,b+m)_1$ of every possible dimension (t,m) for $t \in Z_{\frac{n}{2}}, m \in Z_3$ exactly once.

Now D again contains exactly the edges of $K_{\frac{3n}{2},\frac{3n}{2}}$ and the proof is complete. ∎

4 Conclusion

There are still two missing cases for decomposition of $K_{\frac{3n}{2},\frac{3n}{2}}$ into $(0,j)$-prisms of order $2n$. One of them is the case when $n \equiv 0 \pmod 8$ and $n/\gcd(j,n)$ is odd. The other one is when $n \equiv 0 \pmod 4$ and $j \equiv 2 \pmod 4$.

We also point out that the necessary conditions can be satisfied even when $K_{k,k}$ is not $K_{\frac{3n}{2},\frac{3n}{2}}$. For example, when $n = 50$, then they are met for $K_{60,60}$ or $K_{90,90}$. We do not know whether these graphs can be decomposed into $(0,j)$-prisms of order $2n = 100$.

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References

[1] J.C. Bermond, C. Huang, and D. Sotteau, Balanced cycle and circuit designs: Even case, *Ars. Combinat.* **5** (1978) 293–318.

[2] S. Cichacz, D. Fronček, Factorization of $K_{n,n}$ into $(0, j)$-prisms, *Information Processing Letters* **109** (2009) 932–934.

[3] S. Cichacz, D. Fronček, P. Kovář, Note on decomposition of $K_{n,n}$ into $(0, j)$-prisms, *Lecture Notes in Computer Science* **5874** (2009) 125–133.

[4] S.I. El-Zanati and C. Vanden Eynden, Decompositions of $K_{m,n}$ into cubes, *J. Comb. Designs* **4** (1996) 51–57.

[5] D. Fronček, P. Kovář, M. Kubesa, Decomposition of complete graphs into blown-up cycles $C_m[2]$, to appear in Discr. Math.

[6] R. Häggkvist, Decompositions of complete bipartite graphs, *Surveys in combinatorics*, 1989 (Norwich, 1989), 115–147, London Math. Soc. Lecture Note Ser., 141, Cambridge Univ. Press, Cambridge, 1989.

[7] D. Sotteau, Decomposition of $K_{m,n}$ $(K_{m,n}*)$ into cycles (circuits) of length $2k$, *J. Comb. Theory*, Ser. B **30** (1981) 75–81.

[8] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs* (Internat. Symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris. (1967) 349–355.