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Essays on Auction Mechanisms and Resource Allocation in Keyword Advertising

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Essays on Auction Mechanisms and Resource Allocation in Keyword Advertising

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Essays on Auction Mechanisms and Resource Allocation in Keyword Advertising

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Advances in information technology have created radically new business models, most notably the integration of advertising with keyword-based targeting, or "keyword advertising." Keyword advertising has two main variations: advertising based on keywords employed by users in search engines, often known as "sponsored links," and advertising based on keywords embedded in the content users view, often known as "contextual advertising." Keyword advertising providers such as Google and Yahoo! use auctions to allocate advertising slots. This dissertation examines the design of keyword auctions. It consists of three essays.

The first essay "Ex-Ante Information and the Design of Keyword Auctions" focuses on how to incorporate available information into auction design. In our keyword auction model, advertisers bid their willingness-to-pay per click on their advertisements, and the advertising provider can weigh advertisers' bids differently and require different minimum bids based on advertisers' click-generating potential. We study the impact and design of such weighting schemes and minimum-bids policies. We find that weighting scheme determines how advertisers with different click-generating potential match in equilibrium. Minimum bids exclude low-valuation advertisers and at

the same time may distort the equilibrium matching. The efficient design of keyword auctions requires weighting advertisers' bids by their expected click-through-rates, and requires the same minimum weighted bids. The revenue-maximizing weighting scheme may or may not favor advertisers with low click-generating potential. The revenue-maximizing minimum-bid policy differs from those prescribed in the standard auction design literature. Keyword auctions that employ the revenue-maximizing weighting scheme and differentiated minimum bid policy can generate higher revenue than standard fixed-payment auctions.

The dynamics of bidders' performance is examined in the second essay, "Keyword Auctions, Unit-price Contracts, and the Role of Commitment." We extend earlier static models by allowing bidders with lower performance levels to improve their performance at a certain cost. We examine the impact of the weighting scheme on overall bidder performance, the auction efficiency, and the auctioneer's revenue, and derive the revenue-maximizing and efficient policy accordingly. Moreover, the possible upgrade in bidders' performance levels gives the auctioneer an incentive to modify the auction rules over time, as is confirmed by the practice of Yahoo! and Google. We thus compare the auctioneer's revenue-maximizing policies when she is fully committed to the auction rule and when not, and show that she should give less preferential treatment to low-performance advertisers when she is fully committed.

In the third essay, "How to Slice the Pie? Optimal Share Structure Design in Keyword Auctions," we study the design of share structures in keyword auctions. Auctions for keyword advertising resources can be viewed as share auctions in which the highest bidder gets the largest share, the second highest bidder gets the second largest share, and so on. A share structure problem arises in such a setting regarding how much resources to set aside for the highest bidder, for the second highest bidder, etc. We address this problem under a general specification and derive implications on how the optimal share structure should change with bidders' price elasticity of demand for exposure, their valuation distribution, total resources, and minimum bids.

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Chapter 1

An Introduction to Keyword Advertising and Keyword Auctions

1.1 Introduction

Keyword advertising is a form of targeted online advertising. A basic variation of keyword advertising is "sponsored links" (also known as "sponsored results" and "sponsored search") on search engines. Sponsored links are advertisements triggered by search phrases entered by Internet users on search engines. For example, a search for "laptop" on Google will bring about both the regular search results and advertisements from laptop makers and sellers. Figure 1.1 shows such a search-result page with sponsored links at the top and on the side of the page. Another variation of keyword advertising is "contextual advertising" on content pages. Unlike sponsored links, contextual advertisements are triggered by certain keywords in the content. For example, a news article about "Cisco" is likely to be displayed with contextual advertisements from Cisco network equipment sellers and Cisco training providers.

Both sponsored links and contextual advertisements can target users who are most likely interested in seeing the advertisements. Because of its superior targeting ability, keyword advertising has quickly gained popularity among marketers, and has become a leading form of online advertising. According to a report by Interactive Advertising Bureau and PricewaterhouseCoopers (2008), keyword advertising in the United States reached \$8.5 billion in total revenue in 2007. eMarketer (2007) predicts the market for online advertising will rise from \$16.9 billion in 2006 to \$42 billion in

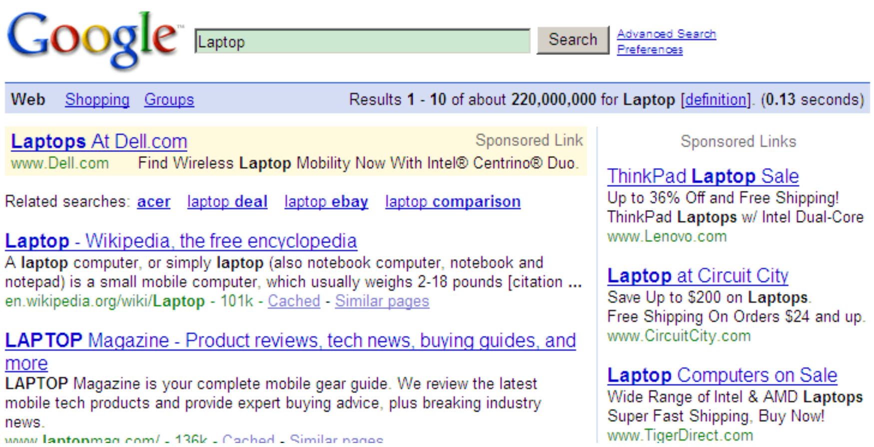

Figure 1.1: Search-based Keyword Advertising

2011, with keyword advertising accounting for about 40% of the total revenue.

A typical keyword advertising market consists of *advertisers* and *publisher*s (i.e., websites), with *keyword advertising providers* in between. There are three key keyword advertising providers in the U.S. keyword advertising market: Google, Yahoo!, and MSN adCenter. Next we use Google's practice to illustrate keyword-advertising business models.

Google has two main advertising programs, Adwords and AdSense. Adwords is Google's flagship advertising program that interfaces with advertisers. Through Adwords, advertisers can submit ads, choose keywords relevant to their businesses, and pay for the cost of their advertising campaigns. Adwords has separate programs for sponsored search (Adwords for search) and for contextual advertising (Adwords for content). In each case, advertisers can choose to place their ads on Google's site only or on publishers' sites that are part of Google's advertising network. Advertisers can also choose to display text, image, or, more recently, video advertisements.

AdSense is another Google advertising program that interfaces with publishers. Publishers from personal blogs to large portals such as CNN.com can participate in

Google's AdSense program to monetize the traffic to their websites. By signing up with AdSense, publishers agree to publish ads and receive payments from Google. Publishers may choose to display text, image, and video advertisements on their sites. They receive payments from Google on either a per-click or per-thousand-impressions basis. AdSense has become the single most important revenue source for many Web 2.0 companies.

Keyword advertising providers use auctions to sell their keyword advertising slots to advertisers. The design of such auctions is the focus of this dissertation. A basic form of keyword auction is as follows. Advertisers choose their willingness-to-pay for a keyword phrase either on a per-click (*pay-per-click*) or on per-impression (*pay-per-impression*) basis. An automated program ranks advertisers and assigns them to available slots whenever a user searches for the keyword or browses a content page deemed relevant to the keyword. The ranking may be based on advertisers' pay-per-click/pay-per-impression only. It may also include other information, such as their historical click-through-rate (CTR), namely the ratio of the number of clicks on the ad to the number of times the ad is displayed. Almost all major keyword advertising providers use automated bidding systems, but they specific designs differ from each other and change over time.

1.2 A Historical Look at Keyword Auctions

Keyword advertising and keyword auctions were born out of practice. They were fashioned to replace the earlier, less efficient market mechanisms and are still being shaped by the accumulative experiences of the keyword advertising industry. In this subsection, we chronicle the design of keyword advertising markets and keyword auctions, and show how they evolved.

1.2.1 Early Internet Advertising Contracts

In early online advertising, advertising space was sold through advance contracts. These contracts were negotiated on a case-by-case basis. As such negotiations were time-consuming, advertising sales were limited to large advertisers (for example, those paying at least a few thousand dollars per month). These advertising contracts were typically priced in terms of the number of thousand-page-impressions (cost-per-mille, or CPM). CPM pricing was borrowed directly from off-line advertising, such as TV, radio, and print, where advertising costs are measured on a CPM basis. The problem with CPM pricing is that it provides no indication as to whether users have paid attention to the advertisement. Advertisers may be concerned that their advertisements are pushed to web users without necessarily generating any impact. The lack of accountability is reflected in the saying among marketing professionals: "Half the money spent on advertising is wasted and you don't even know which half."

1.2.2 Keyword Auctions by GoTo.com

In 1998, a startup company called GoTo.com demonstrated a new proof-of-concept search engine at a technology conference in Monterey, California. At that time, all other search engines sorted search results based purely on algorithm-assessed relevancy. GoTo.com, on the other hand, devised a plan to let advertisers bid on top positions of the search result. Specifically, advertisers can submit their advertisements on chosen words or phrases ("search terms") together with their pay-per-click on these advertisements. Once the submitted advertisements are validated by GoTo.com's editorial team, they will appear as a search result. The highest advertiser will appear at the top of the result list; the second-highest advertiser will appear at the second place of the result list, and so on. Each time a user clicks on an advertisement, the advertiser will be billed the amount of the bid.

GoTo.com's advertising model contains several key innovations. First, advertising based on user-entered search terms represents a new form of targeted advertising that is based on users' behavior. For example, a user who searches "laptop" is highly likely in the process of buying a laptop. Keyword-based search engine advertising opens a new era of behavioral targeted advertising.

Second, by billing advertisers only when users click on the advertisements, GoTo.com provides a partial solution to a longstanding issue of lack of accountability. Clicking on an advertisement indicates online users' interests. Therefore, pay-per-click represents a significant step toward more accountable advertising. The ability to track behaviorial outcomes such as clicks is a crucial difference between online advertising and its off-line counterparts. The act of clicking on an advertisement provides an important clue on advertising effectiveness. Accumulated information on clicking behavior can be further used to fine-tune advertisement placement and content. In such a sense, pay-per-click is a significant leap from the CPM scheme and signifies the huge potential of online advertising.

Finally, the practice of using auctions to sell advertising slots on a continuous, real-time basis is another innovation. These real-time auctions allow advertisements to go online a few minutes after a successful bidding. As there is no pre-set minimum spending, auction-based advertising has the advantage of tapping into the "long tail" of the advertising market, that is, advertisers who have small spending budgets and are more likely to "do-it-yourself."

GoTo.com was re-branded as Overture Services in 2001 and acquired by Yahoo! in 2003. During the process, however, the auction mechanism and the pay-per-click pricing scheme remained largely unchanged.

1.2.3 Subsequent Innovations by Google

Google, among others, made several key innovations to the keyword advertising business model. Some of these have become standard features of today's keyword advertising. In the following, we briefly review these innovations.

Content vs. Advertising. The initial design by GoTo.com features a "paid placement" model: paid advertising links are mixed with organic search results so that users cannot tell whether a link is paid for. Google, when introducing its own keyword advertising in 1998, promoted a "sponsored link" model that distinguished advertisements from organic search results. In Google's design, advertisements are displayed on the side or on top of the result page with a label "sponsored links." Google's practice has been welcomed by the industry and policy-makers and has now become standard practice.

Allocation Rules. Google introduced a new allocation rule in 2002 in its "Adwords Select" program in which listings are ranked not only by bid amount, but also by CTR (later termed as "quality score"). Under such a ranking rule, paying a high price alone cannot guarantee a high position. An advertiser with a low CTR will get a lower position than advertisers who bid the same (or slightly lower) but have higher CTRs. In 2006, Google revised its quality score calculation to include not only advertisers' past CTRs but also the quality of their landing pages. Advertisers with low quality scores are required to pay a high minimum bid or they will become inactive.

Google's approach to allocation gradually gained acceptance. At the beginning of 2007, Yahoo! conducted a major overhaul of its online advertising platform that considers both the CTRs of an advertisement and other undisclosed factors. Microsoft adCenter, which came into use only at the beginning of 2006, used a ranking rule similar to Google's Adwords. Before that, all of the advertisements displayed on the MSN search engine were supplied by Yahoo!.

Payment Rules. In the keyword auctions used by GoTo.com, bidders pay the amount of their bids. This way, any decrease in one's bid will result in less payment. As a result, bidders have incentives to monitor the next highest bids and make sure their own bids are only slightly higher. The benefits from constantly adjusting one's bid create undesirable volatility in the bidding process. Perhaps as a remedy, Google used a different payment rule in their Adwords Select program. In Adwords Select, bidders do not pay the full amount of their bids. Instead, they pay the lowest possible to remain above the next highest competitor. If the next highest competitor's bid drops, Google automatically adjusts the advertiser's payment downward. This feature, termed as "Adwords Discounter," is essentially an implementation of second-price auctions in a dynamic context. One key advantage of such an arrangement is that bidders' payments are no longer directly linked to their bids. This reduces bidders' incentive to game the system. Recognizing this advantage, Yahoo! (Overture) also switched to a similar payment rule.

Pricing Schemes. As of now, Google's Adwords for search offers only pay-per-click advertising. On the other hand, Adwords for content allows advertisers to bid either pay-per-click or pay-per-thousand-impression. Starting spring 2007, Google began beta-testing a new billing metric called pay-per-action with their Adwords for content. Under pay-per-action metric, advertisers pay only for completed actions of choice, such as a lead, a sale, or a page view, after a user has followed through the advertisement to the publisher's website.

1.2.4 Beyond Search Engine Advertising

The idea of using keywords to place most relevant advertisements is not limited to search engine advertising. In 2003, Google introduced an "AdSense" program that allows web publishers to generate advertising revenue by receiving advertisements served by Google. AdSense analyzes publishers' web pages to generate a list of most relevant keywords, which are subsequently used to select the most appropriate advertisements for these pages. The order of advertisements supplied to a page is determined by Adwords auctions. The proceeds of these advertisements are shared between Google and web publishers. Yahoo! has a similar program called Yahoo! Publisher Network.

Chapter 2

Ex-Ante Information and the Design of Keyword Auctions

2.1 Introduction

Keyword advertising is distinguished from offline advertising and other online advertising because it delivers the most relevant advertisement to Internet users, yet in less intrusive ways. The effectiveness of this type of advertising has been demonstrated in its acceptance among marketers. The leading provider of keyword advertising, Google, increased its total revenue 39-fold between 2002 and 2007 to $16.6 billion, mostly from keyword advertising. Keyword advertising has consistently accounted for about 40 percent of the total online advertising revenue in the last few years and will remain the biggest form of online advertising for years to come. It is expected to reach about $16.8 billion by 2011 (eMarketer 2007).

Keyword advertising is undoubtedly enabled by new information technologies. One of the key differences between keyword advertising and traditional forms of advertising such as radio and TV is that keyword advertising providers, with the help of information technology, can better track outcomes of advertisements including how many Internet users click on them and the number that end up making a purchase. The ability to track such outcomes not only allows marketers to better account for their advertising campaigns, but also shapes the design of keyword auctions—a novel mechanism that keyword advertising providers such as Google, Yahoo!, and MSN use to allocate advertising slots. First, it enables outcome-based pricing (or *pay-for-performance*), including the now standard "pay-per-click," in which advertisers pay only when Internet users click on their advertisements, and new variants such as "pay-per-call" (advertisers pay each time an Internet user contacts the advertiser) and "pay-per-purchase" (advertisers pay each time an Internet user follows the advertisement to make a purchase). Second, it allows advertising providers to gather information on advertisers' potential to generate outcomes. For example, in pay-per-click advertising, advertising providers typically accumulate information on advertisers' click-through rates (CTRs)— the number of clicks on an advertisement divided by the number of times displayed—which can be used to infer advertisers' click-generating potential. This paper examines how such information—the ex-ante information on advertisers' potential to generate outcomes—should be integrated into the design of keyword auctions that use outcome-based pricing.

The ex-ante information on advertisers' outcome-generating potential has been gradually integrated into keyword auction designs in terms of ranking rules and minimum-bid policies. The initial keyword auctions, as introduced by Overture (now a subsidiary of Yahoo!), rank advertisers solely by their willingness-to-pay per click (henceforth *unit price*), thus making no use of information on advertisers' click-generating potential. In 2002, Google used such information for the first time by ranking advertisers by the product of unit prices they bid and their historical CTRs so that, everything else being equal, an advertiser with a higher CTR will get a better slot. Later, Google extended the ranking factor from CTRs to a more comprehensive "Quality Score" that also takes into account the quality of the advertisement text and other unannounced relevance factors. Yahoo! adopted a similar ranking rule in its new advertising platform. Recently, advertising providers have begun to make minimum bids depend on advertisers' click-generating potential. For example, Google recently switched from a one-size-fits-all minimum-bid policy to one that requires higher minimum bids for advertisers with low CTRs. These novel designs raise many questions. For example, what is the impact of the weighted ranking rules and differentiated minimum-bid policies on advertisers' equilibrium bidding behavior? How should advertising providers rank advertisers with different CTRs and set minimum bids for them? The goal of this paper is to address these issues.

We address the above issues by studying a model of keyword auctions. In this model, advertisers bid unit prices; the advertising provider not only receives unit-price bids from advertisers but also takes into account the information on the advertisers' click-generating potential. Such information allows the advertising provider to differentiate advertisers with high expected CTRs (h-type) from those with low expected CTRs (l-type). Advertisers, on the other hand, cannot tell another advertiser's CTR-type or valuation-per-click. The advertising provider can assign different weighting factors and impose different minimum bids for advertisers with high and low expected CTRs. Using such a framework, we study how weighting schemes and differentiated minimum-bid policies affect advertisers' equilibrium bidding and how to design such keyword auctions in terms of choosing weighting factors and minimum bids for advertisers with different expected CTRs. Such design issues depart from those in standard auctions where no similar information exists. Our focus on design issues also differentiates our study from studies that focus on equilibrium analysis under given auction rules, such as Edelman et al. (2007) and Varian (2007). More importantly, how to best design weighting schemes and minimum-bid policies is important to the performance of keyword advertising platforms used by search engines.

We study the design of weighting schemes and minimum-bid policies from two perspectives: one that maximizes total expected valuation created (*the efficient* *design*) and one that maximizes the auctioneer's expected revenue (*the revenue-maximizing design*). The efficient design maximizes the "total pie." Such a design is most relevant at the developing stage of the keyword advertising market in which advertising providers are likely to attract advertisers by passing much of the valuation created to them. As the keyword advertising market becomes mature and market shares stabilize, advertising providers will more likely focus on profitability, thereby adopting a revenue-maximizing design.

Our study generates several important insights. We demonstrate that weighting schemes and differentiated minimum bid policies have a significant impact on equilibrium bidding. The weighting scheme determines how advertisers with different expected CTRs match in equilibrium—an advertiser with a low weighting factor compensates by bidding higher (than one with the same valuation-per-click but a higher weighting factor). Minimum bids exclude low-valuation advertisers and, when not equally constraining, can distort the equilibrium matching: some of the less-constrained advertisers will choose not to compete with their more-constrained competitors by bidding low. Despite these nontrivial equilibrium features, the efficient keyword auction design is remarkably simple: it weights advertisers' unit-price bids with their expected CTRs and requires the same minimum weighted score. This implies that one should rank advertisers *as if they bid their true valuation*, and set higher minimum bids for advertisers with lower expected CTRs. The revenue-maximizing design may generate higher revenue than standard fixed-payment auctions, but requires fine balancing between low- and high-CTR advertisers based on their expected CTRs and valuation-per-click distributions. Relative to the efficient weighting scheme, the revenue-maximizing weighting scheme may favor low- or high-CTR advertisers. In choosing revenue-maximizing minimum bids, advertising providers should consider the effect of excluding low-valuation advertisers as well as that of distorted allocations among advertisers with different expected CTRs.

The rest of the paper is organized as follows. In section 2.2 we discuss the related literature. In section 2.3 we lay out our research model. We examine weighting scheme design and differentiated minimum bids design in sections 2.4 and 2.5, respectively. We compare keyword auctions to standard fixed-payment auctions in section 2.6. Section 2.7 discusses some extensions, and section 2.8 concludes the paper.

2.2 Related Literature

How auctioneers should use available information has been an important area of investigation in the auction literature. The early literature focuses on ex-post information. Riley (1988) finds that in common-value auctions, such as drilling-right auctions, auctioneers can increase their revenue by tying winners' payment with the ex-post information on the item's value. McAfee and McMillan (1986) demonstrate that in procurement auctions, making contractors' (bidders') payment partially depend on their ex-post realized costs can reduce the buyer's procurement costs. This paper focuses on how auctioneers can use ex-ante information on bidders' outcome-generating potential.

This research is most related to research on "scoring auctions," or auctions in which bidders are ranked by a score that summarizes multiple underlying attributes. Che (1993) and Asker and Cantillon (2008) study a form of scoring auction used in procurement settings, in which the score is a function of suppliers' non-price attributes (e.g., quality and time-to-completion) minus the price they ask. Ewerhart and Fieseler (2003) study another form of scoring auction, in which a score is a weighted sum of unit-price bids for each input factor (e.g., labor and materials). All three papers show that scoring auctions, though inefficient, can generate higher revenues than efficient mechanisms such as fixed-payment first-price auctions. Keyword auctions in this paper are different from the above scoring auctions in auction rules, equilibrium bidding behavior, and application settings. For example, we study a multiplicative scoring rule that is different from other scoring auctions. The difference in scoring rules also leads to different equilibrium features (e.g., kinks and jumps in our setting). Another important difference is that equilibrium bidding in other scoring auctions is determined by a single parameter, whereas in our paper, equilibrium bidding is determined both by advertisers' valuation-per-click and by their CTR signals. Besides scoring rules, we study differentiated minimum bid policies, which are not discussed in the aforementioned literature.

This paper is closely related to previous studies on ranking rules in keyword auctions. Recall that one approach ranks advertisers only by their unit prices, whereas the other approach ranks advertisers using the product of their unit prices and historical CTRs. Liu and Chen (2006) and Lahaie (2006) study the equilibrium bidding under the two approaches and show that the latter approach is efficient and that the revenues generated by the two approaches are ambiguously ranked. Liu and Chen (2006) study the revenue-maximizing design under a more general class of ranking rules with ranking-by-price and ranking-by-price$\times$CTR as two special cases. They show that neither ranking-by-price nor ranking-by-price$\times$CTR is revenue-maximizing. We extend Liu and Chen (2006) in several ways. First, this paper considers a general multi-slot setting, while Liu and Chen (2006) assume a single slot. Second, this paper allows valuation-per-click to be correlated with CTR signals. Third, for the first time in the literature, this paper studies the use of differentiated minimum bids, together with the weighted ranking rule, as a way of exploiting ex-ante information on advertisers.

Several authors have looked at keyword auctions from different perspectives. Following the "auction of contracts" literature (McAfee and McMillan 1986; Samuelson 1986), a few authors (e.g., Sundararajan 2006) study whether advertisers should pay a fixed payment, a contingent payment, or a combination of the two. Weber and Zheng (2007) study a model of paid referrals in which firms can offer a "bribe" to the search engine in exchange for a higher position. They show that the revenue-maximizing design is a weighted average of the "bribe" and the quality of the product offered by each firm. Feng (2007) studies the optimal allocation of multiple slots when buyers' valuation of slots decreases at different speeds. Edelman et al. (2007) and Varian (2007) examine equilibria of auctions with a "generalized second price (GSP)" payment rule, that is, that winners pay only the lowest price that keeps their positions. They study GSP auctions under a complete-information setting (that is, advertisers know each others' valuation for slots).[1] Edelman et al. (2007) show that GSP auctions under a complete-information setting do not have a dominant-strategy equilibrium, and advertisers will not bid their true valuation. Both Edelman et al. (2007) and Varian (2007) show that GSP auctions admit a range of stable equilibria, and the auctioneer's equilibrium revenue under the GSP rule is at least as high as that under the Vickrey-Clarke-Groves mechanism. While their characterization of the equilibria under the GSP rule applies to both the rank-by-price case and the rank-by-price$\times$CTR case, they do not study what ranking rules advertising providers should choose, nor do they study optimal minimum-bid policies. This paper complements theirs by examining how ranking rules and minimum-bid policies affect equilibrium

[1] Edelman et al. (2007) also study a related "generalized English auction" where advertisers do not have complete information on others' valuation but can observe their previous bids.

bidding and how advertising providers should design such auction dimensions. Also, different from Edelman el al. (2007) and Varian (2007), we model keyword auctions as an incomplete-information game (i.e., advertisers only know a distribution of other advertisers' valuation and click-generating abilities). The real-world keyword auctions may lie between complete-information and incomplete-information. For example, Google does not publish advertisers' bids while Yahoo! does with measures that prevent large-scale automatic harvesting of such information. In either case, advertisers may not know at every minute how much other advertisers value the slots.

2.3 Model Setup

We consider a problem of assigning m advertising slots associated with a keyword to n ($n \geq m$) risk-neutral advertisers. Each advertiser has one advertisement for the keyword and can use only one slot. The number of clicks an advertisement can attract depends both on the quality of the advertisement and on the prominence of the slot the advertisement is assigned to. The quality of an advertisement is considered an attribute of the advertisement and may be determined by the relevance of the advertisement to the keyword, the attractiveness of the advertised product or service, and how well the advertisement is written. For example, for the keyword "refinance," an advertisement from a more reputable lender may generate more clicks than one from a less reputable lender. The prominence of a slot is considered a slot-specific factor and may be determined by the position, size, shape, or media format (text, image, or video) of the slot. For example, an advertisement may attract more clicks when placed on the top of a page than when placed at the bottom of the page. In this light, we assume the number of clicks generated by an advertiser at slot j is $\delta_j q$. δ_j is a deterministic factor that we use to capture the prominence of slot j. We assume

$\delta_1 \geq \delta_2 ... \geq \delta_m > 0$ and normalize $\delta_1 = 1$. q is a stochastic number that we use to capture the quality of the advertisement. We interpret q as the advertiser's CTR in the sense that the higher the quality of the advertisement, the more likely a Web user will click on it. It is important to note that, in general, CTRs are subject to both the advertisement effect and the slot effect. In this paper, an advertiser's CTR refers *exclusively* to the attractiveness of an advertisement, regardless of any slot effect.

Though an advertiser's CTR is realized only after the auction, the advertiser and the auctioneer may have ex-ante information about the advertiser's future CTRs. This is because e-commerce technologies make it easy for advertising providers to track advertisers' past CTRs and to make predictions about their future CTRs. We assume that the auctioneer can observe a signal about each advertiser's future CTR; the same signal is observed by the advertiser but not by other advertisers. For simplicity, we assume that such signal allows the auctioneer to distinguish between two types of advertisers, those with high expected CTRs (h-type) and those with low expected CTRs (l-type). We will extend our model to a multiple CTR-type case in section 2.7. Denote Q_h and Q_l ($Q_h > Q_l > 0$) as the expected CTRs for l-type and h-type advertisers, respectively. We assume the probabilities for advertisers being h-type and l-type are α and $1 - \alpha$, respectively. These probabilities are common knowledge.

Each advertiser has a valuation v for each click, termed the advertiser's *valuation-per-click*. Advertisers may differ in valuation-per-click. For example, for the keyword "refinance," bankone.com may have a higher valuation-per-click than aggregate lender lendingtree.com. The distribution of the valuation-per-click may be correlated with the advertiser's CTR signal such that l- and h-type advertisers may have different valuation-per-click distributions. For example, aggregate lenders (e.g., lendingtree.com) may have higher CTRs but lower valuation-per-click than banks (e.g., bankone.com) for the keyword "refinance." Let $F_l(v)$ and $F_h(v)$ denote the cumulative distribution of valuation-per-click for l- and h-type advertisers, respectively. The realization of an advertiser's valuation-per-click is not known by the auctioneer or other advertisers. But the distributions $F_h(v)$ and $F_l(v)$ are common knowledge.

We assume $F_l(v)$ and $F_h(v)$ have a fixed support $[0, 1]$, and the density functions, $f_l(v)$ and $f_h(v)$, are positive and differentiable everywhere within the support. This assumption can be generalized to $[\underline{v}_l, \bar{v}_l]$ and $[\underline{v}_h, \bar{v}_h]$ for l- and h-type advertisers, respectively. We also assume that one advertiser's valuation-per-click and expected-CTR type are independent of another advertiser's.[2]

We assume advertisers' payoff functions are additive in their total valuation and the payment. In particular, conditional on winning slot j, an advertiser's payoff is

$$vq\delta_j - \text{payment}. \tag{2.1}$$

Advertising slots are sold through a *weighted unit-price auction*, which we describe below. Each advertiser is asked to submit a b that is the advertiser's willingness-to-pay per click, or *unit price*. The auctioneer assigns each advertiser a score based on the advertiser's unit-price bid and CTR signal. The score for an advertiser is

$$s = \begin{cases} b, \text{ if the advertiser is } h\text{-type} \\ wb, \text{ if the advertiser is } l\text{-type} \end{cases}, \tag{2.2}$$

where w is the weighting factor for l-type advertisers, and the weighting factor for h-type advertisers is normalized to 1. The auctioneer allocates the first slot to the

[2]The independent-private-value assumption applies to auctions in which the bidders are buying for their own use and not for resale (McAfee and McMillan 1987). We consider keyword auctions as independent-private-value auctions because advertisers or their advertising agencies bid on slots to display their own advertisements, and slots, once sold, are assigned to specific advertisements and cannot be resold to other advertisers.

advertiser with the highest score, the second slot to the advertiser with the second highest score, and so on. Winners pay for their realized clicks at unit prices they bid.[3] We call such an auction format a *weighted unit-price auction*.

By allowing w to take different values, we can accommodate the following stylized auction formats. When w equals one, the winners are determined solely by the prices they bid. One example is Overture's auction format. When w is less than one, bid prices from l-type advertisers are weighted less than those from h-type advertisers. Google's auction fits in this category because under Google's ranking policy, bids from advertisers with high click-generating potential are weighted more.

We also allow the auctioneer to set different minimum bids (or reserve prices) for advertisers with different CTR signals. In particular, we let $\underline{b}_l$ and $\underline{b}_h$ be the minimum bids for l- and h-type advertisers, respectively.

The auction proceeds as follows. First, the auctioneer announces weighting factors and minimum bids. All advertisers receive signals about their future CTRs and learn their valuation-per-click before the auction. Then, each advertiser submits a unit-price bid, and the auctioneer assigns advertisers to slots based on their unit-price bids and CTR signals according to the announced weighting scheme. Finally, the number of clicks is realized, and advertisers pay the realized clicks at the unit prices they bid.

[3]An alternative payment rule is a "second-score" rule; that is, advertisers will pay a price that matches the next highest score rather than their own scores (all scores are calculated using expected CTRs). We show in the appendix that under our framework, the "second-score" weighted unit-price auctions generate the same expected revenue for the auctioneer as the "first-score" version studied here. The main results in this paper apply also to the "second-score" setting, as these results concern only the expected revenue. We choose to work with the "first-score" format as it permits explicit bidding functions.

2.4 Designing Weighting Scheme

We start by assuming no minimum bids so that we can focus on the design of the weighting scheme. We will first consider how weighting factors affect advertisers' equilibrium bidding. Then, we will examine the efficient and revenue-maximizing weighting schemes.

2.4.1 Weighting Scheme and Equilibrium Bidding

Throughout this paper, we consider a symmetric, pure-strategy Bayesian-Nash equilibrium. By "symmetric," we mean that advertisers with the same valuation-per-click and CTR signal will bid the same.

Let $b_h(v)$ denote the equilibrium bidding function for h-type advertisers, and $b_l(v)$ for l-type advertisers. A bidding function in our setting is a function that associates advertisers' unit-price bids with their valuation-per-click. Because advertisers differ both in valuation-per-click and in expected CTRs, we need a pair of bidding functions to describe our equilibrium. The condition for the pair to be an equilibrium is that an advertiser finds it is optimal to bid according to this pair if all other advertisers bid according to this pair. We conjecture that both bidding functions are strictly increasing (we verify this in the appendix). The following result is key to our analysis.

Lemma 2.4.1. *An l-type advertiser with valuation-per-click v matches an h-type advertiser with valuation-per-click wv in equilibrium. Formally,*

$$b_h(wv) = wb_l(v),\ \forall v, wv \in [0,1]. \tag{2.3}$$

Proof. Unless otherwise noted, all proofs are in the appendix. □

The intuition for Lemma 2.4.1 is as follows. Consider an h-type advertiser with valuation-per-click wv and an l-type advertiser with valuation-per-click v. If the former bids wb and the latter bids b, the two advertisers tie, and therefore their chances of winning each slot are the same. Meanwhile, conditional on winning the same slot, the l-type advertiser's payoff $(Q_l\delta_j(wv - wb))$ differs from the h-type advertiser's $(Q_h\delta_j(v - b))$ only by a scalar. So their total expected payoffs differ only by a scalar, too. Because multiplying the objective of an optimization problem by a scalar does not change the solution to the problem, we conclude that if bidding b is optimal for the l-type advertiser then bidding wb must also be optimal for the h-type advertiser, and vice versa.

We call two advertisers *comparable* if they tie or match (in scores) in equilibrium *without minimum bids.* Lemma 1 greatly simplifies the derivation of advertisers' equilibrium winning probabilities. Let us first consider an l-type advertiser's winning probability against any advertiser, or the advertiser's *one-on-one winning probability*, denoted as $G_l(v)$. Lemma 1 suggests that an l-type advertiser with valuation-per-click v can beat another advertiser, say B, if and only if B is l-type and has valuation-per-click less than v, or B is h-type and has valuation-per-click less than wv. Hence,

$$G_l(v) = \alpha F_h(wv) + (1-\alpha)F_l(v). \tag{2.4}$$

Similarly, an h-type advertiser's winning probability against any advertiser, $G_h(v)$, is

$$G_h(v) = \alpha F_h(v) + (1-\alpha)F_l(v/w). \tag{2.5}$$

We denote $P_l^j(v)$ and $P_h^j(v)$ as l- and h-type advertisers' probabilities of winning the jth slot, respectively. We can write $P_l^j(v)$ and $P_h^j(v)$ as

$$P_\theta^j(v) = \binom{n-1}{n-j} G_\theta(v)^{n-j}\left[1 - G_\theta(v)\right]^{j-1}, \theta \in \{l, h\}. \tag{2.6}$$

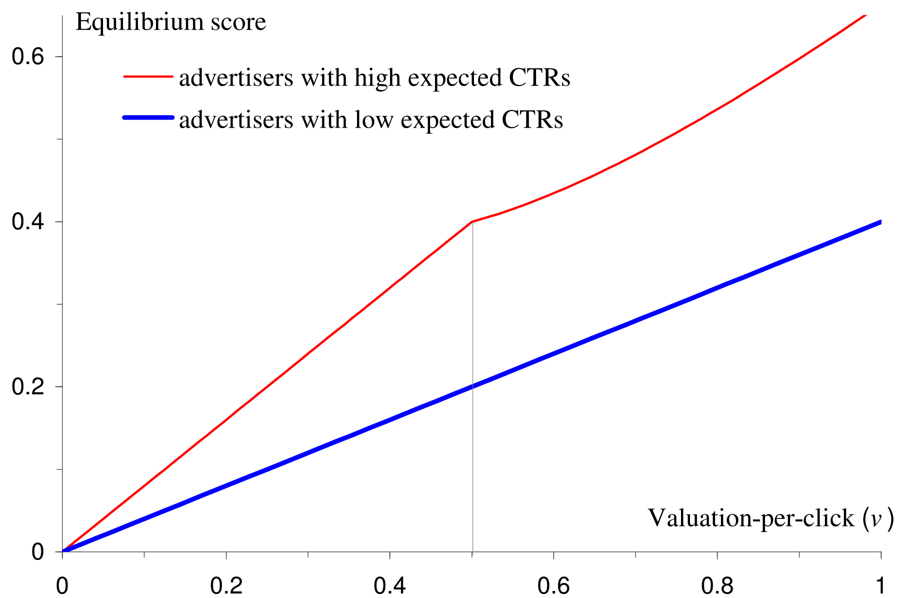

Figure 2.1: Equilibrium Bidding Functions

Proposition 2.4.2. *Given w $(w > 0)$, equilibrium bidding functions are given by*

$$\begin{cases} b_l(v) = v - \frac{\int_0^v \sum_{j=1}^m \delta_j P_l^j(t)dt}{\sum_{j=1}^m \delta_j P_l^j(v)}, \forall v \in [0,1] \\ b_h(v) = v - \frac{\int_0^v \sum_{j=1}^m \delta_j P_h^j(t)dt}{\sum_{j=1}^m \delta_j P_h^j(v)}, \forall v \in [0,1] \end{cases}. \tag{2.7}$$

Proposition 2.4.2 characterizes the equilibrium for a weighted unit-price auction. In Figure 2.1, we plot advertisers' equilibrium scores when l-type's weighting factor is 0.5 and the valuation distributions are uniform. Recall that score is bid times weighting factor. We plot scores instead of unit-price bids because the former better illustrates the equilibrium matching between l- and h-type advertisers. Clearly, an l-type with valuation-per-click 1 ties with an h-type with valuation-per-click 0.5, and h-type advertisers with higher valuation have no comparable l-type advertisers.

Interestingly, the figure shows a kink in h-type advertisers' equilibrium bidding function. Intuitively, this is because h-type advertisers with valuation-per-click less than 0.5 compete with both l- and h-type advertisers, whereas h-type advertisers with valuation-per-click higher than 0.5 complete only with h-type advertisers. The sudden change in the number of competitors causes h-type advertisers with valuation-per-click higher than 0.5 to bid considerably less aggressively than h-type advertisers with valuation-per-click lower than 0.5, thus the kink. Generally speaking, when the weighting factor w for l-type advertisers is less than one, the h-type advertisers' equilibrium bidding function has a kink at w. When w is greater than one, the l-type advertisers' equilibrium bidding function has a kink at $1/w$.[4] These kinks reflect the impact of weighting scheme on the equilibrium matching between l-type and h-type advertisers.

Proposition 2.4.2 has the following implications. Advertisers who receive a high weighting factor are favored in equilibrium allocation, and can win more often with the same unit price. Some advertisers who receive a high weighting factor may out-compete all advertisers who receive a low weighting factor, and thus can benefit from such a situation by bidding less aggressively. Increasing l-type's weighting factor causes the following effects. It increases l-type advertisers' one-on-one winning probability and decreases h-type advertisers' one-on-one winning probability (see (2.4) and (2.5)). Consequently, l-type advertisers are selected more often into high-ranked slots and have a larger total expected winning (defined as the expected value of the slot an advertiser may win, i.e., $\sum_{j=1}^{m} \delta_j P_\theta^j(v)$). Meanwhile, it causes more h-type advertisers to bid aggressively because there are more h-type advertisers with valuation-per-click below w who face competition from both CTR-types.

[4]If we allow more general supports, such as $[\underline{v}_l, \bar{v}_l]$ and $[\underline{v}_h, \bar{v}_h]$, there may be as many as two kinks in the bidding functions. For example, with general supports $[1, z]$ $(2 < z < 4)$ for l-type and $[1, 2]$ for h-type and $w = 0.5$, l-type advertisers' equilibrium bidding function has a kink at 2, and h-type advertisers' equilibrium bidding function has a kink at $z/2$. In some special cases, such as with supports $[2, 4]$ for l-type and $[1, 2]$ for h-type and $w = 0.5$, there is no kink in either type's equilibrium bidding function.

2.4.2 Efficient Weighting Scheme

We measure the efficiency by the total value created. The efficiency criterion, therefore, emphasizes the "total pie," which is important if the auctioneer's objective is to transfer much of the value to advertisers in return for their participation. This is especially true when the keyword advertising market is still nascent, and online advertising providers are still trying to steal market share from the traditional advertising providers. The efficiency criterion may become the criterion of choice for advertising providers who aim at long-term development rather than short-term profits, regardless of their market positions.

We define the *efficient weighting factor*, w_{eff}, as one that maximizes the total expected valuation. We focus on expected valuation (thus ex-ante efficiency) because advertisers' valuation for slots is also determined by the realized CTRs after the auction. The assignment of an advertiser with valuation-per-click v and CTR-signal θ to slot j will generate an expected valuation of $v\delta_j Q_\theta$, $\theta \in \{l, h\}$. Given that the probability of assigning an advertiser to slot j is $P_\theta^j(v)$, the total expected valuation generated by all advertisers is,

$$\begin{aligned} W &= n(1-\alpha) Q_l \int_0^1 v \sum_{j=1}^m \delta_j P_l^j(v) f_l(v)\,dv \\ &\quad + n\alpha Q_h \int_0^1 v \sum_{j=1}^m \delta_j P_h^j(v) f_h(v)\,dv. \end{aligned} \tag{2.8}$$

Proposition 2.4.3. *The efficient weighting factor (for l-type advertisers) is Q_l/Q_h.*

Proposition 2.4.3 suggests that it is efficient to weight advertisers' bids by their expected CTRs (note that the weighting factor pair $(Q_l/Q_h, 1)$ is equivalent to the pair (Q_l, Q_h)). Such a weighting scheme is also efficient if advertisers were to bid their true valuation. In other words, the auctioneer can achieve efficiency by weighting unit-price bids by expected CTRs *as if advertisers are bidding their true valuation*, despite that in our model advertisers generally do not bid their true valuation-per-click. The reason for this lies in the way l- and h-type advertisers are matched in equilibrium. According to Lemma 2.4.1, an l-type advertiser with valuation-per-click v is comparable with an h-type advertiser with valuation-per-click wv. The efficiency criterion requires comparable advertisers to generate the same expected valuation. Hence, the efficient weighting factor must be Q_l/Q_h.

It is worth noting that the efficient weighting factor is independent of the distribution of valuation-per-click and that of CTR types. This feature makes it straightforward to implement an efficient weighting scheme: the auctioneer only needs to estimate the expected CTR for each advertiser-keyword combination and use it to weight the advertiser's unit-price bid. Given that keyword auctions have already been set up to accumulate CTR information for all advertisers and all keywords, it is possible to estimate an advertiser's CTR on a particular keyword and that estimation can be perfected over time.

2.4.3 Revenue-Maximizing Weighting Scheme

Another useful design criterion is revenue-maximization. As the industry matures and the competition for market shares settles, an efficient design toward future growth becomes less appealing, and the auctioneer's objective is likely to transform from maximizing the "total pie" to maximizing the total revenue from existing advertisers. Next, we examine how an auctioneer should choose the weighting factor to maximize the expected revenue.

We define the revenue-maximizing weighting factor, w^*, as one that maximizes the total expected revenue of the auctioneer. We can explicitly derive the auctioneer's expected revenue (π) as (see the appendix for details)

$$\begin{aligned}\pi &= n(1-\alpha)Q_l \int_0^1 \sum_{j=1}^{m} \delta_j P_l^j(v) \left(v - \frac{1-F_l(v)}{f_l(v)}\right) f_l(v)dv \\ &\quad + n\alpha Q_h \int_0^1 \sum_{j=1}^{m} \delta_j P_h^j(v) \left(v - \frac{1-F_h(v)}{f_h(v)}\right) f_h(v)dv. \end{aligned} \tag{2.9}$$

In the above, the total expected revenue consists of the expected revenue from l-type advertisers (the first term) and the expected revenue from h-type advertisers (the second term). Recall that $P_l^j(v)$ is an l-type advertiser's probability of winning the jth slot, and $P_h^j(v)$ is an h-type advertiser's probability. We interpret the terms

$$Q_l \left[v - \frac{1-F_l(v)}{f_l(v)}\right] \text{ and } Q_h \left[v - \frac{1-F_h(v)}{f_h(v)}\right] \tag{2.10}$$

as l-type's and h-type's "revenue contribution" to the auctioneer if they are assigned to a standard slot ($\delta = 1$), respectively. Revenue contribution refers to the revenue captured by the auctioneer, which is usually less than the total valuation created. The difference between advertisers' revenue contribution and their valuation for slots is considered to be the informational rent kept by the advertisers. According to this interpretation, the total expected revenue can be viewed as the total expected revenue contribution of the winners at all slots. The concept of revenue contribution is closely related to the concept of "virtual valuation" introduced by Myerson (1981) in the optimal auction setting. One difference is that we consider revenue contribution across multiple slots, whereas Myerson (1981) studies a single object.

The revenue-maximizing weighting factor can be characterized by the first-order condition of the total expected revenue with respect to the weighting factor. Except in some special cases, the revenue-maximizing weighting factor cannot be expressed in an explicit form. Next, we focus on two issues regarding the revenue-maximizing weighting factor. First, how is it different from the efficient weighting factor? Second, how is it affected by the underlying model primitives, especially valuation-per-click distributions? We first consider a setting in which the valuation-per-click is independent of the CTR signal, so that the valuation-per-click distributions for l- and h-type advertisers are the same (commonly denoted as $F(v)$).

We say a distribution function $F(v)$ is an increasing-hazard-rate (IHR) distribution if its hazard rate $\frac{f(v)}{1-F(v)}$ increases in v throughout the support. Many distributions, including uniform, normal, and exponential, are IHR.

Proposition 2.4.4. *If the valuation-per-click and CTR signals are independent, and $F(v)$ is IHR, then the revenue-maximizing weighting factor w^* must be higher than the efficient weighting factor w_{eff}.*

Proposition 2.4.4 implies that when the distributions of valuation-per-click are the same across l- and h-type advertisers, the revenue-maximizing weighting factor is generally inefficient and discriminates against h-type advertisers relative to the efficient design. The intuition is as follows. For any weighting factor less than w_{eff}, if the valuation-per-click distribution is IHR,

$$Q_l\left(v - \frac{1-F(v)}{f(v)}\right) > Q_h\left(wv - \frac{1-F(wv)}{f(wv)}\right), \text{ for all } v. \tag{2.11}$$

In other words, for any weighting factor less than w_{eff}, the revenue contribution of an l-type advertiser is always higher than that of a comparable h-type advertiser. Thus, the auctioneer can always earn a higher revenue by raising w to allocate the slots more often to l-type advertisers.

When l- and h-type advertisers have different valuation-per-click distributions, however, the revenue-maximizing weighting factor may or may not be higher than the efficient weighting factor, as illustrated by the following example.

Example 2.4.5. *Assume there is only one slot and the valuation-per-click of l- and h-type advertisers are uniformly distributed on $[0, z]$ and $[0, 1]$, respectively. Let $\alpha = 0.5$, $Q_l = 0.5$, $Q_h = 1$, and $n = 5$. We can explicitly solve the revenue-maximizing weighting factor as $w^* = \frac{1}{0.6z+0.8}$, which is lower than $w_{\text{eff}} = 0.5$ when $z > 2$ and higher than w_{eff} when $z < 2$.*

In the above example, when z increases, the valuation-per-click distribution of the l-type advertisers becomes less "tight." As a result, they can claim more informational rent and contribute less to the total revenue. So the auctioneer should allocate the slots less often to them by lowering the weighting factor for l-type advertisers. When l-type advertisers' valuation distribution is loose enough, the revenue-maximizing weighting factor can be less than the efficient weighting factor.

The above example highlights that it is not always best to discriminate against advertisers with high expected CTRs. This is fundamentally because advertisers' revenue contribution is determined both by expected CTRs and valuation distributions. h-type advertisers do not necessarily contribute less to the total revenue than l-type advertisers who have the same total valuation for slots, especially when the former have "tighter" valuation distributions.

2.5 Designing Differentiated Minimum Bids

The optimal auction literature suggests that an optimal design often involves imposing minimum bids to exclude advertisers whose participation reduces the auctioneer's revenue. In our setting, the auctioneer can impose differentiated minimum bids for l- and h-type advertisers because of the information on advertisers' future CTRs.

We say a minimum bid for h-type advertisers is *more constraining* than that for l-type advertisers if the comparable h-type advertiser for the lowest participating l-type advertiser is excluded by the minimum bids. We similarly define the case of a more constraining minimum bid for l-type advertisers. A pair of minimum bids is *equally constraining* if neither bid is more constraining.

Next we will focus on the scenario in which the weighting factor for l-type advertisers is no higher than that for h-type advertisers (assumption A1 below) and the minimum bid for h-type advertisers is equally or more constraining (assumption A2). Analyses of other scenarios—where the weighting factor for l-type is higher and/or the minimum bid for l-type is more constraining—are analogous. We also assume that the minimum bid for h-type advertisers is low enough such that at least some l-type advertisers have comparable participating h-type advertisers (assumption A3). This assumption excludes a trivial case in which l-type advertisers and h-type advertisers each compete with advertisers of their own type. Formally, these assumptions are:

A1) $w \leq 1$. A2) $w\underline{b}_l \leq \underline{b}_h$. A3) $\underline{b}_h < w$.

As in section 2.4, we first examine the impact of differentiated minimum bids on equilibrium bidding and then study the efficient and revenue-maximizing minimum bid design.

2.5.1 Minimum Bids and Equilibrium Bidding

We conjecture that a pure-strategy equilibrium exists. l-type advertisers' equilibrium bidding function must satisfy two criteria: (a) the lowest participating l-type advertiser must have a valuation-per-click of $\underline{b}_l$ and bid his or her true valuation-per-click, (b) the equilibrium bidding function must be strictly increasing. The criterion

(a) is simply the consequence of minimum bids, and the criterion (b) is required by the incentive compatibility condition (see the appendix for a proof). The criteria for h-type advertisers are symmetric.

Since the minimum bid for h-type advertisers is more constraining, some low-valuation l-type advertisers cannot match any participating h-type advertiser in the equilibrium score. But l-type advertisers with high enough valuation-per-click can. We call the lowest valuation-per-click for l-type advertisers to match a participating h-type advertiser the *matching point* for l-type advertisers, denoted as v_0.

If the matching point equals one, no l-type advertiser can match an h-type advertiser in equilibrium. We will focus on the more interesting case of the matching point less than one and assume the condition for that is satisfied (see Proposition 2.5.2 for such a condition).

Will l-type advertisers with valuation-per-click above the matching point match with their comparable h-type advertisers as in the case of no minimum bids? The following lemma shows that they do.

Lemma 2.5.1. *Under assumptions A1-A3, an l-type advertiser with valuation-per-click v above the matching point match an h-type advertiser with valuation-per-click wv in equilibrium. Formally,*

$$b_h(wv) = wb_l(v), \forall v > v_0. \tag{2.12}$$

The question remains where the matching point is. One may conjecture that the matching point will be the valuation-per-click of the l-type advertiser who is comparable with the lowest participating h-type advertiser. However, we show that this may be not the case.

Remark 2.5.1 (Postponed Matching). If the minimum bid for h-type advertisers is more constraining, at least some low-valuation l-type advertisers will bid lower scores than their comparable h-type advertisers.

Suppose the opposite, that is, every l-type advertiser will match the comparable h-type advertiser in equilibrium whenever the latter is not excluded by minimum bids. The first l-type advertiser to have a comparable h-type advertiser is $\underline{b}_h/w$. By definition, $\underline{b}_h/w$ is also the matching point. Since the h-type advertiser with valuation-per-click $\underline{b}_h$ must bid the true valuation (by criterion (a)) and earn zero payoff, the l-type advertiser must also bid his/her true value (by Lemma 2.5.1) and earn zero payoff. However, this cannot be an equilibrium because the l-type advertiser can always earn a positive payoff by bidding less. This contradiction leads us to conclude that the matching point must be higher than $\underline{b}_h/w$. In other words, l-type advertisers avoid matching their comparable h-type ones in equilibrium until their valuation-per-click is high enough.

Given that the minimum bids are not equally constraining, the two bidding functions cannot both be continuous. If both bidding functions were continuous, by the definition of the matching point, the l-type advertiser with valuation-per-click v_0 must match the h-type advertiser with valuation-per-click $\underline{b}_h$ in equilibrium scores and both must earn zero payoff. Our previous argument shows that this cannot be an equilibrium. The following proposition establishes the equilibrium bidding with minimum bids.

Proposition 2.5.2. *Under assumptions A1-A3, the equilibrium bidding functions are given by*

$$b_\theta(v) = v - \frac{\int_{\underline{b}_\theta}^{v} \sum_{j=1}^{m} \delta_j P_\theta^j\left(t\right) dt}{\sum_{j=1}^{m} \delta_j P_\theta^j\left(v\right)}, \forall v \in \left[\underline{b}_\theta, 1\right], \theta \in \{l, h\}, \tag{2.13}$$

where $P_l^j(v)$ *and* $P_h^j(v)$ *are defined in (2.6) and the one-on-one winning probabilities for l- and h-type advertisers are now*

$$G_l(v) = \begin{cases} \alpha F_h(\underline{b}_h) + (1-\alpha) F_l(v) & \text{for } v \in [\underline{b}_l, v_0) \\ \alpha F_h(wv) + (1-\alpha) F_l(v) & \text{for } v \in [v_0, 1] \end{cases} \tag{2.14}$$

$$G_h(v) = \begin{cases} \alpha F_h(v) + (1-\alpha) F_l(v_0) & \text{for } v \in [\underline{b}_h, wv_0) \\ \alpha F_h(v) + (1-\alpha) F_l\left(\frac{v}{w}\right) & \text{for } v \in [wv_0, 1] \end{cases} \tag{2.15}$$

The matching point v_0 *is determined by*

$$w \int_{\underline{b}_l}^{v_0} \sum_{j=1}^{m} \delta_j P_l^j(t)\, dt = \int_{\underline{b}_h}^{wv_0} \sum_{j=1}^{m} \delta_j P_h^j(t)\, dt. \tag{2.16}$$

when $w \int_{\underline{b}_l}^{1} \sum_{j=1}^{m} \delta_j P_l^j(t)\, dt < \int_{\underline{b}_h}^{w} \sum_{j=1}^{m} \delta_j P_h^j(t)\, dt$*, and is 1 otherwise.*

Proposition 2.5.2 characterizes the equilibrium under a weighted unit-price auction with differentiated minimum bids. In Figure 2.2, we show an example of h-type advertisers facing a more constraining minimum bid. In this example, we let $m = 1$, $n = 5$, $\alpha = 0.5$, $F_l(v) = F_h(v) = v$, $w = 0.5$, $\underline{b}_l = 0$, and $\underline{b}_h = 0.1$. Figure 2.2 shows the equilibrium scores for l- and h-type advertisers.

From the figure, l-type advertisers with valuation-per-click lower than the matching point (0.26) bid lower scores than any h-type advertisers. l-type advertisers with valuation-per-click above the matching point match with their comparable h-type advertisers (with valuation-per-click between 0.13 and 0.5). h-type advertisers with valuation-per-click higher than 0.5 beat any l-type advertisers. h-type advertisers with valuation-per-click lower than 0.13 bid lower scores than l-type advertisers below the matching point but bid higher than l-type advertisers above the matching point. As before, the kinks are explained by an abrupt increase/decrease in the number of competing advertisers: the first kink in h-type advertisers' equilibrium bidding function is because l-type advertisers start matching h-type advertisers; the second is because l-type advertisers can no longer match h-type advertisers.

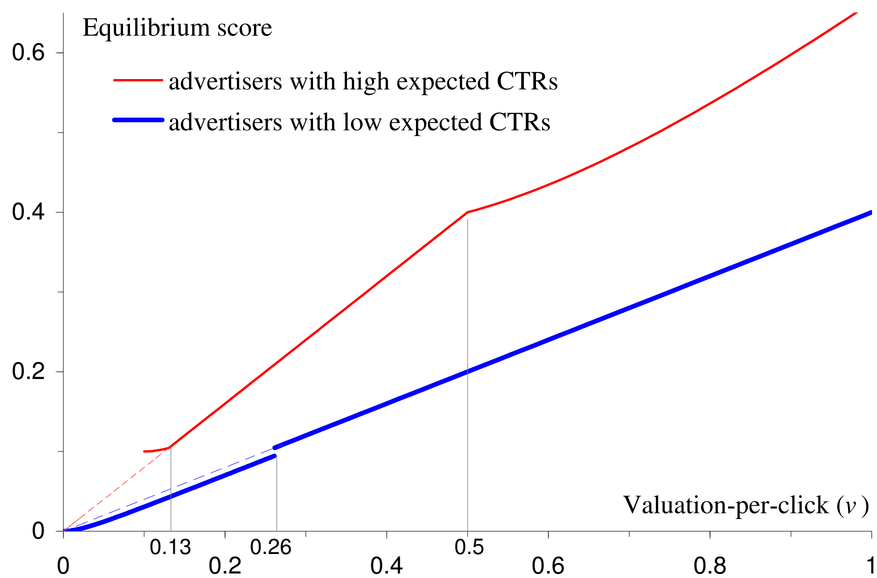

Figure 2.2: Equilibrium Bidding Functions with Minimum Bids

The example confirms the "postponed matching" effect outlined in Remark 2.5.1. In this example, l-type advertisers with valuation-per-click between 0.2 and 0.26 are comparable with h-type advertisers with valuation-per-click between 0.1 and 0.13, but choose not to match the latter. Intuitively, the minimum bid forces h-type advertisers with low valuation-per-click to bid close to their true valuation. Their comparable l-type advertisers, who do not face such a constraint, have the option of bidding significantly lower than their true valuation, which leads to low winning probabilities but high per-click payoffs. Bidding low (and not matching their comparable h-type advertisers) is a dominant strategy for l-type advertisers until their valuation-per-click reaches the matching point.

The jump in l-type advertisers' bidding function at the matching point confirms our earlier argument about discontinuity.[5] At the matching point, l-type ad-

[5]Strictly speaking, at the matching point, the l-type advertiser is indifferent between bidding low and bidding high, and hence could use a mixed strategy. To preserve a pure-strategy equilibrium, in deriving Proposition 2.5.2, we assume that the h-type advertiser always bids high. This assumption

vertisers' bidding strategy changes from not matching h-type advertisers to matching them. The fact the two strategies require quite different unit-price bids explains the jump in the equilibrium bids.

Proposition 2.5.2 has several implications. First, minimum bids exclude low-valuation advertisers and force the participating ones, especially those whose valuation is close to the minimum bids, to bid aggressively. In the above example, h-type advertisers between 0.1 and 0.13 bid higher than they would in the absence of minimum bids (dashed lines indicate their equilibrium scores without minimum bids). Second, if the minimum scores for two CTR-types are the same (in other words, the minimum bid for h-type advertisers is w-times of that for l-type advertisers), two advertisers who would tie without minimum bids remain tying. This is also the reason we call such minimum bids equally constraining. Third, when minimum bids are not equally constraining, advertisers who face a less constraining minimum bid may be better off by choosing not to match their comparable advertisers who face a more-constraining minimum bid, a strategy leading to lower winning odds but a higher per-click payoff. However, advertisers whose valuation is far above minimum bids will choose to match their comparable advertisers, even if the minimum bids are not equally constraining. This later finding is consistent with Google's claim that their differentiated minimum-bids policy only affects a small percentage of advertisers.[6]

As we have mentioned earlier, the intuition in Proposition 2.5.2 carries over to other scenarios (h-type advertisers receive a lower weighting factor, and/or the

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does not affect the equilibrium outcome because the probability measure for an advertiser to be an l-type advertiser with valuation-per-click v_0 is virtually zero.

[6]Google stated in its official blog Inside Adwords (http://adwords.blogspot.com/2006/11/landing-page-quality-update.html) that the introduction of a differentiated minimum-bids policy "will affect a very small portion of advertisers ... However, those who may be providing a low quality user experience will see an increase in their minimum bids."

minimum bid for l-type advertisers is more constraining). For example, when the minimum bid for l-type advertisers is more constraining, h-type advertisers will postpone matching their comparable l-type advertisers, and the jump will occur in h-type's equilibrium bidding function.

2.5.2 Efficient Minimum Bids

We now consider the impact of minimum bids on allocation efficiency. We call a pair of equally constraining minimum bids a *uniform minimum score* policy because they result in identical minimum scores for l- and h-type advertisers. We say a keyword auction is *weakly efficient* if it allocates assets in a way that maximizes the total expected valuation of all participating advertisers. The notion of weak efficiency we use is similar to one discussed by Mark Armstrong (2000). Weak efficiency is different from "strong" efficiency in that weak efficiency concerns the total valuation of participating bidders, whereas strong efficiency concerns the total valuation of all bidders and the auctioneer. Weak efficiency is a necessary condition for strong efficiency.

If the auctioneer uses a uniform minimum score policy, all participating l-type advertisers match their comparable h-type advertisers in equilibrium. Hence, if the weighting factor is Q_l/Q_h, the auction is still efficient by the same argument in Proposition 2.4.3. In fact, such designs are also necessary for the auction to be weakly efficient.

Proposition 2.5.3. *A weighted unit-price auction is weakly efficient if and only if the auctioneer uses the efficient weighting factor and a uniform minimum score.*

Proposition 2.5.3 provides a theoretical justification for using differentiated minimum bids. A uniform minimum-score rule implies that auctioneers should set high minimum unit prices for advertisers with low expected CTRs. This is consistent in principle with Google's recently adopted differentiated minimum-bid practices.

Once again, a uniform minimum-score policy is easy to implement because it does not require knowing the distribution of advertisers' valuation-per-click. Proposition 2.5.3 shows that weighting advertisers' unit-price bids by their expected CTRs, together with a simple uniform minimum-score rule, allows the auctioneer to achieve efficiency among participating advertisers.

2.5.3 Revenue-Maximizing Minimum Bids

Similar to the derivation of (2.9), we can explicitly evaluate the expected revenue of the auctioneer with minimum bids

$$\begin{aligned}\pi &= n(1-\alpha)Q_l \int_{\underline{b}_l}^{1} \sum_{j=1}^{m} \delta_j P_l^j(v) \left(v - \frac{1-F_l(v)}{f_l(v)}\right) f_l(v)dv \\ &\quad + n\alpha Q_h \int_{\underline{b}_h}^{1} \sum_{j=1}^{m} \delta_j P_h^j(v) \left(v - \frac{1-F_h(v)}{f_h(v)}\right) f_h(v)dv. \qquad (2.17)\end{aligned}$$

A pair of minimum bids is revenue-maximizing if this pair is chosen to maximize (2.17). In the appendix we characterize the revenue-maximizing minimum bid policy using a set of first-order conditions. The revenue-maximizing minimum bid policy can be computed numerically. In general, when choosing the revenue-maximizing minimum bids, the auctioneer needs to consider both the *exclusion effect* and the *distortion effect*. The exclusion effect is well-known in the auction design literature. A minimum bid excludes advertisers whose valuation-per-click is lower than the minimum bid, and forces the remaining advertisers to bid higher than they would in the absence of such a minimum bid. The distortion effect is new, however. We have shown earlier that when the minimum bid for h-type advertisers is more constraining, some l-type advertisers will bid lower scores than their comparable h-type advertisers.

The condition for revenue-maximizing minimum bids in our setting is generally different from the "exclusion principle" in standard auctions. The exclusion principle requires that the revenue-maximizing minimum bid should be chosen to admit only the advertisers with positive revenue contribution. In our setting, this would require the revenue-maximizing minimum bids to satisfy, respectively,

$$\underline{b}_l - \frac{1 - F_l(\underline{b}_l)}{f_l(\underline{b}_l)} = 0 \text{ and } \underline{b}_h - \frac{1 - F_h(\underline{b}_h)}{f_h(\underline{b}_h)} = 0. \tag{2.18}$$

The conditions in (2.18) are *not* revenue-maximizing in our setting, however. They ignore the fact that in our setting, minimum bids also cause a distortion effect that has revenue consequences.

The revenue-maximizing minimum bids generally do not have a uniform score either. Intuitively, when we restrict to a uniform minimum-score policy, the distortion effect does not exist. As a result, the revenue-maximizing minimum bid policy should simply exclude advertisers with a negative revenue contribution, that is, one that satisfies (2.18). However, the minimum bid pair determined by (2.18) seldom has a uniform minimum score. For example, if the valuation-per-click distributions for l- and h-type advertisers are the same, conditions in (2.18) lead to the same minimum bid for l- and h-type advertisers, implying different minimum scores for l- and h-type advertisers.

We summarize the above observations in the following remark.

Remark 2.5.2. The revenue-maximizing minimum bid policy is generally not a uniform minimum-score policy or one resulting from a traditional exclusion principle (as determined by (2.18)).

We conclude the above discussion with an example that illustrates how the revenue-maximizing minimum bids in our setting differ from a uniform minimum-

Table 2.1: A Comparison of Revenues under Different Minimum Bid Policies

minimum bid policy	$(\underline{b}_l,\underline{b}_h)$	total expected revenue
revenue-maximizing uniform score	$(0.481, 0.385)$	0.5211
traditional exclusion principle	$(0.333, 0.500)$	0.5213
revenue-maximizing	$(0.334, 0.615)$	0.5309

score policy and from those recommended by the auction design literature. The following example also shows that the auctioneer can achieve a higher revenue with a revenue-maximizing minimum bid policy.

Example 2.5.4. *Assume there are five advertisers and one slot. Let $\alpha = 0.5$, $Q_l = 0.8$, $Q_h = 1$, $w = 0.8$, $F_h(v) = v$, and $F_l(v) = 2v - v^2$. We calculate the minimum bids and expected revenues under three policies: a revenue-maximizing uniform-score policy, a policy using the exclusion principle, and a revenue-maximizing policy (see table 2.1).*

2.6 A Comparison with Fixed-Payment Auctions

Given the results on the efficient and revenue-maximizing designs, we are now able to compare weighted unit-price auctions with traditional auction formats where bidders bid fixed payments. Note that in fixed-payment auctions, winners pay a fixed payment upfront, whereas in unit-price auctions, winners pay ex-post based on realized outcomes. In this sense, advertisers bear less risk in unit-price auctions than in fixed-payment auctions. The risk-sharing feature of unit-price auctions is considered advantageous, for example, by McAfee and McMillan (1986) in the study of procurement auctions. Here we move beyond risk-sharing advantage and focus on comparing weighted unit-price auctions with fixed-payment auctions on allocation efficiency and revenue.

To make a fair comparison, we extend the standard fixed-payment auction to a multi-object setting. We define the *generalized first-price auction* as one in which (1) advertisers bid their total willingness-to-pay b for the first slot, (2) slots are assigned based on the ranking of bids, and (3) if an advertiser wins the j-th slot, he/she will pay $\delta_j b$.

Given our model setting, the probability of an advertiser's expected *total* valuation for the first slot being less than x is

$$\alpha F_h\left(\frac{x}{Q_h}\right) + (1-\alpha)F_l\left(\frac{x}{Q_l}\right). \tag{2.19}$$

When there is only one slot, the generalized first-price auction reduces to a standard first-price auction in which advertisers' valuation for the slot is distributed according to (2.19). Such a standard auction is known to be efficient. In fact, the generalized first-price auction is also efficient. This is because, as in standard auctions, advertisers' bids are monotonically increasing in their valuation (for the first slot) such that slots are allocated efficiently.

Recall that in efficient weighted unit-price auctions, an advertiser is assigned a slot if and only if the advertiser has the highest *total* valuation for the slot among those who have not been assigned a slot (Proposition 2.4.3). This implies that efficient weighted unit-price auctions allocate the same way as generalized first-price auctions and thus generate the same expected revenue to auctioneers. Thus:

Proposition 2.6.1. *The efficient weighted unit-price auction achieves the same efficiency and expected revenue as a generalized first-price auction.*

Because the efficient weighted unit-price auction generates the same expected revenue as the generalized first-price auctions (Proposition 2.6.1) and the revenue-maximizing weighted unit-price auction can generate more revenue than the efficient weighted unit-price auction (Proposition 2.4.4), we immediately have:

Corollary 2.6.2. *Revenue-maximizing weighted unit-price auctions generate more revenue than generalized first-price auctions.*

According to the optimal mechanism design literature, the standard auctions (with an appropriately set reserve price) can achieve the highest revenue among all mechanisms in assigning a single-object setting. The above corollary indicates, however, that weighted unit-price auctions can achieve even higher revenue. The reason lies in that weighted unit-price auctions allow the auctioneer to discriminate advertisers based on information about their expected CTRs, which is not considered in the standard mechanism design setting. Therefore, this corollary illustrates that ex-ante information on bidders' outcome-generating potential can be exploited to enhance the auctioneer's revenue.

We shall note that Proposition 2.6.1 and Corollary 2.6.2 are obtained with the assumption that the auctioneer has the same information on advertisers' future CTRs as advertisers themselves do. In keyword auctions, because advertising providers have full access to advertisers' CTR history, we expect advertisers' information advantage on future CTRs to be small, especially after advertisers have had a long enough history with the advertising provider. However, in other settings where auctioneers have substantially less information on bidders' future outcomes than bidders themselves, fixed-payment auctions may achieve higher allocation efficiency and revenue than weighted unit-price auctions.

2.7 Discussion

In this section, we consider relaxing some of the model assumptions.

The Quality of CTR Information. Given the importance of the information on advertisers' future CTRs, a natural question is how the quality of such information affects our results, which we attempt to address here by perturbing the information quality. One way to do this is to assume that under perfect information, advertisers with high and low expected CTRs can be correctly categorized into h-types and l-types, whereas under imperfect information some of the advertisers may be mis-categorized. Such mis-categorization maintains the same overall expected CTR, but causes the (unbiased) expected CTR for the h-type group to be lower and for the l-type group to be higher; more so as the information quality worsens. By this notion of information quality, we can say one information set (τ) is *less informative* than another (τ') if

$$Q_l' \geq Q_l \text{ and } Q_h' \leq Q_h, \tag{2.20}$$

where superscripts denote parameters under information set τ'. In the extreme case, when the CTR signal is completely uninformative, there is no difference between the expected CTRs of the h-type group and those of the l-type group.

Obviously, our results on equilibrium bidding hold under different information quality in the above sense. The efficient weighting factor for l-type advertisers is higher under lower quality information because of a smaller difference between h-type's and l-type's expected CTRs. The mis-categorization may cause the advertising provider to allocate advertising slots to low valuation advertisers even though higher valuation ones are available, and thus there is a loss of efficiency. The total expected revenue is generally lower because of the decrease in the total valuation created. In sum, deterioration in the quality of information on advertisers' future CTRs generally reduces the efficiency and the expected revenue of weighted unit-price auctions. The following example illustrates such results.

Table 2.2: Impact of Information Quality

	w_{eff}	total expected valuation	w^*	total expected revenue
perfect info	0.50	0.68	0.80	0.44
imperfect info	0.63	0.63	0.85	0.43

Example 2.7.1. *Assume there is one slot , n=5, and* $F_l(v) = F_h(v) = v$ *(uniform distribution). Let* α*=0.5,* $Q_l = 0.5$*, and* $Q_h = 1$ *under perfect information, and* α*=0.45,* $Q_l = 0.591$*, and* $Q_h = 0.944$ *under imperfect information (corresponding to 10% of low-CTR advertisers and 20% of high-CTR advertisers being mis-categorized). Table 2.2 summarizes the changes in efficiency and total expected revenue.*

Multiple CTR-types. The basic intuition of our main results holds for multiple signal types. Suppose there are k CTR-types, indexed by $\theta = 1, 2, ..., k$, and the weight factor for a CTR-type θ is w_θ. We can show, as in Lemma 1, that an advertiser with a CTR-type θ_1 and valuation-per-click v ties with an advertiser with a CTR-type θ_2 and valuation-per-click $w_{\theta_1}v/w_{\theta_2}$ in equilibrium. We can obtain k equilibrium bidding functions in the same way as in Proposition 1, one for each type. Analogous to the two-type case, it is still efficient to weight advertisers' bids by their expected CTRs and to impose a uniform score across different CTR-types. The revenue-maximizing weighting scheme and minimum bid policy are more complex in the multiple CTR-type case because of additional undetermined design parameters; but the basic intuition follows through. For example, the minimum bid policy remains different from a uniform-score policy and from a policy implied by the traditional exclusion principle.

2.8 Conclusion

Information technology gives us the ability to track online behaviors in unprecedented detail. For online advertising, this means that advertisers can monitor how many customers click on their advertisements and how many end up making a purchase. This ability not only enables new outcome-based pricing (also known as "pay-for-performance") models but also allows advertising providers to accumulate information on advertisers' outcome-generating potential. Within this context, we examine how information on advertisers' CTRs can be used in the design of keyword auctions. We evaluate two ways of incorporating advertisers' CTR information into the keyword auction design: by assigning different weighting factors for advertisers with different expected CTRs and by imposing different minimum bids for them. Edelman et al. (2007) and Varian (2007) note that equilibria under rank-by-price and rank-by-price×CTR rules would be different, but do not address the impact of different ranking rules on equilibrium outcome. This paper addresses this question, and also a more general question of how to choose ranking rules and minimum bid policies to best utilize the ex-ante information on advertisers' future CTRs. We study the impact of weighting schemes and differentiated minimum bid policies and how they should be configured to maximize allocation efficiency or total expected revenue.

Although we use pay-per-click keyword auctions as a specific context for our discussion, our model framework and implications can be applied to other outcome-based pricing settings such as pay-per-call and pay-per-purchase advertising auctions. The success of pay-per-click advertising on search engines has inspired innovations in other areas. For example, Google introduced keyword-auction-like mechanisms into TV, online video, and mobile phone advertising. The intuition obtained in this paper can potentially apply to these application areas as well.

Managerial Implications. Our analysis has several implications. First, we gain insight on how weighting schemes and differentiated minimum bids affect equilibrium bidding. We demonstrate that the weighting scheme determines how advertisers with different expected CTRs match in equilibrium: a low-CTR advertiser ties in equilibrium with a high-CTR advertiser when the two have the same *weighted* valuation-per-click—that is, valuation-per-click times the weighting factor. For example, if low-CTR advertisers receive a weighting factor of w, a low-CTR advertiser with valuation-per-click 1 matches a high-CTR advertiser with valuation-per-click w in equilibrium.

As in classic auctions, minimum bids exclude low-valuation advertisers and force others, especially those whose valuation is near minimum bids, to bid closer to their true valuation. Minimum bids in our setting have other effects: When minimum bids are not equally constraining, they distort equilibrium matching between low- and high-CTR advertisers and cause a jump in the less-constrained type's equilibrium bidding. Intuitively, the less-constrained type avoids competing with the more-constrained type, who bids ultra-aggressively because of minimum bids. But for advertisers with valuation well-above the minimum bids, the less-constrained type matches the more-constrained type the same way as the no-minimum-bids case. The jump reflects a transition from avoiding matching to matching among the less-constrained advertisers. These insights, together with ones on the weighting schemes, help advertising providers understand the impact of their auction rules on advertisers. They also provide guidelines for advertisers on how to bid optimally.

Second, the efficient keyword auction design is remarkably simple. It involves weighting advertisers' pay-per-click bids with their expected CTRs, and requires the same minimum score for all advertisers. The former implies lower weighting factors for advertisers with lower expected CTRs. The latter implies higher minimum bids for advertisers with lower expected CTRs. These appear to be consistent with designs used in practice. For example, Google has been using historical CTRs as weighting factors and requiring higher minimum bids for advertisers with low historical CTRs. As we have argued in section 7, the quality of such estimation affects the level of efficiency that keyword auctions can achieve, thus, our results draw attention to the importance of estimating advertisers' future CTRs. Keyword advertising providers may improve the quality of such estimation by acquiring additional information on advertisers' future CTRs and refining their estimation techniques.

Third, we characterize the revenue-maximizing weighting scheme and minimum-bid policy. The revenue-maximizing weighting scheme may favor or disfavor low-CTR advertisers relative to the efficient weighting scheme. If low- and high-CTR advertisers have the same valuation-per-click distribution, advertising providers obtain the highest expected revenue by favoring low-CTR advertisers—the disadvantaged type. But if low-CTR advertisers have a less tight valuation distribution than high-CTR ones, the revenue-maximizing weighting scheme may favor low-CTR advertisers less, possibly even disfavoring them. Such results suggest that we cannot automatically assume that low-CTR advertisers should be favored in a revenue-maximizing design.

Relation to Other Research. This research may have implications for online procurement auctions, which have gained some acceptance in recent years (Snir and Hitt 2003). One of the challenges for online procurement auction designers is to incorporate non-price dimensions such as quality, delivery, and services into auction mechanisms (Beall et al., 2003). Weighted unit-price auctions may provide a framework to do that. Of course, further research is needed to account for special features in procurement settings, such as the cost associated with switching suppliers and the fact that suppliers may misrepresent their non-price attributes.

Our research may also have implications for posted-schedule pricing of information goods and services. A variety of information goods and services such as radio spectrum, network bandwidth, and Internet cache are resources allocated for exclusive use, the use of which may generate trackable outcomes (e.g., number of packets transmitted). Information system researchers have proposed several ways to price these resources (Bapna et al. 2005; Hosanager et al. 2005; Sundararajan 2004). For example, Sundararajan (2004) suggests a nonlinear price schedule that includes a fixed fee and a usage-based fee. Our results may add a new direction for pricing these goods and services, that is, to charge buyers by realized outcomes (such as usage) and differentiate pricing schedules for buyers with different outcome-generating potential (such as usage rates). It will be interesting to compare such an approach to existing ones in the optimal pricing literature.

Limitations and Future Research. This research has certain limitations. We consider CTRs as endowed attributes while in reality advertisers may manipulate their CTRs to gain favorable weighting factors. If the manipulation permanently improves an advertiser's CTR (such as by improving the presentation of the advertisement), then our results apply to the post-manipulation periods. Advertising providers may want to encourage such "manipulation." On the other hand, manipulation that temporarily inflates an advertiser's CTR may be discouraged by a carefully structured CTR-estimation method. For example, manipulation that lasts one period does not have much impact on the weighting factor in a weighting system that emphasizes long-term CTR history. However, coping with various forms of manipulation remains an open issue in keyword auction designs.

Several other issues may be interesting for future research. First, it is not entirely clear whether keyword auctions perform better than alterative mechanisms such as posted-price. A commonly-accepted argument holds that auctions are more suitable than posted-price when bidders' valuation for goods and services is more uncertain (Pinker et al. 2003). This appears to be a plausible explanation for the popularity of auctions in selling keyword advertising slots, given sellers' lack of knowledge on the potential value of keyword advertising slots. Second, an important issue related to this study is how to estimate advertisers' future CTRs. Third, it would be interesting to examine how competition from other keyword advertising providers affects the efficient and revenue-maximizing designs.

Chapter 3

Keyword Auctions, Unit-price Contracts, and the Role of Commitment

3.1 Introduction

Unit-Price Contracts (UPC) are widely used in competitive procurement auctions, such as highway contracting (Stark 1974) and national defense (Samuelson 1983, 1986). In such auctions, bidders submit their bids specifying the unit price for each of the input factors. The auctioneer then calculates a score for each bidder based on both the unit prices and the expected quantities needed. The bidder with the lowest score wins the auction. The final payment to the winner, however, is determined by the number of units that are finally consumed/needed in realization. Moreover, many other regular settings, such as the marketing of publishing rights for books (McAfee and McMillan 1986), can also be interpreted as UPC auctions. In recent years, various formats of UPC auctions have been adopted by major search engines, such as Yahoo!, Google, and MSN, to sell keyword-related advertising slots on their web sites. In these auctions, the advertisers specify their per-click willingness to pay, and the final payments to the search engine are determined by the actual number of click-throughs that their advertisement generates.

Previous literature in UPC procurement auctions reveals that to promote competition, it is beneficial for the auctioneer to give preferential treatment to those bidders with lower efficiency/performance level (McAfee and McMillan 1986, Ewerhart and Fieseler 2003). In practice, UPC auctions often offer a subsidy to contractors with inferior production technologies or lower performance level (Ewerhart and Fieseler 2003). Preferential government procurement policies also affect several hundred billion dollars' worth of trade worldwide each year (Graham 1983). For example, the United Stated Government offers a 50 percent preference for domestic suppliers for military procurement (McAfee and McMillan 1989). These practices are interpreted as either a protectionist devices (Lowinger 1976), or a way to increase bidding competition (McAfee and McMillan 1986).

There is also a fast growing literature studying the "keyword auctions," which is one of the most popular places to apply the UPC auctions on the internet. This stream of literature usually focuses on the impact of varying auction mechanisms on the advertisers' bidding behavior and the auctioneer's revenue. Feng et al. (2007) finds that depending on the correlation between advertisers' willingness to pay and their relevance, Google's (without preferential treatment) and Yahoo's (with preferential treatment) ranking mechanism can out-perform each other under different conditions. Weber and Zheng (2007) study a search model in which advertisers compete for positions in a search engine, and show that the optimal winning rule should put non-zero weight on the advertisers' bids. Liu and Chen (2006) consider a weighted unit-price auction setting where bidders bid on unit prices, and the winner is determined by their bids as well as their past performance.

The above literature assumes that bidders' performance levels are fixed, and does not consider the possible impact of the auction mechanism on bidders' performance level in the long term. It is not obvious, however, whether the practice of preferential treatment works when bidders' performance may change over time for various reasons. For instance, advertisers may invest in their performance for their own interest or reacting to the performance-based ranking policy. In general, the possible upgrade in performance, and hence the change in the performance distribution, calls for adjusting the preferential policy accordingly. In addition, preferential treatment may discourage low-performance bidders from investing in their performance levels, which could negatively impact the auctioneer's revenue. Moreover, under the preferential treatment policy, a less efficient bidder beats more efficient ones with positive probability (Ewerhart and Fieseler 2003). This efficiency loss, especially in the long term, might be significantly detrimental for an industry that is sensitive to its consumer response. In the case of keyword advertising, for example, the long-term user base is a significant basis for search engines' revenue.

In practice, the performance-based ranking mechanisms adopted by the search engines are constantly changing over time, which exhibits an experimental process and may reflect the need to adjust auction policies to dynamic features. For example, Yahoo! used to rank advertisers by their willingness-to-pay per click and has now switched to a new mechanism that also considers click-through rates in its ranking. Google first introduced a design that ranks advertisers by the product of per-click prices they bid and their historical click-through rates in 2002, and is also modifying its "bid $\times$ CTR" algorithm.[1] Besides, instead of announcing how their ranking mechanisms exactly incorporate the performance information as they did earlier, many search engines now keep this ambiguous.

These observations indicate that it is not sufficient to study UPC auctions in a static setting, where the firms' performance levels remain unchanged. On the one hand, the auction mechanism gives bidders with different performance levels different incentives to improve their performances, and this in turn affects the long term average industry performance level. On the other, the updated distribution

[1]https://adwords.google.com/support/bin/answer.py?answer=49174

of firms' performance levels gives the auctioneer an incentive to modify his auction mechanism, so the auctioneer faces a commitment problem. What is the impact of the preferential ranking rules on bidders' performance choice? How should the auctioneer choose the performance-based allocation policy considering advertisers' possible performance upgrade? Should the auctioneer commit to a certain mechanism or it is beneficial to modify its auction mechanism in time?

This paper tries to answer the questions above, by capturing the dynamic effects of bidder performance evolution. We consider a two-period model. In the first period, the auctioneer announces the performance-based auction mechanism and the bidders decide whether or not to improve their performance levels. In the second period, the auction takes place, but may or may not follow the exact auction mechanism announced by the auctioneer in the first period. Bidders bid their unit prices and the winner is chosen based on both their bids and performance levels. This paper relates to literature on investment incentives in procurement auctions (Arozamena and Cantillon 2004, Tan 1992). Most of these studies concern the revenue equivalence under different auction formats (e.g., first-price sealed-bid auctions versus second-price auctions), while our paper focuses on performance-based unit-price format in the context of keyword auctions. To our knowledge, this setup is mostly related to Branco (2002), which studies a procurement auction where two firms compete for a government project and the inefficient firm may improve its technology. However, in his setting, technology choice is unobservable. Instead, we study the case where the performance is observable and thus performance-based allocation is feasible, and obtain different insights.

Under this performance-based unit-price auction framework, we study the bidders' decisions on performance improvement (in the first period) and bid (in the second period), and then examine the impact of performance-based allocation on overall bidder performance, the auction efficiency, and the auctioneer's revenue. We find that the overall performance level is monotonic in the degree of preferential treatment to those bidders with low performance: the more the auctioneer discriminates against low-performance bidders, the higher the overall performance level. The efficient policy involves weighting bidders' unit-price bids by their expected performance. This is consistent with the one in a static case where the performance level is fixed, although the former concerns both allocative efficiency and investment efficiency whereas the later considers allocative efficiency only. We also compare the auctioneer's revenue-maximizing policies when she is fully committed to the auction rule and when not, and show that she should give less preferential treatment to low-performance advertisers when she is fully committed.

This paper is organized as follows. We describe our model in Section 3.2. In Section 3.3, we investigate bidders' bidding decisions, as well as their performance evolution – how bidders with lower performance levels convert to higher performance by paying some cost. In Section 3.4, we study the impact of performance-based allocation on bidders' overall performance. We discuss social welfare in Section 3.5, and explore the revenue-maximizing allocation for an auctioneer in Section 3.6. Section 3.7 concludes the paper.

3.2 Model Setup

We consider an environment where the auctioneer sells a single object to n risk-neutral bidders, and the bidders have independent and private values for the object being sold. We capture the long-term interactions between the auction mechanism and bidders' overall performance levels through a two-period model. In the first period, the auctioneer announces the auction mechanism, and the bidders decide whether or not to invest in improving their performance levels. In the second period, the auctioneer either keeps the announced auction rules, or modifies the rules, and bidders participate in the auction. We refer to the former as a full-commitment case and the latter as a limited-commitment case.

Assume that each bidder is endowed with a yield level y, which measures one's productivity, or efficiency of using the object being auctioned. Denote each bidder's unit valuation of y as v. In keyword advertising, yield level measures the expected number of clicks that an advertiser can generate during a given period of time, and the unit valuation is the advertiser's per-click valuation. In this way, each bidder's total valuation for the object is determined by his yield from the object, y, times his unit valuation, v. We assume that bidders' yield levels are independent of their unit valuations. Therefore, a bidder with a higher yield level is stochastically more efficient than a bidder with a lower yield level. For this reason, we also refer to one's yield level as his performance level.

For simplicity, we assume that y takes a discrete value y_H or y_L, and $y_H > y_L$. We call the bidders with y_H high type bidders, or H types, and bidders with y_L low type bidders, or L types. Denote $\theta \in \{L, H\}$ as a bidder's type. So a θ-type bidder's total valuation of the asset is vy_θ. A L type bidder can invest in and improve his performance level to y_H at a cost c. In keyword advertising, for example, advertisers may engage in extensive marketing research or experiments to improve their web site design, and thus ultimately increase their click-through rates (Schlosser et al. 2006).

We assume that bidders know their own unit valuation v and performance levels y_θ, but not those of others. The unit valuation v is independent and identically-distributed in $[0, 1]$ following a distribution $F(v)$. By the convention of distribution functions, let $F(v) = 1$ for $v > 1$. The corresponding density function is denoted by $f(v)$, which is positive and differentiable everywhere in the support. A bidder's performance level y in the first period is believed to be y_H with probability α, and y_L with probability $1-\alpha$. Both $F(v)$ and α are commonly known. Moreover, each bidder's performance level is known to, or can be verified by the auctioneer after the auction takes place in the second period. For example, in keyword auctions, the number of click-throughs that a certain advertisement generates is not observed by other advertisers, but can be approximately predicted (based on its past performance) and accurately recorded by search engines.

The object is sold through a first-price, sealed-bid unit-price auction. The auctioneer uses a scoring rule to evaluate the bids from bidders with different performance levels. In particular, a weighting factor $w \in (0, 1]$ on the low type bidders' bids is introduced to measure this differential treatment. According to w and bidders' unit-price bids, a score is calculated for a bidder of type θ according to the following formula:

$$s(b, \theta) = \begin{cases} b & \text{if } \theta = H \\ wb & \text{if } \theta = L \end{cases} \tag{3.1}$$

The bidder with the highest score wins and pays for all the realized yield at his unit-price bid b. There is no entry fee or reserve price.

We assume that a bidder's payoff from winning the object is additive in its total valuation for the object, yv, and its payment, yb. So the expected payoff for a θ-type bidder with unit valuation v who placed a bid b is

$$U_\theta(v, b) = y_\theta(v - b)\text{Prob}(\text{win}|b, \theta). \tag{3.2}$$

By allowing w to take different values, we can accommodate different auction formats. When $w = \frac{y_L}{y_H}$, the low- and high-type bidders are treated "fairly," as the winner is chosen based on the ranking of the total revenue created. The Google auction belongs to this category. When $w \in [0, \frac{y_L}{y_H})$, the bid submitted by a low performance bidder is treated unfavorably. When $w \in (\frac{y_L}{y_H}, 1]$, the low-type bidder is treated favorably. The Yahoo!'s earlier auction format determined the winner solely by the bidding amount, which is equivalent to the case when $w = 1$. We denote the case of $w = 1$ as a *standard unit-price auction*, and the case of $w < 1$ as a *performance-based auction*.

The timing is as follows. In the first period, the auctioneer announces the weighting factor w. Then, bidders' performance levels and valuations realize, and the L types decide whether to improve their performance or not. In the second period, all bidders participate in the auction.[2] Each bidder's score is calculated. The object is assigned to the bidder with the highest score, and payment is made to the auctioneer.

We are interested in the impacts of the performance-based allocation on bidders' performance evolution, the resulting social welfare, and the auctioneer's expected revenue.

3.3 Bidding Strategy and Performance Conversion

In this section, we focus on the L types' performance decisions. To derive the subgame perfect equilibrium, we start with the second period, where bidders participate in the auction with a given weighting factor w. We conjecture that in equilibrium, there exists a cutoff value v^*, such that in the first period, all the L types whose unit valuations are above v^* will improve their performance levels by incurring an investment cost c, and all the L types whose unit valuations are below

[2]In the limited commitment case, the auctioneer modifies the auction rule according to the updated distribution of bidders' performances before the auction begins.

v^* will remain at the low performance level. This conjecture is intuitive because, in general, upon winning the object, the L types with higher unit valuations have higher gains from improving their performance ($(y_H - y_L)v$) than those L types with lower unit valuations. As a result, the L types with higher valuations should be more likely to invest in their performance. We will verify later that this is an equilibrium strategy.

3.3.1 Second Period: the Bidding Strategy for a Given w

Denote $F_L(v) \equiv F(v|\theta = L)$ as the distribution function of L types' unit valuation in the second period, and $F_H(v) \equiv F(v|\theta = H)$ as that of H types'. $F_L(v)$ and $F_H(v)$ can be calculated by applying Bayes' rule. In particular, according to the conjecture of cutoff v^* (we use $\Pr(\cdot)$ to represent the probability of an event in the second period),

$$F_L(v) = \frac{\Pr(V \leq v, \theta = L)}{\Pr(\theta = L)} = \begin{cases} \frac{F(v)}{F(v^*)} & \text{if } v \leq v^* \\ 1 & \text{if } v > v^* \end{cases} \tag{3.3}$$

and

$$F_H(v) = \frac{\Pr(V \leq v, \theta = H)}{\Pr(\theta = H)} = \begin{cases} \frac{\alpha F(v)}{1-(1-\alpha)F(v^*)} & \text{if } v \leq v^* \\ \frac{F(v)-(1-\alpha)F(v^*)}{1-(1-\alpha)F(v^*)} & \text{if } v > v^*, \end{cases} \tag{3.4}$$

This is because, under our conjecture, a bidder is an L type in the second period only if he is endowed with low performance in the first period (with probability $(1-\alpha)$) and remains at it (if his unit valuation is less than v^*). Therefore, in the second period, a bidder with probability $(1-\alpha)F(v^*)$ is an L type, and thus with probability $[1-(1-\alpha)F(v^*)]$ is an H type. If we simply denote $P_L \equiv \Pr(\theta = L)$ and $P_H \equiv \Pr(\theta = H)$, that is

$$P_L = (1-\alpha)F(v^*), \text{ and } P_H = 1-(1-\alpha)F(v^*). \tag{3.5}$$

In addition, in the second period, a bidder is an L type with unit-valuation less than v only if he is endowed with low performance and with unit valuation less than v, which occurs with probability $(1-\alpha)F(v)$. This accounts for (3.3). Similarly, a bidder is an H type with unit-valuation less than v (if $v \leq v^*$) with probability $\alpha F(v)$. For the case where $v > v^*$, we need to take into account that the H types in the second period also include those converted from L types, which happens with probability $(1-\alpha)(F(v) - F(v^*))$. Therefore, the probability of one being H type of $v > v^*$ in the second period is $\alpha F(v) + (1-\alpha)(F(v) - F(v^*))$ (Equation 3.4). We denote the corresponding density functions as $f_L(v)$ and $f_H(v)$ accordingly (specified in Appendix A).

We consider a symmetric, pure-strategy, Bayesian-Nash equilibrium. By "symmetric," we mean that bidders with the same unit valuations and performance levels will bid the same amount in equilibrium. Let $b_H(v)$ and $b_L(v)$ denote the bidding functions for the H types and L types in the second period, respectively. We conjecture that the bidding functions are strictly increasing (we will verify this later). Using an approach similar to that in Liu and Chen (2006), we obtain the following result regarding the bidding functions:

Lemma 3.3.1. $b_H(wv) = wb_L(v)$ *for* $v \in [0, v^*]$.

Lemma 3.3.1 shows that in equilibrium, an H type with unit valuation wv and an L type with unit valuation v will obtain the same score (recall the scoring rule (3.1)). The intuition is as follows. Consider an H type with unit valuation wv who bids wb and an L type with unit valuation v who bids b. By the scoring rule, the former has the same probability of winning as the latter. Then, the expected payoff of the former differs from that of the latter only by a scalar according to (3.2). Because multiplying a payoff function by a scalar does not alter the solution to an optimization problem, b is the solution to the L type's optimization problem if and only if wb is the solution to the H type's problem.

By the conjecture of the monotonicity of bidding functions, an L type i with unit-valuation v can beat a bidder j if and only if j is an L type and has a unit valuation less than v, or j is an H type and has a unit-valuation less than wv (by Lemma 3.3.1). So, the probability for an L type with unit-valuation v to beat the other bidders, or his equilibrium winning probability, can be represented by

$$\rho_L(v) \equiv [P_H F_H(wv) + P_L F_L(v)]^{n-1}. \tag{3.6}$$

Similarly,

$$\rho_H(v) \equiv \left[P_H F_H(v) + P_L F_L(\frac{v}{w})\right]^{n-1}. \tag{3.7}$$

Lemma 3.3.2 presents the equilibrium bidding functions.

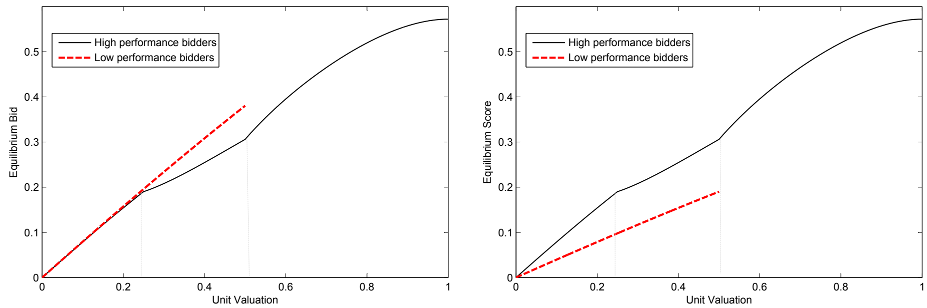
Lemma 3.3.2. *For a given $w \leq 1$ and the cutoff value v^*, the equilibrium bidding functions for* H *types and* L *types are increasing in v, and can be represented as the following:*

$$b_L(v) = v - \frac{\int_0^v \rho_L(t)dt}{\rho_L(v)}, \text{ for } v \in [0, v^*] \tag{3.8}$$

$$b_H(v) = v - \frac{\int_0^v \rho_H(t)dt}{\rho_H(v)}, \text{ for } v \in [0, 1]. \tag{3.9}$$

It is easy to verify that the above bidding functions are indeed strictly increasing (see the appendix for the proof). Also, as indicated by (3.8) and (3.9), a bidder's equilibrium unit-price bid is always less than his true unit valuation, which is common among first-price auctions.

It can be shown that there are two kinks (wv^* and v^*) in the H types' bidding function. This is due to the kinks in the winning probability $\rho_H(t)$ (see Appendix A), as the competition situation faced by the H types is discontinuous. Figure 3.1 illustrates the equilibrium bidding functions and equilibrium scores with $w = 0.5$ and $v^* = 0.5$. By the scoring rule, H types' equilibrium scores are their equilibrium bids,



(a) Equilibrium Bid (b) Equilibrium Score

Figure 3.1: The Equilibrium Bidding Functions and Equilibrium Scores

and L types' are their equilibrium bids adjusted by the weighting factor w. The first kink in the H types' bidding function is because $F_L(\frac{t}{w}) = 1$ for all $t > 0.25$, and the second is due to the kink of $F_H(t)$ at $t = 0.5$. Intuitively, it is still possible for an H type with a relatively low unit valuation ($v \leq 0.25$) to lose to an L type, even though an L type's bid gets discounted by w. However, once the H type's unit valuation exceeds a certain value (0.25), he can always beat an L type, because all the L types' unit valuations are below 0.5. Therefore, there exists a kink at $v = 0.25$. Similarly, an H type with an intermediate unit valuation ($0.25 < v \leq 0.5$) faces different competition pressure from an H type with higher valuation ($v > 0.5$), because an H type with a high unit valuation ($v > 0.5$) can be either an original H type or one converted from an L type. Therefore, the second kink happens at $v = 0.5$.

Denote $V_\theta(v) \equiv U_\theta(v, b_\theta(v))$ as the equilibrium expected payoff of a bidder with type θ and unit valuation v. From the payoff function (3.2) and the bidding functions (3.8) and (3.9),

$$V_\theta(v) = y_\theta(v - b_\theta(v))\rho_\theta(v) = y_\theta \int_0^v \rho_\theta(t)dt. \tag{3.10}$$

by noticing that $\text{Prob}(\text{win}|b_\theta(v), \theta) = \rho_\theta(v)$ (both representing the equilibrium winning probability).

3.3.2 First Period: Performance Conversion for a Given w

Given the equilibrium bidding functions in the second period, we now solve the L types' decisions on performance upgrades in the first period. We show that there does exist such a cutoff value v^* in equilibrium, from which no bidder will unilaterally deviate. An L type benefits from converting to an H type, because both his performance level and his winning probability increase. The overall benefit is represented by the increase in the equilibrium payoffs from changing from an L type to an H type, that is, $V_H(v) - V_L(v)$. However, the investment in the performance incurs a cost c. Given w, whether an L type converts or not depends on this tradeoff.

Proposition 3.3.3. *Given* w $(0 < w \leq 1)$ *and the investment cost* c*, there exists a cutoff value* v^**, such that an* L *type with unit valuation higher than* v^* *converts to an* H *type, and with valuations lower than* v^* *does not, where* v^* *is determined as follows.*

$$\begin{cases} v^* = 0 & \text{if} \quad c = 0 \\ v^* \;\; \text{solves} \;\; \Delta V(v^*) = c & \text{if} \quad 0 < c < \Delta V(1) \\ v^* = 1 & \text{if} \quad c \geq \Delta V(1) \end{cases} \tag{3.11}$$

where

$$\Delta V(v^*) = y_H \int_0^{v^*} \rho_H(t)dt - y_L \int_0^{v^*} \rho_L(t)dt. \tag{3.12}$$

When it is costless for the L types to convert, every L type will convert to an H type ($v^* = 0$) for the benefit from both improved productivity and advantageous position in the scoring rule. Consequently, in the second period, all bidders in the auction are H types. In such a case, the performance-based auction is identical to the standard unit-price auction, and the problem becomes straightforward. When the investment cost is prohibitive (higher than the gain that the L type of the highest unit valuation can get from the conversion), the gain from improving performance cannot compensate for the cost, and no L types will convert ($v^* = 1$). In this case, the problem is identical to the case studied by Liu and Chen (2006).

For the rest of this paper, we focus on the more interesting case where the investment cost is modest and at least some L types are interested in the conversion.

In general, the conversion decision is based on the tradeoff between the gain from conversion and the investment cost. Everything else being equal, higher cost results in fewer L types converting. Moreover, as the cost increases, the L types who choose to remain at the low performance level instead of converting are those with relatively low unit-valuations, since those low-valuation L types could benefit less from conversion than high-valuation ones. Similarly, the difference between the two performance levels affects the L types' decisions on the conversion: given y_L, a higher y_H, which means more gain from conversion, could induce more L types to convert. As a summary,

Corollary 3.3.4. *For a given w, v^* increases in the investment cost (c) and decreases in the high performance level (y_H).*

3.4 Bidders' Overall Performance

In this section, we examine the effect of differential treatment (w) on the winning bidder's overall performance level. First, we observe that the cutoff value is closely related to the weighting factor.

Lemma 3.4.1. *The cutoff value v^* is increasing in the weighting factor w.*

Intuitively, as w increases, and hence the L types are treated more favorably,[3] bidders' benefit from improving their performance levels reduces. As a result, the benefit can no longer compensate for the investment cost c for some L types who

[3] Actually, when $w > \frac{y_L}{y_H}$, L types are given preferential treatment.

would choose to convert. So, as w increases, fewer bidders have incentive to invest in their performance levels. In other words, the more the L types are discriminated against in the auction (the smaller the w), the larger incentive for the L types to improve their performance levels, because otherwise they have little chance to win.

Next, we explore the impact of the weighting factor on the expected winning performance level. The expected winning performance level is determined by the bidders' performance weighted by their winning probabilities, which can be expressed as:

$$y_L P_L \int_0^{v^*} \rho_L(v) f_L(v)\, dv + y_H P_H \int_0^1 \rho_H(v) f_H(v)\, dv. \tag{3.13}$$

Proposition 3.4.2. *The more the* L *types are discriminated against, the higher the overall performance level. That is, the expected winning performance level is decreasing in w.*

To show this, from Lemma 3.4.1, we know that as w decreases, fewer bidders remain at the low performance level. Moreover, as w decreases, L types' equilibrium winning probabilities decrease as well. Both factors drive the expected winning performance level to the same direction: the lower the w, the higher the expected performance level. As a result, the performance-based auctions with $0 < w < 1$, in general, enhance the expected winning performance level compared to the standard UPC auctions (where $w = 1$).

3.5 Auction Efficiency

In this section, we study the efficiency of the UPC auction. The efficiency in our dynamic setting not only means that the object is assigned to the bidder with the highest valuation, but also that the L types make proper investment in their performance levels.

The efficient weighting factor w_{eff} is defined as one that maximizes social welfare, that is, the total expected payoff of the auctioneer and bidders. The efficiency may be in the auctioneer's best interest in the long run. In keyword auctions, especially, the market for keyword advertising is relatively nascent, and search engines are still at an early stage of the use of the UPC auction mechanism. It is sensible for the auctioneer (search engine) to choose an efficient design at this stage to maximize the "total pie." After all, unless advertisers feel that they get fair treatment in the auctions and see high returns, they will not return for more business or allocate larger fractions of their advertising budgets on keyword advertising.

The total welfare for the auctioneer and bidders is their total expected valuation minus the expected investment cost:

$$n\left[y_L P_L \int_0^{v^*} v\rho_L(v) f_L(v)\, dv + y_H P_H \int_0^1 v\rho_H(v) f_H(v)\, dv - (1-\alpha)(1-F(v^*))\, c\right]. \tag{3.14}$$

So, w_{eff} maximizes (3.14).

In a standard static auction, an efficient policy considers allocative efficiency only. An efficient auction allocates the object to the bidder with the highest total valuation. In our dynamic setup, the efficiency also refers to the investment efficiency (whether there exists over-investment or under-investment in performance). As a result, it is not obvious whether the results in the static setting will hold in our case when bidders can update their performance levels.

Proposition 3.5.1. *The efficient weighting factor* $w_{eff} = \frac{y_L}{y_H}$.

Proposition 3.5.1 shows that the efficient policy in this two-stage game involves weighting bidders' unit-price bids by their expected performance, which is the same as in the static one (Liu and Chen 2006). This may look surprising, but the intuition is as follows. First of all, the weighting factor $w = \frac{y_L}{y_H}$ ensures the allocative efficiency in the second stage. As we mentioned earlier, when $w = \frac{y_L}{y_H}$, the L types and H types are treated “fairly” in the sense that the winner is chosen based on the ranking of their total contribution. This is equivalent to the standard first-price auction where bidders submit their total payments for the object, so the allocative efficiency is guaranteed.

Second, the weighting factor $w = \frac{y_L}{y_H}$ also leads to an efficient conversion. In general, whether an investment is efficient or not depends on the tradeoff between the social gain (increase in the expected social welfare) and the investment cost. From the previous discussion, the performance-based auction with $w = \frac{y_L}{y_H}$ is equivalent to a first-price auction, and hence equivalent to a second-price auction by the Revenue Equivalence Theorem (Myerson 1981). Notice that a second-price auction is a special case of the Vickrey-Clarke-Groves (VCG) mechanism (Vickrey 1961, Clarke 1971, Groves 1973), in which one's expected payoff equals the externality that he imposes on the other bidders, or, the improvement of the social welfare that he brings in. Therefore, the social gain from one's conversion equals the increase of his payoff $[V_H(v) - V_L(v)]$. For the marginal bidder, i.e., the L type with unit valuation v^*, the increase of his payoff $\Delta V(v^*)$ is equal to the investment cost c by Eq. (3.11), which implies that the social gain from the marginal bidder's conversion breaks even with the investment cost.For the L types with a higher unit valuation ($v > v^*$), the social gain from conversion is greater than the investment cost. Therefore, it is efficient for them to convert. Similarly, it is efficient for the L types with a lower unit valuation ($v < v^*$) not to convert. So $w = \frac{y_L}{y_H}$ leads to investment efficiency.

For these reasons, we define the performance-based auction with $w = \frac{y_L}{y_H}$ as an *efficient performance-based auction.*

It is worth noting that the above efficiency result is independent of whether the auctioneer can fully commit to the pre-announced auction rule or not. This is because, in the second stage, it is always efficient to choose $w = \frac{y_L}{y_H}$ regardless of bidders' conversion patterns (as argued in the above). As a result, whether the auctioneer can fully commit or not does not influence L types' conversion decision in the first period.

3.6 The Revenue-Maximizing w

In this section, we focus on the weighting factor that maximizes the auctioneer's expected revenue and compare the results in the full commitment case to those in the limited commitment case. We denote w^L_{opt} and w^F_{opt} as the revenue-maximizing weighting factors for the former and the latter case, respectively.

Based on bidders' bidding strategy in the second period, and the L types' decisions of their performance conversion in the first period, for a given w, the auctioneer's expected revenue can be expressed as (see the appendix for the derivation)

$$\begin{aligned} \pi = \quad & n y_L P_L \int_0^{v^*} \rho_L(v) \left(v - \frac{1 - F_L(v)}{f_L(v)} \right) f_L(v) dv \\ & + n y_H P_H \int_0^1 \rho_H(v) \left(v - \frac{1 - F_H(v)}{f_H(v)} \right) f_H(v) dv. \end{aligned} \tag{3.15}$$

Basically, the first term is the expected revenue from the L types in the second period (the "ex post" L types), and the second term is that from the "ex post" H types. Denote $J_\theta(v) \equiv \left[v - \frac{1-F_\theta(v)}{f_\theta(v)} \right]$, $\theta \in \{H, L\}$. In the standard auction literature, $J_\theta(v)$ is called *virtual value* (Myerson 1981) or *marginal revenue* (Bulow and Roberts 1989), which represents the expected revenue a certain bidder can bring to the auctioneer if he wins.

3.6.1 Auctioneer with Limited Commitment

Since some higher-valued, low-performance bidders improve their performance levels in the first period, the distribution of the H types and L types changes at the beginning of the second period. As a result, the original w, which is pre-announced at the beginning of the first period, may no longer be optimal, and the auctioneer has incentive to host the auction following a different rule. In practice, both Yahoo! and Google experience a migration of their ranking mechanisms over time. Moreover, neither of them announce the exact w that they are using now, which gives them more flexibility in altering their ranking mechanism. This corresponds to the case where the auctioneer has limited commitment to her pre-announced policy, and the actual auction rule could differ from an earlier announcement (before the L types make their conversion decision).[4]

When the auctioneer has limited commitment, the L types make their conversion decisions based on their anticipation of the weighting factor that the auctioneer will choose in the second period. With this anticipation, the break-even condition (3.11) holds. The auctioneer chooses an revenue-maximizing weighting factor based on her belief about the L types' conversion decisions. In equilibrium, the beliefs are consistent with the equilibrium choice of v^* and w^L_{opt}. The equilibrium outcome is thus identical to a simultaneous move game where the auctioneer and bidders act at the same time. The revenue-maximizing weighting factor and the cutoff value are determined by the equation system below.

$$\Delta V(v^*) = c \text{ and } \frac{\partial \pi}{\partial w} = 0, \tag{3.16}$$

where the first equation is the break-even condition for the L types, and the second

[4] Or from auction to auction, if considering the case where auctions are held repeatedly.

equation comes from the first-order condition of the expected revenue (3.15) with respect to w. Specifically,

$$\frac{\partial \pi}{\partial w} = n(1-\alpha)y_L \int_0^{v^*} \frac{d\rho_L(v)}{dw} J_L(v)f(v)dv + n\alpha y_H \int_0^{wv^*} \frac{d\rho_H(v)}{dw} J_H(v)f(v)dv. \quad (3.17)$$

Proposition 3.6.1 discusses the revenue-maximizing w.

We say a cumulative distribution function $F(v)$ is *increasing-hazard-rate* (IHR) if its hazard rate $\frac{f(v)}{1-F(v)}$ increases in v within the support. Many distributions, including uniform, normal, and exponential, satisfy the condition of increasing hazard rate.

Proposition 3.6.1. *In the limited commitment case, the revenue-maximizing weighting factor w_{opt}^L is determined by (3.16). Moreover, if $F(v)$ is IHR, this revenue-maximizing weighting factor is greater than the efficient weighting factor, that is, $w_{opt}^L > \frac{y_L}{y_H}$.*

Proposition 3.6.1 shows that it is beneficial for the auctioneer to give more preferential treatment to the L type bidders than the efficient level. It indicates that the recommendation of preferential treatment to the L types in a static setting also applies to the dynamic setting, though with a different magnitude. This is because, in the absence of commitment, the auctioneer chooses the revenue-maximizing weighting factor based on the *ex post* (after conversion) type distribution in the second period only, which is exactly as making the choice in a static case. Favoring L type bidders raises L type bidders' winning probabilities, which in turn increases the competitive pressure on H type bidders and thus force them to bid more aggressively. The former effect tends to lower the auctioneer's expected revenue, but the latter tends to enhance the auctioneer's expected revenue. The resolution of such a tradeoff involves some preferential treatment to L type bidders. Such a result holds regardless of the distribution of types (α) and the exact performance levels (y_H and y_L).

The revenue-maximizing weighting factor, however, is different from that in a static case, which corresponds to a special case of $v^* = 1$ in our setting.[5] Proposition 3.6.2 compares the revenue-maximizing w in the limited commitment case to that in the static case.

Proposition 3.6.2. *When $v \sim U[0, 1]$, the auctioneer should give more preferential treatment to the* L *types when they are able to improve their performance levels than when their performance levels are fixed.*

This is because, after the conversion of some higher-valued, low-performance bidders, the remaining L types are in a more disadvantaged situation. So, the auctioneer should give them more preferential treatment to promote competition.

How does the revenue-maximizing weighting factor impact the equilibrium performance level compared to an efficient weighting factor? Recall that Proposition 3.4.2 indicates that a higher weighting factor leads to lower expected winning performance. So we have,

Corollary 3.6.3. *If $F(v)$ is IHR, the revenue-maximizing weighting factor results in a lower expected winning performance level than the efficient weighting factor does.*

In some instances, the expected winning performance is an important factor for the auctioneer to consider. For example, the performance in keyword auctions measures the relevance of the advertisements to consumers, which could impact consumers' search costs for the information of interest and hence affect the long-term

[5] Fixing $v^* = 1$ and solving the second equation in (3.16) give us the revenue-maximizing weighting factor in the static case.

user base. In this case, subsidizing the lower types is not only at the cost of efficiency, as discussed earlier, but also at the cost of lowering the performance, and therefore is even more costly.

3.6.2 Auctioneer with Full Commitment

When fully committed to the pre-announced auction rules, the auctioneer's decision on w directly affects the L types' performance choices. So the auctioneer has to consider the effect of w on the L types' conversion patterns, as well as on the intensity of competition. The effect on competition is similar to that in the limited commitment case. The effect on the L types' conversion patterns is reflected by the term of $\frac{dv^*}{dw}$ in the first-order condition below.

$$n\left[(1-\alpha)\left[(y_L\rho_L(v^*) - y_H\rho_H(v^*))\,v^* + c\right] f(v^*) + \alpha y_H \int_{wv^*}^{v^*} \frac{d\rho_H\,(v)}{dv^*} J_H(v) f(v)dv\right] \times \frac{dv^*(w)}{dw} + \frac{\partial \pi}{\partial w} = 0 \qquad (3.18)$$

where $\frac{\partial \pi}{\partial w}$ is defined as in (3.17).

Lemma 3.6.4. *The revenue-maximizing weighting factor w_{opt}^F is jointly determined by Equation (3.11) and (3.18).*

Considering the *ex post* (after conversion) competition in the second period, the auctioneer has incentive to subsidize the L types (by setting $w > \frac{y_L}{y_H}$), which forces the H types to bid more aggressively. We call this effect the *competition effect*, which is captured by the term $\frac{\partial \pi}{\partial w}$ in (3.18). However, lower w, or less subsidy to the L types, makes more L types convert, and this could presumably enhance the auctioneer's revenue. We call this effect the *conversion effect*, which is captured by the term in the first square bracket in (3.18). In general, the revenue-maximizing weighting factor is determined by the balance between the competition effect and the conversion effect, which depends on the proportion of H types and L types in the distribution.

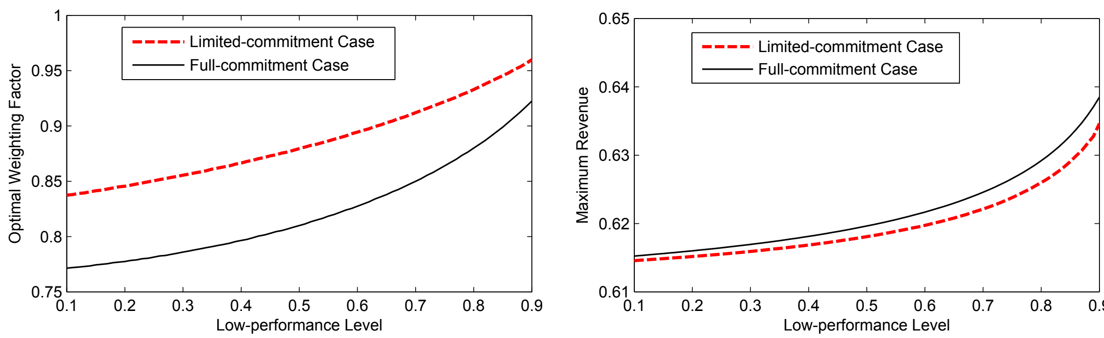
How does the equilibrium differ in the full-commitment case from that in the limited-commitment case?

Proposition 3.6.5. *If the expected revenue in (3.15) is concave with respect to w (i.e., $\partial^2\pi/\partial w^2 < 0$) and $F(v)$ is IHR, $w^F_{opt} < w^L_{opt}$.*

Proposition 3.6.5 shows that an auctioneer who can commit to the auction rule should give less preferential treatment to the low performance bidders than she does when she cannot fully commit. The intuition is as follows. In the full-commitment case, the preferential treatment not only affects the competition in the second period (as in the limited-commitment case), but also impacts L types' conversion decisions. A less preferential treatment to L types gives them more incentive to improve their performance levels, which is beneficial for the auctioneer. However, this objective is not achievable when the auctioneer cannot commit to an auction rule. This is because as any announcement that the auctioneer makes before the second period is not credible, bidders make their performance upgrade decisions based on their own expectations.

The concavity condition per se does not drive the result in Proposition 3.6.5; rather, it ensures that the optimization problem in the limited commitment case is well behaved. In fact, as long as $\partial\pi/\partial w = 0$ (the second equation in (3.16)) has a unique solution, the result in Proposition 3.6.5 holds (see this in the proof of Proposition 3.6.5). Also, we can show that the assumption of concavity of the revenue function can be relaxed to a weaker notion of quasiconcave functions. The concavity condition is satisfied by many common distributions, for instance, uniform distribution. The following is one numerical example.

Example 3.6.6. *Let $F(v) = v$ (uniform distribution at $[0, 1]$), $n = 5$, $\alpha = 0.5$, $y_H = 1$, and y_L change from 0.1 to 0.9. We can derive the revenue-maximizing weighting factors and the maximum revenue with the changes in the y_L in the limited-commitment case and the full-commitment case, respectively.*



(a) Revenue-maximizing Weighting Factors

(b) Maximum Revenues

Figure 3.2: A Comparison of the Cases with Different Commitments

Figure 3.2(a) says that w_{opt} in the full-commitment case is less than that in the limited-commitment case. Also, Figure 3.2(b) shows that a fully committed auctioneer makes more expected revenue than an auctioneer with limited commitment. In fact, whether an auctioneer can fully commit or not has a general impact on her expected revenue, as summarized in the following corollary.

Corollary 3.6.7. *A fully committed auctioneer makes more revenue than an auctioneer with limited commitment.*

This is because with full commitment, the auctioneer can always commit to the revenue-maximizing weighting factor chosen in the limited commitment case and achieve at least the same revenue as in the limited commitment case. But the reverse is not true.

3.7 Conclusion

We study a performance-based unit-price auction model where the low performance bidders can improve their performance level at a cost. We find that there exists a cutoff value in bidders' unit valuations, such that all low-type bidders with unit valuations greater than the cutoff value convert to a high performance level. The efficient allocation in the static case carries on to our dynamic case. The revenue-maximizing weighting factor, however, is different from that in the static case. In addition, whether or not the auctioneer can commit to an auction policy also impacts the revenue-maximizing factor. In the limited-commitment case, the auctioneer has more incentive to subsidize lower performance bidders to promote competition than in the static case where bidders performance levels remain unchanged. Moreover, the revenue-maximizing weighting factor in the full-commitment case is less than that in the limited-commitment case, which indicates that the auctioneer should give less preferential treatment to the low-type bidders to encourage them to improve their performance levels.

Our research generates several managerial implications. First, allocation policies critically affect bidders' performance choices, and auctioneers should handicap low-performance bidders for better overall performance. Improving overall performance should be an important concern for such markets as two-sided networks. For example, in keyword auctions hosted by the search engines, both user traffic and advertising revenue are important for the search engine, and the performance/relevance of the advertisement affect the search engine's long-term user base.

To achieve long-run allocation efficiency, which is probably more important for a start-up company which wishes to establish a good reputation, it is sufficient to follow the efficient policy developed in a static setting. In particular, the auctioneer needs to estimate the performance levels of the bidders and adjust their unit-price bids by the estimated performance in allocating objects.

If the improvement of performance takes a long time, and the short-run profit is also important for the auctioneer, it is beneficial for the auctioneer to bias toward disadvantaged bidders to promote overall competition. Taking into account bidders' possible upgrade in their performance, auctioneers with limited commitment should bias even more than suggested in a static case. It is important to note that the benefit of preferential treatment to low-performance bidders is at the cost of lower efficiency and the market's overall performance. Moreover, auctioneers may be better off if they can fully commit to a pre-announced revenue-maximizing policy. The lack of commitment in keyword auctions may be due to the fact that the market for keyword advertising is still nascent, and both search engines and advertisers are learning from practice. In a well understood market, it is better for an auctioneer to credibly communicate its commitment, e.g., explicitly announce the weighting factor. In doing so, the auctioneer should bias less for low-performance bidders to encourage them to improve performance.

Future extension of this research includes introducing competition among multiple auctioneers. In the traditional procurement setting, it is natural to assume that auctioneers are monopolists, since the objects requested differ from one another. In keyword auctions, however, the competition among keyword advertising providers (mainly, Google, Yahoo!, and Microsoft) is commonly observed and has become an important feature. Open questions remain: How do bidders make the performance choice with an outside option; and how is the revenue-maximizing weighting factor determined in competing auctions?

Chapter 4

How to Slice the Pie? Optimal Share Structure Design in Keyword Auctions

4.1 Introduction

Unlike conventional advertising, keyword advertising providers, such as Google, Yahoo!, and MSN, use auctions to sell keyword advertising resources. A typical keyword auction runs like this: for each keyword, keyword advertising providers offer several advertising slots at the same time. Advertisers compete for these slots by bidding on their willingness-to-pay per impression (*pay-per-impression*) or willingness-to-pay per click (*pay-per-click*). In pay-per-impression auctions, a keyword advertising provider ranks advertisers by their pay-per-impressions and, in pay-per-click auctions, by the product of pay-per-clicks and click-through-rates, which are defined as the number of clicks per thousand page impressions. The keyword advertising provider then fills keyword advertising slots such that the highest ranked advertiser goes to the first slot, the second highest ranked advertiser goes to the second slot, and so on. If we view slots as resources of various sizes, keyword auctions can also be viewed as share auctions in which the auctioneer packages the total resources into various shares and simultaneously sells them to bidders. We study a "share structure" problem arising from such an auction setting, that is, how much resources should keyword advertising providers set aside for the highest bidder, for the second highest bidder, and so on.

We consider resources in the keyword advertising setting to be "effective impressions," or impressions adjusted for effectiveness. This is to account for the fact that different slots on the same page may get different amount of users' attention. By this notion of resources, a top slot offers more effective impressions than than a bottom one. Similarly, a slot on website optimized for advertising offers more effective impressions than one on a website that is not optimized for advertising, for the same number of raw impressions.[1]

An important premise of our study is that keyword advertising providers can choose the share structure, that is, keyword advertising providers can choose the fractions of total available resources assigned to the highest advertiser, the second highest advertiser, and so on. Such flexible shares can be implemented with existing infrastructures. For example, keyword advertising providers can tailor the amount of effective impressions allocated to k-th highest bidder by randomizing between different slots and by controlling the number of websites on which the advertisement is shown. In the end, advertisers do not get fixed slots, but receive certain amount of effective impressions that may come from multiple slots.[2]

There are several reasons for keyword advertising providers to choose share structures rather than take them for granted. First, assigning the jth slot to the jth highest bidder may be a natural choice when there is a single website. However, with the advent of "AdSense," many websites become available for advertisers. Simply assigning jth slots on every website to the j-th highest advertiser may not always make sense because, for one, first slots on all websites may be too much resources for any single advertiser. Second, there are cases where the number of available

[1]In practice, the effectiveness of slots can be inferred from natural experiments that involve identical advertisements being displayed on different slots. keyword advertising providers may also conduct such experiments.

[2]In fact, even in the current system, advertisers are not guaranteed a particular slot. Google provides advertisers an estimation of expected number of impressions and clicks, along with their expected slots, in its Traffic Estimator software.

slots on any single website may be too few to satisfy many advertisers at the same time. As a result, keyword advertising providers may want to allocate a "fraction" of a slot to an advertiser (e.g., by time-sharing with other advertisers). Third, how advertising slots are located on a page is often up to the publishers. There may be no meaningful ranking among slots (e.g., horizontally-arranged slots may have roughly the same value to advertisers). In sum, as more and more advertisers and publishers participating in keyword auctions,[3] it is imperative for keyword advertising providers to optimize on the way advertising resources are allocated among winning bidders. This requires keyword advertising providers to choose share structures, rather than letting them dictated by the (natural) distribution of slots on and across pages.

To our knowledge, this is the first paper to investigate the optimal share structure problem associated with keyword auctions. We address the optimal share-structure problem in the following setting. The auctioneer has a fixed amount of resources and divides them into multiple shares. Bidders' valuation for a share is determined by the size of the share and a private signal (we shall call it the bidder's *type*). Bidders are invited to bid their willingness-to-pay for per unit resource (or *unit price*). The auctioneer allocates shares to bidders in a way such that the bidder who offers to pay the highest unit price gets the largest share, the bidder who offers to pay the second highest unit price gets the second largest share, and so on. In this setting, we first characterize the optimal share structure that maximizes the auctioneer's expected revenue. We then examine how the optimal share structure is

[3]keyword advertising providers are also actively seeking expansion of their keyword advertising services to other media, including mobile devices, radio, online video, and print advertising. For example, in February 2006, Google announced a deal with global operator Vodafone to include its search engine on Vodafone Live! mobile internet service. Google experimented with classified advertising in the *Chicago Sun-Times* in fall 2005. On January 17, 2006, Google agreed to acquire dMarc Broadcasting, an automated booking and scheduling service for radio advertisements. In August 2007, Google began a new type of video advertisement on popular video site YouTube.

affected by various factors, including bidders' price elasticity of demand (defined as the percentage change in demand due to one percent change in price), the distribution of bidders' type, total resources available, and whether a minimum bid is used.

Our analysis generate several insights on designing optimal share structures in keyword auctions. First, we characterize the optimal share structure under our model setting. We conclude that when advertisers' valuation is linear or convex in share sizes, a single grand share, or winner-take-all, is optimal. When advertisers' valuation is sufficiently concave, multiple shares may become optimal. The optimal share structure generally consists of a series of what we call "plateaus," or consecutive shares with identical size. The starting and ending ranks of plateaus are determined mainly by the distribution of bidders' type. For a special group of single-peaked type distributions (such as uniform, normal, or exponential distributions), plateaus degenerate to single shares so that the optimal share structure consists of a series of strictly decreasing shares.

We also offer several insights on how the optimal share structures should change when the underlying demand or supply factors change. We say a share structure is *steeper* if it has larger high-ranked shares and smaller low-ranked ones. First, as bidders' demands for advertising resources become less elastic, the auctioneer should use a less steep share structure. When bidders have near-perfect or perfect elastic demand (the valuation function is close to linear), the auctioneer should allocate as much resources as possible to the highest advertisers. Second, a change in the type distribution affects the optimal share structure in the following way. The optimal share structure should remain the same in the case of "scaling" (i.e., multiplying each advertiser's type by a common factor), and less steep when the type distribution is "shifted" to the right (i.e., increasing each advertiser's type by a common factor). Third, when total resource increases, the absolute share sizes increase. But changes in percentage values of shares depend on how advertisers' price elasticity of demand changes in total resources assigned to them: high-ranked shares should increase by a larger percentage when bidders' price elasticity of demand increases in the amount of resources allocated, and the converse is true when bidders' price elasticity decreases. This result reinforces our previous intuition that a more elastic demand should be associated with a steeper share structure. Finally, we show that the auctioneer should use a less steep share structure when imposing an optimally set minimum bid (and a steeper share structure when not). Together the above results provide useful guidelines for keyword advertising providers on how to fit share structures to the micro-market conditions for each keyword.

The rest of the paper is organized as follows. In Section 4.2 we discuss the related literature. We set forth our model in Section 4.3. In Section 4.4, we derive general results on optimal share structures. Section 4.5 discusses how the optimal share structure is affected by the underlying supply and demand factors. In Section 4.6, we extend our analysis to a case with minimum bids. Section 4.7 concludes the paper.

4.2 Literature Review

Our research problem bears some connections to the prize-allocation problem in contests (Glazer and Hassin 1988, Moldovanu and Sela 2001, Liu et al. 2007). As in keyword auctions, prizes are allocated by the rank of contestants. Contests are often viewed as all-pay-auctions (all-pay auctions are auctions in which bidders forfeit their bids whether they win or not). Glazer and Hassin (1988) are the first to study the prize-allocation problem using the all-pay auction framework. One of their findings is that winner-take-all is optimal when contestants' cost is a linear function of their effort and skill is uniformly distributed. Moldovanu and Sela (2001) generalize Glazer and Hassin (1988)'s result by showing that winner-take-all is *always* optimal for general distributions as long as the disutility function is linear; on the other hand, when the disutility function is convex, multiple prizes may be optimal. Our winner-take-all result under the linear valuation setting echoes the finding of Moldovanu and Sela (2001) under the contest setting. However, there are noticeable differences between this study and Moldovanu and Sela's. Moldovanu and Sela (2001) model all-pay auctions whereas we study keyword auctions in which only winners pay. We characterize the optimal share structure for n-shares whereas Moldovanu and Sela's analysis is limited to two prizes (generalizing their results to more than two prizes is nontrivial). Our findings on the existence of plateaus in the optimal share structures and its relationship to the underlying type distribution are novel. Moreover, we offer several managerial insights that are nonexistent in Moldovanu and Sela (2001), including how bidders' type distribution and total resources available affect optimal share structures. Liu et al. (2007) also study how type distributions affect allocation of total prize sum, but they do so in a consumer contest setting in which the contest designer has a different objective than ours.

Prior research have examined keyword auctions from several other perspectives. A few authors study Google's novel way of ranking advertisers, that is, by weighting advertisers' pay-per-click bids by their historical click-through-rates (Weber and Zheng 2007, Liu and Chen 2006, Lahaie 2007). They found that such Google's ranking rule achieves efficiency in a unit-price auction setting (Liu and Chen 2006, Lahaie 2007). Edelman et al. (2007) and Varian (2007) both study the "second-price-like" feature of keyword auctions, that is, an advertiser pays not his or her own pay-per-click but the next highest one. Edelman et al. (2007) show that such an auction mechanism does not have a dominant-strategy equilibrium and truth-telling is not an equilibrium. All of the above studies treat share structures as exogenously given. This paper complements the above studies by exposing the share structure decisions facing keyword advertising providers.

Keyword auctions depart from traditional auctions for divisible goods known as "share auctions." The earliest paper to study share auctions is dated back to Wilson (1979). Share auctions have been used in selling important economic resources, such as electricity, pollution permits, and treasury notes (Wang and Zender 2002). In share auctions, auctioneers ask each buyer to report both a price and a share of the total resources that the buyer desires at this price. Auctioneers subsequently solve the market-clearance problem based on buyers' price/share quotes. Whereas in keyword auctions, auctioneers "pre-package" resources into shares, and bidders only bid on prices. Our research on keyword auctions contributes to the literature on divisible-goods auctions by studying a novel auction format.

4.3 Research Model

A risk-neutral auctioneer has an exogenous amount of divisible resources, with normalized size 1. The auctioneer auctions the resources in prepackaged shares to n risk-neutral bidders (advertisers), indexed by $i = 1, 2, ..., n$. The total resources are packaged in as many as n shares (i.e., the number of shares is no more than the number of bidders). Denote s_j as the size of the jth largest share (jth share for short). We term vector $\mathbf{s} = (s_1, s_2, ..., s_n)$ as a *share structure*. We denote $\mathbb{S}$ as the set of *feasible* share structures, that is, those satisfying the following two conditions:

$$s_1 \geq s_2 \geq ... \geq s_n \geq 0 \text{ and } \sum_{j=1}^{n} s_j \leq 1. \tag{4.1}$$

The first condition comes from the definition of s_j's. The second condition requires that the sum of all shares does not exceed available resources.

The auctioneer uses a unit-price auction to allocate all shares simultaneously. In particular, the auctioneer invites bidders to bid on their willingness-to-pay for per-unit resource, or the *unit prices*, and assigns the largest share s_1 to the highest bidder, the second largest share s_2 to the second highest bidder, and so on. Bidders pay for their assigned shares at the unit prices they bid.[4]

Our assumption that advertisers are ranked by their unit-price bids is consistent with prevailing ways of allocating advertising resources. As we have mentioned earlier, one approach is to rank advertisers by their willingness-to-pay per impression, which corresponds directly to our assumption. Another approach is to rank advertisers by the product of their pay-per-clicks and historical click-through-rates. If an advertiser's historical click-through-rate is an unbiased estimator for the advertiser's future click-through-rates, then this approach essentially ranks advertisers by their *expected* pay-per-impressions, thus largely consistent with our model assumption. In fact, our model can also be applied in pay-per-action advertising (i.e., advertisers pay only when a user performs a designated action, such as purchase or registration), as long as keyword advertising providers rank advertisers by the product of pay-per-action and action rates (corresponds to click-through-rates in pay-per-click advertising).

[4]The results on share structures hold also in the case where bidders pay the next highest unit price. This is because the problem of optimal share structure is fully determined by the auctioneer's expected revenue and the auctioneer's expected revenue is the same under two specifications. The proof of the latter is analogous to a proof of the revenue equivalence theorem (Myerson 1981).

We assume that bidder i's valuation of a share is a function of a parameter v_i, termed as bidder i's *type*, and the size of the share. Specifically, bidder i's valuation for the jth share takes the following form,

$$u\left(v_i, s_j\right) = v_i Q\left(s_j\right), \tag{4.2}$$

where $Q(0) = 0$ and $Q'(\cdot) > 0$. Type v captures the difference in bidders' valuation for resources. $Q(\cdot)$ captures how bidders' valuation changes in share sizes.

We assume that each bidder's type is drawn from a common distribution $F(v)$, $v \in [\underline{v}, \bar{v}]$. Each bidder's type is private information, but the distribution $F(v)$ is common knowledge. We assume $F(v)$ is twice-differentiable, and its density function, $f(v)$, is positive anywhere on $[\underline{v}, \bar{v}]$. In the keyword advertising setting, advertisers have different valuation for effective impressions for several reasons. First, everything else being equal, users may tend to click on one advertisement over another. Second, even when users tend to click on two advertisements equally, advertisements may differ in their power to generate follow-up activities such as purchasing or signing up.

Each bidder's expected payoff is the expected valuation minus expected payment to the auctioneer. In particular, if we denote $p_j(b)$ as the probability of winning the jth share by placing bid b, the expected payoff of a bidder of type v is

$$U(v, b) = \sum_{j=1}^{n} p_j(b)\left(vQ(s_j) - bs_j\right) \tag{4.3}$$

The auctioneer's revenue is the sum of payments from all bidders. Because the auctioneer does not know bidders' types or their bids ex ante, the auctioneer's expected revenue is the expected payment from all bidders:

$$\pi = nE\left[b\sum_{j=1}^{n} p_j(b)s_j\right] \tag{4.4}$$

In sum, the game proceeds as follows. First, the auctioneer announces a share structure $(s_1, s_2, ..., s_n)$. All bidders learn their types and compete for shares by bidding their unit prices. The auctioneer then allocates the shares based on the ranking of the bids. A bidder's problem is to maximize, for any announced share structure, the expected payoff (4.3) by choosing a unit price b. The auctioneer's problem is to maximize the expected revenue (4.4) by choosing a share structure $\mathbf{s} \in \mathbb{S}$. How to design the share structure from the auctioneer's perspective is the focus of this paper.

Notice that given the valuation function (4.2), we can define an induced demand function of bidder i for a fixed-price setting

$$D_i(p) = \arg\max_s \{v_i Q(s) - ps\} \tag{4.5}$$

Correspondingly, we can calculate the *(price) elasticity of demand* as $-\frac{Q'(s)}{Q''(s)s}$. Clearly, $Q(\cdot)$ determines the elasticity of bidders' demand. Specifically, a bidder's demand is perfectly elastic if $Q(\cdot)$ is linear, and the elasticity of demand decreases as $Q(\cdot)$ becomes more concave.

4.4 General Results on the Optimal Share Structure

In this section, we study the keyword advertising provider's problem of choosing an optimal share structure. We assume there is no minimum bid to focus on the impact of underlying supply and demand factors. We examine the interaction between the optimal share structure and the minimum bid policy in Section 4.6. In the following, we first derive some basic concepts and results under a relatively simple linear-valuation setting. We then use these concepts and results to examine the optimal share structures under more general valuation functions.

4.4.1 Equilibrium Bidding Function and Revenue

Using methods outlined in the auction literature (e.g., McAfee and McMillan 1987), we can derive bidders' bidding function (see the appendix for details) as

$$\beta(v) = v\frac{\sum_{j=1}^{n} P_j(v)Q(s_j)}{\sum_{j=1}^{n} P_j(v)s_j} - \frac{\sum_{j=1}^{n} Q(s_j)\int_{\underline{v}}^{v} P_j(x)dx}{\sum_{j=1}^{n} P_j(v)s_j},\ v \in [\underline{v}, \bar{v}] \tag{4.6}$$

where

$$P_j(v) \equiv \binom{n-1}{n-j} F(v)^{n-j}(1-F(v))^{j-1} \tag{4.7}$$

is the equilibrium probability for a bidder of type v to win the jth share .

The auctioneer's expected revenue is the sum of expected payments from all bidders, which can be written as (see the appendix for details)

$$\pi \;=\; n\sum_{j=1}^{n} Q(s_j)\int_{\underline{v}}^{\bar{v}} P_j(v)\left[v - \frac{1-F(v)}{f(v)}\right] f(v)dv \tag{4.8}$$

We denote

$$J(v) \equiv v - \frac{1-F(v)}{f(v)}, \tag{4.9}$$

and term $J(v)$ the *marginal revenue of type* v.[5] The term $Q(s_j)J(v)$ represents a bidder's contribution to the auctioneer's revenue if the bidder's type is v and is assigned the jth share. It is clear from (4.9) that the marginal revenue from a bidder is less than the bidder's true type v (except for bidders with $v = \bar{v}$). The difference reflects the bidder's informational rent.

We further denote

$$\alpha_j \equiv n\int_{\underline{v}}^{\bar{v}} P_j(v)J(v)f(v)dv,\ j = 1, 2, ..., n \tag{4.10}$$

[5]The term "marginal revenue" is also used by Bulow and Roberts (1989) and Klemperer (1999) under a single-object auction setting.

and term α_j the *return factor for the jth share.* Noting that $P_j(v)$ is the equilibrium probability of winning the jth share, the return factor is the expected marginal revenue generated by winners of the jth share. By this definition, the expected revenue (4.8) can be written as

$$\pi = \sum_{j=1}^{n} \alpha_j Q(s_j) \tag{4.11}$$

Intuitively, a share with a high return factor yields higher revenue for the auctioneer than a same-sized share with a low return factor. The auctioneer's problem is to maximize (4.11) subject to conditions in (4.1).

The relative magnitude of the return factors holds special importance in the optimal share structure problem. The return factor of a share is generally determined by the distribution of types and the number of bidders, but is independent of the share structure $\mathbf{s}$ or the valuation function $Q(\cdot)$. The following lemma provides some insight on the ranking of α_j's.

We say a distribution satisfies the monotone-hazard-rate (MHR) condition if its hazard rate, $\frac{f(v)}{1-F(v)}$, is monotonically increasing within its support. The MHR property is satisfied by most commonly used single-peaked distributions, such as uniform, normal, and logistic.

Lemma 4.4.1. *(a)* $\alpha_1 > \alpha_j$*, for all* $j > 1$*, and* $\alpha_1 > 0$*. (b) Under the MHR condition,* $\alpha_1 > \alpha_2 > ... > \alpha_n$*.*

Lemma 4.4.1 (a) shows that the first share is superior than any other share in terms of return factors, regardless of the number of bidders or the type distribution. The intuition for this result is as follows. According to (4.10), the return factor of the jth largest share is the expected marginal revenue of the winner of the jth largest share. Notice that the marginal revenue is the highest at $\bar{v}$ regardless of the type distribution. This implies that the marginal revenue at the neighborhood of $\bar{v}$ is higher than other values of v (due to the continuity). Meanwhile, the bidders at that neighborhood of $\bar{v}$ are more likely to win the first share. Therefore, the first share has an advantage of getting the bidders with the highest marginal-revenue, therefore generating the highest return.

The intuition for (b) is as follows. In general, a high-type bidder is more likely to win a high-ranked share than a low-ranked share, and the reverse is true for a low-type bidder. The MHR condition, which implies that a high-type bidder has higher marginal revenue than a low-type bidder, reinforces the advantage of offering a high-ranked share (in terms of the revenue generated), thus ensuring the decreasing order among return factors.

Lemma 4.4.1 has immediate implications for the linear-valuation case. With linear valuation $Q(s) = s$, an advertiser's valuation of a share becomes $u(v, s) = vs$. So v can be interpreted as advertisers' marginal valuation. The problem of choosing an optimal share structure (4.11) becomes the following constrained linear program:

$$\max_{s_1, s_2, \ldots, s_n} \sum_{j=1}^{n} \alpha_j s_j, \text{ subject to (4.1)} \tag{4.12}$$

Since α_1 is the largest among all the return factors by Lemma 4.4.1, the auctioneer should allocate all the resources to the first share, leading to a winner-take-all share structure. Under the MHR condition, return factors decrease. The winner-take-all result can be strengthened to a greedy allocation, that is, to fill up the jth share before the $(j+1)$th share. The strengthened result is useful when, say, the first share must be less than 1. The following proposition summarizes the above intuition.

Proposition 4.4.2. *If advertisers' valuation is linear in share size, (a) it is optimal to provide one grand share of size 1 (whenever possible). (b) When the MHR condition* *holds, the auctioneer should allocate the resources in a greedy way, that is, to fill up jth share before moving on to $(j+1)$th share.*

Proof. omitted. □

Recall that linear valuation corresponds to the setting in which advertisers' demand for effective impressions is perfectly elastic. This setting may hold approximately when supply of advertising resources is small compared with the demand – if we believe advertisers' marginal valuation starts to decline beyond a certain point. Proposition 4.4.2 suggests that when advertisers' valuation is perfectly elastic (supply is relatively small), it is optimal to use a winner-take-all share structure.

Our analysis shows that a high-ranked share promotes higher bids from high-type bidders than a low-ranked share, provided that two shares have the same size. The opposite is true for low-type bidders. So the optimal choice of share structure generally involves trading off between best motivating high-type bidders and best motivating low-type ones. One would naturally think it depends on how bidders' type is distributed. Surprisingly, the above proposition shows that it is *always* optimal to best-motivate the high-type ones, regardless of the distribution of bidder types or the number of bidders.

It is also worth pointing out that when the MHR condition is violated, return factors may not follow a descending order, and therefore the greedy allocation specified in (b) may not be optimal, as proved by the following example.

Example 4.4.3. *Let $F(v) = (v-1)^{1/4}$, $v \in [1, 2]$, and $n = 3$. Assume that share sizes cannot exceed* 0.5*. Calculation shows that the return factors are* 1.14*,* 0.91*, and* 0.94*, respectively. A greedy allocation,* $(0.5, 0.5, 0)$*, generates an expected revenue of* 1.029

whereas the optimal share structure, $(0.5, 0.25, 0.25)$*, generates an expected revenue of* 1.036*.*

4.4.2 Concave Valuation

Proposition 4.4.2 suggests that the auctioneer should allocate all the resources to the largest share. In the keyword advertising setting, this would require one advertisement to appear at all available advertising slots for a particular keyword. However, we seldom see this. One explanation could be that advertisers' valuation is nonlinear in share sizes. We address the nonlinear valuation case in this section.

When $Q(\cdot)$ is convex, bidders' marginal valuation, $vQ'(s)$, increases with the share size. Hence, the auctioneer has stronger incentive to create a larger share than in the linear valuation case. Consequently, the results in proposition 4.4.2 continue to hold. Thus, we will focus on the case of concave $Q(\cdot)$.

In keyword advertising, $Q(\cdot)$ might be concave for a few reasons. First, consumers' attention devoted to an advertisement may be less than twice as much if we double the amount of time it is shown. Second, the unit cost to fulfill consumers' requests may rise because of limited production/service capacity. Thus, advertisers' marginal valuation for effective impression may decrease as the total effective impressions increase. Casual observation shows that smaller e-commerce websites start losing some customers because of the congestion problem as the traffic to their website becomes very high.

In the case of concave valuation, bidders' unit valuation decreases with share size, and so does the unit price they are willing to pay. As a result, the auctioneer has additional incentive to offer the advertising resources in smaller shares. The following example illustrates an optimal share structure with multiple shares. The optimal share structure consists of several groups of same-sized shares. We call each such group a *plateau* (a plateau is *nontrivial* if it consists of more than one share).

Example 4.4.4. *Let $F(v) = (v-1)^{1/4}$, $v \in [1,2]$, $n = 3$, and $Q(s) = \sqrt{s}$. Under this specification, the optimal share structure is $(0.4311, 0.2845, 0.2845)$, where the first plateau consists of the first share, and the second plateau consists of the second and the third shares.*

When do nontrivial plateaus occur in optimal share structures? What determines the boundaries of plateaus (i.e., their starting and ending ranks)? The following lemma shows that the boundaries are intimately related to return factors.

We let $j_0 = 0$ and define index j_k as one that maximizes the average return factor of shares starting from $(j_{k-1}+1)$.[6] Formally, let

$$j_k = \arg \max_{j \in \{j_{k-1}+1, \ldots, n\}} \left\{ \frac{1}{j - j_{k-1}} \sum_{l=j_{k-1}+1}^{j} \alpha_l \right\} \tag{4.13}$$

Lemma 4.4.5. *Under an arbitrary concave function $Q(\cdot)$, (j_k+1)th to (j_{k+1})th shares must be equal in size in the optimal share structure.*

Lemma 4.4.5 says that the optimal share structure is segmented into a series of plateaus with j_k being their ending ranks. Denote the *average return factor for the k-th plateau* as

$$\bar{\alpha}_k \equiv \frac{1}{j_k - j_{k-1}} \sum_{j=j_{k-1}+1}^{j_k} \alpha_j \tag{4.14}$$

By definition of j_k's, the $\bar{\alpha}_k$'s must strictly decrease; otherwise, $\bar{\alpha}_k$ is less than or equal to $\bar{\alpha}_{k+1}$, and j_k cannot be the maximizer for the average return factor starting from

[6] If there are multiple maximums, we define j_k as the largest one.

$j_{k-1}+1$. Moreover, for a similar reason, the average return factor of the first l shares within a plateau is no greater than that of the remaining shares in that plateau.

The rationale behind this lemma is as follows. Suppose (j_k+l)-th and (j_k+l+1)-th shares are both in a same plateau and the former is assigned more resources than the latter. The auctioneer can always profitably shift a small amount of resource from each of the first l shares and spread equally among remaining shares in the plateau. This is because our definition of j_k's guarantees the average return factor of the first l shares in a plateau to be lower than that of the remaining ones in the same plateau. This process can continue till all shares within the plateau are equal in size.

Note that from Lemma 4.4.1, the return factor of the first share is higher than that of any other share. Hence the return factor of the first share is higher than any average return factors starting from the first share. This implies the first plateau must end at rank 1. In other words, the first plateau must consist of the first share only.

Example 4.4.6. *Continue with the previous example. Calculation shows that* $\alpha_1 = 1.14$, $\alpha_2 = 0.91$, *and* $\alpha_3 = 0.94$. *By definition of* j_k*'s (4.13),* $j_1 = 1$ *and* $j_2 = 3$. *Thus, the first plateau consists of the first share, and the second plateau consists of the second and the third shares, confirming the previous example.*

Based on the result in Lemma 4.4.5, we characterize the optimal share structure in the following proposition.

Proposition 4.4.7. *Given an arbitrary concave function* $Q(\cdot)$ *and* j_k*'s and* $\bar{\alpha}_k$*'s as defined in (4.13) and (4.14), respectively, the optimal share structure is given by*

$$\mathbf{s}^* = (z_1\ ,\ \underbrace{z_2, ..., z_2}_{j_2-j_1}\ ,\ \underbrace{z_3, ..., z_3}_{j_3-j_2}\ ,\ \dots\ ,\ \underbrace{z_k, ..., z_k}_{j_{k^*}-j_{k^*-1}}\ ,\ \underbrace{0, ..., 0}_{n-j_{k^*}}) \tag{4.15}$$

where the number of positive plateaus k^* and their sizes $z_1, ..., z_{k^*}$ are determined by

$$\bar{\alpha}_1 Q'(z_1) = \bar{\alpha}_2 Q'(z_2) = ... = \bar{\alpha}_{k^*} Q'(z_{k^*}) \geq \bar{\alpha}_{k^*+1} Q'(0) \tag{4.16}$$

and $\sum_{k=1}^{k^*} (j_k - j_{k-1}) z_k = 1$*. Furthermore,* $z_1 > z_2 > ... > z_{k^*} > 0$*.*

Proposition 4.4.7 shows that the optimal share structure consists of a series of decreasing plateaus. This is a result of decreasing average return factors associated with these plateaus. The beginning and ending ranks of each plateau are determined only by the return factors (and thus by the type distribution and the number of bidders), whereas the number of plateaus and the share sizes within plateaus are jointly determined by the return factors and the shape of the valuation function.

Under the MHR condition, the return factors monotonically decrease. As a result, the $\bar{\alpha}_k$'s are identical to α_j's, and each plateau consists of a single share. Hence, we have the following corollary.

Corollary 4.4.8. *If the MHR condition holds, the optimal share structure* $\mathbf{s}^*$*, together with the optimal number of positive shares,* k^**, is determined by*

$$\alpha_1 Q'(s_1) = \alpha_2 Q'(s_2) = ... = \alpha_{k^*} Q'(s_{k^*}) \geq \alpha_{k^*+1} Q'(0). \tag{4.17}$$

and $\sum_{k=1}^{k^*} s_k = 1$*. Furthermore,* $s_1 > s_2 > ... > s_{k^*} > 0$*.*

Proposition 4.4.7 and Corollary 4.4.8 indicate that the average marginal returns ($\alpha_k Q'(s_k)$) should be equal across plateaus. This is because otherwise the auctioneer can always profitably shift the resources from a plateau with a low average return to ones with high average returns. Plateaus will degenerate to single shares if the MHR condition holds (Corollary 4.4.8). Otherwise, an optimal share structure may contain plateaus of at least two shares wherever the underlying average returns is non-monotonic. It is worth noting that the optimal share structure characterized in Proposition 4.4.7 holds for general concave valuations and type distributions.

4.5 The Relationship between Optimal Share Structures and Underlying Supply and Demand Factors

In this section, we carry out a series of comparative-static analysis on the optimal share structure. The optimal share structure may be affected by several factors, including the shape of the bidders' valuation function $Q(\cdot)$, the distribution of type v, and the total resources available. A comparative-static analysis on these underlying factors provides rich managerial implications on how to set different share structures for different market settings.

To carry out the comparative-static analysis, we need a way of comparing share structures. We propose a "steepness" order defined as follows.

Definition 4.5.1 (Steepness Order)**.** Let $\mathbf{s} = (s_1, ..., s_n)$ and $\hat{\mathbf{s}} = (\hat{s}_1, ..., \hat{s}_n)$ be two feasible share structures with $\sum_{j=1}^{n} s_j = \sum_{j=1}^{n} \hat{s}_j$. We say $\hat{\mathbf{s}}$ is less steep (or flatter) than $\mathbf{s}$ if there exists c, $c \in \{1, 2, ..., n\}$, such that

$$\hat{s}_j < s_j, \forall j < c \text{ and } \hat{s}_j \geq s_j, \forall j \geq c. \tag{4.18}$$

We say $\hat{\mathbf{s}}$ is strictly less steep (or strictly flatter) than $\mathbf{s}$ if $c > 1$.[7]

Intuitively, a steeper share structure has larger high-ranked shares and smaller low-ranked shares. By the definition of the steepness order, the steepest share structure is one grand share, or *winner-take-all*, and the least steep one is n equal shares.

The steepness order can be measured by the widely used Herfindahl index, defined as the sum of squares of shares. We can easily verify that the steeper the share structure the higher the Herfindahl index, but the converse is not necessarily

[7]When $c = 1$, $\hat{s}_j \geq s_j$ for any j, implying $\hat{\mathbf{s}} = \mathbf{s}$ because the total allocated resources are the same.

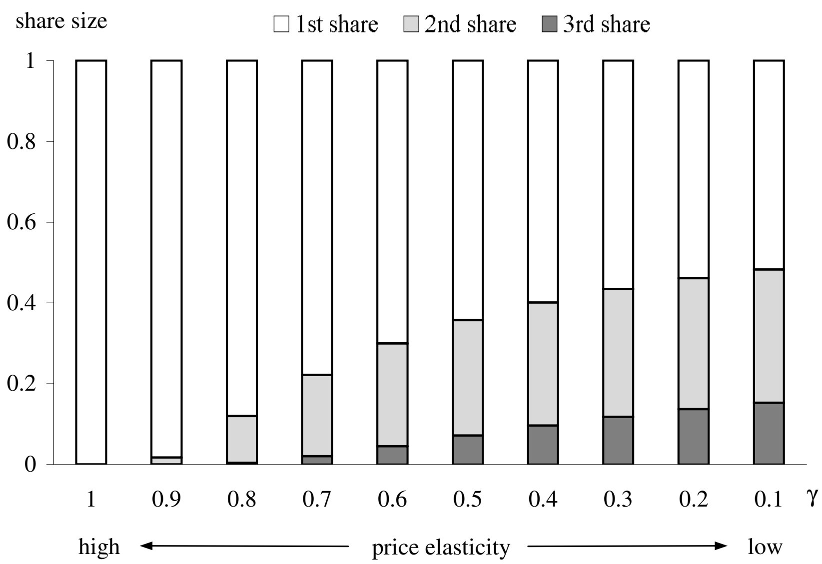


Figure 4.1: Optimal Share Structure as a Function of γ, $Q(s) = s^\gamma$

true. For example, $(0.6, 0.2, 0.2)$ has a higher Herfindahl index than $(0.5, 0.35, 0.15)$, but the former is neither steeper nor flatter than the latter.

4.5.1 Concavity of the Valuation Function

We next consider the impact of the concavity of the valuation function on the optimal share structure, starting with an example.

Example 4.5.1. *Let v be uniformly distributed on $[1, 2]$, $n = 3$, and $Q(s_j) = s_j^\gamma$. Figure 4.1 shows the optimal sizes of the first, the second, and the third shares as functions of γ.*

Example 4.5.1 shows that as the concavity increases (γ decreases), the optimal share structure moves away from winner-take-all toward a more egalitarian share structure. Specifically, this involves a gradual shift of resources away from the first share to the second and the third shares. Can this example be generalized to a broader setting? Below we show that the answer is yes.

We adopt a typical concavity measure using the *concave transformation* (Mas-Colell et al. 1995). Let $Q(\cdot)$ and $\hat{Q}(\cdot)$ be strictly increasing and concave functions defined on $X \subseteq R^{+}$. We say $\hat{Q}(\cdot)$ is more concave than $Q(\cdot)$ if there exists an increasing concave function $\psi(\cdot)$ such that $\hat{Q}(x) = \psi(Q(x))$ for every x.

Proposition 4.5.2. *The optimal share structure becomes less steep as the concavity of $Q(\cdot)$ increases. Furthermore, if the optimal share structure under $Q(\cdot)$ is not winner-take-all, the optimal share structure becomes strictly less steep as the concavity of $Q(\cdot)$ increases.*

By Proposition 4.5.2, as the valuation function becomes more concave, the optimal share structure moves away from winner-take-all toward n equal shares. It is worth noting that the result in Proposition 4.5.2 holds regardless of the type distribution.

The process prescribed by Proposition 4.5.2 has a few properties. First, as the concavity of valuation function increases, the number of shares weakly increases. This is because if $\hat{\mathbf{s}}$ is flatter than $\mathbf{s}$, the number of (positive) shares in $\hat{\mathbf{s}}$ is no less than those in $\mathbf{s}$. Second, the new optimal share structure, provided it is different from the previous one, always has a smaller first share. Otherwise, the steepness order implies the remaining shares would be at least as large as those in the previous optimal share structure, which is unlikely except when the two share structures are exactly the same. Third, the low-ranked shares will not become smaller as valuation functions become more concave. For example, if the current optimal share structure is $(0.5, 0.25, 0.25)$, $(0.45, 0.35, 0.2)$ must not be optimal for more concave valuation functions.

We are the first, to our knowledge, to establish the relationship between the concavity of bidders' valuation (which can also be interpreted as the price elasticity of their demands) and the steepness of optimal share structure. Moldovanu and Sela (2001) suggested that it may become optimal to award multiple prizes when contestants' disutility functions become concave. But they have not gone further to show how a multiple-prize structure should evolve as the disutility functions become more concave. We show that share structures become less steep as the valuation function becomes more concave for a general type distribution. One must note that it is not entirely clear whether similar results hold under the contest setting because of differences in the problem structures.

The implication of Proposition 4.5.2 is highly actionable. Keyword advertising providers can estimate the elasticity of advertisers' demands for a particular keyword, which is possible given the bidding history of advertisers and the experimentation opportunities in keyword auctions. Then, based on whether the elasticity is high or low, keyword advertising providers decide whether to provide steep share structures (e.g., via featured listings) or flat ones (e.g., via randomizing in slot assignment).

4.5.2 Type Distribution

The distribution of types varies across different keywords. Some keywords (e.g., "mortgage") are more expensive than others (e.g., "CD"). The distribution of willingness-to-pay per impression may also differ from one keyword to another. For example, keywords with generic appeal may attract advertisers from different industries with wildly different willingness-to-pay; whereas more specific ones may attract advertisers of the same narrowly defined industry with nearly identical willingness-to-pay. The question is how the keyword advertising providers should tailor the share structure offerings from different type distributions.

The type distribution affects the optimal share structure via the return factors.

To study the effect of the type distribution, we first must understand how the return factors impact steepness of the optimal share structure. The next lemma associates the steepness with return factors.

Lemma 4.5.3. *Let $Q(\cdot)$ be an arbitrary concave function and denote* $\mathbf{s}$ *and* $\hat{\mathbf{s}}$ *as the optimal share structures under return factor vectors* $\alpha = (\alpha_1, ..., \alpha_n)$ *and* $\hat{\alpha} = (\hat{\alpha}_1, ..., \hat{\alpha}_n)$*, respectively. Suppose that the plateau boundaries under* α *and* $\hat{\alpha}$ *are the same.*[8] $\hat{\mathbf{s}}$ *is less steep than* $\mathbf{s}$ *if*

$$\frac{\hat{\bar{\alpha}}_{k+1}}{\hat{\bar{\alpha}}_k} \geq \frac{\bar{\alpha}_{k+1}}{\bar{\alpha}_k}, \text{ for any } k \text{ such that } z_{k+1} > 0. \tag{4.19}$$

$\hat{\mathbf{s}}$ *is strictly less steep than* $\mathbf{s}$ *if at least one strict inequality holds in (4.19).*

Lemma 4.5.3 shows that if $\hat{\bar{\alpha}}_{k+1}/\hat{\bar{\alpha}}_k$ is closer to 1 or, in other words, the average return factors are more equal among different plateaus, then the optimal share structure is flatter.

We say the type distribution is "scaled" if each bidder's type parameter is multiplied by w, $w > 0$. We say bidders' type distribution is "shifted" to the right if each bidder's type parameter increases by w, $w > 0$. The following proposition summarizes the impact of scaling, shifting, and a change in the underlying type distribution termed as *marginal-revenue-ratio compression.*

[8] This result can be extended to share structures with different plateau boundaries. To do so, we can "iron out" the peaks in the return factors in each share structure by defining a normalized return factor vector $\beta \equiv (\beta_1, ..., \beta_n)$ such that $\beta_{j_{k-1}+1} = ... = \beta_{j_k} = \bar{\alpha}_k$, for each k. The normalized return factor vector retains the original plateau boundaries and average return factors, and is nonincreasing. The optimal share structure under the normalized return factor vector is the same as under the original one, because by Proposition 4.4.7, all that matters to the optimal share structure problem is the average return factors. We can then compare two share structures based on normalized return factors using the following result: $\hat{\mathbf{s}}$ is less steep than $\mathbf{s}$ if $\frac{\hat{\beta}_{j+1}}{\hat{\beta}_j} \geq \frac{\beta_{j+1}}{\beta_j}$, for any j such that $s_{j+1} > 0$ (the proof is similar to the proof of Lemma 4.5.3).

Proposition 4.5.4. *We use α_j and $\hat{\alpha}_j$ to denote the return factors under type distributions F and $\hat{F}$, respectively.*

(a) *(scaling) When $\hat{F}$ is F scaled by a factor w, the return factors $\hat{\alpha}_j = w\alpha_j$, and the optimal share structure remains the same.*

(b) *(shifting) When $\hat{F}$ is F shifted to the right by w, the return factors $\hat{\alpha}_j = \alpha_j + w$, and the optimal share structure becomes less steep.*

(c) *(marginal-revenue-ratio compression) Assume both type distributions are regular and have positive marginal revenues. When the ratio of marginal revenues at percentile y and percentile x ($y > x$) under $\hat{F}$ is smaller than that under F, that is,*

$$\frac{\hat{J}(\hat{F}^{-1}(y))}{\hat{J}(\hat{F}^{-1}(x))} \leq \frac{J(F^{-1}(y))}{J(F^{-1}(x))}, 0 \leq x < y \leq 1, \tag{4.20}$$

$\hat{\mathbf{s}}$ *is less steeper than* $\mathbf{s}$ *(strictly steeper if the above holds in strict inequality).*

Proof. (a) See Appendix for a proof for $\hat{\alpha}_j = w\alpha_j$. Given $\hat{\alpha}_j = w\alpha_j$, the j_k sequence remains the same as before by definition, suggesting the boundaries of plateaus will be the same. In addition, the optimal solution to (4.16) is invariant to a scaling of $\hat{\alpha}_j$'s, suggesting the optimal share size for each plateau is also the same.

(b) See Appendix for a proof for $\hat{\alpha}_j = w + \alpha_j$. Given $\hat{\alpha}_j = w + \alpha_j$, the j_k sequence remains the same as before by definition, suggesting the boundaries of plateaus will be the same. However, the ratio of average return factors decreases because for given $w > 0$, $\frac{\bar{\alpha}_{k+1}}{\bar{\alpha}_k} < \frac{\bar{\alpha}_{k+1}+w}{\bar{\alpha}_k+w}$. Therefore, the optimal share structure becomes less steep (Lemma 4.5.3).

(c) See the appendix for a proof. □

The intuition for Proposition 4.5.4(a) is as follows. When all bidders' valuation is scaled by factor w, their equilibrium bids are scaled by the same factor, but their probabilities of winning each share remain the same. As a result, all the return factors are scaled by the same factor. Because what matters to the optimal share structure are the *relative* sizes of return factors, the optimal share structure should remain the same.

Proposition 4.5.4(b) suggests shifting to the right causes a flatter optimal share structure. Again, shifting does not change bidders' winning probabilities. But it causes all bidders' to increase their bids. The increase in low-type bidders' bids is more significant as their type v has increased by a larger proportion. This in turn reduces the ratios between return factors of high-ranked shares and those of low, because the low-ranked shares are more likely assigned to low-type bidders. As the return factors for low-ranked shares increase relative to those for high-ranked shares, the optimal share structure should be less steep (Lemma 4.5.3).

$J\left(F^{-1}\left(x\right)\right)$ measures the marginal revenue at percentile x. Proposition 4.5.4 (c) suggests that the optimal share structure should be less steep if marginal revenue ratios between any two percentiles are compressed. This result highlights that the optimal share structure is particularly associated with marginal revenues, and what matters is the *ratios* of marginal revenues at different percentiles rather than the absolute marginal revenues.

Returning to the questions we raised at the beginning of the section, we can now say that the optimal share structure for expensive keywords may be steeper or flatter than less expensive ones. If the type distribution (approximately interpreted as willingness-to-pay) for more expensive keywords is simply a rescaling of that for the less expensive ones, such as from a uniform distribution on $[1, 2]$ to a uniform distribution on $[2,4]$, keyword advertising providers should apply the same share structure for these keywords. If the type distribution shifts from $[1,2]$ to $[4,5]$, the marginal revenue ratio between two percentiles is compressed, and keyword advertising providers should use a flatter share structure (Proposition 4.5.4 (b) and (c)).

4.5.3 Total Resources

So far we have assumed the total resource is fixed and normalized to 1. In this subsection, we allow the total resources to change and examine its impact on optimal share structures.

The total resources available to advertisers change with the number of searches conducted (in search-based advertising) and the number of page views (in contextual advertising). The total resources may change over time for many reasons, including a change in the popularity of the websites, increased searches on particular keywords because of special events, or new additions to the advertising networks. For example, when Google signed a contract with AOL.com to serve online advertisements on AOL.com, Google's keyword advertising resources surged. How should keyword advertising providers adjust their share offerings according to the total resources available? We address this question below.

First, note that our characterization of the optimal share structures in Propositions 4.4.2 and 4.4.7 still holds except that total available resources are no longer 1. Moreover, when total resource is the only changing factor, the return factors are the same, and therefore the boundaries of plateaus should remain the same (Lemma 4.4.5). Therefore, we can concentrate on how the total resources are allocated among different plateaus. We observe the following trend in the sizes and the number of shares in the optimal structure.

Lemma 4.5.5. *Under a concave $Q(\cdot)$, as the total resources increase, the size of each positive share increases and the number of positive shares weakly increases.*

In general, the marginal return of resource decreases in the amount of resources allocated because of the concave valuation function. The size of each share must increase because otherwise it will generate higher marginal return than previously, contradicting the optimal condition.

From the above analysis we know that all shares will increase. But how about the proportions of shares? When all shares increase by the same percentage, the proportions of all shares (relative to the total resources) should remain constant. When high-ranked shares increase by a larger (smaller) percentage than low-ranked shares, the proportion of high-ranked share should increase (decrease). The following examples show that the proportions of shares may or may not be the same.

Example 4.5.6. *Assume there are two bidders. Let $Q(s) = \sqrt{s}$ and assume $\alpha_1 > \alpha_2 > 0$. According to (4.17), the optimal size of the first share s_1^* must satisfy $-\frac{1}{2}(s_1^*)^{-0.5}\alpha_1 = -\frac{1}{2}(s_2^*)^{-0.5}\alpha_2$, which implies that the proportion of share 1, $\frac{s_1^*}{s_1^*+s_2^*} = \frac{\alpha_1^2}{\alpha_1^2+\alpha_2^2}$, is a constant.*

Example 4.5.7. *Continue with the previous example. Let $Q(s) = ln(s+1)$ instead and assume $\alpha_1 < 2\alpha_2$ (to rule out the winner-take-all case). By (4.17), we have $\frac{s_1^*+1}{s_2^*+1} = \frac{\alpha_1}{\alpha_2}$. Using simple algebra we can conclude that the proportion of the first share $(\frac{s_1^*}{s_1^*+s_2^*})$ decreases as the total resources increase.*

The following proposition suggests that whether some shares will increase relatively faster than others depends on how bidders' price elasticities change with the resources allocated to them.

Proposition 4.5.8. *As the total resources increase, a high-ranked share will increase by a smaller (bigger, or the same) percentage than a low-ranked one, if bidders' price elasticity decreases (increases, or remains constant) in the amount of resources allocated to them.*

The above result has an intuitive explanation. When bidders have a decreasing price elasticity, their demand becomes less elastic as shares become larger. Thus, when the total resources increase, the auctioneer should increase high-ranked shares by a smaller percentage. Conversely, when bidders have increasing (constant) price elasticity, the auctioneer should increase high-ranked shares by a larger (equal) percentage.

To compare share structures by their proportions, we can normalize the total resources available to one by dividing each share by $\sum_{j=1}^{n} s_j$. After such a normalization, the above proposition effectively says that as the total resources increase, the optimal (normalized) share structure becomes flatter (becomes steeper, or remains constant), if bidders' price elasticity decreases (increases, or remains constant) in the amount of resources allocated to them. This result reveals a clear connection between the steepness of optimal share structures and bidders' demand elasticities through changes in total resources and their impact on bidders' demand elasticities. In example 4.5.6, it can be easily verified that we have constant elasticity of demand, which accounts for the same increase speeds for shares or the same steepness as the total resource increases.

4.6 The Interaction between Optimal Share Structures and Minimum Bids

In this section, we extend our model to the case where minimum bids are imposed. Most of keyword advertising providers use a minimum bid policy to screen advertisers. This is consistent with the general result in the optimal auction design literature that an optimally set minimum bid can help auctioneers to achieve higher revenue. In the following, we focus on how the introduction of minimum bids affects the optimal share structure. We start by examining the optimal minimum bid policy under our model setting.

We assume the auctioneer imposes a minimum bid r. As a result, some low-type bidders may no longer participate in the auction, because bidding higher than r would result in negative payoffs for them. Therefore, there exists a corresponding marginal type, v_0, who is indifferent between participating and not participating.

Using a similar approach as in section 4.4.1, we can derive the equilibrium bidding function for bidders and the expected revenue for the auctioneer under a minimum bid r, as follows.

$$\beta(v) = v\frac{\sum_{j=1}^{n} P_j(v)Q(s_j)}{\sum_{j=1}^{n} P_j(v)s_j} - \frac{\sum_{j=1}^{n} Q(s_j)\int_{v_0}^{v} P_j(x)dx}{\sum_{j=1}^{n} P_j(v)s_j},\ v \in [v_0, \bar{v}] \quad (4.21)$$

$$\pi = n\sum_{j=1}^{n} Q(s_j)\int_{v_0}^{\bar{v}} P_j(v)J(v)f(v)dv \quad (4.22)$$

where the marginal type v_0 is determined by

$$r = v_0\frac{\sum_{j=1}^{n} P_j(v_0)Q(s_j)}{\sum_{j=1}^{n} P_j(v_0)s_j}. \quad (4.23)$$

The bidding function and the auctioneer's revenue are similar to those without minimum bids, except that the lower bound of integral changes from $\underline{v}$ to the marginal type v_0.

The auctioneer should choose the optimal marginal type v_0^* to maximize the total expected revenue (see the proof of Proposition 4.6.1 for the derivation of v_0^*).[9] We assume $v_0^* > \underline{v}$ to avoid the trivial case. It is worth noting that although the optimal marginal type does not depend on the share structure, the corresponding minimum bid does, as seen from (4.23).

We redefine the return factors as

$$\alpha_j(v_0) \equiv n \int_{v_0}^{\bar{v}} P_j(v) J(v) f(v) dv, \tag{4.24}$$

which are the same as before except that they are now functions of the marginal type. We show in the appendix (in Lemma C.0.1) that Lemma 4.4.1 (b) carries over to this case; that is, under the MHR condition, the return factors decrease in rank. This is because excluding low-type bidders reinforces the advantages of high ranked shares in generating returns. Immediately, it follows that Proposition 4.4.2 (b) carries over too; that is, when the MHR condition holds and the valuation is linear, the auctioneer should use a greedy allocation among shares.

Given any marginal type, the characterizations of optimal share structures in Proposition 4.4.7 and Corollary 4.4.8 continue to hold. So do the results on the impact of concavity of valuation function (Proposition 4.5.2) and of total resources (Proposition 4.5.8) on optimal share structures. In addition, the impact of scaling and shifting of type distribution in Proposition 4.5.4 continues to hold as long as the marginal type is adjusted accordingly (multiplied by w in scaling case and increased by w in shifting case).

[9] Under reasonable conditions, v_0^* is unique and is invariant to the share structure. For example, under the MHR condition, the marginal revenue function $J(v)$ crosses zero (from below) at most once, and thus it is optimal to set the optimal marginal type to the crossing point or $\underline{v}$, whichever is higher. When the MHR condition is not satisfied, $J(v)$ may cross zero multiple times. But in many cases, the optimal marginal type is the same crossing point across different share structures.

We next examine the impact of minimum bids on optimal share structures. Properly set minimum bids will exclude low-type bidders with negative marginal revenue, and thus improve the return factors of all shares. However, the improvements may differ across shares. Therefore, the optimal share structure may change as a result of imposing minimum bids, as suggested by the following proposition.

Proposition 4.6.1. *If the MHR condition holds, the optimal share structure under the optimal minimum bid* r^* *is flatter than under no minimum bid.*

Intuitively, minimum bids exclude low-type bidders who have negative marginal revenues, and thus increase the total expected revenue. These bidders, if permitted in the auction, would be more likely to win low ranked shares than high ranked ones. Therefore, excluding them results in relatively larger improvements on the marginal returns of low ranked shares than those of high ranked ones. This implies that resources should be shifted toward low ranked shares, and the optimal share structure should be flatter.

In sum, the optimal share structure interacts with the optimal minimum bid policy. On one hand, changes in the share structures call for adjustments of the optimal minimum bids. On the other hand, the introduction of minimum bids requires a flatter optimal share structure.

4.7 Conclusion

The innovative use of auctions in the development of e-commerce challenges our understanding. Much research has been done on auctions used in business-to-consumer environments such as eBay auctions (Bapna et al. 2003) or in business-to-business environments (Jap 2007). In comparison, less has been done on another major application of auctions in electronic commerce, namely keyword advertising auctions. This paper aims to address a novel problem in keyword advertising auction settings: how an auctioneer should choose the optimal share structure to maximize revenue.

The above issue is important because keyword advertising providers face the issue of optimal share structure design routinely. The demand and supply of keyword advertising resources are highly dynamic. On one hand, the supply of advertising resources fluctuate as new websites join keyword advertising providers' advertising network and existing websites may lose their draw of online users. On the other hand, the demand for advertising on particular keywords shifts constantly in response to changes in underlying market trends. Therefore, keyword advertising providers must constantly manage their share offerings to best respond to the changes in demand and supply. To do so, keyword advertising providers need a good understanding of how the optimal share structures are affected by changes in demand and supply factors.

We study the optimal share structure problem in a setting where bidders' valuation for a share is jointly determined by their private value for effective impressions (their type) and a common valuation function that captures their elasticity of demand. The auctioneer prepackages resources into shares and allocates the shares using a unit-price auction. Using this framework, we characterize the optimal share structures and investigate how the optimal share structures change with several underlying primitives.

While we use keyword auctions as the setting of our discussion, the unit-price auctions with prepackaged shares may be used for other divisible goods, such as network bandwidth and grid-computing power. In fact, Google filed a proposal on May 21, 2007, to the Federal Communications Commission calling on using keyword-auction-like mechanisms to allocate radio spectrum.

4.7.1 Implications for Managers

Our characterization of the optimal share structure, together with the comparative static analysis of the underlying demand and supply factors, generates several insights for keyword advertising providers on share structure designs.

First, one key determinant of the sharing structure is advertisers' price elasticities. In our setting, advertisers' price elasticity is defined as the percentage change in their demand for effective impressions due to one percent change in price per unit effective impression. When facing a perfect or near-perfect elastic demand, keyword advertising providers should allocate as much exposure as possible to the highest-paying advertisers. As advertisers' demand for exposure becomes less elastic, keyword advertising providers should use flatter share structures – meaning that they should move advertising resources away from highest bidders to lower-ranked bidders.

Another important determinant is return factors of shares. Return factors reflect the difference in returns from different shares for the same amount of resource allocated. Return factors are determined mainly by the distribution of bidders' types. When return factors strictly decrease (such as under uniform, normal, or exponential distributions), the optimal share structure should strictly decrease. Otherwise, the optimal share structures may have plateaus.

Because return factors are determined by the distribution of valuation, changes in the underlying valuation distribution may lead to different optimal share structures. The rule of thumb is that the ratio of return factors rather than absolute values determine the optimal share structure. For example, when the valuation distribution is scaled by a common factor, all return factors are also scaled by the same factor, and the optimal share structure remains the same. When the valuation distribution shifts (to the right) by a common factor, return factors for low-ranked shares increase by a higher percentage than those for high-ranked shares, and the optimal share structure should be flatter. In other words, it is not the absolute difference in advertisers' valuation for keywords but the relative difference in their valuations that affects the optimal share structures. Keyword advertising providers may estimate return factors for different ranks by conducting controlled experiments (e.g., systematically varying the amount of resources allocated to each share rank).

Third, keyword advertising providers should react to changes in total advertising resources based on bidders' price elasticity. Specifically, keyword advertising providers should allocate proportionally less resources to high-ranked shares when bidders' price elasticities decreases with exposure levels; and keyword advertising providers should allocate proportionally more to high-ranked shares when bidders' price elasticities increase. Keyword advertising providers may estimate advertisers' price elasticities by examining the accumulated data on advertisers' willingness-to-pay and their budgets or by conducting market research with a sample of advertisers.

Finally, keyword advertising providers should coordinate between the minimum bids and the optimal share structures. In general, keyword advertising providers should use flatter share structures when they use minimal bids than when they do not. If keyword advertising providers decide to raise minimum bids, they should generally offer flatter structures at the same time.

4.7.2 Implications for Future Research

We have abstracted away some of the details in keyword auctions as we focus on factors that are most relevant to the share structure problem. Some of these details may deserve their own attention in future studies. For example, we have assumed that bidders' valuation for a slot can be captured by a one-dimensional variable "effective impression." It is conceivable that there are cases where participants of the same auction may have "horizontally differentiated" tastes for slots. It would be interesting to look at how keyword advertising providers should package their resources in this heterogenous setting.

This paper focuses on the issue of optimal share structure design, assuming the auctioneer has decided to use the keyword auction. Why keyword auctions have become the mechanism of choice is interesting in itself. There may be a few reasons for such auctions to become mainstream in online advertising markets. Keyword auctions are simpler than conventional divisible good auctions. Given that the online advertising is designed to facilitate participation of thousands of small advertisers, keeping the auction mechanism simple and easy to understand is essential. Winner determination in the keyword auction is straightforward, making it for real-time environments. Nevertheless, it remains interesting to compare keyword auctions with alternative mechanisms for divisible goods such as the conventional discriminatory-price and uniform-price auctions (Wilson 1979; Wang and Zender 2002). Our paper facilitates such comparison because one would need to pick an optimal share structure for keyword auctions to make a meaningful comparison.

Appendices

Appendix A

Proofs of the Results in Chapter 2

Throughout the appendix, we denote

$$\rho_\theta(v) \equiv \sum_{j=1}^{m} P_\theta^j(v)\delta_j \tag{A.1}$$

Proof of Lemma 2.4.1.

Consider an h-type advertiser with valuation-per-click wv who bids wb and a lower-CTR advertiser with v who bids b. Both advertisers get a score wb, and their payoff functions are

$$U_l(v,b) = Q_l(v-b)\sum\nolimits_{j=1}^{m} \delta_j \Pr(wb \text{ ranks } j\text{th}) \tag{A.2}$$

$$U_h(wv,wb) = Q_h(wv-wb)\sum\nolimits_{j=1}^{m} \delta_j \Pr(wb \text{ ranks } j\text{th}) \tag{A.3}$$

It is easy to establish that

$$U_h\left(wv,wb\right) = \frac{wQ_h}{Q_l}U_l(v,b). \tag{A.4}$$

For $b_l(v)$ and $b_h(v)$ to be equilibrium bidding functions, at any v, $b_l(v)$ must maximize $U_l(v,b)$ and $b_h(v)$ must maximize $U_h(v,b)$. So, (A.4) suggests that if bidding b is the best for an l-type advertiser with valuation-per-click v, bidding wb must be the best for an h-type advertiser with valuation-per-click wv, which implies $b_h(wv)$ equals $wb_l(v)$.

Proof of the Revenue Equivalence between First- and Second-Score Weighted Unit-price Auctions.

First, we show that the same relationship as in (2.3) holds between l- and h-type advertisers' bidding functions under the second-score setting. To see, we denote $s_{j:n-1}$ as the random variable for jth highest score among $n-1$ advertisers in equilibrium. Consider an h-type advertiser with valuation-per-click wv bidding wb and an l-type advertiser with valuation-per-click v bidding b. So both advertisers get a score $s = wb$.

$$U_h(wv, wb) = Q_h \left[\sum_{j=1}^{m} \delta_j \Pr(wb \text{ ranks jth}) \left(wv - E\left[s_{j:n-1} | s_{j:n-1} \leq s < s_{j-1:n-1}\right]\right) \right]$$

$$U_l(v, b) = Q_l \left[\sum_{j=1}^{m} \delta_j \Pr(wb \text{ ranks jth}) (v - \frac{1}{w} E\left[s_{j:n-1} | s_{j:n-1} \leq s < s_{j-1:n-1}\right]) \right]$$

Similar to the proof of Lemma 1, we have $U_h(wv, wb) = \frac{wQ_h}{Q_l} U_l(v, b)$ and $b_h(wv) = wb_l(v)$. $b_h(wv) = wb_l(v)$ implies the l-type advertiser with valuation-per-click v and the h-type advertiser with wv will tie in both first- and second-score weighted unit-price auctions. Therefore, first- and second-score weighted unit-price auctions allocate the slots in the same way. By the revenue equivalence theorem (e.g., Proposition 14.1 in Krishna 2002), the two formats must generate the same amount of revenue.

Proof of Proposition 2.4.2.

Denote the inverse bidding functions as $b_l^{-1}(b)$ and $b_h^{-1}(b)$, respectively, which are strictly increasing given the monotonicity of the bidding functions. Lemma 1 implies that $b_h^{-1}(wb) = wb_l^{-1}(b)$ for $b \in [0, b_l(1)]$. Substituting this into (A.2) and (A.3), we can uniformly write the payoff functions as

$$U_\theta(v, b) = Q_\theta(v - b)\rho_\theta\left(b_\theta^{-1}(b)\right) \tag{A.5}$$

where $\rho_\theta(v)$ is defined in (A.1).

We denote

$$V_\theta(v) \equiv U_\theta(v, b_\theta(v)) = Q_\theta(v - b_\theta(v))\rho_\theta\left(b_\theta^{-1}(b_\theta(v))\right) \tag{A.6}$$

as the equilibrium payoff of an advertiser with valuation-per-click v.

$$\begin{aligned}\frac{dV_\theta(v)}{dv} &= \frac{\partial U_\theta\left(v, b_\theta(v)\right)}{\partial v} + \frac{\partial U_\theta\left(v, b_\theta(v)\right)}{\partial b}\frac{db_\theta(v)}{dv} \\ &= \frac{\partial U_\theta\left(v, b_\theta(v)\right)}{\partial v} = Q_\theta \rho_\theta(v)\end{aligned}$$

where the second equality is due to $\frac{\partial U_\theta(v,b_\theta(v))}{\partial b} = 0$ (the first-order condition). Applying the boundary condition $V_\theta(0) = 0$, we get

$$V_\theta(v) = Q_\theta \int_0^v \rho_\theta(t)dt \tag{A.7}$$

Combining (A.6) (note $b_\theta^{-1}\left(b_\theta(v)\right) = v$) and (A.7), we can solve the equilibrium bidding function as

$$b_\theta(v) = v - \frac{\int_0^v \rho_\theta(t)dt}{\rho_\theta(v)} \tag{A.8}$$

Now we show that $\frac{db_\theta(v)}{dv} > 0$.

$$\frac{db_\theta(v)}{dv} = \frac{\rho_\theta'(v)\int_0^v \rho_\theta(t)dt}{\rho_\theta^2(v)} \tag{A.9}$$

The sign of the above first-order derivative is solely determined by that of $\rho_\theta'(v)$. It is sufficient to show $\rho_\theta'(v) > 0$, or $\sum_{j=1}^m \delta_j P_\theta^{j\prime}(v) > 0$.

$$P_\theta^{j\prime}(v) = \tbinom{n-1}{n-j} G_\theta(v)^{n-j-1}\left(1 - G_\theta(v)\right)^{j-2}\left[(n-j) - (n-1)G_\theta(v)\right]G_\theta'(v) \tag{A.10}$$

Notice that $P_\theta^{1\prime}(v) \geq 0$ and $P_\theta^{n\prime}(v) \leq 0$ for all v; $P_\theta^{j\prime}(v)$ $(1 < j < n)$ crosses zero only once from positive to negative on $(0,1)$. The crossing point, v_j^c, is the solution to $G_\theta(v_j^c) = \dfrac{n-j}{n-1}$. It is clear that $0 < v_{n-1}^c < ... < v_3^c < v_2^c < 1$. Thus, for a given $v \in (0,1)$, there exists $j_v \in \{1, 2, ..., n-1\}$ such that

$$P_\theta^{j\prime}(v) > 0, \text{ for } j = 1, ..., j_v, \text{ and } P_\theta^{j\prime}(v) \leq 0, \text{ for } j = j_v + 1, ..., n. \tag{A.11}$$

Let $\delta_{m+1} = \delta_{m+2} = ... = \delta_n = 0$. We have

$$\sum_{j=1}^m \delta_j P_\theta^{j\prime}(v) = \sum_{j=1}^n \delta_j P_\theta^{j\prime}(v) > \delta_{j_v} \sum_{j=1}^n P_\theta^{j\prime}(v) = 0 \tag{A.12}$$

where the inequality is due to $\delta_1 \geq \delta_2 \geq ... \geq \delta_n$ and (A.11), and the last equality is due to the fact that $\sum_{j=1}^{n} P_\theta^j(v) = (G_\theta(v) + 1 - G_\theta(v))^{n-1} = 1$.

Proof of Proposition 2.4.3.

First note that $G_h(wv) = G_l(v)$ and $\frac{dG_h(v)}{dw}|_{wv} = -\frac{1-\alpha}{\alpha w}\frac{f_l(v)}{f_h(wv)}\frac{dG_l(v)}{dw}$. We can establish

$$\frac{d\rho_h(v)}{dw}|_{wv} = -\frac{1-\alpha}{\alpha w}\frac{f_l(v)}{f_h(wv)}\frac{d\rho_l(v)}{dw}. \tag{A.13}$$

Using the same technique in Proof of Proposition 1, we can show

$$\frac{d\rho_l(v)}{dw} > 0 \tag{A.14}$$

Taking the first-order derivative of (2.8) with respect to w yields

$$(1-\alpha) Q_l \int_0^1 v\frac{d\rho_l(v)}{dw}f_l(v)\,dv + \alpha Q_h \int_0^1 v\frac{d\rho_h(v)}{dw}f_h(v)\,dv \tag{A.15}$$

If $w \leq 1$, noting $\frac{d\rho_h(v)}{dw} = 0$ for $v > w$, we can re-organize (A.15) as

$$\begin{aligned}
& (1-\alpha) Q_l \int_0^1 v\frac{d\rho_l(v)}{dw}f_l(v)\,dv + \alpha Q_h \int_0^w v\frac{d\rho_h(v)}{dw}f_h(v)\,dv \\
= \; & (1-\alpha) Q_l \int_0^1 v\frac{d\rho_l(v)}{dw}f_l(v)\,dv - (1-\alpha) w Q_h \int_0^1 v\frac{d\rho_l(v)}{dw}f_l(v)\,dv \\
= \; & (1-\alpha)(Q_l - wQ_h) \int_0^1 v\frac{d\rho_l(v)}{dw}f_l(v)\,dv
\end{aligned} \tag{A.16}$$

where the second equality is due to integration by substitution and (A.13). Because $\frac{d\rho_l(v)}{dw} > 0$ by (A.14), the above first order derivative is positive if $w < \frac{Q_l}{Q_h}$ and negative if $w > \frac{Q_l}{Q_h}$. So $w = \frac{Q_l}{Q_h}$ maximizes the social welfare among all $w \in [0, 1]$.

Using a similar logic, we can verify $w > 1$ cannot maximize the social welfare. So, $w_{\text{eff}} = \frac{Q_l}{Q_h}$.

The Derivation of Expected Revenue.

The expected payment from an advertiser is equal to the advertiser's total expected valuation upon winning minus the advertiser's expected payoff.

$$Q_\theta v \rho_\theta(v) - V_\theta(v) = Q_\theta \left[v\rho_\theta(v) - \int_0^v \rho_\theta(t)dt \right] \tag{A.17}$$

where the equality is due to (A.7).

The expected payment from one advertiser (with probability α being h-type and with probability $(1-\alpha)$ being l-type) is

$$\begin{aligned}
& \alpha E\left[Q_h v \rho_h(v) - V_h(v)\right] + (1-\alpha) E\left[Q_l v \rho_l(v) - V_l(v)\right] \\
= & \alpha Q_h \int_0^1 \left[v\rho_h(v) - \int_0^v \rho_h(t)dt \right] f_h(v)dv \\
& +(1-\alpha)Q_l \int_0^1 \left[v\rho_l(v) - \int_0^v \rho_l(t)dt \right] f_l(v)dv \\
= & \alpha Q_h \int_0^1 \rho_h(v) \left[v - \frac{1-F_h(v)}{f_h(v)} \right] f_h(v)dv \\
& +(1-\alpha)Q_l \int_0^1 \rho_l(v) \left[v - \frac{1-F_l(v)}{f_l(v)} \right] f_l(v)dv
\end{aligned} \tag{A.18}$$

The total expected revenue from all advertisers is n times the above.

Proof of Proposition 2.4.4.

Taking the first order derivative of the expected revenue (2.9) with respect to w yields

$$\begin{aligned}
\frac{d\pi}{dw} = & (1-\alpha) Q_l \int_0^1 \frac{d\rho_l(v)}{dw} \left(v - \frac{1-F_l(v)}{f_l(v)} \right) f_l(v)dv \\
& +\alpha Q_h \int_0^1 \frac{d\rho_h(v)}{dw} \left(v - \frac{1-F_h(v)}{f_h(v)} \right) f_h(v)dv
\end{aligned} \tag{A.19}$$

We only need to check the sign of $\frac{d\pi}{dw}$ for $0 < w \leq \frac{Q_l}{Q_h}$. For $0 < w \leq \frac{Q_l}{Q_h}$,

$$\begin{aligned}
\frac{d\pi}{dw} &= (1-\alpha) Q_l \int_0^1 \frac{d\rho_l(v)}{dw}\left(v - \frac{1-F_l(v)}{f_l(v)}\right) f_l(v)dv \\
&\quad + \alpha Q_h \int_0^w \frac{d\rho_h(v)}{dw}\left(v - \frac{1-F_h(v)}{f_h(v)}\right) f_h(v)\, dv \\
&= (1-\alpha) Q_l \int_0^1 \frac{d\rho_l(v)}{dw}\left(v - \frac{1-F_l(v)}{f_l(v)}\right) f_l(v)dv \\
&\quad - (1-\alpha) Q_h \int_0^1 \frac{d\rho_l(v)}{dw}\left(wv - \frac{1-F_h(wv)}{f_h(wv)}\right) f_l(v)dv \\
&= (1-\alpha) \int_0^1 \frac{d\rho_l(v)}{dw} f_l(v)\left[Q_l\left(v - \frac{1-F_l(v)}{f_l(v)}\right) - Q_h\left(wv - \frac{1-F_h(wv)}{f_h(wv)}\right)\right] dv \\
&= (1-\alpha) \\
&\quad \times \int_0^1 \frac{d\rho_l(v)}{dw} f_l(v)\left[v\left(Q_l - Q_h w\right) + Q_h\frac{1-F_h(wv)}{f_h(wv)} - Q_l\frac{1-F_l(v)}{f_l(v)}\right] dv \quad \text{(A.20)}
\end{aligned}$$

where the first equality is because for $v > w$, $\frac{d\rho_h(v)}{dw} = 0$ and the second equality is due to (A.13).

Note that $\frac{d\rho_l(v)}{dw} > 0$ by (A.14). Clearly, for $0 < w \leq \frac{Q_l}{Q_h}$, $v\left(Q_l - Q_h w\right) \geq 0$. By the IHR property (note that $F_l(\cdot) = F_h(\cdot) = F(\cdot)$),

$$Q_h\frac{1-F_h(wv)}{f_h(wv)} - Q_l\frac{1-F_l(v)}{f_l(v)} > 0, \tag{A.21}$$

So, (A.20) is greater than 0, which implies $w^* > \frac{Q_l}{Q_h}$.

Proof of the Strict Monotonicity of Bidding Functions.

Take l-type advertisers as an example. The incentive compatibility conditions requires that for any $v'' > v'$.

$$\begin{aligned}
\left[v' - b_l(v')\right]\rho_l(v') &\geq \left[v' - b_l(v'')\right]\rho_l(v'') &\text{(A.22)}\\
\left[v'' - b_l(v'')\right]\rho_l(v'') &\geq \left[v'' - \rho_l(v')\right]\rho_l(v') &\text{(A.23)}
\end{aligned}$$

Combining (A.22) and (A.23), we get $\rho_l(v'') \geq \rho_l(v')$, which implies $b_l(v'') \geq b_l(v')$. Next we show $b_l(v') \neq b_l(v'')$. Actually, if $b_l(v') = b_l(v'') \equiv b$, then $b_l(v) = b$ for all

$v \in [v', v'']$. an l-type advertiser with valuation-per-click v'' is better off by bidding $b + \epsilon$ (ϵ is an infinitesimal positive number) since the advertiser loses only ϵ for per-unit resource awarded, yet improves its probability of beating another advertiser by a significant amount. This contradicts the equilibrium condition. Therefore $b_l(v)$ must be strictly increasing.

Proof for Lemma 2.5.1.

We suppose there exists a mapping $\Lambda : (v_0, 1] \to [\underline{b}_h, 1]$ such that $wb_l(v) = b_h(\Lambda(v))$. That is, an l-type advertiser with v will tie with an h-type advertiser $\Lambda(v)$ in equilibrium. Similarly, we define $P_\theta^j(v) \equiv \binom{n-1}{n-j} [G_\theta(v)]^{n-j} [1 - G_\theta(v)]^{j-1}$ and $\rho_\theta(v) \equiv \sum_{j=1}^m P_\theta^j(v)\, \delta_j, \theta \in \{l, h\}$, where

$$\begin{aligned} G_l(v) &= [(1-\alpha) F_l(v) + \alpha F_h(\Lambda(v))], \text{ for all } 1 \geq v > v_0 \\ G_h(v) &= \left[(1-\alpha) F_l\left(\Lambda^{-1}(v)\right) + \alpha F_h(v)\right], \text{ for all } \Lambda(1) \geq v > \Lambda(v_0) \end{aligned}$$

We can then solve the equilibrium bidding for each advertiser type as

$$b_l(v) = v - \frac{U_l^0/Q_l + \int_{v_0}^{v} \rho_l(t)\, dt}{\rho_l(v)} \quad \text{(A.24)}$$

$$b_h(v) = v - \frac{U_h^0/Q_h + \int_{\Lambda(v_0)}^{v} \rho_h(t)\, dt}{\rho_h(v)} \quad \text{(A.25)}$$

where U_l^0 and U_h^0 are equilibrium payoff of an l-type advertiser with valuation-per-click v_0 and equilibrium payoff of an h-type advertiser with valuation-per-click $\Lambda(v_0)$, respectively. By $wb_l(v) = b_h(\Lambda(v))$,

$$\begin{aligned} w\left[v - \frac{U_l^0/Q_l + \int_{v_0}^{v} \rho_l(t)\, dt}{\rho_l(v)}\right] &= \Lambda(v) - \frac{U_h^0/Q_h + \int_{\Lambda(v_0)}^{\Lambda(v)} \rho_h(t)\, dt}{\rho_h(\Lambda(v))} \\ &= \Lambda(v) - \frac{U_h^0/Q_h + \int_{v_0}^{v} \rho_l(t)\, \Lambda'(t)\, dt}{\rho_l(v)} \quad \text{(A.26)} \end{aligned}$$

where the second step is due to $\rho_l(v) = \rho_h(\Lambda(v))$. We multiply both sides of (A.26) by $\rho_l(v)$ and take the first-order derivative with respect to v,

$$w\left[v\rho_l' + \rho_l - \rho_l\right] = \Lambda'\rho_l + \rho_l'\Lambda - \rho_l\Lambda' \tag{A.27}$$

so we get $\Lambda(v) = wv$.

Proof of Proposition 2.5.2.

Our analysis in section 2.5.1 implies that an l-type advertiser with $v \in [\underline{b}_h, v_0)$ participates but cannot compete with any participating h-type advertisers. The probability for such an l-type advertiser to beat any other advertiser is $G_l(v) = \alpha F_h(\underline{b}_h) + (1-\alpha) F_l(v)$. For an l-type advertiser with $v \in [v_0, 1]$ (who competes with both l-type advertisers and h-type advertisers), $G_l(v) = \alpha F_h(wv) + (1-\alpha) F_l(v)$. Similarly, we can obtain the probability of beating any other advertiser for h-type advertisers with valuation-per-click in $[\underline{b}_h, wv_0]$ (who beat any l-type advertisers in $[\underline{b}_h, v_0)$ but none of the l-type advertisers in $[v_0, 1]$), in $[wv_0, w]$ (who compete both with h-type advertisers and l-type advertisers), and in $(w, 1]$ (who beat any l-type advertisers). The equilibrium winning and the equilibrium bidding functions follow naturally. The only undetermined variable is v_0. Notice that Lemma 2.5.1 implies for any $v \in [v_0, 1]$,

$$V_h(wv) = Q_h(wv - b_h(wv))\rho_h(wv) = Q_h w(v - b_l(v))\rho_l(v) = \frac{wQ_h}{Q_l}V_l(v) \tag{A.28}$$

Meanwhile, we have (by a similar process in the proof of Proposition 2.4.2)

$$V_l(v_0) = Q_l \int_{\underline{b}_l}^{v_0} \rho_l(t)dt \text{ and } V_h(wv_0) = Q_h \int_{\underline{b}_h}^{wv_0} \rho_h(t)dt. \tag{A.29}$$

Evaluating (A.28) at $v = v_0$ and substituting the above, we immediately have $w \int_{\underline{b}_l}^{v_0} \rho_l(t)dt = \int_{\underline{b}_h}^{wv_0} \rho_h(t)dt$, which determines v_0. We can verify that the bidding strategies obtained in the above process constitute an equilibrium.

Proof of Proposition 2.5.3.

In the following proof, we only consider the non-trivial case in which at least some participating l-type advertisers can match h-type ones in valuation; i.e., $\underline{b}_h < Q_l/Q_h$.

(Only-if part) We first show that a weighted unit-price auction with unequally constraining minimum bids is inefficient. When the minimum bid for h-type advertisers is more constraining, any weighting factor that results in a matching point being 1 for l-type advertisers is not efficient, since an l-type advertiser with valuation-per-click 1 would lose to an h-type advertiser with valuation-per-click $\underline{b}_h$ despite having higher expected valuation. If the matching point is less than 1, by Lemma 2.5.1, an l-type advertiser with valuation-per-click $v > v_0$ will tie with an h-type advertiser with valuation-per-click wv (provided that $wv < 1$). By the same argument in Proposition 2.4.3, the allocation *among these advertisers* is efficient only if the weighting factor is Q_l/Q_h. However, if the weighting factor is Q_l/Q_h and the minimum bid for h-type advertisers is more constraining, by Proposition 2.5.2, h-type advertisers with valuation-per-click between $\underline{b}_h$ and wv_0 are unmatched by any l-type advertisers, implying that the h-type advertisers are inefficiently favored under the current minimum bids. So, it is not possible to achieve allocation efficiency with a more constraining minimum bid for h-type advertisers. By a similar argument, we can show that nor is it possible with a less constraining minimum bid for h-type advertisers.

One cannot achieve efficiency with equally-constraining minimum bids but an inefficient weighting factor either. If minimum bids are equally constraining, an l-type advertiser with valuation-per-click v always ties with an h-type advertiser with valuation-per-click wv. By the argument in Proposition 2.4.3, one can achieve efficiency only by setting the weighting factor to Q_l/Q_h. In sum, a weighted unit-price auction is weakly efficient only if the weighting factor is efficient and minimum bids are equally constraining.

The Derivation of Revenue-maximizing Minimum Bids.

Define $J_\theta(v) = v - \frac{1-F_\theta(v)}{f_\theta(v)}$ and $\rho_l\left(v_0^-\right) \equiv \lim_{v\to v_0^-} \rho_l(v) = \rho_h\left(\underline{b}_h\right)$. Taking the partial derivative of of (2.17) with respect to $\underline{b}_h$ and $\underline{b}_l$, respectively, we obtain the first-order conditions (note that v_0 is a function of $\underline{b}_h$ and $\underline{b}_l$)

$$
\begin{aligned}
&(1-\alpha) Q_l\rho_l\left(v_0^-\right) J_l\left(v_0\right) f_l\left(v_0\right) \frac{\partial v_0}{\partial \underline{b}_h} + (1-\alpha) Q_l \int_{\underline{b}_l}^{v_0} \frac{d\rho_l(v)}{d\underline{b}_h} J_l\left(v\right) f_l\left(v\right) dv \\
&- (1-\alpha) Q_l\rho_l\left(v_0\right) J_l\left(v_0\right) f_l\left(v_0\right) \frac{\partial v_0}{\partial \underline{b}_h} - \alpha Q_h\rho_h\left(\underline{b}_h\right) J_h\left(\underline{b}_h\right) f_h\left(\underline{b}_h\right) \\
&+\alpha Q_h \int_{\underline{b}_h}^{wv_0} \frac{d\rho_h\left(v\right)}{dv_0}\frac{\partial v_0}{\partial \underline{b}_h} J_h\left(v\right) f_h\left(v\right) dv = 0 \qquad \text{(A.30)}
\end{aligned}
$$

$$
\begin{aligned}
&- (1-\alpha) Q_l\rho_l\left(\underline{b}_l\right) J_l\left(\underline{b}_l\right) f_l\left(\underline{b}_l\right) + (1-\alpha) Q_l\rho_l\left(v_0^-\right) J_l\left(v_0\right) f_l\left(v_0\right) \frac{\partial v_0}{\partial \underline{b}_l} \\
&- (1-\alpha) Q_l\rho_l\left(v_0\right) J_l\left(v_0\right) f_l\left(v_0\right) \frac{\partial v_0}{\partial \underline{b}_l} \\
&+\alpha Q_h \frac{\partial v_0}{\partial \underline{b}_l} \int_{\underline{b}_h}^{wv_0} \frac{d\rho_h\left(v\right)}{dv_0} J_h\left(v\right) f_h\left(v\right) dv = 0 \qquad \text{(A.31)}
\end{aligned}
$$

where $\frac{\partial v_0}{\partial \underline{b}_h}$ and $\frac{\partial v_0}{\partial \underline{b}_l}$ can be derived from the partial derivatives of both sides of equation (2.16) with respect to $\underline{b}_h$ and $\underline{b}_l$, respectively:

$$
\begin{aligned}
w\rho_l\left(v_0^-\right) \frac{\partial v_0}{\partial \underline{b}_h} + w \int_{\underline{b}_l}^{v_0} \frac{d\rho_l(t)}{d\underline{b}_h} dt &= w\rho_h\left(wv_0\right) \frac{\partial v_0}{\partial \underline{b}_h} - \rho_h\left(\underline{b}_h\right) + \frac{\partial v_0}{\underline{b}_h} \int_{\underline{b}_h}^{wv_0} \frac{d\rho_h\left(t\right)}{dv_0} dt \\
-w\rho_l\left(\underline{b}_l\right) + w\rho_l\left(v_0^-\right) \frac{\partial v_0}{\partial \underline{b}_l} &= w\rho_h\left(wv_0\right) \frac{\partial v_0}{\partial \underline{b}_l} + \frac{\partial v_0}{\partial \underline{b}_l} \int_{\underline{b}_h}^{wv_0} \frac{d\rho_h\left(v\right)}{dv_0} dv
\end{aligned}
$$

The system of equations above allows us to solve the revenue-maximizing minimum bids for l-type advertisers ($\underline{b}_l^*$) and h-type advertisers ($\underline{b}_h^*$). For example, solving (A.30) we can get $\underline{b}_h = \underline{b}_h^*\left(\underline{b}_l\right)$. Substituting $\underline{b}_h^*\left(\underline{b}_l\right)$ into (A.31), we can derive $\underline{b}_l^*$.

Proof of Proposition 2.6.1.

Denote $H(x) \equiv \alpha F_h\left(\frac{x}{Q_h}\right) + (1-\alpha)F_l\left(\frac{x}{Q_l}\right)$. Using the similar approach as in Proof for Proposition 1, we can derive the equilibrium bidding function for the generalized first-price auction as

$$b\left(x\right) = x - \frac{\sum_{j=1}^{m}\delta_j \int_0^x \binom{n-1}{n-j} H(t)^{n-j}\left[1-H(t)\right]^{j-1} dt}{\sum_{j=1}^{m}\delta_j \binom{n-1}{n-j} H(x)^{n-j}\left[1-H(x)\right]^{j-1}}$$

If this bid is from an l-type advertiser, let $v = x/Q_l$. Noting that $H(Q_l v) = \alpha F_h\left(\frac{Q_l}{Q_h}v\right) + (1-\alpha)F_l\left(v\right) = \alpha F_h\left(w_{\text{eff}}v\right) + (1-\alpha)F_l\left(v\right) = G_l(v)$, we have

$$\begin{aligned} b\left(x\right) &= Q_l v - \frac{\sum_{j=1}^{m}\delta_j \int_0^{Q_l v} \binom{n-1}{n-j} H(t)^{n-j}\left[1-H(t)\right]^{j-1} dt}{\sum_{j=1}^{m}\delta_j \binom{n-1}{n-j} H(Q_l v)^{n-j}\left[1-H(Q_l v)\right]^{j-1}} \\ &= Q_l v - Q_l \frac{\int_0^v \rho_l(t)dt}{\rho_l(v)} = Q_l b_l(v) \end{aligned}$$

which means the total payment the advertiser bids is exactly the unit price he/she would bid under efficient weighted unit-price auctions times his/her expected CTR. Similar argument holds if the bid is from an h-type advertiser. Therefore, efficient weighted unit-price auctions are revenue-equivalent to generalized first-price auctions.

Appendix B

Proofs of the Results in Chapter 3

We can specify $f_\theta(v)$ as follows.

$$f_L(v) = \frac{f(v)}{F(v^*)}, \text{ if } v \in [0, v^*]. \tag{B.1}$$

$$f_H(v) = \begin{cases} \frac{\alpha f(v)}{1-(1-\alpha)F(v^*)}, & \text{if } v \in [0, v^*], \\ \frac{f(v)}{1-(1-\alpha)F(v^*)}, & \text{if } v \in (v^*, 1]. \end{cases} \tag{B.2}$$

Substituting (3.3), (3.4), and (3.5) into (3.6) and (3.7), respectively, we have

$$\rho_L(v) = [\alpha F(wv) + (1-\alpha)F(v)]^{n-1}, \text{ for } v \in [0, v^*], \tag{B.3}$$

and

$$\rho_H(v) = \begin{cases} \left[\alpha F(v) + (1-\alpha)F\left(\frac{v}{w}\right)\right]^{n-1}, & \text{for } v \in [0, wv^*], \\ [\alpha F(v) + (1-\alpha)F(v^*)]^{n-1}, & \text{for } v \in (wv^*, v^*], \\ [F(v)]^{n-1}, & \text{for } v \in (v^*, 1]. \end{cases} \tag{B.4}$$

For $v \in [0, v^*]$, it is easy to establish

$$\rho_L(v) = \rho_H(wv), \tag{B.5}$$

and

$$\frac{d\rho_L(v)}{dw} = -w\frac{\alpha}{1-\alpha}\frac{f(wv)}{f(v)}\frac{d\rho_H(v)}{dw}|_{wv}. \tag{B.6}$$

Proof of Lemma 3.3.1.

Consider an L type with v who bids b and an H type with unit-valuation wv who bids wb. Both bidders get a score wb, and their payoff functions are

$$U_L(v, b) = y_L(v-b)\Pr(wb \text{ is the highest score}) \tag{B.7}$$

and

$$U_H(wv, wb) = y_H(wv - wb)\mathrm{Pr}(wb \text{ is the highest score}). \tag{B.8}$$

It is easy to establish that

$$U_H(wv, wb) = \frac{wy_H}{y_L} U_L(v, b). \tag{B.9}$$

For $b_L(v)$ and $b_H(v)$ to be equilibrium bidding functions, at any v, $b = b_L(v)$ must maximize $U_L(v, b)$ and $b = b_H(v)$ must maximize $U_H(v, b)$. So, (B.9) suggests that if bidding b is the best choice for an L type with unit-valuation v, bidding wb must be the best choice for an H type with unit-valuation wv, which implies $b_H(wv) = wb_L(v)$.

Proof of Lemma 3.3.2.

We denote $V_\theta(v) \equiv U_\theta(v, b_\theta(v))$ as the equilibrium expected payoff of a bidder with type θ and unit valuation v. By (3.2),

$$V_\theta(v) \equiv U_\theta(v, b_\theta(v)) = y_\theta(v - b_\theta(v))\mathrm{Prob}(\mathrm{win}|b_\theta(v), \theta). \tag{B.10}$$

By the Envelope Theorem (see, e.g., Mas-Colell et al. 1995, p. 965), $\frac{dV_\theta(v)}{dv} = \frac{\partial U_\theta(v, b_\theta(v))}{\partial v}$. So

$$\frac{dV_\theta(v)}{dv} = y_\theta \times \mathrm{Prob}(\mathrm{win}|b_\theta(v), \theta) = y_\theta \rho_\theta(v), \tag{B.11}$$

where the last step is due to $\mathrm{Prob}(\mathrm{win}|b_\theta(v), \theta) = \rho_\theta(v)$ (both representing one's equilibrium winning probability). Applying the boundary condition $V_\theta(0) = 0$ (i.e., the bidder with the lowest valuation gets zero payoff), we get

$$V_\theta(v) = y_\theta \int_0^v \rho_\theta(t)dt. \tag{B.12}$$

Combining (B.12) and (B.10) (noting $\mathrm{Prob}(\mathrm{win}|b_\theta(v), \theta) = \rho_\theta(v)$), we can solve the bidding function as

$$b_\theta(v) = v - \frac{\int_0^v \rho_\theta(t)dt}{\rho_\theta(v)}. \tag{B.13}$$

It is easy to see that $b_\theta(v)$ is indeed monotonically increasing, since

$$\frac{db_\theta(v)}{dv} = 1 - 1 + \frac{\rho'_\theta(v)\int_0^v \rho_\theta(t)dt}{\rho_\theta^2(v)} > 0, \tag{B.14}$$

where the inequality is due to $\rho'_\theta(v) > 0$.

Proof of Proposition 3.3.3.

For a given v^* and any $v < v^*$, by (3.10) and (3.12),

$$\Delta V(v^*) - [V_H(v) - V_L(v)] = y_H \int_v^{v^*} \rho_H(t)dt - y_L \int_v^{v^*} \rho_L(t)dt. \tag{B.15}$$

Because $y_H > y_L$ and $\rho_H(t) \geq \rho_L(t)$ for all $t \in [v, v^*]$, the above in (B.15) must be positive. Therefore, by substituting $\Delta V(v^*) = c$ in (B.15),

$$V_H(v) - V_L(v) < c. \tag{B.16}$$

If $c = 0$, all L types convert to H types for the benefit from the performance increase and the scoring rule (3.1), and hence $v^* = 0$. If $\Delta V(1) \leq c$, then $v^* = 1$ must be an equilibrium because (B.16) implies that the benefit from deviating to convert cannot compensate for the investment cost for any L type with $v \in [0, 1)$.

If $0 < c < \Delta V(1)$, there is an unique solution to $\Delta V(v^*) = c$, since $\Delta V(v^*)$ is continuous and monotonically increasing in v^* (by checking the first order derivative). And, v^* determined by $\Delta V(v^*) = c$ must be an equilibrium for the following reasons. First, for an L type with unit valuation $v \in [0, v^*)$, it is unprofitable to deviate to convert due to (B.16). Second, for an L type with valuation $v \in (v^*, 1]$, if he remained at the low performance (instead of conversion), his best bidding strategy in the second period was to bid $b_H(wv)/w$ by a similar argument to the one for Lemma 3.3.1. Therefore, by (B.9), his maximum expected payoff could be

$$U_L(v, b_H(wv)/w) = \frac{y_L}{wy_H} U_H(wv, b_H(wv)) = \frac{y_L}{wy_H} V_H(wv), \tag{B.17}$$

which can be shown to be less than $V_H(v) - c$, the net equilibrium payoff. In fact, $[V_H(v) - \frac{y_L}{wy_H} V_H(wv)]$ increases in v (by checking the first-order derivative), and

$$c = \Delta V(v^*) = V_H(v^*) - V_L(v^*) = V_H(v^*) - \frac{y_L}{wy_H} V_H(wv^*), \tag{B.18}$$

where the last step is due to the mapping in (B.9) at equilibrium (recall $U_\theta(v, b_\theta(v)) \equiv V_\theta(v)$). Therefore, $V_H(v) - c > \frac{y_L}{wy_H} V_H(wv)$ for any $v \in (v^*, 1]$, implying that it is unprofitable for an L type of this v to deviate.

Proof of Corollary 3.3.4.

In the equilibrium, we have

$$\Delta V(v^*) = y_H \int_0^{v^*} \rho_H(t)dt - y_L \int_0^{v^*} \rho_L(t)dt = c, \tag{B.19}$$

where v^* can be regarded as a function of the related parameters. Applying the Implicit Function Theory (see, e.g., Mas-Colell et al. 1995, p. 940) to (B.19) with respect to c and noticing $\rho_H(t)$ is a function of v^* from (B.4), we have

$$\left[y_H \rho_H(v^*) - y_L \rho_L(v^*) + y_H \int_{wv^*}^{v^*} \frac{d\rho_H(t)}{dv^*} dt \right] v^{*\prime}(c) = 1, \tag{B.20}$$

Since the coefficient of $v^{*\prime}(c)$ in the above is positive, $v^{*\prime}(c) > 0$.

Similar argument leads to that v^* decreases in y_H.

Proof of Lemma 3.4.1.

Applying the Implicit Function Theory to (B.19) with respect of w and noticing $\rho_\theta(v)$ is a function of w from (B.3) and (B.4), we have

$$\begin{aligned} &\left[y_H \rho_H(v^*) - y_L \rho_L(v^*) + y_H \int_{wv^*}^{v^*} \frac{d\rho_H(t)}{dv^*} dt \right] \frac{dv^*(w)}{dw} \\ &\qquad = y_L \int_0^{v^*} \frac{d\rho_L(t)}{dw} dt - y_H \int_0^{wv^*} \frac{d\rho_H(t)}{dw} dt. \end{aligned}$$

Since the right hand side is positive (due to $\frac{d\rho_L(t)}{dw} > 0$ and $\frac{d\rho_H(t)}{dw} < 0$) and the coefficient of $\frac{dv^*(w)}{dw}$ is positive, $\frac{dv^*(w)}{dw} > 0$.

Proof of Proposition 3.4.2.

Substituting in (3.5), (B.1), and (B.2), we can re-organize (3.13) as

$$(1-\alpha)\,y_L \int_0^{v^*} \rho_L(v)\,f(v)\,dv + \alpha y_H \int_0^{v^*} \rho_H(v)\,f(v)\,dv + y_H \int_{v^*}^{1} \rho_H(v)\,f(v)\,dv. \tag{B.21}$$

Taking the first order derivative of the above with respect to w, we have (recall $\rho_H(v)$ is a step function as specified in (B.4))

$$(1-\alpha)\left[y_L\rho_L(v^*) - y_H\rho_H(v^*)\right] f(v^*)\,\frac{dv^*(w)}{dw} + (1-\alpha)\,y_L \int_0^{v^*} \frac{d\rho_L(v)}{dw} f(v)\,dv$$
$$+\alpha y_H \int_0^{wv^*} \frac{d\rho_H(v)}{dw} f(v)\,dv + \alpha y_H \frac{dv^*(w)}{dw} \int_{wv^*}^{v^*} \frac{d\rho_H(v)}{dv^*} f(v)\,dv.$$

By integration by substitution and then applying (B.6),

$$\int_0^{wv^*} \frac{d\rho_H(v)}{dw} f(v)\,dv = \int_0^{v^*} \frac{d\rho_H(v)}{dw}|_{wz} f(wz)\,w dz = -\frac{1-\alpha}{\alpha} \int_0^{v^*} \frac{d\rho_L(v)}{dw} f(v)\,dv. \tag{B.22}$$

Substituting in (B.22) and integrating $\int_{wv^*}^{v^*} \frac{d\rho_H(v)}{dv^*} f(v)\,dv$, we can re-write the above first-order derivative as

$$\begin{aligned}
& (1-\alpha)\left[y_L\rho_L(v^*) - y_H\rho_H(v^*)\right] f(v^*)\,\frac{dv^*(w)}{dw} + (1-\alpha)\,y_L \int_0^{v^*} \frac{d\rho_L(v)}{dw} f(v)\,dv \\
& -(1-\alpha)\,y_H \int_0^{v^*} \frac{d\rho_L(v)}{dw} f(v)\,dv + (1-\alpha)\,y_H \left(\rho_H(v^*) - \rho_H(wv^*)\right) f(v^*)\,\frac{dv^*(w)}{dw} \\
= \; & (1-\alpha)\left(y_L - y_H\right)\rho_L(v^*)\,f(v^*)\,\frac{dv^*(w)}{dw} \\
& \qquad + (1-\alpha)\left(y_L - y_H\right) \int_0^{v^*} \frac{d\rho_L(v)}{dw} f(v)\,dv,
\end{aligned} \tag{B.23}$$

where the equality is due to $\rho_H(wv^*) = \rho_L(v^*)$ from (B.5). Since $\frac{dv^*(w)}{dw} > 0$ (by Lemma 3.4.1) and $\frac{d\rho_L(v)}{dw} > 0$, the first order derivative is negative, which implies that the expected performance decreases in w.

Proof of Proposition 3.5.1.

Substituting in (3.5), (B.1), and (B.2), we can re-organize (3.14) as

$$\begin{aligned} & n\left[(1-\alpha)\, y_L \int_0^{v^*} v\rho_L(v)\, f(v)\, dv \right. \\ & \quad \left. +\alpha y_H \int_0^{v^*} v\rho_H(v)\, f(v)\, dv + y_H \int_{v^*}^1 v\rho_H(v)\, f(v)\, dv \right] \\ & -n(1-\alpha)(1-F(v^*))\, c. \end{aligned} \tag{B.24}$$

Taking the first order derivative of (B.24) with respect to w (and removing the constant n),

$$\begin{aligned} & (1-\alpha)\left[y_L\rho_L(v^*) - y_H\rho_H(v^*)\right] v^* f(v^*) \frac{dv^*(w)}{dw} + (1-\alpha)\, f(v^*) \frac{dv^*(w)}{dw} c \\ & +(1-\alpha)\, y_L \int_0^{v^*} v \frac{d\rho_L(v)}{dw} f(v)\, dv \\ & +\alpha y_H \int_0^{wv^*} v \frac{d\rho_H(v)}{dw} f(v)\, dv + \alpha y_H \frac{dv^*(w)}{dw} \int_{wv^*}^{v^*} v \frac{d\rho_H(v)}{dv^*} f(v)\, dv. \end{aligned} \tag{B.25}$$

By integration by parts,

$$\alpha y_H \int_{wv^*}^{v^*} v \frac{d\rho_H(v)}{dv^*} f(v)\, dv = (1-\alpha)\, y_H f(v^*) \left(\rho_H(v)\, v\big|_{wv^*}^{v^*} - \int_{wv^*}^{v^*} \rho_H(v)\, dv\right). \tag{B.26}$$

Also, notice $\rho_H(wv^*) = \rho_L(v^*)$ from (B.5). By substituting in (B.26) and applying integration by substitution similar to the one in (B.22), (B.25) can re-organized as

$$\begin{aligned} & (1-\alpha)(y_L - wy_H) \int_0^{v^*} v \frac{d\rho_L(v)}{dw} f(v)\, dv \\ & +(1-\alpha)(y_L - wy_H)\, \rho_L(v^*)\, v^* f(v^*) \frac{dv^*(w)}{dw} \\ & +(1-\alpha)\, f(v^*) \frac{dv^*(w)}{dw} [c - y_H \int_{wv^*}^{v^*} \rho_H(v)\, dv], \end{aligned} \tag{B.27}$$

Using $\rho_H(wv) = \rho_L(v)$ by (B.5), we can re-write the equilibrium condition (B.19) as

$$(y_H w - y_L) \int_0^{v^*} \rho_L(x)\, dx + y_H \int_{wv^*}^{v^*} \rho_H(x)\, dx = c \tag{B.28}$$

By substituting into (B.28), the first-order derivative (B.27) can be re-organized as

$$(1-\alpha)\left[f\left(v^{*}\right)\frac{dv^{*}(w)}{dw}(v^{*}\rho_{L}\left(v^{*}\right)-\int_{0}^{v^{*}}\rho_{L}\left(v\right)dv)+\int_{0}^{v^{*}}v\frac{d\rho_{L}\left(v\right)}{dw}f\left(v\right)dv\right]\left(y_{L}-wy_{H}\right). \tag{B.29}$$

Since the coefficient of $(y_L - wy_H)$ is positive, $w = \frac{y_L}{y_H}$ is the only solution making (B.29) zero. Therefore, $w^* = \frac{y_L}{y_H}$.

Derivation of the Expected Revenue.

The expected payment from a θ-type bidder of unit valuation v is the difference between his expected valuation and his expected payoff; that is

$$y_\theta v\rho_\theta(v) - V_\theta(v) = y_\theta\left[v\rho_\theta(v) - \int_0^v \rho_\theta(t)dt\right]. \tag{B.30}$$

The expected payment from one bidder (with probability P_H being an H type and with probability P_L being an L type) is

$$\begin{aligned}
& P_L E\left[y_L v\rho_L(v) - V_L(v)\right] + P_H E\left[y_H v\rho_H(v) - V_H(v)\right] \\
= \; & y_L P_L \int_0^{v^*}\left[v\rho_L(v) - \int_0^v \rho_L(t)dt\right] f_L(v)dv + y_H P_H \int_0^1 \left[v\rho_H(v) - \int_0^v \rho_H(t)dt\right] f_H(v)dv \\
= \; & y_L P_L \int_0^{v^*} \rho_L(v)\left[v - \frac{1-F_L(v)}{f_L(v)}\right] f_L(v)dv + y_H P_H \int_0^1 \rho_H(v)\left[v - \frac{1-F_H(v)}{f_H(v)}\right] f_H(v)dv.
\end{aligned}$$

The expected revenue from all bidders is n times the above.

Proof of Proposition 3.6.1.

Given v^*, the first-order derivative of π in (3.15) with respect to w can be organized as (substituting in (3.5), (B.1), and (B.2))

$$\begin{aligned}
\frac{\partial \pi}{\partial w} &= n(1-\alpha)y_L \int_0^{v^*}\frac{d\rho_L\left(v\right)}{dw}J_L(v)f(v)dv + n\alpha y_H \int_0^{wv^*}\frac{d\rho_H\left(v\right)}{dw}J_H(v)f(v)dv \quad \text{(B.31)} \\
&= n(1-\alpha)y_L \int_0^{v^*}\frac{d\rho_L\left(v\right)}{dw}J_L(v)f(v)dv - n(1-\alpha)y_H \int_0^{v^*}\frac{d\rho_L\left(v\right)}{dw}J_H(wv)f(v)dv,
\end{aligned}$$

where the second step is by applying integration by substitution similar to the one in (B.22). Substituting in $J_\theta(v)$, we have

$$\frac{\partial \pi}{\partial w} = n(1-\alpha)\int_0^{v^*} \frac{d\rho_L(v)}{dw}\left[(y_L - y_H w)\, v + \left(y_H \frac{1-F_H(wv)}{f_H(wv)} - y_L \frac{1-F_L(v)}{f_L(v)}\right)\right] f(v)dv. \tag{B.32}$$

Notice that

$$\frac{1-F_L(v)}{f_L(v)} = \frac{F(v^*)-F(v)}{f(v)} \leq \frac{1-F(v)}{f(v)}$$

and

$$\frac{1-F(wv)}{f(wv)} \leq \frac{\frac{1-(1-\alpha)F(v^*)}{\alpha} - F(wv)}{f(wv)} = \frac{1-F_H(wv)}{f_H(wv)}.$$

If $\frac{f(v)}{1-F(v)}$ is increasing in v, then $\frac{1-F(v)}{f(v)} \leq \frac{1-F(wv)}{f(wv)}$ and thus $\frac{1-F_L(v)}{f_L(v)} \leq \frac{1-F_H(wv)}{f_H(wv)}$. Therefore, for all $w \in [0, \frac{y_L}{y_H}]$, we have $\frac{\partial \pi}{\partial w} > 0$, implying $w^L_{opt} > \frac{y_L}{y_H}$.

Proof of Proposition 3.6.2.

Substituting $F(v) = v$ and $f(v) = 1$ into (B.32),

$$\begin{aligned}\frac{\partial \pi}{\partial w} &= n(1-\alpha)(n-1)(\alpha w + 1 - \alpha)^{n-2}\alpha \\ &\quad \times \int_0^{v^*} v^{n-1}\left[2(y_L - w y_H)\, v + \left(\frac{1-(1-\alpha)v^*}{\alpha} y_H - v^* y_L\right)\right] dv. \end{aligned} \tag{B.33}$$

By integration, we can obtain the solution to $\partial\pi/\partial w = 0$:

$$w^L_{opt} = \frac{(n+1)\, y_H \frac{1-(1-\alpha)v^*}{\alpha} + (n-1)\, y_L v^*}{2n y_H v^*}.$$

It is easy to see that w^L_{opt} decreases in v^*. Notice that in the static case $v^* = 1$ and in our dynamic (nontrivial) case with limited commitment $v^* < 1$. So the revenue-maximizing weighting factor in a dynamic limited-commitment case is greater than in a static case.

Proof of Lemma 3.6.4.

In the full commitment case, we have

$$\frac{d\pi}{dw} = \frac{\partial \pi}{\partial w} + \frac{\partial \pi}{\partial v^*}\frac{dv^*(w)}{dw}, \tag{B.34}$$

where $\partial\pi/\partial w$ is the same as specified in (B.31). To get $\partial\pi/\partial v^*$, we re-organize the expected revenue π in (3.15) as below by substituting in P_θ, $f_\theta(v)$, and $F_\theta(v)$ (specified in (3.5), (B.1), (B.2), (3.3), and (3.4)).

$$\begin{aligned}
& n(1-\alpha)\, y_L \int_0^{v^*} \rho_L(v)\, J_L(v)\, f(v)\, dv \\
& + n y_H \left[\int_0^{v^*} \rho_H(v)\, J_H(v)\, \alpha f(v)\, dv + \int_{v^*}^{1} \rho_H(v)\, J_H(v)\, f(v)\, dv \right] \\
= \; & n(1-\alpha)\, y_L \int_0^{v^*} \rho_L(v)\left[v f(v) - \left(F(v^*) - F(v)\right)\right] dv \\
& + n y_H \left[\int_0^{v^*} \rho_H(v)\left[\alpha v f(v) - \left(1-(1-\alpha)F(v^*)\right) + \alpha F(v)\right] dv \right. \\
& \qquad \left. + \int_{v^*}^{1} \rho_H(v)\left[v f(v) - (1-F(v))\right] dv \right]
\end{aligned}$$

Then, we have

$$\begin{aligned}
\frac{\partial \pi}{\partial v^*} = \; & n(1-\alpha)\left[y_L\rho_L(v^*) - y_H\rho_H(v^*)\right] v^* f(v^*) - n(1-\alpha) f(v^*)\, y_L \int_0^{v^*} \rho_L(v)\, dv \\
& + n(1-\alpha) f(v^*)\, y_H \int_0^{v^*} \rho_H(v)\, dv + n y_H \int_{wv^*}^{v^*} \frac{d\rho_H(v)}{dv^*} J_H(v)\, \alpha f(v)\, dv
\end{aligned} \tag{B.35}$$

which leads to (3.18) by noticing that $y_H \int_0^{v^*} \rho_H(v)\, dv - y_L \int_0^{v^*} \rho_L(v)\, dv = c$.

Proof of Proposition 3.6.5.

Notice that $J_H(v)\alpha f(v) = \left[\alpha v f(v) - \left(1-(1-\alpha)F(v^*)\right) + \alpha F(v)\right]$ for $v \in [wv^*, v^*]$. By substituting (B.26) in (B.35),

$$\begin{aligned}
\frac{\partial \pi}{\partial v^*} = \; & n(1-\alpha) f(v^*) \left[\left(y_L\rho_L(v^*) - w y_H \rho_H(wv^*)\right) v^* + y_H \int_0^{wv^*} \rho_H(v) dv - y_L \int_0^{v^*} \rho_L(v) dv \right] \\
& + n\left[-y_H \int_{wv^*}^{v^*} \frac{d\rho_H(v)}{dv^*} \left(1-(1-\alpha)F(v^*) - \alpha F(v)\right) dv \right].
\end{aligned}$$

Notice that the term in the second square bracket is negative. The term in the first square bracket, by integration by substitution for $\int_0^{wv^*} \rho_H(v)dv$ and applying $\rho_H(wv) = \rho_L(v)$ from (B.5), can be simplified to $\left[(y_L - wy_H)\left(\rho_L(v^*)v^* - \int_0^{v^*} \rho_L(v)\,dv\right)\right]$, which is negative for $w > y_L/y_H$. Therefore, $\partial\pi/\partial v^* < 0$ for all $w \geq y_L/y_H$.

Notice that $\frac{dv^*(w)}{dw} > 0$ by Lemma 3.4.1. Therefore, for a certain $\tilde{w} \geq y_L/y_H$, we have $\frac{d\pi}{dw}|_{\tilde{w}} < 0$ by (B.34) as long as $\frac{\partial\pi}{\partial w}|_{\tilde{w}} \leq 0$. In particular, if $F(v)$ is IHR, $w^L_{opt} > y_L/y_H$; if $\frac{\partial^2\pi}{\partial w^2} < 0$, $\frac{\partial\pi}{\partial w}|_{w^L_{opt}} = 0$ leads to $\frac{\partial\pi}{\partial w} < 0$ for $w \in (w^L_{opt}, 1]$. So we have $\frac{d\pi}{dw} < 0$ for for $w \in (w^L_{opt}, 1]$, implying $w^F_{opt} < w^L_{opt}$.

Appendix C

Proofs of the Results in Chapter 4

Proof of Lemma 4.4.1.

(a) For $j = 1, 2, ..., n-1$,

$$\begin{aligned}
\alpha_j &= n\int_{\underline{v}}^{\bar{v}} P_j(v)\left[vf(v) - (1-F(v))\right]dv = n\int_{\underline{v}}^{\bar{v}} P_j(v)\,d\left[-v(1-F(v))\right] \\
&= n\int_{\underline{v}}^{\bar{v}} v(1-F(v))\,dP_j(v) \\
&= n\int_{\underline{v}}^{\bar{v}} \binom{n-1}{n-j} F(v)^{n-j-1}(1-F(v))^{j-1}\left[(n-j)-(n-1)F(v)\right]vf(v)\,dv
\end{aligned}$$

where the third step is due to integration by parts. We can easily verify $\alpha_1 > 0$.

For $j = 2, 3, ..., n-1$,

$$\begin{aligned}
&\alpha_1 - \alpha_j = n\int_{\underline{v}}^{\bar{v}} \{(n-1)F(v)^{n-2}(1-F(v)) - \\
&\binom{n-1}{n-j} F(v)^{n-j-1}(1-F(v))^{j-1}\left[(n-j)-(n-1)F(v)\right]\}vf(v)\,dv \quad \text{(C.1)}
\end{aligned}$$

Denoting $A(v) \equiv (n-1)F(v)^{j-1} - \binom{n-1}{n-j}(1-F(v))^{j-2}\left[(n-j)-(n-1)F(v)\right]$, we can rewrite (C.1) as $\alpha_1 - \alpha_j = n\int_{\underline{v}}^{\bar{v}}[1-F(v)]F(v)^{n-j-1}A(v)vf(v)\,dv$. We argue that $A(v)$ single-crosses zero from below on $[\underline{v}, \bar{v}]$. To see, let v^0 be the solution to $(n-j)-(n-1)F(v) = 0$. We can verify that $A(\underline{v}) < 0$, $A(v)$ increases in v for $v \leq v^0$, and $A(v)$ is positive for all $v > v^0$. Thus $A(v)$ crosses zero only once from below, implying $[1-F(v)]F(v)^{n-j-1}A(v)f(v)$ also single-crosses zero from below on $(\underline{v}, \bar{v})$. Denoting the crossing point of the latter as v^c, we have

$$\begin{aligned}
\alpha_1 - \alpha_j &= n \int_{\underline{v}}^{\bar{v}} (1 - F(v)) F(v)^{n-j-1} A(v) f(v) v dv \\
&> nv^c \int_{\underline{v}}^{\bar{v}} (1 - F(v)) F(v)^{n-j-1} A(v) f(v) dv \\
&= nv^c \int_{\underline{v}}^{\bar{v}} [P_2(v) - jP_{j+1}(v) + (j-1) P_j(v)] f(v) dv \qquad \text{(C.2)}
\end{aligned}$$

where the last equality results from substituting the definition of $A(v)$ and rearranging terms. The right side of (C.2) is zero because for $j = 1, ..., n$,

$$\begin{aligned}
\int_{\underline{v}}^{\bar{v}} P_j(v) f(v) dv &= \binom{n-1}{n-j} \int_{\underline{v}}^{\bar{v}} F(v)^{n-j} [1 - F(v)]^{j-1} dF(v) \\
&= \binom{n-1}{n-j} \int_0^1 x^{n-j} (1-x)^{j-1} dx \\
&= \binom{n-1}{n-j} \binom{n-1}{n-j}^{-1} \frac{1}{n} = \frac{1}{n} \qquad \text{(C.3)}
\end{aligned}$$

where the second step is due to integration by substitution and the third step is due to repeated integration by parts. Therefore, $\alpha_1 - \alpha_j > 0$ for $j = 2, 3, ..., n-1$.

We next show that $\alpha_1 - \alpha_n > 0$.

$$\begin{aligned}
\alpha_1 - \alpha_n &= n \int_{\underline{v}}^{\bar{v}} \left\{ F(v)^{n-1} - [1 - F(v)]^{n-1} \right\} d[-v(1 - F(v))] \\
&= -n\underline{v} + n \int_{\underline{v}}^{\bar{v}} v [1 - F(v)] (n-1) \left[F(v)^{n-2} + (1 - F(v))^{n-2} \right] f(v) dv \\
&> -n\underline{v} + n\underline{v} \int_{\underline{v}}^{\bar{v}} [1 - F(v)] (n-1) \left[F(v)^{n-2} + (1 - F(v))^{n-2} \right] f(v) dv \\
&= -n\underline{v} + n\underline{v} \left(\frac{1}{n} + \frac{n-1}{n} \right) = 0
\end{aligned}$$

where the second step is due to integration by parts and the fourth step is due to (C.3).

(b) Denote $h_j(x) \equiv nP_j(x) f(x)$. By (C.3), $\int_{\underline{v}}^{\bar{v}} h_j(x) dx = 1$. Thus we can regard $h_j(x)$ as a probability density function. We next show that for $j = 1, 2, ..., n-1$,

$h_j(x)$ first-order stochastically dominates $h_{j+1}(x)$.

$$\begin{aligned} & h_j(x) - h_{j+1}(x) \\ &= nf(x)\left\{\binom{n-1}{n-j} F(x)^{n-j}[1-F(x)]^{j-1} - \binom{n-1}{n-j-1} F(x)^{n-j-1}[1-F(x)]^{j}\right\} \\ &= \binom{n}{j} f(x) F(x)^{n-j-1}[1-F(x)]^{j-1}[nF(x)-(n-j)] \end{aligned}$$

Denote v_j^c as the solution to $nF(x)-(n-j)=0$. Because $h_j(x) < h_{j+1}(x)$ for any $x \in (\underline{v}, v_j^c)$, $\int_{\underline{v}}^{v} h_j(x)\,dx < \int_{\underline{v}}^{v} h_{j+1}(x)\,dx$ for $v \in (\underline{v}, v_j^c)$. Because $h_j(x) > h_{j+1}(x)$ for any $x \in (v_j^c, \bar{v})$, $\int_v^{\bar{v}} h_j(x)\,dx > \int_v^{\bar{v}} h_{j+1}(x)\,dx$ for $v \in (v_j^c, \bar{v})$, which implies $\int_{\underline{v}}^{v} h_j(x)\,dx < \int_{\underline{v}}^{v} h_{j+1}(x)\,dx$ for $v \in (v_j^c, \bar{v})$ (note that $\int_{\underline{v}}^{v} h_j(x)\,dx = 1 - \int_v^{\bar{v}} h_j(x)\,dx$). In all, we have $\int_{\underline{v}}^{v} h_j(x)\,dx < \int_{\underline{v}}^{v} h_{j+1}(x)\,dx$ for any $v \in (\underline{v}, \bar{v})$, implying that $h_j(x)$ first-order stochastically dominates $h_{j+1}(x)$. According to the property of first-order stochastic dominance (e.g., Proposition 6.D.1 at page 195 of Mas-Colell et al. (1995)), if $J(x)$ is an increasing function of x, $\int_{\underline{v}}^{\bar{v}} h_j(x) J(x)\,dx > \int_{\underline{v}}^{\bar{v}} h_{j+1}(x) J(x)\,dx$. Therefore $\alpha_j > \alpha_{j+1}$.

Proof of Lemma 4.4.5.

Assume the optimal share structure is $(s_1^*, s_2^*, ..., s_n^*)$. Denote $\sum_{j=j_k+1}^{j_{k+1}} s_j^* \equiv \sigma$ and notice that $s_{j_k}^* \geq \frac{1}{j_{k+1}-j_k}\sigma \geq s_{j_{k+1}+1}^* \geq 0$ because of the size-order constraint. $(s_{j_k+1}^*, s_{j_k+2}^*, ..., s_{j_{k+1}}^*)$ must be the solution to the following maximization problem:

$$\max \sum_{j=j_k+1}^{j_{k+1}} \alpha_j Q(s_j), \text{ subject to: } \quad s_{j_k+1} \geq ... \geq s_{j_{k+1}} \text{ and } \sum_{j=j_k+1}^{j_{k+1}} s_j \leq \sigma \quad \text{(C.4)}$$

$$s_{j_k}^* \geq s_{j_k+1} \text{ and } s_{j_{k+1}} \geq s_{j_{k+1}+1}^* \quad \text{(C.5)}$$

We will work on the related maximization problem without constraint (C.5) and check (C.5) later. The Lagrangian function then can be written as

$$L = \sum_{j=j_k+1}^{j_{k+1}} \alpha_j Q(s_j) + \mu\left(\sigma - \sum_{j=j_k+1}^{j_{k+1}} s_j\right) + \sum_{j=j_k+1}^{j_{k+1}-1} \gamma_j (s_j - s_{j+1})$$

where μ and γ_j are Lagrange multipliers. Hence, the Kuhn-Tucher conditions are (let $\gamma_{j_k} \equiv 0$, $\gamma_{j_{k+1}} \equiv 0$)

$$\alpha_j Q'(s_j) - \mu + \gamma_j - \gamma_{j-1} = 0, \text{ for } j = j_k + 1, ..., j_{k+1} \tag{C.6}$$

Averaging (C.6) for the first l shares and the remaining shares, respectively, we have

$$\frac{1}{l}\left(\sum\nolimits_{j=j_k+1}^{j_k+l} \alpha_j Q'(s_j) + \gamma_{j_k+l}\right) = \frac{1}{j_{k+1} - j_k - l}\left(\sum\nolimits_{j=j_k+l+1}^{j_{k+1}} \alpha_j Q'(s_j) - \gamma_{j_k+l}\right). \tag{C.7}$$

By definition, j_{k+1} is the maximizer for the average return factor starting from $j_k + 1$, so

$$\frac{1}{l}\sum\nolimits_{j=j_k+1}^{j_k+l} \alpha_j \leq \frac{1}{j_{k+1} - j_k - l}\sum\nolimits_{j=j_k+l+1}^{j_{k+1}} \alpha_j \tag{C.8}$$

Also note that $Q'(s_j)$ is nondecreasing in j. Therefore, we have $\frac{1}{l}\sum_{j=j_k+1}^{j_k+l} \alpha_j Q'(s_j) \leq \frac{1}{j_{k+1}-j_k-l}\sum_{j=j_k+l+1}^{j_{k+1}} \alpha_j Q'(s_j)$. If $\gamma_{j_k+l} = 0$, (C.7) can hold only if (C.8) holds in equality and $s_{j_k+1} = ... = s_{j_{k+1}}$. In other words, if any $\gamma_j = 0$ ($j_k < j < j_{k+1}$), we must have $s_{j_k+1} = ... = s_{j_{k+1}}$. Otherwise, we have $\gamma_j > 0$ for all $j_k < j < j_{k+1}$, which implies $s_{j_k+1} = ... = s_{j_{k+1}}$ by the Kuhn-Tucker condition. So, regardless, we have $s_{j_k+1} = ... = s_{j_{k+1}} = \frac{1}{j_{k+1}-j_k}\sigma$, which naturally satisfies constraint (C.5).

Proof of Proposition 4.4.7.

Suppose that we have m plateaus. By Lemma 4.4.5, shares are equal in size within a plateau. Denote $n_k \equiv j_k - j_{k-1}$ as the number of shares in plateau k and z_k as the size of a share in that plateau. Recall that $\bar{\alpha}_k$ decreases in k. Without loss of generality, we assume there exists $k_0 \in \{1, 2, ..., m\}$ such that $\bar{\alpha}_{k_0} > 0 \geq \bar{\alpha}_{k_0+1}$. Clearly, it is never optimal to allocate resources to plateaus with non-positive average return factors. Therefore, $z_{k_0+1} = ... = z_m = 0$ in the optimal share structure. The optimal share structure problem becomes

$$\max_{\{z_1,...,z_{k_0}\}} \sum_{k=1}^{k_0} n_k \bar{\alpha}_k Q(z_k), \text{ subject to: } z_1 \geq ... \geq z_{k_0} \geq 0 \text{ and } \sum_{k=1}^{k_0} n_k z_k \leq 1 \tag{C.9}$$

The Lagrangian function for the above is (notice that $z_{k_0+1} = 0$)

$$L(\mathbf{s}, \mu, \lambda) = \sum_{k=1}^{k_0} n_k \bar{\alpha}_k Q(z_k) + \mu \left(1 - \sum_{k=1}^{k_0} n_k z_k\right) + \sum_{k=1}^{k_0} \lambda_k (z_k - z_{k+1}) \qquad \text{(C.10)}$$

where μ and λ_k are Lagrange multipliers. Hence, the Kuhn-Tucker conditions are (define $\lambda_0 \equiv 0$)

$$n_k \bar{\alpha}_k Q'(z_k) - n_k \mu + \lambda_k - \lambda_{k-1} = 0, \text{ for } k = 1, 2, ..., k_0 \qquad \text{(C.11)}$$

If $\lambda_1 = ... = \lambda_{k_0} = 0$, we immediately have, by (C.11), that $\bar{\alpha}_1 Q'(z_1) = ... = \bar{\alpha}_{k_0} Q'(z_{k_0}) \geq \bar{\alpha}_{k_0+1} Q'(0)$. Otherwise, there must exist k ($k < k_0$) such that $\lambda_0 = ... = \lambda_k = 0$ and $\lambda_{k+1} > 0$. From (C.11),

$$\bar{\alpha}_{k+1} Q'(z_{k+1}) - \mu + \frac{\lambda_{k+1} - \lambda_k}{n_{k+1}} = \bar{\alpha}_{k+2} Q'(z_{k+2}) - \mu + \frac{\lambda_{k+2} - \lambda_{k+1}}{n_{k+2}}. \qquad \text{(C.12)}$$

Note that $\bar{\alpha}_{k+1} > \bar{\alpha}_{k+2} > 0$ and $z_{k+1} = z_{k+2}$ (because $\lambda_{k+1} > 0$). (C.12) requires $\lambda_{k+2} > \lambda_{k+1} > 0$. Using the similar logic repeatedly, we can get $\lambda_{k_0} > ... > \lambda_{k+2} > \lambda_{k+1} > 0$, which implies $z_{k_0+1} = z_{k_0} = ... = z_{k+1} = 0$ (because $z_{k_0+1} = 0$). Substituting the λ-sequence into (C.10), we have

$$\bar{\alpha}_1 Q'(z_1) = \bar{\alpha}_2 Q'(z_2) = ... = \bar{\alpha}_k Q'(z_k) \geq \bar{\alpha}_{k+1} Q'(0), \qquad \text{(C.13)}$$

which implies $z_1 > z_2 > ... > z_k$ (because $\bar{\alpha}_k$ decreases and $Q(\cdot)$ is concave).

In addition, we have $\mu > 0$ from (C.11) when $k = 1$, which implies $\sum_{k=1}^{k_0} n_k z_k = 1$, that is, all the resources are offered in the optimal share structure.

Proof of Proposition 4.5.2.

Assume that $\hat{Q}(\cdot)$ is more concave than $Q(\cdot)$; that is, there exists a concave function $\psi(\cdot)$ such that $\hat{Q}(\cdot) = \psi(Q(\cdot))$. Notice that $\frac{\hat{Q}'(x)}{Q'(x)} = \psi'(Q(x))$, which decreases in x. Therefore, for any x_1 and x_2 ($x_1 < x_2$),

$$\frac{\hat{Q}'(x_1)}{\hat{Q}'(x_2)} > \frac{Q'(x_1)}{Q'(x_2)} \qquad \text{(C.14)}$$

Denote $\mathbf{s}$ and $\hat{\mathbf{s}}$ as the optimal share structures under $Q(\cdot)$ and $\hat{Q}(\cdot)$, respectively. To show $\hat{\mathbf{s}}$ is less steep than $\mathbf{s}$, it is sufficient to show that

$$\hat{s}_j \geq s_j \Rightarrow \hat{s}_{j+1} \geq s_{j+1}, \forall j \tag{C.15}$$

First, note that $\mathbf{s}$ and $\hat{\mathbf{s}}$ have identical plateau boundaries by Lemma 4.4.5. If j and $j+1$ are located in the same plateau, $s_j = s_{j+1}$ and $\hat{s}_j = \hat{s}_{j+1}$, and $\hat{s}_{j+1} \geq s_{j+1}$ holds trivially. Otherwise, we assume j and $j+1$ are located in plateau k and $k+1$, respectively. We focus the nontrivial case $s_{j+1} > 0$. By Proposition 4.4.7, we have $\bar{\alpha}_k \hat{Q}'(\hat{s}_j) \geq \bar{\alpha}_{k+1} \hat{Q}'(\hat{s}_{j+1})$ and $\bar{\alpha}_k Q'(s_j) = \bar{\alpha}_{k+1} Q'(s_{j+1})$, and hence

$$\frac{\hat{Q}'(\hat{s}_{j+1})}{\hat{Q}'(\hat{s}_j)} \leq \frac{Q'(s_{j+1})}{Q'(s_j)} \tag{C.16}$$

Combining (C.16) with that $\frac{\hat{Q}'(\hat{s}_{j+1})}{\hat{Q}'(s_j)} \leq \frac{\hat{Q}'(\hat{s}_{j+1})}{\hat{Q}'(\hat{s}_j)}$ (because of the concavity of $\hat{Q}(\cdot)$) and $\frac{Q'(s_{j+1})}{Q'(s_j)} < \frac{\hat{Q}'(s_{j+1})}{\hat{Q}'(s_j)}$ (because of (C.14)), we have $\frac{\hat{Q}'(\hat{s}_{j+1})}{\hat{Q}'(s_j)} < \frac{\hat{Q}'(s_{j+1})}{\hat{Q}'(s_j)}$, which implies that $\hat{s}_{j+1} > s_{j+1}$.

We now show $\hat{s}_1 < s_1$, if $s_2 > 0$. If $\hat{s}_1 \geq s_1$, we know from the above proof that $\hat{s}_2 > s_2$ (note that s_1 and s_2 are not in the same plateau by Proposition 4.4.7) and $\hat{s}_j \geq s_j$ for $j > 2$, which contradicts that $\sum_{j=1}^{n} s_j = \sum_{j=1}^{n} \hat{s}_j$.

Proof of Lemma 4.5.3.

It is sufficient to show that if $\hat{s}_j \geq s_j$ then $\hat{s}_{j+1} \geq s_{j+1}$, for any j. If j and $j+1$ are located in the same plateau, $\hat{s}_j = \hat{s}_{j+1}$ and $s_j = s_{j+1}$, and $\hat{s}_{j+1} \geq s_{j+1}$ holds trivially. So we assume j and $j+1$ are located in plateau k and $k+1$, respectively. If $s_{j+1} = 0$, the result holds trivially. Suppose $s_{j+1} > 0$ (so that $\hat{s}_j \geq s_j > s_{j+1} > 0$). By Proposition 4.4.7, we have $\hat{\bar{\alpha}}_k Q'(\hat{s}_j) \geq \hat{\bar{\alpha}}_{k+1} Q'(\hat{s}_{j+1})$ and $\bar{\alpha}_k Q'(s_j) = \bar{\alpha}_{k+1} Q'(s_{j+1})$. Together with condition (4.19), we have

$$\frac{Q'(\hat{s}_j)}{Q'(\hat{s}_{j+1})} \geq \frac{\hat{\bar{\alpha}}_{k+1}}{\hat{\bar{\alpha}}_k} \geq \frac{\bar{\alpha}_{k+1}}{\bar{\alpha}_k} = \frac{Q'(s_j)}{Q'(s_{j+1})} \tag{C.17}$$

which implies $\frac{Q'(\hat{s}_j)}{Q'(s_j)} \geq \frac{Q'(\hat{s}_{j+1})}{Q'(s_{j+1})}$. Note that $\frac{Q'(\hat{s}_j)}{Q'(s_j)} \leq 1$ by concavity of $Q(\cdot)$ and $\hat{s}_j \geq s_j$. So we have $\frac{Q'(\hat{s}_{j+1})}{Q'(s_{j+1})} \leq 1$, which implies $\hat{s}_{j+1} \geq s_{j+1}$.

Proof of Proposition 4.5.4.

(a) Denote $\hat{F}(wv)$ as the distribution function after scaling. Clearly, $\hat{F}(wv) = F(v)$, and hence $\hat{P}_j(wv) = P_j(v)$. It is easy to verify $\hat{f}(wv) = \frac{f(v)}{w}$ and $\hat{J}(wv) = wJ(v)$. Based on these relationships, we have

$$\hat{\alpha}_j = n\int_{w\underline{v}}^{w\bar{v}} \hat{P}_j(x)\hat{J}(x)\hat{f}(x)dx = n\int_{w\underline{v}}^{w\bar{v}} P_j(\frac{x}{w})J(\frac{x}{w})f\left(\frac{x}{w}\right)dx = wn\int_{\underline{v}}^{\bar{v}} P_j(v)J(v)f(v)dv$$

where the third step is due to integration by substitution.

(b) Denote $\hat{F}(v+w)$ as the distribution function after shifting. Clearly, $\hat{F}(v+w) = F(v)$, and hence $\hat{P}_j(v+w) = P_j(v)$. It is easy to verify $\hat{f}(v+w) = f(v)$ and $\hat{J}(v+w) = J(v) + w$. Based on these relationships, we have

$$\begin{aligned}\hat{\alpha}_j &= n\int_{\underline{v}+w}^{\bar{v}+w} \hat{P}_j(x)\hat{J}(x)\hat{f}(x)dx = n\int_{\underline{v}+w}^{\bar{v}+w} P_j(x-w)\left(J(x-w)+w\right)f(x-w)dx \\ &= wn\int_{\underline{v}}^{\bar{v}} P_j(v)f(v)dv + n\int_{\underline{v}}^{\bar{v}} P_j(v)J(v)f(v)dv = w + \alpha_j \qquad \text{(C.18)}\end{aligned}$$

where the third step is due to integration by substitution and the last step is due to (C.3).

(c) Under the regular condition, $\bar{\alpha}_j$ sequence coincides with α_j sequence. So, by Lemma 4.5.3, a sufficient condition for $\hat{\mathbf{s}}$ to be less steeper than $\mathbf{s}$ is

$$\frac{\hat{\alpha}_{j+1}}{\hat{\alpha}_j} \geq \frac{\alpha_{j+1}}{\alpha_j}, \text{ whenever } s_{j+1} > 0 \qquad \text{(C.19)}$$

Denote $P_j^n(x) \equiv \binom{n-1}{n-j} x^{n-j}(1-x)^{j-1}$ and $\mathcal{J}(x) = J(F^{-1}(x))$. We can write that $\alpha_j = n\int_0^1 P_j^n(F(v))J(v)f(v)dv = n\int_0^1 P_j^n(x)\mathcal{J}(x)dx$. Noting that $\hat{\alpha}_{j+1} > 0$ and $\alpha_{j+1} > 0$ (from $\hat{\mathcal{J}}(x) > 0$ and $\mathcal{J}(x) > 0$), we can rewrite (C.19) as

$$\int_0^1 P_{j+1}^n(x)\mathcal{J}(x)dx\int_0^1 P_j^n(x)\hat{\mathcal{J}}(x)dx \leq \int_0^1 P_j^n(x)\mathcal{J}(x)dx\int_0^1 P_{j+1}^n(x)\hat{\mathcal{J}}(x)dx$$

By Athey (2000), the above holds if the following two conditions are satisfied:

$$P_{j+1}^{n}(x)\,\eth(x)\cdot P_{j}^{n}(y)\,\hat{\eth}(y) \leq P_{j}^{n}(y)\,\eth(y)\cdot P_{j+1}^{n}(x)\,\hat{\eth}(x)\,,\ \forall\ x<y \quad \text{(C.20)}$$

$$P_{j+1}^{n}(x)\,\eth(x)\cdot P_{j}^{n}(y)\,\hat{\eth}(y) \leq P_{j}^{n}(x)\,\eth(x)\cdot P_{j+1}^{n}(y)\,\hat{\eth}(y)\,,\ \forall\ x>y \quad \text{(C.21)}$$

(C.21) simplifies to $P_{j+1}^{n}(x)\,P_{j}^{n}(y) \leq P_{j}^{n}(x)\,P_{j+1}^{n}(y)$, which holds naturally for $x>y$. (C.20) can be written as

$$\frac{\hat{\eth}(y)}{\hat{\eth}(x)} \leq \frac{\eth(y)}{\eth(x)}, \forall x<y. \quad \text{(C.22)}$$

If (C.22) holds in strict inequality, so does (C.19), suggesting $\hat{\mathbf{s}}$ is strictly less steeper than $\mathbf{s}$.

Proof of Lemma 4.5.5.

We prove by contradiction. Suppose that we have m plateaus. Let $\hat{\mathbf{s}}$ denote the new optimal share structure after increasing the total resources. Notice that $\bar{\alpha}_j$'s remain the same. Assume $\hat{z}_l \leq z_l$ for some $l \in \{1,2,...,k^*\}$. Then it must be that $\hat{z}_k \leq z_k$, for all $k \in \{1,2,...,m\}$. Otherwise, assume $\hat{z}_j > z_j \geq 0$ for some $j \in \{1,2,...,m\}$. Because $\hat{z}_j > 0$ and $z_l > 0$, by the first order condition (4.16), $\bar{\alpha}_j Q'(\hat{z}_j) \geq \bar{\alpha}_l Q'(\hat{z}_l)$ and $\bar{\alpha}_l Q'(z_l) \geq \bar{\alpha}_j Q'(z_j)$. Since $\bar{\alpha}_l Q'(\hat{z}_l) \geq \bar{\alpha}_l Q'(z_l)$, we thus have $\bar{\alpha}_j Q'(\hat{z}_j) \geq \bar{\alpha}_j Q'(z_j)$, which contradicts with $\hat{z}_j > z_j$. But if $\hat{z}_k \leq z_k$ for all k, the available resources are not fully allocated, which cannot be optimal. So all shares with positive sizes must increase with the total resources. As a result, the number of positive shares weakly increases.

Proof of Proposition 4.5.8.

Let $\mathbf{s}$ and $\hat{\mathbf{s}}$ denote the share structure before and after increasing the total resources, respectively. Consider $z_{k+1} > 0$ (so that $\hat{z}_{k+1} > 0$ by Lemma 4.5.5). By the optimal condition (4.16), $\frac{Q'(z_k)}{Q'(z_{k+1})} = \frac{\bar{\alpha}_{k+1}}{\bar{\alpha}_k} = \frac{Q'(\hat{z}_k)}{Q'(\hat{z}_{k+1})}$. Noting that $\int_a^b \frac{Q''(x)}{Q'(x)}dx = \ln\frac{Q'(b)}{Q'(a)}$, we have,

$$\int_{z_{k+1}}^{z_k} \frac{Q''(x)}{Q'(x)} dx = \int_{\hat{z}_{k+1}}^{\hat{z}_k} \frac{Q''(x)}{Q'(x)} dx. \tag{C.23}$$

Denote $r(x) \equiv -\frac{Q'(x)}{Q''(x)x}$. Note that

$$\int_a^b \frac{Q''(x)}{Q'(x)} dx = -\int_a^b \frac{1}{r(x)x} dx = -\frac{1}{r(\xi)} \int_a^b \frac{1}{x} dx = -\frac{1}{r(\xi)} \ln \frac{b}{a}, \text{ for some } \xi \in (a,b). \tag{C.24}$$

From Lemma 4.5.5, we know $z_{k+1} < \hat{z}_{k+1}$ and $z_k < \hat{z}_k$. Suppose $z_k \le \hat{z}_{k+1}$. Substituting (C.24) into both sides of (C.23), we have

$$\frac{1}{r(\xi)} \ln \frac{z_k}{z_{k+1}} = \frac{1}{r(\hat{\xi})} \ln \frac{\hat{z}_k}{\hat{z}_{k+1}}, \text{ where } z_{k+1} < \xi < z_k \le \hat{z}_{k+1} < \hat{\xi} < \hat{z}_k.$$

It is straightforward that $r'(\cdot) > (<,=)\, 0$ implies $\frac{z_k}{z_{k+1}} < (>,=) \frac{\hat{z}_k}{\hat{z}_{k+1}}$, or $\frac{\hat{z}_{k+1}}{z_{k+1}} < (>,=) \frac{\hat{z}_k}{z_k}$. If $\hat{z}_{k+1} < z_k$, (C.23) can be rewritten as,

$$\int_{z_k}^{\hat{z}_k} \frac{Q''(x)}{Q'(x)} dx = \int_{z_{k+1}}^{\hat{z}_{k+1}} \frac{Q''(x)}{Q'(x)} dx$$

By the same logic, $r'(\cdot) > (<,=)\, 0$ implies $\frac{\hat{z}_{k+1}}{z_{k+1}} < (>,=) \frac{\hat{z}_k}{z_k}$.

Lemma C.0.1 (Ranking of $\alpha_j(v_0)$)**.** *Under the MHR condition, for any marginal type $v_0 \in (\underline{v}, \bar{v})$, if $\alpha_j(v_0) > 0$, $\alpha_j(v_0) > \alpha_{j+1}(v_0)$; if $\alpha_j(v_0) \le 0$, $\alpha_{j+1}(v_0) < 0$.*

Proof. For the case $v_0 \in (\underline{v}, \bar{v})$, define $h_j(x|x \ge v_0) \equiv \frac{h_j(x)}{\int_{v_0}^{\bar{v}} h_j(t)dt}$. Following steps in the proof of Lemma 4.4.1 (b), we can similarly show that $h_i(x|x \ge v_0)$ first-order stochastically dominates $h_{i+1}(x|x \ge v_0)$. Thus,

$$\int_{v_0}^{\bar{v}} h_j(x|x \ge v_0) J(x) dx > \int_{v_0}^{\bar{v}} h_{j+1}(x|x \ge v_0) J(x) dx, \text{ for } j = 1, 2, ..., n-1. \tag{C.25}$$

Substituting $h_j(x|x \ge v_0)$ with $\frac{h_j(x)}{\int_{v_0}^{\bar{v}} h_j(t)dt}$ and rearranging, we have

$$\alpha_j(v_0) = \int_{v_0}^{\bar{v}} h_j(x) J(x) dx > \frac{\int_{v_0}^{\bar{v}} h_j(t) dt}{\int_{v_0}^{\bar{v}} h_{j+1}(t) dt} \int_{v_0}^{\bar{v}} h_{j+1}(x) J(x) dx = \frac{\int_{v_0}^{\bar{v}} h_j(t) dt}{\int_{v_0}^{\bar{v}} h_{j+1}(t) dt} \alpha_{j+1}(v_0). \tag{C.26}$$

Suppose $\alpha_j(v_0) > 0$. If $\alpha_{j+1}(v_0) \geq 0$, from $\int_{v_0}^{\bar{v}} h_j(t)\,dt > \int_{v_0}^{\bar{v}} h_{j+1}(t)\,dt > 0$ (because $h_j(t)$ first-order stochastically dominates $h_{j+1}(t)$), we have $\alpha_j(v_0) > \alpha_{j+1}(v_0)$. If $\alpha_{j+1}(v_0) < 0$, it is easy to see $\alpha_j(v_0) > \alpha_{j+1}(v_0)$.

If $\alpha_j(v_0) \leq 0$, (C.26) implies $\alpha_{j+1}(v_0) < 0$. □

Proof of Proposition 4.6.1.

First, we derive v_0^*. The first-order derivative of the expected revenue (4.22) with respect to the marginal type (v_0) is $-n\sum_{j=1}^{n} Q(s_j)P_j(v_0)J(v_0)f(v_0)$. The optimal marginal type is either an interior solution of $J(v) = 0$ or one of the two corner solutions ($\underline{v}$ or $\bar{v}$). Notice that $\bar{v}$ cannot be optimal since the expected revenue decreases at the neighborhood of $v = \bar{v}$, implied by $J(\bar{v}) > 0$. Under the regular condition, v_0^* is either the solution to $J(v_0) = 0$ or $\underline{v}$, whichever is higher.

We prove Proposition 4.6.1 for any $v_0 \in (\underline{v}, v_0^*]$. First note that $\alpha_j(v_0) = \alpha_j(\underline{v}) - \int_{\underline{v}}^{v_0} h_j(v)\,J(v)\,dv$. According to Lemma 4.5.3, a sufficient condition is that for any j such that $s_{j+1}^* > 0$ (s_{j+1}^* denotes the $(j+1)$-th share in the optimal share structure under no minimum bid),

$$\frac{\alpha_{j+1}(v_0)}{\alpha_j(v_0)} = \frac{\alpha_{j+1}(\underline{v}) - \int_{\underline{v}}^{v_0} h_{j+1}(v)\,J(v)\,dv}{\alpha_j(\underline{v}) - \int_{\underline{v}}^{v_0} h_j(v)\,J(v)\,dv} > \frac{\alpha_{j+1}(\underline{v})}{\alpha_j(\underline{v})} \tag{C.27}$$

Following steps in the proof of Lemma 4.4.1 (b), we can similarly show that $\int_{\underline{v}}^{v_0} h_j(v)\,dv < \int_{\underline{v}}^{v_0} h_{j+1}(v)\,dv$ and $h_j(v) / \int_{\underline{v}}^{v_0} h_j(v)\,dv$ first-order stochastically dominates $h_{j+1}(v) / \int_{\underline{v}}^{v_0} h_{j+1}(v)\,dv$. Therefore,

$$\frac{\int_{\underline{v}}^{v_0} h_j(v)\,J(v)\,dv}{\int_{\underline{v}}^{v_0} h_j(v)\,dv} > \frac{\int_{\underline{v}}^{v_0} h_{j+1}(v)\,J(v)\,dv}{\int_{\underline{v}}^{v_0} h_{j+1}(v)\,dv} \tag{C.28}$$

By the MHR condition, $J(v) \leq 0$ for $v \leq v_0^*$, so we have

$$0 > \int_{\underline{v}}^{v_0} h_j(v) J(v)\, dv > \frac{\int_{\underline{v}}^{v_0} h_j(v)\, dv}{\int_{\underline{v}}^{v_0} h_{j+1}(v)\, dv} \int_{\underline{v}}^{v_0} h_{j+1}(v) J(v)\, dv > \int_{\underline{v}}^{v_0} h_{j+1}(v) J(v)\, dv \tag{C.29}$$

where the last inequality is due to $0 < \int_{\underline{v}}^{v_0} h_j(v)\, dv < \int_{\underline{v}}^{v_0} h_{j+1}(v)\, dv$.

Given $s_{j+1}^* > 0$, we have $\alpha_{j+1}(\underline{v}) > 0$. The result (C.27) follows from the fact that $\alpha_j(\underline{v}) > \alpha_{j+1}(\underline{v}) > 0$ (Lemma 4.4.1) and that $0 > \int_{\underline{v}}^{v_0} h_j(v) J(v)\, dv > \int_{\underline{v}}^{v_0} h_{j+1}(v) J(v)\, dv$ (by (C.29)).

Derivation of the Bidding Function.

Let $\beta(v)$ denote bidders' bidding function. We consider the case with a strictly increasing bidding function, and thus the inverse bidding function, $\beta^{-1}(b)$, exists and is strictly increasing.

If a bidder's rivals bid according to $\beta(v)$, the bidder's probability of winning jth share by placing bid b is $p_j(b) \equiv \binom{n-1}{n-j} F\left(\beta^{-1}(b)\right)^{n-j} \left(1 - F\left(\beta^{-1}(b)\right)\right)^{j-1}$. Since in equilibrium the bidder bids $b = \beta(v)$, the equilibrium probability of winning jth share is $P_j(v) \equiv p_j(\beta(v)) = \binom{n-1}{n-j} F(v)^{n-j} (1 - F(v))^{j-1}$.

Denote $V(v) \equiv U(v, \beta(v))$ as the equilibrium payoff of a bidder of type v.

$$V(v) = U(v, \beta(v)) = \sum_{j=1}^{n} p_j(\beta(v)) (vQ(s_j) - \beta(v)s_j) \tag{C.30}$$

We have $\frac{dV(v)}{dv} = \frac{\partial U(v,\beta(v))}{\partial v} + \frac{\partial U(v,\beta(v))}{\partial b} \frac{d\beta(v)}{dv}$. According to the first-order condition, $\frac{\partial U(v,\beta(v))}{\partial b} = 0$. So

$$\frac{dV(v)}{dv} = \frac{\partial U(v, \beta(v))}{\partial v} = \sum_{j=1}^{n} p_j(\beta(v)) Q(s_j) = \sum_{j=1}^{n} P_j(v) Q(s_j) \tag{C.31}$$

Moving dv to the right hand side, integrating both sides from $\underline{v}$ to v, and assuming $V(\underline{v}) = 0$ (the lowest type gets zero payoff),we get

$$V(v) = \sum_{j=1}^{n} Q(s_j) \int_{\underline{v}}^{v} P_j(x)dx, \text{ for } v \in [\underline{v}, \bar{v}]. \tag{C.32}$$

Combining (C.30) and (C.32), we can solve the equilibrium bidding function as

$$\beta(v) = v\frac{\sum\limits_{j=1}^{n} P_j(v)Q(s_j)}{\sum\limits_{j=1}^{n} P_j(v)s_j} - \frac{\sum\limits_{j=1}^{n} Q(s_j) \int_{\underline{v}}^{v} P_j(x)dx}{\sum\limits_{j=1}^{n} P_j(v)s_j}, v \in [\underline{v}, \bar{v}] \tag{C.33}$$

Derivation of the Expected Revenue.

The expected payment from a bidder of type v is $\beta(v) \sum\limits_{j=1}^{n} s_j P_j(v)$. The expected payment from one bidder is

$$\begin{aligned}
E\left[\beta(v) \sum_{j=1}^{n} s_j P_j(v)\right] &= \int_{\underline{v}}^{\bar{v}} \left[\beta(v) \sum_{j=1}^{n} s_j P_j(v)\right] f(v)dv \\
&= \int_{\underline{v}}^{\bar{v}} \left[v \sum_{j=1}^{n} Q(s_j)P_j(v) - \sum_{j=1}^{n} Q(s_j) \int_{\underline{v}}^{v} P_j(t)dt\right] f(v)dv \\
&= \int_{\underline{v}}^{\bar{v}} \left[v \sum_{j=1}^{n} Q(s_j)P_j(v)f(v) - (1 - F(v)) \sum_{j=1}^{n} Q(s_j)P_j(v)\right] dv \\
&= \sum_{j=1}^{n} Q(s_j) \int_{\underline{v}}^{\bar{v}} P_j(v) \left(v - \frac{1 - F(v)}{f(v)}\right) f(v)dv \qquad \text{(C.34)}
\end{aligned}$$

The total expected revenue from all bidders is n times the above.

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