Discretisation of Maxwell-Bloch dynamical systems

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Abstract — **We investigate the smallest possible physically meaningful RMB model, which models propagation through a thin slab of resonant atomic material placed in a recirculating cavity, which has 2 degrees of freedom in the spatial distribution of the field . The model may be used to determine the stationary states of the cavity field, which are then subjected to stability analysis.**

1 INTRODUCTION

The Maxwell-Bloch dynamical system consists of Maxwell's equations for an electromagnetic field coupled to a quantum-electronic system to represent a resonant or near-resonant polarisation induced in the propagation medium. While the full 3D Maxwell equations along with arbitrary quantum models form excellent models of nonlinear optics in polarisable media, such general models are analytically insoluble and computationally intractable. A very minimal Maxwell Bloch model, however, is completely integrable, and the integrable solutions already exhibit complex dynamical states that are comparable with experimental models. The integrable reduced Maxwell-Bloch (RMB) model consists of plane waves (3D to 1D reduction), one-way propagation (order reduction of the wave equation) and a near-resonant 2-state atom for the quantum-electronics (quantum state reduction) [1, 2, 3]. Such a heavy reduction of the degrees of freedom of the RMB system while preserving the physical reality of the model encourages a more systematic investigation of the question of the active degrees of freedom.

The RMB equations, like any other integrable system, possess an infinite sequence of conserved quantities that can be explicitly determined directly from the equations. However these are in general infinite in number, and it is not at all easy to determine the angle variables that correspond with the actions of the integrals, and so invert the representation from action-angle variables to the physical fields.

2 MAXWELL-BLOCH EQUATIONS

Maxwell's equations for 1-dimensional propagation (plane waves) along the spatial z-axis are

$$\partial_t E + c^2 \partial_z B = -\epsilon^{-1} \partial_t P \tag{1}$$

$$\partial_z E + \partial_t B = 0 \tag{2}$$

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for electric field E, magnetic flux B, and light velocity c. P is the electronic polarisation induced in the dielectric medium, and is generically described by a quantum electronic model

$$P = -Ne \langle Q \rangle \tag{3}$$

$$\partial_t \langle Q \rangle = (\mathrm{i}\hbar)^{-1} \langle [Q, H] \rangle \tag{4}$$

$$\partial_t \Gamma = (\mathrm{i}\hbar)^{-1} [H, \Gamma] \tag{5}$$

$$\langle Q \rangle = \mathrm{tr}\{\Gamma Q\} \tag{6}$$

where $H = H_0 + eEQ$ is the quantum Hamiltonian of electrons in the medium, Γ is the density operator, and Q is a suitable operator for the electronic dipole induced in the medium, N is the number density of polarised centres (atoms), $-e$ is the electron charge, and tr$\{.\}$ represents the matrix or operator trace of its argument. The very simplest such system occurs when each polarisation centre is assigned a Hamiltonian consisting of a diagonal 2×2 matrix $H_0 = -\frac{1}{2}\hbar\Omega_0\sigma_3$ and the dipole operator Q is assigned an off-diagonal symmetric 2×2 matrix $Q = q_0\sigma_2$; this minimal system of polarisation is then called a Bloch model, or 2-level atom [5] .

The second Maxwell equation (2) implies that a potential A may be introduced so that $E = -\partial_t A, B = \partial_z A$, and then the first Maxwell equation (1) implies that

$$c^2 \partial_z^2 A - \partial_t^2 A = -\epsilon^{-1} \partial_t P \tag{7}$$

which is the wave equation for an electromagnetic potential driven by a dielectric polarisation. An alternative form for the dipole interaction involving the potential A rather than the electric field E is given by

$$H = H_0 + (e/m)\Xi A \tag{8}$$

where Ξ is the operator of electron momentum. In a 2-level Bloch system, $\Xi = p_0\sigma_2$ where p_0 is the *momentum parameter*. From the Maxwell-Bloch system one derives easily *Poynting's Theorem*

$$\partial_z(EB) = -\tfrac{1}{2}c^{-2}\partial_t(E^2 + c^2B^2) - \mu_0 \partial_t \langle H_0 \rangle \tag{9}$$

$$\Rightarrow \partial_z \int_{-\infty}^{\infty} EB \mathrm{d}t = 0 \tag{10}$$

for suitable boundary conditions at $t \to \pm\infty$. The Poynting flux EB is an example of a conserved flux.

3 REDUCED MAXWELL-BLOCH SYSTEMS

A further reduction of the MB equations results after introducing the one-way wave approximation, leading to

$$\partial_z A + c^{-1}\partial_t A = \tfrac{1}{2}\mu_0 P \tag{11}$$

with the same equations ( ) for the polarisation. After the time-shifted variable $\tau = t - c^{-1}z$ is introduced

$$\partial_z A \quad = \quad \tfrac{1}{2}\mu_0 P \tag{12}$$

with τ replacing t in the dynamical equations for P. It is also convenient to introduce the normalised field $\psi = (ep_0/m)A$, which has the dimensions of frequency. Then the normalised form of the RMB equations is

$$\partial_z \psi \quad = \quad -K\mathrm{tr}\{\Gamma\sigma_1\} \tag{13}$$

$$\partial_\tau \Gamma \quad = \quad [U_1, \Gamma] \tag{14}$$

with $U_1 = -\mathrm{i}(-\frac{1}{2}\Omega_0\sigma_3 + \psi\sigma_2)$ and K is a dimensionalised coupling constant

$$K = \frac{N}{2c\epsilon_0\hbar}\left(\frac{ep_0}{m\Omega_0}\right)^2. \tag{15}$$

The reduced Maxwell-Bloch system with a 2×2 Hamiltonian H for the polarisation is an example of a completely integrable evolutionary system [1]. If we define the two matrices

$$U \quad = \quad -\mathrm{i}(-\tfrac{1}{2}\zeta\Omega_0\sigma_3 + \psi\sigma_2) \tag{16}$$

$$V \quad = \quad \frac{\mathrm{i}K}{\zeta^2 - 1}(\zeta\rho_1\sigma_1 + \rho_2\sigma_2 + \zeta\rho_3\sigma_3) \tag{17}$$

where

$$\Gamma \quad = \quad \tfrac{1}{2}(\sigma_0 + \rho_1\sigma_1 + \rho_2\sigma_2 + \rho_3\sigma_3) \tag{18}$$

with a free spectral parameter ζ, then the RMB equations are equivalent to the zero-curvature condition

$$\partial_z U \quad = \quad \partial_t V - [U, V] \tag{19}$$

identically at all ζ. The zero-curvature condition is the compatibility condition for the two ODEs

$$\partial_t v \quad = \quad Uv \tag{20}$$

$$\partial_z v \quad = \quad Vv \tag{21}$$

with a complex 2-component vector v at each (z, t).

4 DISCRETISED MAXWELL-BLOCH EQUATIONS

The discretisation of the spatial variable z may be contemplated in two circumstances; either, as a sampling of the dependent variables on the underlying continuum of z, or as a model for discrete polarisation induced in concentrated points of the z-axis, with free-space propagation in the space between the polarisation centres. Either of these scenarios leads to the same canonical discretised MB system.

In the second of these scenarios, of a set of concentrated polarisation sources located at discrete points along the z-axis, consider the following spatially periodic problem. Concentrated polarisation centres are located at the points $z = z_k = kL, k = 0, 1, \ldots$, and an initial condition is prescribed for the field $\psi(0, t)$ which we assume also to be periodic in time with period T, so that $\psi(0, t + T) = \psi(0, t)$. This field propagates freely between the polarisation centres so that $\psi(z, \tau) = \psi(z_k, \tau)$, but undergoes a jump change at each polarisation centre so that

$$\psi(z_{k+1}, \tau) - \psi(z_k, \tau) = \tfrac{1}{2}\mu_0 L P_{k+1}(\tau)$$

where $P_k(\tau)$ is the polarisation induced in the concentrated dielectric at $z = z_k$. The discrete Maxwell-Bloch system is then

$$\psi_k - \psi_{k-1} \quad = \quad -KL\mathrm{tr}\{\Gamma_k\sigma_1\} \tag{22}$$

$$\mathrm{i}\hbar\partial_\tau\Gamma_k \quad = \quad [H_k, \Gamma_k] \tag{23}$$

where $H_k(\tau)$ is the Hamiltonian for the electrons at $z = z_k$. This system is a model of propagation through a sequence of periodically placed polarising sheets, which can also be regarded as an unfolded ring cavity containg a single polarising sheet *per* period of the cavity. In order to apply this model in the ring cavity context it is further required to determine what initial condition $\psi_0(\tau)$ for the *unfolded* sequence will result in a stationary field in the *folded* cavity. This completes the specification of an *electromagnetic* Maxwell-Bloch system. It is necessary then to address the question of what *canonical integrable* RMB system closely or exactly models the electromagnetic one.

When space is discretised a *canonical integrable* RMB system has a compatibility (zero-curvature) condition

$$\partial_\tau v_k \quad = \quad U_k v_k \tag{24}$$

$$v_{k+1} \quad = \quad \Phi_k v_k \tag{25}$$

where Φ_k is a unitary operator, and the compatibiity of these two equations requires that

$$U_{k+1} \quad = \quad \Phi_k U_k \Phi_k^{-1} + \partial_\tau \Phi_k \Phi_k^{-1} \tag{26}$$

which is the equivalent of the zero-curvature condition. This condition has the form of a gauge transformation for advancing the matrix U_k from step k to step $k + 1$.

In order to match the canonical integrable system to the electromagnetic one, we have available the choice of the matrices U and Φ. It is natural to choose the matrix U to be identical to that for the continuous-z case, as previously, and to construct a Φ-matrix to 'best approximate' the electromagnetic model. Let us present Φ in the form

$$\Phi = aI + bV \tag{27}$$

where a and b are scalars, and V is anti-hermitian so that $V^\dagger = -V$.

When both time and space are discretised in the compatible matrix equations we have

$$v_k^{m+1} = \Psi_k^m v_k^m \tag{28}$$

$$v_{k+1}^m = \Phi_k^m v_k^m \tag{29}$$

with Φ_k^m and Ψ_k^m two unitary operators. For compatibility of these two equations it is required that v_{k+1}^{m+1} is the same for both routes of computing it, leading to

$$\Psi_{k+1}^m \Phi_k^m = \Phi_k^{m+1} \Psi_k^m \tag{30}$$

A suitable candidate for the Ψ operator of this pair is

$$\Psi_k^m = (1 + \tfrac{1}{2}\Delta_t U_k^m)(1 - \tfrac{1}{2}\Delta_t U_k^m)^{-1}. \tag{31}$$

Since $U^2 = (\zeta^2 + \psi^2)I$ the Cayley-Hamilton theorem can be applied to obtain

$$\Psi_k^m = \cosh\theta_k^m + U_k^m \frac{\sinh\theta_k^m}{\theta_k^m} \tag{32}$$

with $\theta_k^m = \sqrt{\zeta^2 + (\psi_k^m)^2}$.

5 DISCRETE INTEGRABLE EQUATIONS

The integrability of any discrete evolution system is embedded in the Lax equations

$$L_{k+1} = G_k L_k G_k^{-1} \tag{33}$$

$$v_{k+1} = G_k v_k \tag{34}$$

where, for $k = 0, 1, \ldots$, L_k is a linear operator acting in some Hilbert space, v_k is an eigenvector in the spectrum of L_k, and the operator G_k effects a gauge transformation in the Hilbert space. The first equation guarantees that any eigenvalue of L_k, λ_k is preserved by the evolution from step k to step $k+1$. Generally, L_k is a differential or pseudo-differential operator and the Hilbert space is a space of functions of a transverse variable τ, physically identified with time. For the RMB system of integrable equations the Lax operator for continuous τ is

$$L = \partial_\tau^2 + (\mathrm{i}\partial_t\psi + \psi^2) \tag{35}$$

which is similar to that for the mKdV hierarchy of integrable equations [4]. The eigenvalue problem $Lv = \lambda v$ is equivalent to the matrix system

$$\partial_t \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -\mathrm{i} \begin{pmatrix} \psi & \zeta \\ \zeta & -\psi \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{36}$$

with $\zeta^2 = -\lambda$, which is itself equivalent by gauge transformation to

$$\partial_t \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = -\mathrm{i} \begin{pmatrix} \zeta & -\mathrm{i}\psi \\ \mathrm{i}\psi & -\zeta \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix}. \tag{37}$$

If, in addition, the time variable τ is discretised to a finite number of sample times τ_m, then the Lax operator L is a matrix acting on discrete vectors v in a space of dimension M, and has a finite number M of discrete eigenvalues λ. This finite number of eigenvalues forms a generating function for the invariants of the doubly discretised system in the form of the polynomial

$$\det(\lambda - L_k) = \sum_{j=1}^{M} I_j \lambda^j$$

as the coefficients of powers of λ [4]. If the elements of the matrix L are values of the field ψ at discrete times, then the invariants I_j are polynomials of these field values.

The appropriate Lax operator for discretised systems of RMB type is obtained by replacing the derivative ∂_τ by a suitable finite difference operator, and constructing a compatible set of operators Φ. The smallest nontrivial physically meaningful system of this type occurs when time is discretised to three values with periodic boundary conditions.

References

[1] J. M. Arnold, 'Dynamical systems in nonlinear optics: reduced Maxwell-Bloch models', *Optical and Quantum Electronics*, **40**, 787, 2009.

[2] J. M. Arnold, 'A completely integrable 3-level Maxwell-Bloch system with rotational degeneracy', *International Conference on Electromagnetics in Advanced Applications (ICEAA)*, Torino, Italy, September 10-14, 2001.

[3] J. M. Arnold, 'Integrability of Maxwell-Bloch systems', *URSI General Assembly*, 2005.

[4] G. B. Segal, Integrable systems and inverse scattering, in Hitchen, Segal, Ward, *Integrable Systems: Twistors, Loop Groups and Riemann Surfaces*, Oxford Graduate Texts in Mathematics **4**, Clarendon Publications, 1999.

[5] L. Allen and J. H. Eberly, *Optical resonance and 2-level atoms*, Dover, 1987.