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Pretzel Knots of Length Three with Unknotting Number One

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Pretzel Knots of Length Three with Unknotting Number One

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To my loved ones.

Acknowledgments

First I would like to thank my advisor Cameron Gordon. He has been generous, patient, enthusiastic, and caring. I am eternally grateful for his support during my time at UT. I would also like to thank those who took the time to answer my questions and engage in valuable conversations.

I would also like to my coauthors in London. Dorothy Buck and Julian Gibbons were instrumental in strengthening our theorems. It should be noted that the work in Sections 1, 2, 3, and 5 were completed independently by Dorothy and Julian. The work in Section 4 is entirely my own. A stronger version of the main theorem found in Chapter 4 can be found in a paper by Dorothy, Julian, and myself (currently submitted.)

Finally I would like to give thanks to my friends and family who have encouraged me through graduate school. Your love and support has meant the world to me.

Pretzel Knots of Length Three with Unknotting Number One

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The University of Texas at Austin, 2012

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This thesis provides a partial classification of all 3-stranded pretzel knots $K = P(p, q, r)$ with unknotting number one. Scharlemann-Thompson, and independently Kobayashi, have completely classified those knots with unknotting number one when p, q, and r are all odd. In the case where $p = 2m$, we use the signature obstruction to greatly limit the number of 3-stranded pretzel knots which may have unknotting number one. In Chapter 3 we use Greene's strengthening of Donaldson's Diagonalization theorem to determine precisely which pretzel knots of the form $P(2m, k, -k-2)$ have unknotting number one, where $m \in \mathbb{Z}$, $m > 0$, and $k > 0$, odd. In Chapter 4 we use Donaldson's Diagonalization theorem as well as an unknotting obstruction due to Ozsváth and Szabó to partially classify which pretzel knots $P(2, k, -k)$ have unknotting number one, where $k > 0$, odd. The Ozsváth-Szabó obstruction is a consequence of Heegaard Floer homology. Finally in Chapter 5 we explain why the techniques used in this paper cannot be used on the remaining cases.

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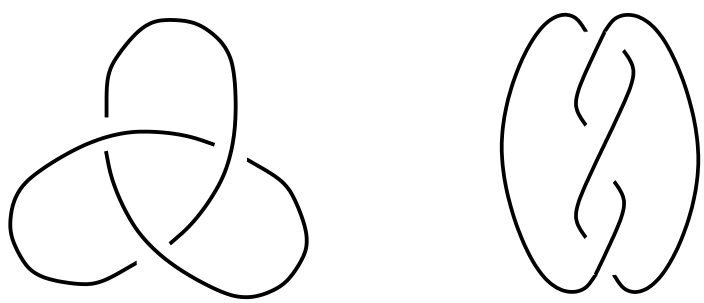
Chapter 1

Introduction

A *knot* K is an embedding of a circle into a 3-manifold M. We will only consider knots embedded into S^3. A *diagram* of K, D_K, is a projection of K onto a plane such that the overcrossing and undercrossing at each crossing are distinguished. There is an infinite number of knot diagrams for every knot. Figure 1.1 shows two such diagrams of the trefoil:



(a) (b)

Figure 1.1: Two diagrams of the trefoil.

Two knot diagrams represent the same knot if they are *ambient isotopic*. The knots represented by two knot diagrams D_K and D'_K are ambient isotopic if and only if they are connected by a sequence of moves called Reidemeister moves.

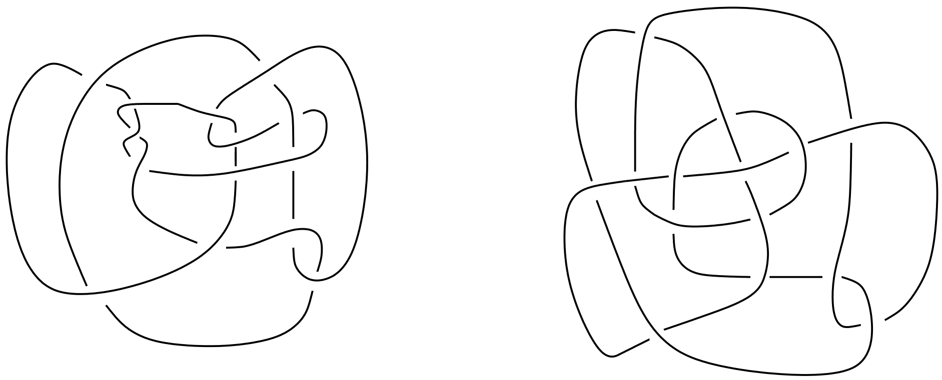
In order to show two knot diagrams represent different knots, however, one cannot simply appeal to knot diagrams. Instead we use knot invariants. A *knot invariant* is an algebraic quantity, such as a number, polynomial, or homology theory, assigned to a knot diagram that only depends on the knot represented by the diagram. Examples of knot invariants include knot signature, Alexander polynomial, Jones polynomial, and knot Floer homology. If the values of a knot invariant are different for two different diagrams, then the diagrams represent two different knots.

One such knot invariant is the unknotting number. The *unknotting number* of K, $u(K)$, is the minimal number of times a knot must cross itself in order to unknot it. Equivalently, let D_K be a knot diagram, and $u(D_K)$ be the minimal number of times the diagram must cross itself in order to unknot it. We can then define the unknotting number as $u(K) := min\{u(D_K)|D_K$ is a diagram of $K\}$. Although it is easy to understand, computing the unknotting number can be quite difficult. For example, in [26] Stoimenow shows that the knot 14_{36750} has a 14-crossing diagram with unknotting number 2 (Figure 1.2(a)), as well as a 14-crossing diagram with unknotting number 1 (Figure 1.2(b)).

The unknotting crossing in Figure 1.2(b) can be found in [26].

Even more striking is the example by Bleiler in [1] (and independently Nakanishi in [18]). All minimal crossing diagrams of the knot 10_8 have unknotting number three (see Figure 1.3(a)), but 10_8 has a 14-crossing diagram for which the unknotting number is two (see Figure 1.3(b)).

Finding an upper bound for $u(K)$ can be done by computing the un-



(a) (b)

Figure 1.2: Two 14-crossing diagrams of the knot 14_{36750}.

(a) (b)

Figure 1.3: Two diagrams of the knot 10_8.

knotting number for well chosen diagrams. For example, from the diagram in Figure 1.4 we can show $u(11_{328}) \leq 2$.

However, unless one can find a diagram D_K such that $u(D_K) = 1$, finding a lower bound can be difficult. A classical lower bound for the unknotting number is given by $|\sigma(K)| \leq 2u(K)$ (see [16]), where $\sigma(K)$ is the signature of K. Similar results use the 4-ball genus of a knot, namely $g_4(K) \leq 2u(K)$ (see

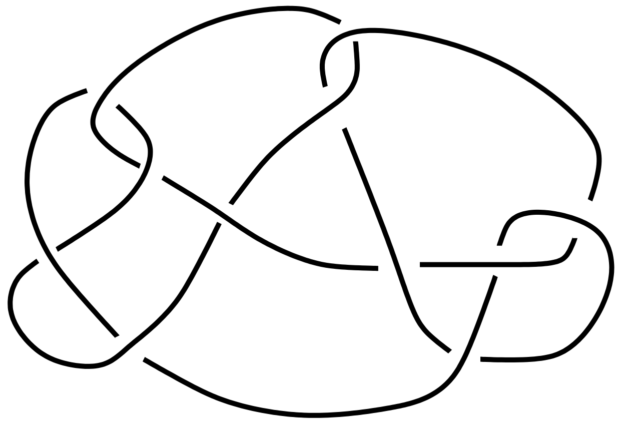


Figure 1.4: The knot 11_{328}

[16]), and Rasmussen's s-invariant, namely $|s(K)| \leq 2u(K)$ (see [23]).

Much more is known about topological obstructions to a knot having unknotting number one. Given a knot $K \subset S^3$, let $\Sigma(K)$ be the double branched cover of S^3 branched along K. An important obstruction in this paper is the *Montesinos trick*: if $u(K) = 1$ then $\Sigma(K)$ arises as a half integral surgery on some knot $\kappa \in S^3$, ie $\Sigma(K) \cong S^3_{D/2}(\kappa)$, where $D = |\det(K)|$ [14]. In particular, if $u(K) = 1$, then $H_1(\Sigma(K))$ is cyclic [17]. In [25], Scharlemann shows that unknotting number one knots are prime. There is also a linking form obstruction due to Lickorish [12].

This work was motivated by the following question: Which algebraic knots, in the sense of Conway, have unknotting number equal to one? A complete treatment of algebraic knots can be found in [27] and [6]. The three distinct types of algebraic knots are *2-bridge, large algebraic*, and *Montesinos length 3*. They are characterized by their double branched covers. To wit, if K is 2-bridge then $\Sigma(K)$ is a lens space; if K is large algebraic then $\Sigma(K)$ is a graph manifold, which is toroidal; and if K is Montesinos length 3 then $\Sigma(K)$ is a Seifert fibered space over S^2 with three exceptional fibers. Kanenobu and Murakami describe in [9] those 2-bridge knots with unknotting number one. Gordon and Luecke describe in [6] all large algebraic knots with unknotting number one in terms of the algebraic tangles of K. The double branched cover of a Montesinos knot of length 3, however, is neither a lens space nor toroidal, so the results of [9] and [6] do not apply. It is then natural to ask:

Question 1.0.1. *Which Montesinos knots of length three have unknotting number one?*

Torisu gives a conjecture to the above question in [28]. Using the notation therein,

Conjecture 1.0.2 (Torisu)**.** *Let K be a Montesinos knot of length three. Then $u(K) = 1$ if and only if $K = \mathcal{M}(0; (p, -r), (q, s), (2mn \pm 1, 2n^2))$, where p, q, r, s, m, and n are non-zero integers, m and n are coprime, and $ps - rq = 1$.*

The conjecture would be true if it were known that a standard diagram of $K = \mathcal{M}(0; (p, -r), (q, s), (2mn \pm 1, 2n^2))$ realizes the unknotting operation for knots with $u(K) = 1$. The theorem would also be true if the *Seifert Fibering Conjecture* is true.

Conjecture 1.0.3 (Seifert Fibering Conjecture)**.** *For a knot $K \subset S^3$ which is neither a torus knot nor a cable on a torus knot, only integral slopes can yield a Seifert fibered space under Dehn surgery.*

For the time being we have only Torisu's Theorem:

Theorem 1.0.4 (Torisu)**.** *Let K be a Montesinos knot of length three and suppose the unknotting operation is realized by a crossing change in a standard diagram. Then $u(K) = 1$ if and only if $K = \mathcal{M}(0; (p, -r), (q, s), (2mn \pm 1, 2n^2))$, where p, q, r, s, m, and n are non-zero integers, m and n are coprime, and $ps - rq = 1$.*

We focus on a subset of Montesinos knots of length three called pretzel knots of length three. In chapters 3 and 4, respectively, will prove the following theorems:

Theorem 3.0.1. *Let $K = P(2m, k, -k-2)$ be a three stranded pretzel knot, where $m \in \mathbb{Z}$ and$m > 0$. If $u(K) = 1$, then up to reflection $K = P(2m, 1, 3)$.*

Theorem 4.1.1. *Let $K = P(2, k, -k)$ be a three stranded pretzel knot and $k > 3$. If k is a prime power, then $u(K) > 1$.*

We should note that our results can be viewed as a partial proof to Conjecture 1.0.2.

Chapter 2

Preliminaries

2.1 Pretzel Knots

A *pretzel link of length n*, $K = P(a_1, a_2, \ldots, a_n)$, where $a_i \in \mathbb{Z}$, is a link which has the following form:

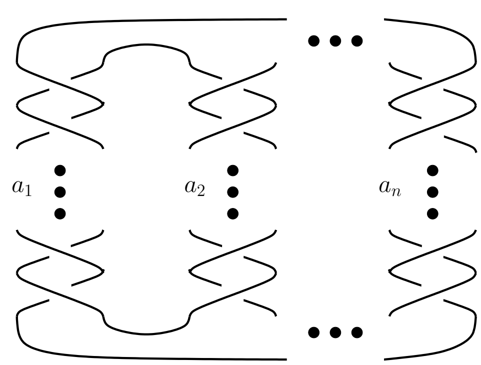


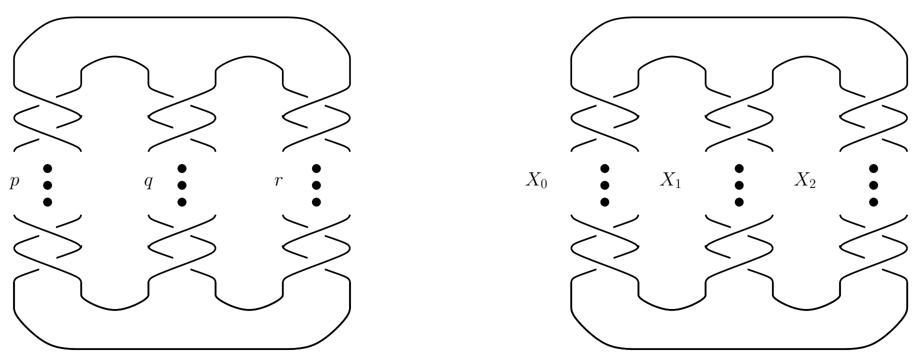
Figure 2.1: The pretzel knot $P(a_1, a_2, \ldots, a_n)$.

In Figure 2.1, a_1 is negative whereas a_2 and a_n are positive. Pretzel knots of length one are unknots. Pretzel knots of length two are 2-bridge knots. From [9], it follows that the only such knots with unknotting number one are pretzels $P(k, 3-k)$ where $k \in \mathbb{Z}$. Pretzel knots of length four or greater are large algebraic knots. In [15], Motegi shows that no pretzel knots of length four or greater (and more generally, no Montesinos knot of length four or greater) have unknotting number one.

Let $K = P(p, q, r)$ denote the 3-stranded pretzel link with p, q, and r half twists, as in Figure 2.2(a). The knot $K = P(p, q, r)$ is a knot if at least two of $\{p, q, r\}$ are odd. Otherwise it is a link. If any of p, q, or r equals ± 1, the corresponding pretzel link $P(p, q, r)$ is a 2-bridge link. Since unknotting number one 2-bridge links are already known, we restrict ourselves to $|p|$, $|q|$, $|r| > 1$. All 3-stranded pretzel links satisfy the following relations:

$$P(p,q,r) = P(r,p,q) \qquad P(p,q,r) = P(q,p,r) \qquad \overline{P(p,q,r)} = P(-p,-q,-r), \tag{2.1}$$

where $\overline{K}$ is the mirror image of K. Clearly $u(K) = u(\overline{K})$, and so without loss of generality we can assume at least two of $\{p, q, r\}$ are positive.



(a) (b)

Figure 2.2: (a) The pretzel link $P(p, q, r)$, where p is negative and q, r are positive. (b) A pretzel link with shaded regions X_0, X_1, and X_2.

Next we reduce the number of 3-stranded pretzel knots which might have unknotting number one by using 'classical' techniques. When p, q, and r

are all odd, K is a genus one knot. Scharlemann and Thompson showed in [24] (and independently Kobayashi in [10]) which genus one knots have unknotting number one. As a corollary, they show a genus one 3-stranded pretzel knot $P(p, q, r)$ has unknotting number one if and only if the set $\{p, q, r\}$ contains one of $\pm\{1, 1\}$ or $\pm\{3, -1\}$. Note that these knots are 2-bridge. Most 3-stranded pretzel knots with an even strand, however, are not genus one, and so [24] (and [10]) does not apply. Therefore we consider 3-stranded pretzel knots of the form $K = P(2m, q, r)$, $m \in \mathbb{Z}$ and q, r odd.

Recall from Chapter 1 that the signature of a knot gives a lower bound for the unknotting number: $|\sigma(K)| \leq 2u(K)$. In [5], Gordon and Litherland prove that the signature of a knot can be computed from any regular projection of K. In particular, the signature of a knot can be computed from the signature of a Goeritz matrix of K, $G(D_K)$, plus a certain correction term, $\mu(D_K)$. Given these quantities, Gordon and Litherland show:

Theorem 2.1.1 (Gordon-Litherland)**.** $\sigma(K) = Sign(G(D_K)) - \mu(D_K)$.

By shading and labeling the three regions of the projection of K as in figure 2.2(b), we see a Goeritz matrix of K is

$$G(D_K) = \begin{pmatrix} 2m+q & -q \\ -q & q+r \end{pmatrix}.$$

Since $G(K)$ is a 2×2 matrix, $\mathrm{Sign}(G(K)) \in \{-2, 0, 2\}$. Since a knot with unknotting number one must have $\sigma(K) = 0$ or ± 2, we can restrict $\mu(K)$ to $\{-4, -2, 0, 2, 4\}$. According to a result in [5], the correction term $\mu(K)$

for a regular projection of $P(2m, q, r)$ equals $q + r$. Since $|p|, |q|, |r| > 1$ and $\mu(D_K) \not\geq 6$, q and r cannot both be positive. We can assume $q > 0$ and $r < 0$. Furthermore it follows from our assumption immediately after Equation 2.1 that $p = 2m > 0$. The reader will note that such knots are *nonalternating*.

Relabel the knot $K = P(2m, k, -k+n)$, where $m \in \mathbb{N}$, $k \in \mathbb{N}$ odd, and $n \in \{-4, -2, 0, 2, 4\}$. A Goeritz matrix of the regular projection of K equals

$$G(K) = \begin{pmatrix} 2m+k & -k \\ -k & n \end{pmatrix}.$$

From this it follows that knots of the form $P(2m, k, -k-4)$, and $P(2m, k, -k+4)$ with $\det(K) < 0$, have the property $|\sigma(K)| = 4$. Therefore these knots do not have unknotting number one.

Knots which may have unknotting number one fall into 5 distinct cases:

Case 1) If $n = -2$ then Sign($G(K)$)=0, and so $\sigma(K) = 2$.

Case 2) If $n = 0$ then Sign(($G(K)$)=0, and so $\sigma(K) = 0$.

Case 3) If $n = 2$ and $\det(G(K)) < 0$ then Sign(($G(K)$)=0, and so $\sigma(K) = -2$.

Case 4) If $n = 2$ and $\det(G(K)) > 0$ then Sign(($G(K)$)=2, and so $\sigma(K) = 0$.

Case 5) If $n = 4$ and $\det(G(K)) > 0$ then Sign(($G(K)$)=2, and so $\sigma(K) = -2$.

It follows from [16] that knots with signature equal to ± 4 do not have unknotting number one. Theorem 3.0.1 determines which knots in **Case 1** have unknotting number one, and Theorem 4.1.1 partially determines which knots in **Case 2** have unknotting number one.

2.2 Plumbing Diagrams

Given a knot $K = P(p,q,r) \in S^3$ and $|p|,|q|,|r| \geq 2$, $\Sigma(K)$ is a Seifert fibered space over S^2 with three exceptional fibers, $S^2(p,q,r)$. In [19], Neumann and Raymond provide a method for constructing a four-manifold X from a plumbing diagram such that $\partial X = \Sigma(K)$. First find a continued fraction expansion for $p/(p-1)$, $q/(q-1)$, $r/(r-1)$:

$$\frac{p}{p-1} = [p_1, p_2, \ldots, p_i] \qquad \frac{q}{q-1} = [q_1, q_2, \ldots, q_j] \qquad \frac{r}{r-1} = [r_1, r_2, \ldots, r_k],$$

where

$$[x_1, x_2, \ldots, x_n] = x_1 - \cfrac{1}{x_2 - \cfrac{1}{\ddots - \frac{1}{x_n}}}.$$

Let $\widetilde{G}(p,q,r)$ be the weighted graph as in Figure 2.3, where each vertex v has weight $w(v)$. To associate a smooth 4-manifold $\widetilde{X} = W(\widetilde{G}(p,q,r))$ to the graph $\widetilde{G}(p,q,r)$, start with disk bundles over the 2-sphere, one for each vertex, of Euler number $w(v)$. Next plumb together those disk bundles which correspond to adjacent vertices. This manifold has $H_2(X)$ free abelian and is generated by the homology classes $[v]$. The boundary of the resulting smooth 4-manifold is the double branched cover of K, $\Sigma(K) = \partial(\widetilde{X})$. Furthermore the intersection form of $\widetilde{X}$, given in terms of the basis of spheres used in the construction, is the incidence matrix of $\widetilde{G}(p,q,r)$. Explicitly, $[v] \cdot [v] = w(v)$, $[v] \cdot [v'] = 1$ if the two distinct vertices are connected by an edge, and 0 otherwise.

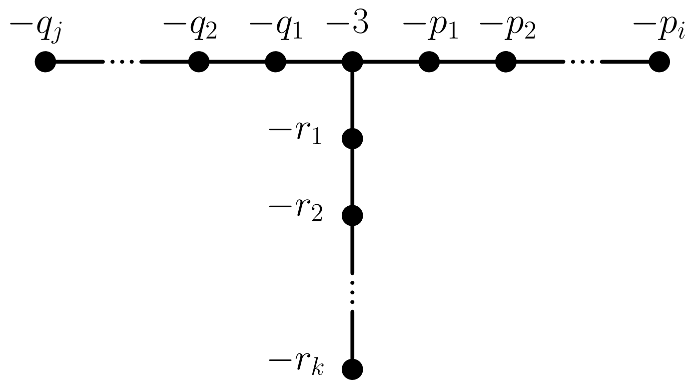


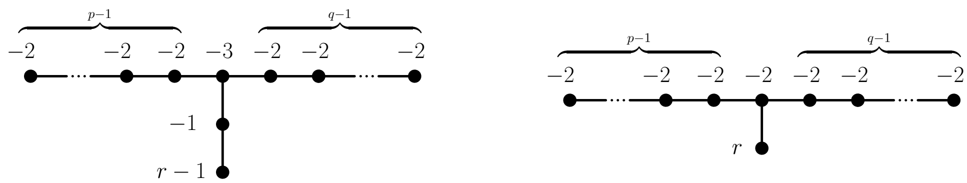
Figure 2.3: A weighted graph $\widetilde{G}(p,q,r)$.

The obstructions used in the proofs of Theorems 3.0.1 and 4.1.1 require the manifold $\widetilde{X}$ to be negative definite. In order to determine when this is the case, suppose $p, q > 0$, $r < 0$, and consider the continued fraction expansion of $\frac{p}{p-1}$, $\frac{q}{q-1}$, and $\frac{r}{r-1}$:

$$\frac{p}{p-1} = \overbrace{[2,2,\ldots,2]}^{p-1} \qquad \frac{q}{q-1} = \overbrace{[2,2,\ldots,2]}^{q-1} \qquad \frac{r}{r-1} = [1, -r+1].$$

The plumbing diagram for $\widetilde{X}$ is given by the graph $\widetilde{G} = \widetilde{G}(p,q,r)$, as in figure 2.4(a). By blowing down the -1 framed vertex of $\widetilde{G}$ we obtain the graph $G = G(p,q,r)$ as in figure 2.4(b).

Although the manifolds $W(\widetilde{G})$ and $W(G)$ have identical boundary, the following lemma of Greene and Jabuka shows why we will use the graph in Figure 2.4(b).



(a) (b)

Figure 2.4: (a) The weighted graph $\widetilde{G}(p,q,r)$ and (b) the weighted graph $G(p,q,r)$.

Lemma 2.2.1 (Greene-Jabuka [8])**.** *The incidence matrix of the weighted graph G from Figure 2.4(b) is negative definite if and only if p, q, and r satisfy*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 0.$$

2.3 Donaldson's Diagonalization Obstruction

One approach to showing a knot does not have unknotting number one is due to Cochran-Lickorish (see [3]) and uses Donaldson's Diagonalization Theorem. The obstruction is most easily stated in terms of the Signed Montesinos Trick [7]:

Proposition 2.3.1 (Signed Montesinos Trick)**.** *Suppose that K is a knot with unknotting number one, and reflect it as necessary so that it can be unknotted by changing a negative crossing to a positive one. Then $\Sigma(K) = S^3_{-\epsilon D/2}(\kappa)$ for some knot $\kappa \subset S^3$, where $\epsilon = (-1)^{\sigma(K)/2}$ and $D = \det(K)$.*

For any knot $K \subset S^3$ and positive integer $D = 2n-1$, the space ${S^3}_{-D/2}(K)$ is the boundary of an oriented 4-manifold, W. In particular W is obtained by attaching a handle to K with framing $-n$ and a handle to a meridian μ of K with framing -2. As knots in $\partial B^4 = S^3$, orient K and μ so they have linking number one. The intersection pairing of W with respect to the basis of $H_2(W)$, $\{x, y\}$, implied by these handle attachments is given by the negative definite form:

$$R_n = R = \begin{pmatrix} -n & 1 \\ 1 & -2 \end{pmatrix}.$$

For a knot K satisfying the conditions of Proposition 2.3.1, either $\sigma(K) = 0$ or $\sigma(K) = 2$. If $\sigma(K) = 0$, then $-\Sigma(\overline{K}) = \Sigma(K) = {S^3}_{-D/2}(\kappa)$, whereas if $\sigma(K) = 2$, then $-\Sigma(K) = {S^3}_{-D/2}(\overline{\kappa})$. From Proposition 2.3.1 we conclude the following:

Proposition 2.3.2. *Assume that $\sigma(K) = 2$ and K can be unknotted by changing a negative crossing, or $\sigma(K) = 0$ and K can be unknotted by changing a positive crossing. Then $-\Sigma(K)$ is the oriented boundary of a compact 4-manifold W_K with negative definite intersection pairing given by R_n.*

Now suppose that K is a pretzel knot to which we assign a plumbing diagram as described in Section 2.2. Furthermore suppose K satisfies Lemma 2.2.1. Then $\Sigma(K)$ is the oriented boundary of a compact, negative-definite manifold X_K with $H_2(X_K)$ torsion-free, $\pi_1(X_K) = 0$, and has an intersection pairing which is given in a suitable basis $\{v_1, v_2, \ldots, v_k\}$ by the incidence matrix of the graph in Figure 2.4(b).

By gluing X_K and W along their common boundary we obtain a closed, smooth, oriented, simply connected, negative-definite manifold, $X = X_K \cup_{\Sigma(K)} W$. This allows us to use Donaldson's Diagonalization Theorem:

Theorem 2.3.3 (Donaldson, [4]). *Let X be a closed, oriented, simply connected, smooth 4-manifold. If the intersection form Q_X is negative-definite, then there exists an integral matrix A such that $-AA^T = Q_X$.*

It is then obvious how to use the obstruction on those knots in **Case 1** of Section 2.1. However, for all pretzel knots $P(k, -k-2, 2m)$ and corresponding Q_X there does exist an integral matrix A which satisfies Theorem 2.3.3.

2.4 Heegaard Floer Homology Obstruction

Recent advances in the study of unknotting number come from Heegaard Floer homology. Ozsváth and Szabó developed such an obstruction in [22]. In this work they showed certain Montesinos knots of length three, including $10_{125} = P(2, 5, -3)$ and $10_{126} = P(2, 3, -5)$, do not have unknotting number one. Here we give a brief summary of Heegaard Floer homology as well as their obstruction.

Let Y be an oriented 3-manifold or 4-manifold. The space $\text{Spin}^c(Y)$ of spin^c structures on Y is an affine space over the cohomology group $H^2(Y, \mathbb{Z})$. Each spin^c structure has a first Chern class, $c_1(\mathfrak{s})$, in $H^2(Y; \mathbb{Z})$ related by the formula $c_1(\mathfrak{s}+h) = c_1(\mathfrak{s}) + 2h$ for any $h \in H^2(Y;\mathbb{Z})$. If X is a 4-manifold, an element $v \in H^2(X,\mathbb{Z})$ is called *characteristic* if

$$\langle v, x\rangle \equiv \langle x, x\rangle \pmod 2 \text{ for all } x \in H_2(X).$$

The set of characteristic classes is denoted by $\text{Char}(X)$. If we further assume that $|H^2(X;\mathbb{Z})|$ is odd, then there is a (non-canonical) bijection between $\text{Spin}^c(X)$ and $H^2(X,\mathbb{Z})$.

In [20] Ozsváth and Szabó associate to an oriented rational homology 3-sphere Y equipped with a spin^c structure $\mathfrak{s}$ of Y a rational number $d(Y,\mathfrak{s})$. This numerical invariant is called a *correction term*. Ozsváth and Szabó prove the following theorem in [20]:

Theorem 2.4.1 (Ozsváth and Szabó)**.** *Let Y be a rational homology 3-sphere that bounds a negative definite 4-manifold X. Then for all $\mathfrak{s} \in Spin^c(X)$,*

$$c_1(\mathfrak{s})^2 + b_2(X) \leq 4d(Y, \mathfrak{s}|_Y) \tag{2.2}$$

and

$$c_1(\mathfrak{s})^2 + b_2(X) \equiv 4d(Y, \mathfrak{s}|_Y) \ (\text{mod } 2)\ . \tag{2.3}$$

This means the correction terms of Y can be used as an obstruction to Y bounding a negative definite 4-manifold. A detailed description of how to compute $d(Y,\mathfrak{s})$ can be found in [22]. What follows are the necessary details for our purposes. Let Y be a rational homology 3-sphere and X a smooth, simply connected 4-manifold such that $\partial X = Y$ and $|H^2(Y;\mathbb{Z})|$ is odd. After fixing a basis for $H_2(X,\mathbb{Z})$ we have an isomorphism $H_2(X,\mathbb{Z}) \cong \mathbb{Z}^r$, where r is the second betti number of X. Let Q_X be the matrix of the intersection pairing of X.

The elements in $\mathrm{Char}(Q_X)$ are covectors $v \in \mathbb{Z}^r$ such that $v_i \equiv Q_{X_{i,i}}$ (mod 2). The squares of the first chern classes, $c_1(\mathfrak{s})^2$, are computed using the pairing induced by Q on $H^2(X,\mathbb{Z})$; $v^T Q_X^{-1} v$ with our choice of basis. Define a function

$$M_Q : \mathbb{Z}/D\mathbb{Z} \to \mathbb{Q} \tag{2.4}$$

$$M_Q(i) = \max\left(\frac{v^T Q_X^{-1} v + r}{4}\bigg| v \in \mathrm{Char}(Q), [v] = i\right) \tag{2.5}$$

The expression in (2.4) has a maximum because Q_X is negative definite. Theorem 2.4.1 can be restated as

Theorem 2.4.2. *Let Y be a rational homology 3-sphere which is the boundary of a simply connected negative definite 4-manifold X with $|H^2(Y,\mathbb{Z})|$ odd. If the intersection pairing of X is represented in a basis by the matrix Q_X and* $\det(Q_X) = D$ *then there exists a group isomorphism*

$$\phi : \mathbb{Z}/D\mathbb{Z} \to Spin^c(Y)$$

with

$$M_Q(i) \leq d(Y, \phi(i)) \tag{2.6}$$

$$M_Q(i) \equiv d(Y, \phi(i)) \ (mod\ 2)\ . \tag{2.7}$$

In [22] Ozsváth and Szabó define an *L-space* as a rational homology 3-sphere with the property rank $\widehat{HF}(Y) = |H_1(Y)|$. Furthermore, a *sharp* manifold is defined in [22] by Ozsváth and Szabó as follows:

Definition 2.4.3 ([22], Definition 2.5)**.** *A negative-definite smooth 4-manifold X with L-space boundary Y is* sharp *if, for every $\mathfrak{t} \in Spin^c(Y)$, there is some $\mathfrak{s} \in Spin^c(X)$ with $\mathfrak{s}|_Y = \mathfrak{t}$ and equality holds in the inequality (2.2) and (2.6).*

The obstruction then works as follows: given a knot K, we hope to construct a sharp, negative definite 4-manifold X such that $\partial X = \Sigma(K)$. If $u(K) = 1$, then $\Sigma(K) = S^3_{D/2}(\kappa)$. Therefore $\Sigma(K)$ must bound a 4-manifold with intersection form R (see Section 2.3). The terms $M_Q(i)$ and $M_R(i)$ must satisfy the following conditions:

Theorem 2.4.4 (Ozsváth-Szabó)**.** *Let K be a knot with $\det(K) = D$ such that $\Sigma(K)$ bounds a 4-manifold X with a negative definite plumbing diagram. Let $M_Q(\Sigma(K), \mathfrak{s})$, as in Theorem 2.4.2, be computed using the intersection form of X. Assume $u(K) = 1$. Then there exists an isomorphism $\phi : \mathbb{Z}/D\mathbb{Z} \to \mathbb{Z}/D\mathbb{Z}$ and $\epsilon \in \{\pm 1\}$ with the properties that for all $i \in \mathbb{Z}/D\mathbb{Z}$:*

$$-\epsilon M_Q(\phi(i)) - M_R(i) \equiv 0 \ (mod\ 2) \tag{2.8}$$

$$-\epsilon M_Q(\phi(i)) - M_R(i) \geq 0. \tag{2.9}$$

where R is the intersection form of the 4-manifold coming from the Montesinos trick. Furthermore, if Y is an L-space, X is sharp, and $|M_Q(0)| \leq \frac{1}{2}$, then there is a choice of ϵ and ϕ which satisfies (2.8) *and* (2.9)*, and the following symmetry condition:*

$$-\epsilon M_Q(\phi(i)) - M_R(i) = -\epsilon M_Q(\phi(2l - i)) - M_R(2l - i) \tag{2.10}$$

for $1 \leq i < l$ when $D = 4l - 1$ and for $0 \leq i < l$ when $D = 4l + 1$.

Remark In [22], Theorem 2.4.4 assumes K is alternating, for which $\Sigma(K)$ is known to satisfy the aforementioned conditions. To follow the convention of Ozsváth and Szabó we will let $T_{\phi,\epsilon}(i) = -\epsilon M_Q(\phi(i)) - M_R(i)$. Equation 2.10 can be restated as:

$$T_{\phi,\epsilon}(i) = T_{\phi,\epsilon}(2l - i) \tag{2.11}$$

2.5 A Strengthening of Donaldson's Obstruction

Using Heegaard Floer homology, Greene strengthened Theorem 2.3.3:

Theorem 2.5.1 (Greene). *Suppose K is a knot in S^3, $\Sigma(K)$ is an L-space, and $u(K) = 1$. Suppose also that either $\sigma(K) = 0$ and K is undone by changing a positive crossing, or that $\sigma(K) = 2$. If X_K is a smooth, sharp, simply connected 4-manifold with rank r negative-definite intersection form Q_K, and X_K is bounded by $\Sigma(K)$, then there exists an integral matrix A such that $-AA^T = Q_K \oplus R_n$, and A can be chosen such that the last two rows are $(0, 1, x_3, \dots, x_{r+2})$ and $(1, -1, 0, \dots, 0)$. Furthermore the values $x_3, \dots, x_{r+2}$ are non-negative integers and obey the condition*

$$x_3 \leq 1, \;\; x_i \leq x_3 + \cdots + x_{i-1} + 1 \;\; \text{for } 3 < i < r+2 \tag{2.12}$$

and the upper right $r \times r$ matrix of A has determinant ± 1.

We will make use of this theorem in the next section.

Chapter 3

The Case $n = -2$

In this chapter we will prove the following theorem:

Theorem 3.0.1. *Suppose* $K = P(2m, k, -k-2)$*, where* $m \in \mathbb{Z}$*,* $m > 0$*,* k *odd, and* $k > 0$*, is a 3-stranded pretzel knot. If* $u(K) = 1$*, then* $K = (2m, 1, -3)$*.*

First, observe in Figure 3.1 that knots of the form $P(2m, 1, -3)$ have unknotting number one:

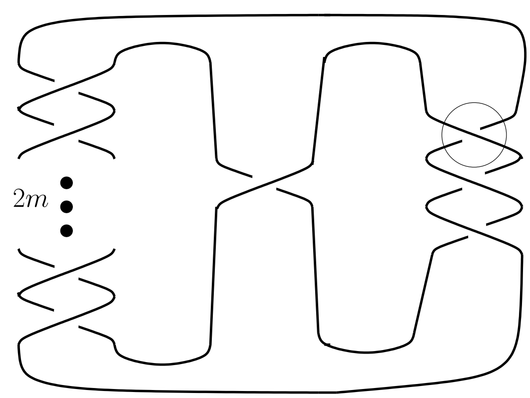


Figure 3.1: The knot $P(2m, 1, -3)$, with an unknotting crossing circled.

Next, in order to show that all other knots of the form $P(2m, k, -k-2)$ have unknotting number greater than one, we make use of Theorem 2.5.1. The manifold X_K is constructed from a plumbing diagram as described in Section

2.2. Using Figure 2.4(b), the diagram in Figure 3.2 describes a manifold X_K such that $\partial X_K = \Sigma(K)$:

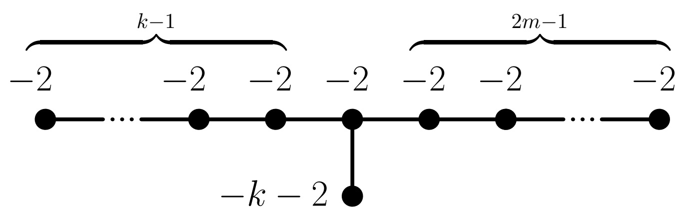


Figure 3.2: A weighted graph $\widetilde{G}(2m, -k-2, k)$.

In order to use Theorem 2.5.1 we must show three things; 1) $\Sigma(K)$ is an L-space, 2) X is negative-definite, and 3) X is sharp. First we use the following theorem to determine when $\Sigma(K)$ is an L-space:

Theorem 3.0.2 (Champanerkar-Kofman, [2]). *Let $L = P(p_1, p_2, -q)$ with $p_1, p_2, q \geq 2$. The space $\Sigma(L)$ is an L-space if and only if:*
(1) $q \geq min\{p_1, p_2\}$ or
(2) $q = min\{p_1, p_2\} - 1$ and $max\{p_1, p_2\} \leq 2q + 1$.

Corollary 3.0.3. *If $K = P(2m, k, -k-2)$, $m \in \mathbb{Z}$, $m > 0$, and $k \geq 3$, then $\Sigma(K)$ is an L-space.*

Next it is easy to show the following is a corollary of Lemma 2.2.1:

Corollary 3.0.4. *Let $K = P(2m, k, -k-2)$, $m \in \mathbb{Z}$, $m > 0$, $k \geq 3$ and k odd. Then the 4-manifold X described in Figure 3.2 is negative definite.*

Proof. Using Lemma 2.2.1, note that:

$$\begin{aligned}\tfrac{1}{p}+\tfrac{1}{q}+\tfrac{1}{r} &= \tfrac{1}{2m}+\tfrac{1}{k}+\tfrac{1}{-k-2} \\ &> \tfrac{1}{2m} \\ &> 0\end{aligned}$$

$\square$

Finally, in order to show X is sharp, we use a theorem in [21]. Let $w(v)$ be the weight of each vertex, and $d(v)$ be the number of edges which contain v. A vertex v is called a *bad* vertex if $w(v) > -d(v)$.

Theorem 3.0.5 (Ozsváth-Szabó)**.** *Let G be a plumbing diagram, and X the associated 4-manifold. If G is a negative-definite graph with at most 2 bad points, then X is sharp.*

Corollary 3.0.6. *The 4-manifold associated with the plumbing diagram in Figure 3.2 is sharp.*

To show the matrix A described in Theorem 2.5.1 does not exist, begin by writing down the $k+2m+2 \times k+2m+2$ intersection form of $X = X_K \cup_{\Sigma(K)} W_K$:

$$
Q = \left(\begin{array}{cccccccc|cc}
-2 & 1 & 0 & & \cdots & & & 0 & & \\
1 & -2 & 1 & & & & & & & \\
0 & 1 & -2 & & & & & & & \\
 & & & & \ddots & & & 1 & & \\
\vdots & & & \ddots & \ddots & & & & & \\
 & & & & & & 1 & & & \\
 & & & & & 1 & -2 & 0 & & \\
0 & & & 1 & & & 0 & -k-2 & & \\
\hline
 & & & & & & & & -n & 1 \\
 & & & & & & & & 1 & -2
\end{array}\right)
$$

where $Q_{k+2m,k} = Q_{k,k+2m} = 1$. For our convenience, label the i^{th} row of A as v_i. Using this notation, note that $Q_{X_{i,j}} = -(AA^T)_{i,j} = -v_i \cdot v_j$.

We now show the matrix A guaranteed in Theorem 2.5.1 does not exist. Begin by noting that since $|v_i \cdot v_i| = 2$ for $i \neq k+2m$ and $k+2m+1$, $k+2m$ rows of A have exactly two nonzero entries. Without loss of generality, set $v_1 = (1, -1, 0, \ldots, 0)$. Making this choice, and taking into considering the two row conditions of Theorem 2.5.1, forces A to take the following form:

$$
A = \left(\begin{array}{cccccc|cc}
1 & -1 & & & & & & \\
 & 1 & -1 & & & & & \\
 & & \ddots & \ddots & & & & \\
 & & & & & & & \\
 & & & & 1 & -1 & & \\
* & * & \cdots & \cdots & * & * & a & b \\
\hline
* & * & \cdots & \cdots & * & * & & 1 \\
 & & & & & & 1 & -1
\end{array}\right).
$$

Next, let $A_{2m+k+1,1} = \alpha$. Since $v_{2m+k+1} \cdot v_i = 0$ for $i = 1, 2, \ldots, 2m+k$, the first $2m+k$ terms of $v_{2m+k+1} = \alpha$:

$$A = \left(\begin{array}{cccccc|cc} 1 & -1 & & & & & & \\ & 1 & -1 & & & & & \\ & & \ddots & \ddots & & & & \\ & & & & & & & \\ & & & & 1 & -1 & & \\ * & * & \dots & \dots & * & * & a & b \\ \hline \alpha & \alpha & \dots & \dots & \alpha & \alpha & & 1 \\ & & & & & & 1 & -1 \end{array}\right).$$

According the Theorem 2.5.1, $\alpha = 0$ or 1.

a) If $\alpha = 0$, then $v_{2m+k+1} \cdot v_{2m+k+1} = 1 = n = \frac{1}{2}(\det(K)+1)$, and so $\det(K) = 1$. However, using the Goeritz matrix one can show $\det(P(2m, k, -k-2)) = |k^2 + 2k + 4m|$, a contradiction.

b) If $\alpha = 1$, then $v_{2m+k+1} \cdot v_{2m+k+1} = k + 2m + 1 = n = \frac{1}{2}(\det(K) + 1)$, and so $2k + 4m + 1 = \det(K) = k^2 + 2k + 4m$. This is true only when $k^2 = 1$, a contradiction.

The reader will note that when $k = 1$ there exists a matrix A which satisfies Theorem 2.5.1, as expected.

Chapter 4

The Case n=0

4.1 Donaldson's Theorem and $P(2, k, -k)$

In this section we will prove the following theorem:

Theorem 4.1.1. *Let $K = P(2, k, -k)$ be a three standed pretzel knot and $k > 3$. If k is a prime power, then $u(K) > 1$.*

We will show that Theorem 2.4.4 and Theorem 2.3.3 can be used as an unknotting number one obstruction for those knots described in Theorem 4.1.1. Begin by constructing the plumbing diagram described in Section 2.2, in particular the diagram described in Figure 2.4(b). Associate to the diagram the 4-manifold X, and let Q_X be the intersection form of X.

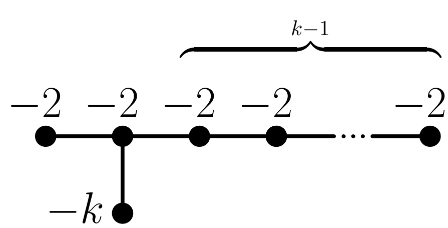


Figure 4.1: A weighted graph $G(2, -k, k)$.

In order to use Theorems 2.4.4 and 2.3.3, we must show three things;

1) $\Sigma(K)$ is an L-space, 2) X is negative-definite, and 3) X is sharp. We can show these conditions are met using the techniques in Section 3.

Corollary 4.1.2. *If $K = P(2, k, -k)$ and $k \geq 3$, then $\Sigma(K)$ is an L-space.*

Corollary 4.1.3. *Let $K = P(2, k, -k)$, $k \geq 3$ and k odd. Then the 4-manifold X described in Figure 4.1 is negative definite.*

Corollary 4.1.4. *The 4-manifold associated with the plumbing diagram in Figure 4.1 is sharp.*

Now we have shown pretzel knots in Theorem 4.1.1 satisfy the necessary conditions. To prove Theorem 4.1.1 we first show there exists a ϕ and ϵ which satisfy (2.8) and (2.9). Then we will show that these ϕ and ϵ are unique for prime k, and finally they fail condition (2.11).

First we mimic the proof of Theorem 3.0.1 to prove the following Lemma:

Lemma 4.1.5. *If $K = P(2, k, -k)$ has unknotting number one, and $k \geq 3$ is odd, then it must be undone by changing a negative crossing.*

Proof. First recall from the plumbing construction above that $\Sigma(K) = \partial X$, and the intersection form of X is negative definite. Next, from Proposition 2.3.2 if K can be unknotting by changing a positive crossing, then $-\Sigma(K)$ is the boundary of a negative-definite 4-manifold. Therefore we can glue X and W_K together along $\Sigma(K)$ to form a closed 4-manifold with negative-definite intersection form Q. Therefore there must exist an integral $k+4 \times k+4$ matrix

A such that $-AA^T = Q$.

Similar to Section 3, the matrix A must have the form:

$$A = \left(\begin{array}{ccccccc|cc}
-1 & 1 & & & & & & & \\
 & -1 & 1 & & & & & & \\
 & & -1 & 1 & & & & & \\
 & & & \ddots & \ddots & & & & \\
\vdots & \vdots & \vdots & & -1 & 1 & & & \\
 & & & & & -1 & 1 & & \\
a & a & a & \dots & a & b & b & c & c \\
\hline
d & d & d & \dots & d & d & d & 1 & \\
 & & & & & & & 1 & -1
\end{array}\right)$$

Denote the rows by v_i, with a total of $k+4$ rows. Then $-v_k \cdot v_{k+2} = 1$, so $b = a - 1$. Then $v_{k+2} \cdot v_{k+2} = k$ implies

$$ka^2 + 2(a-1)^2 + 2c^2 = k \tag{4.1}$$

whence evidently we must have $a = -1, 0, 1$ (else the LHS is too big). We split the cases:

1. If $a = -1$, then from (4.1) we have $2c^2 + 8 = 0$.
2. If $a = 0$, then from (4.1) we have $2c^2 + 2 = k$. Since k is odd, this cannot happen.
3. If $a = 1$, then $c = 0$ (from (4.1)). Since $v_{k+2} \cdot v_{k+3} = 0$, $d = 0$. The fact that $v_{k+3} \cdot v_{k+3} = n$ yields us $n = 1$, so $k^2 = 1$, contradicting $k \geq 3$.

This completes the proof.

□

Corollary 4.1.6. *If $K = P(2, k, -k)$ has unknotting number one, then $\epsilon = -1$ in Theorem 2.4.4.*

4.2 The values M_Q

Using the Goeritz matrix of $P(2, k, -k)$, it is easy to see $|\det(K)| = k^2$. Therefore for knots of the form $P(2, k, -k)$, each characteristic covector v corresponds to an element $i \in \mathbb{Z}/k^2\mathbb{Z}$ according to the incident matrix Q of $\widetilde{G}(2, k, -k)$. In order to determine which vectors $v \in \text{Char}(Q)$ correspond to which $i \in \mathbb{Z}/k^2\mathbb{Z}$, we prove the following proposition:

Proposition 4.2.1. *Let $K = P(2, k, -k)$, k odd, $k \geq 3$. If $v = (v_1, \ldots, v_{k+2}) \in$ Char(Q), then $[v] = v_1 + 2v_2 + \cdots + kv_k + \frac{k^2+k}{2}v_{k+1} + \frac{k^2+k+2}{2}v_{k+2}$ (mod k^2).*

Proof. Up to an ordering of the vertices of Figure 4.1, the incidence matrix of $\widetilde{G}(2, k, -k)$ is

$$Q = \begin{pmatrix} -2 & 1 & & & & & \\ 1 & \ddots & \ddots & & & & \\ & \ddots & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 & 1 \\ & & & & 1 & -2 & 0 \\ & & & & 1 & 0 & -k \end{pmatrix}.$$

Row operations of Q correspond to linear combinations of relations of the group $H^2(\Sigma(K), \mathbb{Z})$. Using this we can simplify Q:

$$\longrightarrow \begin{pmatrix} -2 & 1 & & & & & \\ -3 & & 1 & & & & \\ \vdots & & & \ddots & & & \\ -k & & & & 1 & & \\ k-1 & & & & -2 & 1 & 1 \\ 0 & & & & 1 & -2 & 0 \\ 0 & & & & 1 & & -k \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} -2 & 1 & & & & & \\ -3 & & 1 & & & & \\ \vdots & & & \ddots & & & \\ -k & & & & 1 & & \\ -k-1 & & & & 0 & 1 & 1 \\ k & & & & & -2 & 0 \\ k & & & & & & -k \end{pmatrix} \longrightarrow$$

$$\begin{pmatrix} -2 & 1 & & & & & \\ -3 & & 1 & & & & \\ \vdots & & & \ddots & & & \\ -k & & & & 1 & & \\ -k-1 & & & & 0 & 1 & 1 \\ -k-2 & & & & & & 2 \\ k & & & & & & -k \end{pmatrix}.$$

Add $\frac{k+1}{2}$ copies of the $k+1$ row to the $k+2$ row.

$$\begin{pmatrix} -2 & 1 & & & & & \\ -3 & & 1 & & & & \\ \vdots & & & \ddots & & & \\ -k & & & & 1 & & \\ -k-1 & & & & 0 & 1 & 1 \\ -k-2 & & & & & & 2 \\ \frac{-k^2-k-2}{2} & & & & & & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} -2 & 1 & & & & & \\ -3 & & 1 & & & & \\ \vdots & & & \ddots & & & \\ -k & & & & 1 & & \\ \frac{k^2-k}{2} & & & & 0 & 1 & \\ k^2 & & & & & & \\ \frac{-k^2-k-2}{2} & & & & & & 1 \end{pmatrix}.$$

$\square$

The calculations of M_Q rely heavily upon the matrix Q^{-1}.

Proposition 4.2.2. *If the matrix*

$$Q = \begin{pmatrix} -2 & 1 & & & & & \\ 1 & \ddots & \ddots & & & & \\ & \ddots & -2 & 1 & & & \\ & & 1 & -2 & 1 & & \\ & & & 1 & -2 & 1 & 1 \\ & & & & 1 & -2 & 0 \\ & & & & 1 & 0 & -k \end{pmatrix},$$

then

$$Q^{-1} = \frac{-1}{k^2} \begin{pmatrix} k^2-k+2 & \dots & 3k-2 & 2k & k & 2 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 4k-4 & \dots & (k-2)(3k-2) & 2k(k-2) & k(k-2) & 2(k-2) \\ 3k-2 & \cdots & (k-1)(3k-2) & 2k(k-1) & k(k-1) & 2(k-1) \\ 2k & \cdots & 2k(k-1) & 2k^2 & k^2 & 2k \\ k & \cdots & k(k-1) & k^2 & k^2 & k \\ 2 & \cdots & 2(k-1) & 2k & k & k+2 \end{pmatrix}.$$

Of particular interest are the diagonal entries of Q^{-1}. Note that:

$$(-k^2)Q^{-1}_{i,i} = \begin{cases} ik^2 - i^2k + 2i^2 & \text{if} \quad 1 \leq i \leq k \\ k^2 & \text{if} \quad i = k+1 \\ k+2 & \text{if} \quad i = k+2. \end{cases} \tag{4.2}$$

Theorem 4.2.3. *Let K, Q, $T_{\phi,\epsilon}$ be as in Section 2.4. Then $\phi \in Aut(\mathbb{Z}/k^2\mathbb{Z})$, $x \mapsto \frac{k^2-k-4}{2}x$ is an automorphism such that $T_{\phi,-1}(2) = M_Q(\phi(2)) - M_R(2) \geq 2$.*

Proof. According to [22] the vector $(5, -2)$ corresponds to the element $2 \in \mathbb{Z}/k^2\mathbb{Z}$. Using Equation 2.4 we see that $M_R(2) = \frac{-8}{k^2}$. According to proposition 4.2.1 the vector $v = (0, \ldots, 0, k-4)$ corresponds to $k^2 - k - 4$ in $\mathbb{Z}/k^2\mathbb{Z}$. Therefore $[v] = \phi(2)$. Finally using equation (2.4),

$$\begin{aligned} M_Q(\phi(2)) &\geq \frac{v^T Q^{-1} v + (k+2)}{4} \\ &= \frac{\frac{-1}{k^2}(k+2)(k-4)^2 + (k+2)}{4} \\ &= \frac{1}{4k^2}(-k^3 + 6k^2 - 32 + k^3 + 2k^2) \\ &= \frac{2k^2 - 8}{k^2} \end{aligned}$$

Therefore $M_Q(\phi(2)) - M_R(2) \geq \frac{2k^2-8}{k^2} - \frac{-8}{k^2} = 2$, as desired.

□

According to the symmetry condition of Theorem 2.4.4, $T_{\phi,\epsilon}(2) = T_{\phi,\epsilon}(2l - 2)$, where l depends on $\det(G(K)) = k^2$. For odd k, $k^2 = 4l + 1$, and so $l = \frac{k^2-1}{4}$.

Theorem 4.2.4. *Let K, G, $T_{G,\epsilon}$ be as above. Then $\phi \in Aut(\mathbb{Z}/k^2\mathbb{Z})$, $x \mapsto \frac{k^2-k-4}{2}x$ is an automorphism such that* $T_{\phi,-1}\left(\frac{k^2-5}{4}\right) = M_Q\left(\phi\left(\frac{k^2-5}{2}\right)\right) - \gamma_D\left(\left(\frac{k^2-5}{2}\right)\right) = 0$.

The proof comes in 3 parts. First, we will show $T_{\phi,-1}\left(\frac{k^2-5}{4}\right)$ is an even integer greater than zero. Next, in Section 4.3 we show that the vector v in Theorem 4.2.4 maximizes $v^TQ^{-1}v$ over all $v \in \mathrm{Char}(Q)$ and $[v] \equiv \frac{1}{4}(k^2+5k+20) \pmod{k^2}$. Finally, in Section 4.4, we show that ϕ is the only automorphism of $\mathbb{Z}/k^2\mathbb{Z}$ such that $T_{\phi,\epsilon}(2)$ is even.

First we consider the case when $k \equiv 3 \pmod 4$. According to [22], the vector $(-5,0)$ corresponds to the element $\frac{k^2-5}{2}$ and $M_R\left(\frac{k^2-5}{2}\right) = \frac{k^2-25}{2k^2}$. Let $v = \left(0,\ldots,0,-2,0,\ldots,0,\frac{13-k}{2}\right)$, where $v_{\frac{3k+3}{4}} = -2$. Then note that the following are equal:

$$\begin{aligned}
[v] &= \left(\tfrac{13-k}{2}\right)\left(\tfrac{k^2+k+2}{2}\right) + \left(\tfrac{3k+3}{4}\right)(-2) \\
&= \tfrac{1}{4}(-k^3+12k^2+5k+20) \\
&\equiv \tfrac{1}{4}(-k^3+5k+20) \pmod{k^2} \\
&= \tfrac{1}{4}(-k^3-k^2+k^2+5k+20) \\
&= -k^2\left(\tfrac{k+1}{4}\right) + \tfrac{1}{4}(k^2+5k+20) \\
&\equiv \tfrac{1}{4}(k^2+5k+20) \pmod{k^2},
\end{aligned}$$

$$
\begin{aligned}
\phi\left(\tfrac{k^2-5}{2}\right) &= \left(\tfrac{k^2-5}{2}\right)\left(\tfrac{k^2-k-4}{2}\right) \\
&= \tfrac{1}{4}(k^4-k^3-9k^2+5k+20) \\
&= \tfrac{1}{4}(-k^3+5k+20)+k^2\left(\tfrac{k^2-9}{4}\right) \\
&\equiv \tfrac{1}{4}\left(-k^3-k^2+k^2+5k+20\right) \pmod{k^2} \\
&\equiv k^2\left(\tfrac{-k-1}{4}\right)+\tfrac{1}{4}\left(k^2+5k+20\right) \pmod{k^2} \\
&\equiv \tfrac{1}{4}(k^2+5k+20) \pmod{k^2}.
\end{aligned}
$$

Finally, note that

$$
\begin{aligned}
M_Q\left(\phi\left(\tfrac{k^2-5}{2}\right)\right) &\geq \tfrac{v^TQ^{-1}v+(k+2)}{4} \\
&= \tfrac{-k-50/k^2+(k+2)}{4} \\
&= \tfrac{k^2-25}{2k^2}
\end{aligned}
$$

In the case $k \equiv 1 \pmod 4$, a similar argument holds.

4.3 Equality in Theorem 4.2.4

Theorem 4.2.4 is not complete until we show equality. The vector $v \in \mathrm{Char}(Q)$ which achieves the maximum in (2.4) has the form $v = (v_1, v_2, \ldots, v_{k+2})$, where $v_i = -2$ or 0 for $i = 1, 2, \ldots, k+1$, and $v_{k+2} = -k, -k+2, \ldots,$ or $k-2$. It is not practical to determine $[w]$ for all $w \in \mathrm{Char}(Q)$ and then compute M_Q for all w such that $[w] = \frac{1}{4}(k^2+5k+20) \pmod{k^2}$, and hope that vector w which achieves the maximum is indeed v. Instead, first note that maximizing M_Q is equivalent to maximizing $v^TQ^{-1}v$. Next let $w(i) = (0, \ldots, 0, -k+2i)$.

Then

$$
\begin{aligned}
[v]-[w(i)] &\equiv \tfrac{1}{4}(k^2+5k+20)-(-k+2i)\left(\tfrac{k^2+k+2}{2}\right) \pmod{k^2} \\
&= \tfrac{1}{4}(k^2+5k+20)+k^2\left(\tfrac{k+1}{2}\right)-ik^2+k-ik-2i \\
&\equiv \tfrac{1}{4}(k^2+5k+20)+k-ik-2i \pmod{k^2} \\
&= \tfrac{1}{4}\left(k^2+9k+20-4ik-8i\right)
\end{aligned}
$$

We only need to consider $i=0,1,\ldots,k-1$. It will be helpful to set $i=\frac{k+9+4j}{4}$, so $j=\frac{-k-9}{4},\frac{-k-5}{4},\ldots,\frac{3k-13}{4},\frac{3k-9}{4}$. By doing so we can rewrite

$$
[v]-[w(i)]=\left(\frac{-k(2j+1)-4j+1}{2}\right)
$$

From this we can construct Table 4.1:

For each i under consideration, $[w(i)]\neq[v]$. However changing the j^{th} entry of $w(i)$ adds $2j$ to the value of $[v]-[w(i)]$ for $j=1,2,\ldots,k$, and changing the $k+1$ entry from 0 to -2 adds k to the value of $[v]-[w(i)]$. By changing the right entries of v we can obtain many vectors v' such that $[v]=[v']$. However explicitly computing $v'^TQ^{-1}v'$ for every such v' is unreasonable. The following two lemmas greatly reduce the number of vectors which much be checked against $v^TQ^{-1}v$.

Lemma 4.3.1. *Let v, $w\in Char(Q)$, where $v_i=w_i$ for $i=1,2,\ldots,k+2$ except $v_m=v_n=-2$ and $v_{m+1}=v_{n-1}=0$, whereas $w_m=w_n=0$ and*

i	$-k+2i$	j	$[v]-[w(i)] (\text{mod } k^2)$
0	$-k$	$\frac{-k-9}{4}$	$\frac{k^2+9k+20}{4}$
1	$-k+2$	$\frac{-k-5}{4}$	$\frac{k^2+5k+16}{4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\frac{k+5}{4}$	$\frac{-k+5}{2}$	-1	$\frac{k+5}{2}$
$\frac{k+9}{4}$	$\frac{-k+9}{2}$	0	$\frac{-k+1}{2}$
$\frac{k+13}{4}$	$\frac{-k+13}{2}$	1	$\frac{-3k-3}{2}$
$\frac{k+17}{4}$	$\frac{-k+17}{2}$	2	$\frac{-5k-7}{2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-1$	$k-2$	$\frac{3k-13}{4}$	$\frac{-3k^2+9k+24}{4}$
k	k	$\frac{3k-9}{4}$	$\frac{-3k^2+5k+20}{4}$

Table 4.1:

$w_{m+1} = w_{n-1} = -2$ *for* m, n *such that* $n - m \geq 3$, $n \leq k$. *Then* $[v] = [w]$ *and* $(-k^2)v^T Q^{-1} v < (-k^2) w^T Q^{-1} w$.

Proof. It is easy to see that $[v] = [w]$. To show the inequality we must consider two types of terms, $(-k^2)v_n^2 Q_{n,n}$ and $(-k^2)v_m v_n Q_{m,n}$, $n \neq m$. First, recall the diagonal entries of $(-k^2)Q^{-1}_{i,i} = ik^2 - i^2 k + 2i^2$ for $1 \leq i \leq k$. Then note:

$$-k^2(Q^{-1}_{m,m} + Q^{-1}_{n,n}) = (m+n)k^2 - (m^2+n^2)(k) + +2(m^2+n^2),$$

and

$$
\begin{aligned}
-k^2(Q^{-1}_{m+1,m+1}+Q^{-1}_{n-1,n-1}) &= (m+n)k^2-(m^2+n^2+2m-2n+2)(k)+\\
&\quad +2(m^2+n^2+2m-2n+2),
\end{aligned}
$$

and so

$$
\begin{aligned}
-k^2[(Q^{-1}_{m,m}+Q^{-1}_{n,n})-(Q^{-1}_{m+1,m+1}+Q^{-1}_{n-1,n-1})] &= (-2n+2m+2)k+2(2n-2m-2)\\
&= 2k(-n+m+1)-4(-n+m+1)\\
&= (2k-4)(-n+m+1)\\
&< 0,
\end{aligned}
$$

and therefore $(-k^2)(Q^{-1}_{m,m}+Q^{-1}_{n,n}) < (-k^2)(Q^{-1}_{m+1,m+1}+Q^{-1}_{n-1,n-1})$.

Next we consider terms of the form $v_m v_n Q_{m,n}$, $m \neq n$. This term is non-zero when $v_m = v_n = -2$, so we need only consider the value of $(-k^2)Q^{-1}_{m,n}$. First observe $(-k^2)Q^{-1}_{m,n} < (-k^2)Q^{-1}_{m+1,n-1}$.

Next we consider what happens when $m < m+1 < l < n-1 < n$ and $v_l \neq 0$. Therefore:

- $(-k^2)Q^{-1}_{m,l} < (-k^2)Q^{-1}_{m+1,l}$ and
- $(-k^2)Q^{-1}_{l,n} < (-k^2)Q^{-1}_{l,n-1}$

Finally consider what happens when $l < m$ and $v_l \neq 0$. Unlike before, $(-k^2)Q^{-1}_{m,l} > (-k^2)Q^{-1}_{m+1,l}$. However note that $Q^{-1}_{m,l}, Q^{-1}_{m-1,l}, Q^{-1}_{n-1,l}, Q^{-1}_{n,l}$ all lie on the same row of Q^{-1}. Since the terms of the columns (and rows) of Q^{-1} decrease at a constant rate for $i \leq k$, $(-k^2)Q^{-1}_{l,m} + (-k^2)Q^{-1}_{l,n} = (-k^2)Q^{-1}_{l,m+1} + (-k^2)Q^{-1}_{l,n-1}$. A similar thing happens when $n < l$ and $v_l \neq 0$.

□

Lemma 4.3.2. *Let v, $w \in Char(Q)$, where $v_i = w_i$ for $i = 1, 2, \ldots, k+2$ except $w_j = -2$ and $w_{j-1} = w_1 = 0$ whereas $v_{j-1} = v_1 = -2$ and $v_j = 0$ for $j \leq k$. Then $[v] = [w]$ and $w^T Q^{-1} w < v^T Q^{-1} v$.*

Proof. It is easy to see that $[v] = [w]$. To show the inequality we must consider terms of the form $v_m^2 Q_{m,m} \neq 0$ and terms of the form $v_m v_n Q_{m,n}$, $m \neq n$. First, recall from the matrix Q^{-1} that $(-k^2)Q^{-1}_{i,i} = ik^2 - i^2 k + 2i^2$ for $1 \leq i \leq k$. Then note:

$$(-k^2)Q^{-1}_{n,n} = nk^2 - kn^2 + 2n^2$$

and

$$\begin{aligned}(-k^2)Q^{-1}_{n-1,n-1} + (-k^2)Q^{-1}_{1,1} &= (n-1)k^2 - (n^2 - 2j + 1)k + (k^2 - k + 2) + \\ &\quad +2(n^2 - 2n + 1), \\ &= nk^2 - kn^2 + 2nk - 2k + 2n^2 - 4n + 4\end{aligned}$$

and so

$$\begin{aligned}(-k^2)[Q^{-1}_{n,n} - \left(Q^{-1}_{n-1,n-1} + Q^{-1}_{1,1}\right)] &= nk^2 + 2n^2 - \\ &\quad -(nk^2 + 2nk - 2k + 2n^2 - 4n + 4) \\ &= -2nk + 2k + 4n - 4 \\ &= ((2k-4)(1-n)) \\ &< 0\end{aligned}$$

Next we consider terms of the form $v_m v_n Q_{m,n}$, $m \neq n$. First $(-k^2)v_{m-1} v_1 Q^{-1}_{m-1,1} > 0$, whereas $0 \cdot w_m Q^{-1}_{m,l} = 0$.

Finally suppose $v_l \neq 0$, $l \neq 1, m, m+1$. If $1 < l < m$, then $(-k^2)Q^{-1}_{m,l} > (-k^2)Q^{-1}_{m+1,l}$, and consequently $(-k^2)Q^{-1}_{m,l} + (-k^2)Q^{-1}_{1,l} > (-k^2)Q^{-1}_{m+1,l}$. On the other hand if $m+1 < l$, then $(-k^2)Q^{-1}_{1,l} + (-k^2)Q^{-1}_{m,l} = (-k^2)Q^{-1}_{m+1,l}$. $\square$

Corollary 4.3.3. *The vector v which maximizes $v^TQ^{-1}v$ over all $[v] = n \in \mathbb{Z}/k^2\mathbb{Z}$ with fixed v_{k+2} is the vector with $v_k = \cdots = v_l = -2$, at most one $v_1, \ldots, v_{l-1} = -2$, where $l = 2, 3, \ldots,$ or k, and $v_{k+1} = 0$ or -2, depending v_{k+2}.*

We will use $w'(i)$ to be the unique vector which satisfies Lemma 4.3.3. Next we need to determine the minimal number of entry changes to $w(i)$ so that $[w'(i)] = [v]$. First assume j is odd. We can create Table 4.2 and Table 4.3:

Table 4.2: $k \equiv 7 \pmod 8$

i	$-k+2i$	j	$[v]-[w(i)](\text{mod } k^2)$	$C(w(i))$	$L(w(i))$
1	$-k+2$	$\frac{-k-5}{4}$	$\frac{k^2+5k+12}{4}$	–	–
3	$-k+6$	$\frac{-k+3}{4}$	$\frac{k^2-3k-4}{4}$	–	–
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\frac{k+5}{4}$	$\frac{-k+5}{2}$	-1	$\frac{k+5}{2}$	–	–
$\frac{k+13}{4}$	$\frac{-k+13}{2}$	1	$\frac{-3k-3}{2}$	1	$\frac{3k+3}{4}$
$\frac{k+21}{4}$	$\frac{-k+21}{2}$	3	$\frac{-7k-11}{2}=-2k+\frac{-3k-11}{2}$	2	$\frac{3k+11}{4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-2$	$k-4$	$\frac{3k-17}{4}$	$\frac{-3k^2+9k+36}{4}$	–	–
k	k	$\frac{3k-9}{4}$	$\frac{-3k^2+k+20}{4}$	–	–

Here, $C(w(i))$ is the minimal number of changes to $w(i)$ so that $[w'(i)] = [v]$. Similarly, $L(w(i))$ is the guaranteed leftmost term of $w'(i)$ which is non-zero. The fourth and sixth columns are only true for certain j and k. For example when $j = 3$ and $k = 7$, the sixth column claims that the leftmost non-zero term is w_8, which is false. The numbers in the sixth column, $\frac{3k+\frac{1}{2}(j^2+4j+1)}{4}$, only hold true when

$$\frac{6k+j^2+4j+1}{8} \leq k - \frac{j-1}{2}, \tag{4.3}$$

which can we rewritten as

$$j \leq \sqrt{2k+19} - 4. \tag{4.4}$$

Table 4.3: $k \equiv 3 \pmod 8$

i	$-k+2i$	j	$[v]-[w(i)] \pmod{k^2}$	$C(w(i))$	$L(w(i))$
0	$-k$	$\frac{-k-9}{4}$	$\frac{k^2+9k+20}{4}$	$-$	$-$
2	$-k+4$	$\frac{-k-1}{4}$	$\frac{k^2+k+4}{4}$	$-$	$-$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\frac{k+5}{4}$	$\frac{-k+5}{2}$	-1	$\frac{k+5}{2}$	$-$	$-$
$\frac{k+13}{4}$	$\frac{-k+13}{2}$	1	$\frac{-3k-3}{2}$	1	$\frac{3k+3}{4}$
$\frac{k+21}{4}$	$\frac{-k+21}{2}$	3	$\frac{-7k-11}{2} = (-2k) + \left(\frac{-3k-11}{2}\right)$	2	$\frac{3k+11}{4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-3$	$k-6$	$\frac{3k-21}{4}$	$\frac{-3k^2+13k+44}{4}$	$-$	$-$
$k-1$	$k-2$	$\frac{3k-13}{4}$	$\frac{-3k^2+5k+28}{4}$	$-$	$-$

If condition (4.4) is true then $w_{\frac{6k+j^2+4j+1}{8}} = -2$ in step 4. Otherwise $w_{\frac{6k+j^2+4j+1}{8}} = 0$. Step 3 of (4.5) will increase the approximation of $(-k^2)\left(v^TQ^{-1}v - w^TQ^{-1}w\right)$ whenever $-k+2i > 0$, or equivalently $j > \frac{k-9}{4}$, and decrease when $j < \frac{k-9}{4}$. Finally note that $j < \frac{3}{4}k$ for all relevant k. Therefore there are only 7 cases to check:

Case 1	$0 < \frac{k-9}{4} \leq \sqrt{2k+19} - 4 \leq j < \frac{3}{4}k$
Case 2	$0 < \frac{k-9}{4} \leq j < \sqrt{2k+19} - 4 < \frac{3}{4}k$
Case 3	$0 < j < \frac{k-9}{4} \leq \sqrt{2k+19} - 4 < \frac{3}{4}k$
Case 4	$0 < \sqrt{2k+19} - 4 < \frac{k-9}{4} < j < \frac{3}{4}k$
Case 5	$0 < \sqrt{2k+19} - 4 < j < \frac{k-9}{4} < \frac{3}{4}k$
Case 6	$0 < j < \sqrt{2k+19} - 4 < \frac{k-9}{4} < \frac{3}{4}k$
Case 7	$j < 0$

Cases 1, 2, and 3 only hold when $\frac{k-9}{4} \leq \sqrt{2k+19}-4$, which is true for $k \leq 27$. Tables 4.6, 4.7, 4.8, 4.9, 4.10, and 4.11 are Tables 4.2 and 4.3 for $k = 7$, 11, 15, 19, 23, and 27. In each case the w which minimizes $(-k^2)w^T Q^{-1} w$ is $w = v$, as claimed.

Still the vectors $w'(i) = (w'_1(i), \ldots, w'_{k+2}(i))$ can be too complicated to compute $w'(i)^T G^{-1} w(i)$ without a computer. Therefore we will estimate the value of $v^T G^{-1} v - w'(i)^T G^{-1} w(i)$ in five steps:

$$\begin{aligned}
&\text{Step 1)}\ \ (-k^2)(v^T Q^{-1} v) = (k^3 - 6k^2 + 32)\\
&\text{Step 2)}\ -(k^2)(w_{k+2}(i))Q^{-1}_{k+2,k+2}(w_{k+2}(i)) = -(k^3+2k^2-4ik^2-8ik+4i^2k+8i^2)\\
&\text{Step 3) twice the sum of the terms of the form } -(-k^2)w'_l(i)Q^{-1}_{l,k+2}w'_{k+2}(i), l \neq k+2\\
&\text{Step 4) the sum of the terms of the form } -(-k^2)w'_l(i)Q^{-1}_{l,l}w'_l(i), l \neq k+2\\
&\text{Step 5) twice the sum of the terms of the form } -(-k^2)w'_l(i)Q^{-1}_{l,h}w'_h(i), l < h, h \neq k+2
\end{aligned} \tag{4.5}$$

Case 4 First compute steps 1 and 2:

$$
\begin{aligned}
(-k^2)\left(v^TQ^{-1}v - w(i)^TQ^{-1}w(i)\right) &= (k^3+50) - (-k+2i)^2(k+2) \\
&= k^3 + 50 - (k^3 + 2k^2 - 4ik^2) - \\
&\quad -(-8ik + 4i^2k + 8i^2) \\
&= 4ik^2 - 4i^2k - 2k^2 + 8ik - 8i^2 + 50 \\
&= 4\left(\tfrac{k+9+4j}{4}\right)k^2 - 4\left(\tfrac{k+9+4j}{4}\right)^2 k - 2k^2 + \\
&\quad +8\left(\tfrac{k+9+4j}{4}\right)k - 8\left(\tfrac{k+9+4j}{4}\right)^2 + 50 \\
&= \tfrac{3}{4}k^3 + 2jk^2 - 4j^2k + 4k^2 - 14jk - \\
&\quad -8j^2 - \tfrac{45}{4}k - 36j + \tfrac{19}{2}.
\end{aligned}
\tag{4.6}
$$

When $\frac{k-9}{4} < j$ we must take into account step 3. Given j, the w which minimizes $(-k^2)w^TQ^{-1}w$ has at least $\frac{j+1}{2}$ entries equal to -2. Therefore we can overestimate the value of step 3:

$$
\begin{aligned}
-\left(2(-2)(-k+2i)\textstyle\sum_{l=0}^{\frac{j-1}{2}} 2(k-l)\right) &= 8(-k+2i)\left(\textstyle\sum_{l=0}^{\frac{j-1}{2}} k - \sum_{l=0}^{\frac{j-1}{2}} l\right) \\
&= 8\left(-k + 2\left(\tfrac{k+9+4j}{4}\right)\right)\left(k\tfrac{j+1}{2} - \tfrac{j^2-1}{8}\right) \\
&= 8\left(\tfrac{-k+9+4j}{2}\right)\left(\tfrac{4kj+4k-j^2+1}{8}\right)
\end{aligned}
$$

$$
= -2jk^2 + \frac{17}{2}j^2k - 2j^3 - 2k^2 + 26jk - \frac{9}{2}j^2 + \frac{35}{2}k + 2j + \frac{9}{2} \tag{4.7}
$$

When $\sqrt{2k+19} - 4 < j$, $w_{\frac{6k+j^2+4j+1}{8}} = 0$, so for step 4 we can assume $w_k = w_{k-1} = \cdots = w_{k-\frac{j-1}{2}} = -2$. An underestimate of step 4 is:

$$-(-2)^2\cdot\sum_{l=0}^{\frac{j-3}{2}}(-k^2)Q^{-1}_{k-l,k-l} = -4\cdot\sum_{l=0}^{\frac{j-3}{2}}(l+2)k^2-(l^2+4l)k+2l^2$$

$$= -4\left(\left(\tfrac{j^2+4j-5}{8}\right)k^2-\left(\tfrac{j^3+6j^2-37j+30}{24}\right)k+ + \left(\tfrac{j^3-6j^2+11j-6}{12}\right)\right)$$

$$= -\frac{1}{2}j^2k^2+\frac{1}{6}j^3k-2jk^2+j^2k-\frac{1}{3}j^3+\frac{5}{2}k^2-\frac{37}{6}jk+2j^2+5k-\frac{11}{3}j^3+2 \quad (4.8)$$

Finally we need to include some terms from step 5 into the approximation. Since w has at least $\frac{j-1}{2}$ consecutive entries equal to -2, the entries $Q^{-1}_{k-1,k},\ldots,Q^{-1}_{k-1-\frac{j-5}{2},k-\frac{j-5}{2}}$ along the superdiagonal (and subdiagonal) contribute to the approximation. To simplify calculation and ensure the estimation is not too large (in absolute value):

$$\begin{aligned} -2\left((-2)(-2)\textstyle\sum_{i=0}^{\frac{j-5}{2}}Q^{-1}_{k-1-i,k-i}\right) &< -8\textstyle\sum_{i=0}^{\frac{j-5}{2}}Q^{-1}_{k-1,k} \\ &= -8\left(\tfrac{j-3}{2}\right)(2k^2-2k) \qquad (4.9) \\ &= -8jk^2+24k^2+8jk-24k \end{aligned}$$

By combining (4.6), (4.7), (4.8), and (4.9) we obtain the estimate:

$$
\begin{aligned}
-k^2(v^TQ^{-1}v - w^TQ^{-1}w) &\le -\tfrac{1}{2}j^2k^2 + \tfrac{1}{6}j^3k + \tfrac{3}{4}k^3 - 10jk^2 + \\
&\quad + \tfrac{11}{2}j^2k - \tfrac{7}{3}j^3 + \tfrac{57}{2}k^2 + \tfrac{83}{6}jk - \\
&\quad - \tfrac{21}{2}j^2 - \tfrac{51}{4}k - \tfrac{113}{3}j + 16 \\
&= \tfrac{1}{8}j^2k(-k + \tfrac{4}{3}j) + \tfrac{11}{2}j^2k(-\tfrac{1}{31}k + 1) + \\
&\quad + (-\tfrac{3}{4}k + 16) - \tfrac{49}{248}j^2k^2 + \tfrac{3}{4}k^3 - \\
&\quad - 10jk^2 + \tfrac{57}{2}k^2 + \tfrac{83}{6}jk \\
&< \tfrac{1}{8}j^2k(-k + \tfrac{4}{3}\left(\tfrac{3k-9}{4}\right)) - \tfrac{49}{248}j^2k^2 + \\
&\quad + \tfrac{3}{4}k^3 - 10\left(\tfrac{k-7}{4}\right)k^2 + \tfrac{57}{2}k^2 + \tfrac{83}{6}jk \\
&< -\tfrac{3}{8}j^2k - \tfrac{49}{248}j^2k^2 - \tfrac{7}{4}k^3 + 46k^2 + \tfrac{83}{6}jk \\
&< jk\left(-\tfrac{49}{248}(5)(31) + \tfrac{83}{6}\right) + \\
&\quad + k^2\left(-\tfrac{7}{4}(31) + 46\right) \\
&< 0
\end{aligned}
$$

Case 5 Since $j < \frac{k-9}{4}$, $-k + 2i < 0$, step 3 decreases the value of the approximation. In case 4 the value of step 3 was positive, and therefore made (4.7) an overestimation. In this case the value of step 3 is negative so we must be sure not to take away too much from the estimation. To do this we only consider the $\frac{j-1}{2}$ consecutive -2s which appear in w.

$$
\begin{aligned}
-2(-2)(-k+2i)\sum_{l=0}^{\frac{j-3}{2}} 2(k-l) &= 8(-k+2i)\left(\sum_{l=0}^{\frac{j-3}{2}} k - \sum_{l=0}^{\frac{j-3}{2}} l\right) \\
&= 8\left(-k + 2\left(\tfrac{k+9+4j}{4}\right)\right)\left(k\tfrac{j-1}{2} - \tfrac{j^2-4j+3}{8}\right) \\
&= 8\left(\tfrac{-k+9+4j}{2}\right)\left(\tfrac{4kj+4k-j^2+4j-3}{8}\right)
\end{aligned}
$$

$$
= -2jk^2 + \frac{17}{2}j^2k - 2j^3 - 2k^2 + 24jk + \frac{7}{2}j^2 + \frac{39}{2}k + 12j - \frac{27}{2} \qquad (4.10)
$$

The approximation resulting from combining (4.6), (4.8), (4.10), and (4.9) is

$$
\begin{aligned}
-k^2(v^TQ^{-1}v - w^TQ^{-1}w) &\leq -\tfrac{1}{2}j^2k^2 + \tfrac{1}{6}j^3k + \tfrac{3}{4}k^3 - 10jk^2 + \\
&\quad + \tfrac{11}{2}j^2k - \tfrac{11}{3}j^3 + \tfrac{57}{2}k^2 + \tfrac{35}{4}jk - \\
&\quad - \tfrac{5}{2}j^2 - \tfrac{43}{4}k - \tfrac{83}{3}j - 2 \\
&\leq \tfrac{1}{24}j^2k\left(-k+4j\right) + jk\left(-\tfrac{11}{8}k + \tfrac{11}{2}j\right) + \\
&\quad + \tfrac{3}{4}k^3 - \tfrac{69}{8}jk^2 + \tfrac{57}{2}k^2 + \tfrac{35}{4}jk - \\
&\quad - \tfrac{11}{24}j^2k^2 \\
&\leq \tfrac{1}{24}j^2k\left(-k + 4\left(\tfrac{k-11}{4}\right)\right) + \tfrac{3}{4}k^3 - \tfrac{69}{8}jk^2 + \\
&\quad + jk\left(-\tfrac{11}{8}k + \tfrac{11}{2}\left(\tfrac{k-11}{4}\right)\right) + \tfrac{57}{2}k^2 + \tfrac{35}{4}jk - \\
&\quad - \tfrac{11}{24}j^2k^2 \\
&\leq -\tfrac{11}{24}j^2k - \tfrac{121}{8}jk - \tfrac{11}{24}j^2k^2 + \tfrac{3}{4}k^3 - \\
&\quad - \tfrac{69}{8}jk^2 + \tfrac{57}{2}k^2 + \tfrac{35}{4}jk \\
&\leq -\tfrac{11}{24}k^2(\sqrt{2k+19}-4)^2 + \tfrac{3}{4}k^3 - \\
&\quad - \tfrac{69}{8}(\sqrt{2k+19}-4)k^2 + \tfrac{57}{2}k^2 - \tfrac{51}{8}jk \\
&\leq -\tfrac{11}{24}k^2(2k+35-8\sqrt{2k+19}) + \tfrac{3}{4}k^3 - \\
&\quad - \tfrac{69}{8}k^2\sqrt{2k+19} + \tfrac{69}{2}k^2 + \tfrac{57}{2}k^2 \\
&\leq -\tfrac{1}{6}k^3 - \tfrac{119}{24}k^2\sqrt{2k+19} + \tfrac{1127}{24}k^2 \\
&< -\tfrac{1}{6}k^2(31) - \tfrac{119}{24}k^2\sqrt{2(31)+19} + \tfrac{1127}{24}k^2 \\
&< 0
\end{aligned}
$$

Case 6 Since $j \leq \sqrt{2k+19} - 4$, $w_{\frac{6k+j^2+4j+1}{8}} = -2$.

$$-(-2)(-2)\left(\left(\tfrac{6k+j^2+4j+1}{8}\right)k^2-\left(\tfrac{6k+j^2+4j+1}{8}\right)^2k+2\left(\tfrac{6k+j^2+4j+1}{8}\right)^2\right)=$$

$$\begin{array}{c}=\tfrac{1}{16}j^4k+\tfrac{1}{4}j^2k^2+\tfrac{1}{2}j^3k-\tfrac{1}{8}j^4-\tfrac{3}{4}k^3+jk^2-\tfrac{3}{8}j^2k-\\ -j^3-\tfrac{17}{4}k^2-\tfrac{11}{2}jk-\tfrac{9}{4}j^2-\tfrac{23}{16}k-j-\tfrac{1}{32}.\end{array} \tag{4.11}$$

The approximation resulting from combining (4.6), (4.8), (4.10), (4.9), and (4.11) is:

$$\begin{array}{rcl}-k^2(v^TQ^{-1}v-w^TQ^{-1}w) & < & \tfrac{1}{16}j^4k-\tfrac{1}{4}j^2k^2+\tfrac{2}{3}j^3k-\tfrac{1}{8}j^4-9jk^2+\tfrac{41}{8}j^2k-\\ & & -\tfrac{14}{3}j^3+\tfrac{97}{4}k^2+\tfrac{13}{4}jk-\tfrac{19}{4}j^2-\tfrac{195}{16}k-\tfrac{86}{3}j-\tfrac{65}{32}\\ & < & \tfrac{1}{16}j^2k(2k+35-8\sqrt{2k+19})-\tfrac{1}{8}j^2k^2+\\ & & +\tfrac{2}{3}jk(2k+35-8\sqrt{2k+19})-9jk^2+\\ & & +\tfrac{41}{8}j^2k+\tfrac{97}{4}k^2+\tfrac{13}{4}jk\\ & < & -\tfrac{1}{2}j^2k\sqrt{2(31)+19}-\tfrac{1}{4}j^2k^2+\tfrac{319}{12}jk+\tfrac{117}{16}j^2k-\\ & & -\tfrac{16}{3}jk\sqrt{2(31)+19}-\tfrac{23}{3}jk^2+\tfrac{97}{4}k^2\\ & < & -\tfrac{9}{2}j^2k-\tfrac{1}{4}j^2k^2-48jk-\tfrac{23}{3}jk^2+\tfrac{117}{16}j^2k+\\ & & +\tfrac{97}{4}k^2+\tfrac{319}{12}jk\\ & < & (-\tfrac{1}{10}j^2k^2+\tfrac{45}{16}j^2k)+(-\tfrac{3}{20}j^2k^2-\tfrac{23}{3}jk^2+\\ & & +\tfrac{97}{4}k^2)-\tfrac{257}{12}jk\\ & < & (-\tfrac{31}{10}j^2k+\tfrac{45}{16}j^2k)+(-\tfrac{27}{20}k^2-23k^2+\tfrac{97}{4}k^2)\\ & < & 0\end{array}$$

Case 7 When $j<0$, $[v]-[w(i)]$ is odd. So unlike the previous six cases, $w_{k+1}=-2$, which adds k to the value $[v]-[w(i)]$. When $j=\frac{-k-5}{4}$

(resp. $\frac{-k-9}{4}$), setting $w_{k+1} = -2$ changes $[v]-[w(i)]$ to $\frac{-3k^2+9k+12}{4}$ (resp. $\frac{-3k^2+13k+20}{4}$), which is less than the value of $[v]-[w(i)]$ when $i = k-2$ (resp. $i = k-3$). Therefore when $j = \frac{-k-5}{4}$ (resp. $\frac{-k-9}{4}$), there are $\frac{1}{2}\left(\frac{3k-21}{4}\right)$ (resp. $\frac{1}{2}\left(\frac{3k-25}{4}\right)$) consecutive terms of w equal to -2, namely $w_{k+1}, w_k, \ldots, w_{k-\frac{3k-25}{4}}$ (resp. $w_{k+1}, w_k, \ldots, w_{k-\frac{3k-29}{4}}$). In general:

$$\begin{aligned}
\text{If } k \equiv 7 \pmod 8 \quad &\Rightarrow \quad w \text{ has } \tfrac{3k-21}{8} + 1 + \tfrac{i-1}{2} \text{ consecutive entries equal to } -2 \\
&\Rightarrow \quad w \text{ has at least } \tfrac{4k-8+4j}{8} \text{ consecutive entries equal to } -2. \\
\text{If } k \equiv 3 \pmod 8 \quad &\Rightarrow \quad w \text{ has } \tfrac{3k-25}{8} + 1 + \tfrac{i}{2} \text{ consecutive entries equal to } -2 \\
&\Rightarrow \quad w \text{ has at least } \tfrac{4k-8+4j}{8} \text{ consecutive entries equal to} -2.
\end{aligned}$$

To estimate $(-k^2)(v^TQ^{-1}v - w^TQ^{-1}w)$ comes in three parts. First comes from (4.6). The second comes from $w_{k-1} = -2$, which decreases the expression by $4k^2$. The final part comes from $w_k = \cdots = w_{\frac{4k+16+4j}{8}} = -2$. These decrease the expression by:

$$\begin{aligned}
4 \cdot \sum_{l=0}^{\frac{4k-16+4j}{8}-1} (-k^2)Q^{-1}_{k-l,k-l} &= 4 \cdot \sum_{l=0}^{\frac{k+j}{2}-3} (l+2)k^2 - (l^2-4l)k + 2l^2 \\
&= 4\left(\tfrac{k^2+2jk+j^2-2k-2j-8}{8}\right)k^2 - \\
&\quad -4\left(\tfrac{(k+j)^3-3(k+j)^2-46(k+j)-168}{24}\right)k + \\
&\quad +4\left(\tfrac{(k+j)^3-15(k+j)+74(k+j)-120}{12}\right)
\end{aligned}$$

$$= \frac{1}{3}k^4 + \frac{1}{2}jk^3 - \frac{1}{6}j^3k - \frac{1}{6}k^3 + jk^2 + \frac{3}{2}j^2k + \frac{1}{3}j^3 - \frac{4}{3}k^2 - \frac{7}{3}jk - 5j^2 - \frac{10}{3}k + \frac{74}{3}j - 40 \tag{4.12}$$

After combining (4.6), (4.12), and $-4k^2$, and recalling $-\frac{k}{2} < j \leq -1$ and $k \geq 31$:

$$\begin{aligned}
-k^2(v^TQ^{-1}v - w^TQ^{-1}w) &< -\tfrac{1}{3}k^4 - \tfrac{1}{2}jk^3 + \tfrac{1}{6}j^3k + \tfrac{11}{12}k^3 + jk^2 - \tfrac{11}{2}j^2k - \\
&\quad -\tfrac{1}{3}j^3 + \tfrac{4}{3}k^2 - \tfrac{35}{3}jk - 3j^2 - \tfrac{95}{12}k - \tfrac{182}{3}j + \tfrac{99}{2} \\
&< -\tfrac{1}{4}k^3\left(k+2j\right) + k^3\left(-\tfrac{1}{24}k + \tfrac{11}{12}\right) - \tfrac{11}{2}j^2k + \\
&\quad +j^3\left(\tfrac{1}{24}k - \tfrac{1}{3}\right) + k^2\left(j+1\right) + \tfrac{1}{96}k^2\left(k^2 - 32\right) + \\
&\quad +\tfrac{1}{8}j^3k - \tfrac{1}{96}k\left(k^3 + 12j\right) - 3j^2 - 7k + \\
&\quad +\tfrac{1}{96}\left(k^4 + 61j\right) - \left(\tfrac{1}{96}k^4 + 100\right) \\
&< 0
\end{aligned}$$

as desired.

Next assume j is even. First construct tables similar to Table 4.2 and Table 4.3:

Unlike tables 4.2 and 4.3, when $j > 0$ the value of $[v] - [w(i)]$ is odd. This means the first entry of $w(i)$ changed to -2 is w_{k+1}. Once again the fourth and sixth columns are only true for certain j and k. The numbers in the sixth column, $\frac{6k+j^2+2j+6}{8}$ only hold true when

$$1 \leq \frac{6k + j^2 + 2j + 6}{8} \leq k - \frac{j-2}{2} \tag{4.13}$$

The right side of the inequality simplifies to

$$j \leq \sqrt{2k + 11} - 3, \tag{4.14}$$

Table 4.4: $k \equiv 7 \pmod 8$

i	$-k+2i$	j	$[v]-[w(i)] \pmod{k^2}$	$C(w(i))$	$L(w(i))$
0	$-k$	$\frac{-k-9}{4}$	$\frac{k^2+9k+20}{4}$	$-$	$-$
2	$-k+4$	$\frac{-k-1}{4}$	$\frac{k^2+k+4}{4}$	$-$	$-$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\frac{k+1}{4}$	$\frac{-k+1}{2}$	-2	$\frac{3k+9}{2}$	$-$	$-$
$\frac{k+9}{4}$	$\frac{-k+9}{2}$	0	$\frac{-k+1}{2}$	$-$	$-$
$\frac{k+17}{4}$	$\frac{-k+17}{2}$	2	$\frac{-5k-7}{2} = (-k) + \frac{-3k-7}{2}$	2	$\frac{3k+7}{4}$
$\frac{k+25}{4}$	$\frac{-k+25}{2}$	4	$\frac{-9k-15}{2} = (-k) + (-2k) + \frac{-3k-15}{2}$	3	$\frac{3k+15}{4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-3$	$k-6$	$\frac{3k-21}{4}$	$\frac{-3k^2+13k+44}{4}$	$-$	$-$
$k-1$	$k-2$	$\frac{3k-13}{4}$	$\frac{-3k^2+5k+28}{4}$	$-$	$-$

If condition (4.14) is true then $w_{\frac{6k+j^2+2j+6}{8}} = -2$ in step 4. Otherwise $w_{\frac{6k+j^2+4j+1}{8}} = 0$. Step 3 of (4.5) will increase the approximation of $(-k^2)\left(v^TQ^{-1}v - w^TQ^{-1}w\right)$ whenever $-k+2i > 0$, or equivalently $j > \frac{k-9}{4}$, and decrease when $j < \frac{k-9}{4}$. Finally note that $j < \frac{3}{4}k$ for all relevant k. This leaves 7 cases to check:

Case 1 $0 < \frac{k-9}{4} \leq \sqrt{2k+11} - 3 \leq j < \frac{3}{4}k$
Case 2 $0 < \frac{k-9}{4} \leq j < \sqrt{2k+11} - 3 < \frac{3}{4}k$
Case 3 $0 < j < \frac{k-9}{4} \leq \sqrt{2k+11} - 3 < \frac{3}{4}k$
Case 4 $0 < \sqrt{2k+11} - 3 < \frac{k-9}{4} < j < \frac{3}{4}k$
Case 5 $0 < \sqrt{2k+11} - 3 < j < \frac{k-9}{4} < \frac{3}{4}k$
Case 6 $0 < j < \sqrt{2k+11} - 3 < \frac{k-9}{4} < \frac{3}{4}k$
Case 7 $j \leq 0$

Table 4.5: $k \equiv 3 \pmod 8$

i	$-k+2i$	j	$[v]-[w(i)] \pmod{k^2}$	$C(w(i))$	$L(w(i))$
1	$-k+2$	$\frac{-k-5}{4}$	$\frac{k^2+5k+12}{4}$	$-$	$-$
3	$-k+6$	$\frac{-k+3}{4}$	$\frac{k^2-3k-4}{4}$	$-$	$-$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\frac{k+1}{4}$	$\frac{-k+1}{2}$	-2	$\frac{3k+9}{2}$	$-$	$-$
$\frac{k+9}{4}$	$\frac{-k+9}{2}$	0	$\frac{-k+1}{2}$	$-$	$-$
$\frac{k+17}{4}$	$\frac{-k+17}{2}$	2	$\frac{-5k-7}{2} = (-k) + \frac{-3k-7}{2}$	2	$\frac{3k+7}{4}$
$\frac{k+25}{4}$	$\frac{-k+25}{2}$	4	$\frac{-9k-15}{2} = (-k) + (-2k) + \frac{-3k-15}{2}$	3	$\frac{3k+15}{4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-2$	$k-4$	$\frac{3k-17}{4}$	$\frac{-3k^2+9k+36}{4}$	$-$	$-$
k	k	$\frac{3k-9}{4}$	$\frac{-3k^2+k+20}{4}$	$-$	$-$

Cases 1, 2, and 3 only hold when $\frac{k-9}{4} \leq \sqrt{2k+11} - 3$, which is true for $k \leq 31$. Tables 4.12, 4.13, 4.14, 4.15, 4.16, 4.17, and 4.18 are Tables 4.2 and 4.3 for $k = 7$, 11, 15, 19, 23, 27, and 31. In each case the w which minimizes $(-k^2)w^TQ^{-1}w$ is $w = v$, as claimed.

Case 4 Since $j > \frac{k-9}{4}$, $-k+2i$ is positive and we must consider step 3. Given j, the w which minimizes $(-k^2)w^TQ^{-1}w$ has at least $\frac{j+2}{2}$ entries equal to -2. One of these is the w_{k+1} entry. Therefore we can overestimate the value of step 3:

$$
\begin{aligned}
-(-4)(-k+2i))\left(k+\textstyle\sum_{l=0}^{\frac{j-2}{2}} 2(k-l)\right) &= -(2(-2)(-k+2i))\,k+ \\
&\quad +8(-k+2i)\left(\textstyle\sum_{l=0}^{\frac{j-2}{2}} k-\sum_{l=0}^{\frac{j-2}{2}} l\right) \\
&= 4k\left(-k+2\left(\tfrac{k+9+4j}{4}\right)\right)+ \\
&\quad +8\left(-k+2\left(\tfrac{k+9+4j}{4}\right)\right)\cdot \\
&\quad \cdot\left(k\tfrac{j}{2}-\tfrac{j^2-j}{8}\right) \\
&= (-2k^2+18k+8jk)+ \\
&\quad +(-2jk^2+\tfrac{17}{2}j^2k-2j^3+ \\
&\quad +\tfrac{35}{2}jk-\tfrac{5}{2}j^2+\tfrac{9}{2}j)
\end{aligned}
$$

$$
= -2jk^2+\frac{17}{2}j^2k-2j^3-2k^2+\frac{51}{2}jk-\frac{5}{2}j^2+18k+\frac{9}{2}j \qquad (4.15)
$$

For step 4 use $w_{k+1}=\cdots=w_{k-\frac{j-2}{2}}=-2$,

$$
\begin{aligned}
-(-2)^2\left(k^2+\textstyle\sum_{l=0}^{\frac{j-2}{2}}(-k^2)Q^{-1}_{k-l,k-l}\right) &= -4k^2-4\textstyle\sum_{l=0}^{\frac{j-2}{2}}(l+2)k^2- \\
&\quad -(l^2+4l)k+2l^2 \\
&= -4k^2-4\left(\left(\tfrac{j^2+6j}{8}\right)k^2-\right. \\
&\quad \left.-\left(\tfrac{j^3+9j^2-22j}{24}\right)k+\left(\tfrac{j^3-3j^2+2j}{12}\right)\right)
\end{aligned}
$$

$$
-\frac{1}{2}j^2k^2+\frac{1}{6}j^3k-3jk^2+\frac{3}{2}j^2k-\frac{1}{3}j^3-\frac{11}{3}jk+j^2-\frac{2}{3}j \qquad (4.16)
$$

Finally we need to include some terms from step 5 into the approximation. Since w has at least $\frac{j-2}{2}$ consecutive terms equal to -2, the entries

$Q^{-1}_{k,k+1}, \ldots, Q^{-1}_{k-\frac{j-4}{2}, k+1-\frac{k-4}{2}}$ along the superdiagonal (and subdiagonal) contribute to the approximations. To simplify the calculation and to ensure the estimation is not too large (in absolute value).

$$\begin{aligned} -2\left((-2)(-2)\sum_{l=0}^{\frac{l-4}{2}}(-k^2)Q^{-1}_{k-1-l,k-l}\right) &< -8\left(Q^{-1}_{k,k+1} + \sum_{l=1}^{\frac{j-4}{2}} Q^{-1}_{k-1,k}\right) \\ &= -8\left((k^2 + (\tfrac{j-4}{2})(2k^2 - 2k)\right) \end{aligned}$$

$$= -8jk^2 + 24k^2 + 8jk - 32k \tag{4.17}$$

By combining (4.6), (4.15), (4.16), and (4.17) we obtain the estimate

$$\begin{aligned} -k^2\left(v^TQ^{-1}v - w^TQ^{-1}w\right) &< -\tfrac{1}{2}j^2k^2 + \tfrac{1}{6}j^3k + \tfrac{3}{4}k^3 - 11jk^2 + 6j^2k - \tfrac{7}{3}j^3 + \\ &\quad +26k^2 + \tfrac{95}{6}jk - \tfrac{19}{2}j^2 - \tfrac{101}{4}k - \tfrac{193}{6}j + \tfrac{19}{2} \\ &< \tfrac{1}{8}j^2k(-k + \tfrac{4}{3}j) + 6j^2k(-\tfrac{1}{35}k + 1) + \\ &\quad +(-k + \tfrac{19}{2}) - \tfrac{57}{280}j^2k^2 + \tfrac{3}{4}k^3 - \\ &\quad -11\left(\tfrac{k-9}{4}\right)k^2 + 26k^2 + \tfrac{95}{6}jk \\ &< -\tfrac{57}{280}j^2k^2 - 2k^3 + \tfrac{203}{4}k^2 + \tfrac{95}{6}jk \\ &< jk(-\tfrac{57}{280}(7)(35) + \tfrac{95}{6}) + k^2(-2(35) + \tfrac{203}{4}) \\ &< 0 \end{aligned}$$

Case 5 Since case 3 decreases the approximation we cannot use (4.15). Instead we change (4.15) slightly so that it underestimates the actual contribution of step 3. So we only consider the $\frac{j}{2}$ consecutive -2s which appear in w.

$$
\begin{aligned}
-(2(-2)(-k+2i))\left(k+\sum_{l=0}^{\frac{j-4}{2}} 2(k-l)\right) &= 4\left(\left(-k+2\left(\tfrac{k+9+4j}{4}\right)\right)\right)k+ \\
&\quad +8(-k+2\left(\tfrac{k+9+4j}{4}\right))\cdot \\
&\quad \cdot\left(\sum_{l=0}^{\frac{j-4}{2}} k-\sum_{l=0}^{\frac{j-4}{2}} l\right) \\
&= -2k^2+18k+8jk+ \\
&\quad +(-4k+36+16j)\cdot \\
&\quad \cdot\left(\tfrac{j-2}{2}k-\tfrac{j^2-6j+8}{8}\right) \\
&= -2k^2+18k+8jk+ \\
&\quad +\tfrac{1}{2}(-k+8+4j)\cdot \\
&\quad \cdot(4jk-8k-j^2+6j-8)
\end{aligned}
$$

$$
-2jk^2+\frac{17}{2}j^2k-2j^3+2k^2+5jk+8j^2-10k+8j-32 \tag{4.18}
$$

The approximation of resulting from combining (4.6), (4.16), (4.17), and (4.18) is

$$
\begin{aligned}
-k^2(v^TQ^{-1}v-w^TQ^{-1}w) &\leq -\tfrac{1}{2}j^2k^2+\tfrac{1}{6}j^3k+\tfrac{3}{4}k^3-11jk^2+6j^2k- \\
&\quad -\tfrac{7}{3}j^3+30k^2-\tfrac{14}{3}jk+j^2- \\
&\quad -\tfrac{213}{4}k-\tfrac{86}{3}j-\tfrac{45}{2} \\
&< (-\tfrac{1}{24}j^2k^2+\tfrac{1}{6}j^3k)+(-\tfrac{7}{3}j^3+j^2)- \\
&\quad -\tfrac{11}{24}j^2k^2+\tfrac{3}{4}k^3-11jk^2+6j^k2+30k^2 \\
&< j^2k\left(-\tfrac{1}{24}k+\tfrac{1}{6}\left(\tfrac{k-9}{4}\right)\right)-11jk^2+6j^2k- \\
&\quad -\tfrac{11}{24}(2k+20-6\sqrt{2k+11})k^2+\tfrac{3}{4}k^3+30k^2 \\
&< (-\tfrac{1}{6}(35)k^2+\tfrac{125}{6}k^2)+(-\tfrac{45}{8}jk^2+\tfrac{45}{8}j^2k)+ \\
&\quad +(\tfrac{11}{4}k^2\sqrt{2k+11}-\tfrac{11}{4}jk^2)-\tfrac{21}{8}jk^2 \\
&< 15k^2-\tfrac{21}{8}(9)k^2 \\
&< 0
\end{aligned}
$$

Case 6 Since $j \leq \sqrt{2k+11} - 3$, $w_{\frac{6k+j^2+2j+6}{8}} = -2$.

$$-(-2)(-2)\left(\left(\tfrac{6k+j^2+2j+6}{8}\right)k^2 - \left(\tfrac{6k+j^2+2j+6}{8}\right)^2 k + 2\left(\tfrac{6k+j^2+2j+6}{8}\right)^2\right) =$$

$$= \tfrac{1}{16}j^4k + \tfrac{1}{4}j^2k^2 + \tfrac{1}{4}j^3k - \tfrac{1}{8}j^4 - \tfrac{3}{4}k^3 + \tfrac{1}{2}jk^2 - \tfrac{1}{2}j^2k -$$

$$= \tfrac{1}{2}j^3 - 3k^2 - \tfrac{3}{2}jk - j^2 - \tfrac{27}{4}k - 3j - \tfrac{9}{2} \tag{4.19}$$

Equation (4.17) only holds for $j > 2$, so consider only equations (4.6), (4.18), (4.16), and (4.19). Then

$$\begin{aligned}
-k^2(v^TQ^{-1}v - w^TQ^{-1}w) &\leq \tfrac{1}{16}j^4k - \tfrac{1}{4}j^2k^2 + \tfrac{5}{12}j^3k - \tfrac{1}{8}j^4 - \tfrac{5}{2}jk^2 + \tfrac{11}{2}j^2k - \\
&\quad - \tfrac{17}{6}j^3 + 3k^2 - \tfrac{85}{6}jk - 28k - \tfrac{95}{3}j - 27 \\
&< \tfrac{1}{16}j^2k(2k + 20 - 6\sqrt{2k+11}) - \tfrac{1}{4}j^2k^2 + \\
&\quad + \tfrac{5}{12}jk(2k + 20 - 6\sqrt{2k+11}) - \tfrac{5}{2}jk^2 + \\
&\quad + \tfrac{11}{2}j^2k + 3k^2 - \tfrac{85}{6}jk \\
&< -\tfrac{3}{8}j^2k\sqrt{2(35)+11} - \tfrac{1}{8}(2)^2k^2 - \tfrac{35}{6}jk - \\
&\quad - \tfrac{5}{2}jk\sqrt{2k+11} - \tfrac{5}{3}jk^2 + \tfrac{27}{4}j^2k + 3k^2 \\
&< -\tfrac{5}{2}jk\sqrt{2k+11} - \tfrac{5}{4}(2)k^2 + \tfrac{5}{2}k^2 - \tfrac{35}{6}jk - \\
&\quad - \tfrac{5}{12}jk^2 + \tfrac{7}{8}j^2k + \tfrac{5}{2}jk\sqrt{2k+11} \\
&< jk(-\tfrac{5}{12}k + \tfrac{7}{8}\sqrt{3k+11} - \tfrac{35}{6}) \\
&< 0
\end{aligned}$$

Case 7 When $j \leq 0$, $[v] - [w(i)]$ is even. So unlike the previous six cases, $w_{k+1} = 0$. When $j = \frac{3k-13}{4}$ (resp. $\frac{3k-9}{4}$) and w_{k+1} is changed from a 0 to -2, then value of $[v] - [w(i)]$ changes to $\frac{-3k^2+9k+28}{4}$ (resp. $\frac{-3k^2+5k+20}{4}$). This is less than the value of $[v]-[w(0)]$, so when $j=\frac{-k-9}{4}$ ($\frac{-k-5}{4}$ resp.) there are at least $\frac{3k-13}{8}$ ($\frac{3k-9}{8}$ resp.) consecutive terms of w equal to -2. In general

$$\begin{aligned}
\text{If } k \equiv 7 \pmod 8 &\Rightarrow \quad w \text{ has } \tfrac{3k-13}{8}+\tfrac{i}{2} \text{ consecutive entries equal to } -2\\
&\Rightarrow \quad w \text{ has at least } \tfrac{4k-4+4j}{8} \text{ consecutive entries equal to } -2.\\
\text{If } k \equiv 3 \pmod 8 &\Rightarrow \quad w \text{ has } \tfrac{3k-9}{8}+\tfrac{i-1}{2} \text{ consecutive entries equal to } -2\\
&\Rightarrow \quad w \text{ has at least } \tfrac{4k-4+4j}{8} \text{ consecutive entries equal to} -2.
\end{aligned}$$

The estimate of $(-k^2)(v^TQ^{-1}v-w^TQ^{-1}w)$ comes in two parts. The first comes from (4.6). The second part comes from the $\frac{4k-4+4j}{8}$ consecutive entries equal to -2. These decrease the expression by:

$$\begin{aligned}
4\sum_{l=0}^{\frac{4k-4+4j}{8}-1}(-k^2)Q^{-1}_{k-l,k-l} &= 4\sum_{l=0}^{\frac{k-3+j}{2}}(l+2)k^2-(l^2-4l)k+2l^2\\
&= 4\left(\tfrac{k^2+2jk+j^2+4k+4j-5}{8}\right)k^2-\\
&\quad -4\left(\tfrac{k^3+3jk^2+3j^2k+3j^3-18k^2-36jk-18j^2+59k+59j-42}{24}\right)k+\\
&\quad +4\left(\tfrac{k^3+3jk^2+3j^2k+3j^3-6k^2-12jk-6j^2+11k+11j-6}{12}\right)
\end{aligned}$$

$$=\frac{1}{3}k^4+\frac{1}{2}jk^3-\frac{1}{6}j^3k+\frac{16}{3}k^3+9jk^2+4j^2k+\frac{1}{3}j^3-\frac{43}{3}k^2-\frac{83}{6}jk-2j^2+\frac{32}{3}k+\frac{11}{3}j-2 \tag{4.20}$$

Combine (4.6) and (4.20), and recall $-\frac{k}{2}<j\leq 0$ and $k\geq 35$:

$$
\begin{aligned}
-k^2(v^TQ^{-1}v - w^TQ^{-1}w) &< -\tfrac{1}{3}k^4 - \tfrac{1}{2}jk^3 + \tfrac{1}{6}j^3k - \tfrac{73}{12}k^3 - 13jk^2 - 2j^2k - \\
&\quad -\tfrac{1}{3}j^3 + \tfrac{49}{3}k^2 - \tfrac{1}{6}jk - 6j^2 - \tfrac{263}{12}k - \tfrac{119}{3}j + \tfrac{23}{2} \\
&< -\tfrac{1}{3}k^4 - \tfrac{1}{2}jk^3 - \tfrac{73}{12}k^3 - 13jk^2 - \tfrac{23}{12}j^2k - \tfrac{1}{3}j^3 + \\
&\quad +(-\tfrac{1}{12}j^2k - \tfrac{1}{6}jk) - (-\tfrac{119}{6}k - \tfrac{119}{3}j) + \\
&\quad +(-\tfrac{25}{12}k + \tfrac{23}{2}) + \tfrac{49}{3}k^2 \\
&< -\tfrac{1}{12}k^4 - \tfrac{1}{2}jk^3 - \tfrac{61}{12}k^3 - 13jk^2 - \tfrac{20}{12}j^2k + \\
&\quad +(-k^3 + \tfrac{49}{3}k^2) + (-j^2k - \tfrac{1}{3}j^3) + \\
&\quad +(-\tfrac{1}{4}k^4 - \tfrac{1}{2}jk^3) \\
&< -\tfrac{1}{12}(35)k^3 + (-\tfrac{61}{12}k^3 - \tfrac{61}{6}jk^2) - \tfrac{17}{6}jk^2 \\
&< -\tfrac{35}{12}k^3 - \tfrac{17}{6}jk^2 \\
&< 0
\end{aligned}
$$

4.4 The Uniqueness of ϕ

The final step is to show ϕ is the automorphism for which $T_{\phi,-1}(2)$ is even. We compute $M_Q(v) - \gamma((a,0))$ and $M_Q(v) - \gamma((a,-2))$ for $a = 0, 1, \ldots, 4k$. If this value is integral then there may exist an automorphism which satisfies Theorem 2.4.4. There are four cases to check.

Case 1: $M_Q(v) - \gamma((a,0))$

$$\begin{aligned}\gamma((a,0)) &= \tfrac{1}{4}\left((a,0)\begin{pmatrix} -n & 1 \\ 1 & -2 \end{pmatrix}^{-1}\begin{pmatrix} a \\ 0 \end{pmatrix} + 2\right) \\ &= \tfrac{1}{4}\left(\tfrac{-2a^2}{2n-1} + 2\right) \\ &= \tfrac{2p^2-2a^2}{4p^2} \\ &= \tfrac{p^2-a^2}{2p^2}\end{aligned}$$

Then $M_Q(v) - \gamma((a,0)) = \frac{2p^2-8}{p^2} - \frac{p^2-a^2}{2p^2} = \frac{4p^2-16+a^2-p^2}{2p^2} = \frac{a^2-16}{2p^2} + \frac{3}{2}$. We must determine for what p, a does there exist $k \in \mathbb{Z}$ such that

$$\frac{a^2-16}{2p^2} + \frac{3}{2} = 2k \iff a^2 - 16 = p^2(4k-3) \iff (a-4)(a+4) = p^2(4k-3)$$

Proposition 4.4.1. *Equation $(a-4)(a+4) = p^2(4k-3)$ has no solutions for $p \geq 5$, p prime.*

First we will show that either $p^2|(a-4)$ or $p^2|(a+4)$. Otherwise $p|(a-4)$ and $p|(a+4)$, and therefore $p|8$, a contradiction.

Now suppose $p^2|(a-4) \iff a-4 = kp^2$, $k \in \mathbb{Z}$. Since a is odd the equation has no solutions for even k. If $k=1$, then $a = p^2+4$. Recall that we only need to consider odd a, $1 \leq a \leq \frac{p^2+1}{2}$. So $a = p^2+4$ falls outside the bounds of a. The same argument holds for all $k > 1$.

Finally suppose $p^2|(a+4) \iff a+4 = kp^2$, $k \in \mathbb{Z}$. As before the equation has no solutions for even k. If $k \geq 1$, then $a = kp^2 - 4$. If k=1, then the inequality $1 \leq \frac{p^2+1}{2} \leq a = p^2 - 4$ only holds for prime $p = 3$, which contradicts the assumptions on p.

Case 2: $M_Q(v) - \gamma((a,-2))$. Recalling that $p^2 = 2n-1$,

$$\begin{aligned}
\gamma((a,-2)) &= \tfrac{1}{4}\left((a,-2)\begin{pmatrix} -n & 1 \\ 1 & -2 \end{pmatrix}^{-1}\begin{pmatrix} a \\ -2 \end{pmatrix} + 2\right) \\
&= \tfrac{1}{4}\left(\tfrac{-2a^2+4a-4n}{2n-1} + 2\right) \\
&= \tfrac{-2a^2+4a-2}{4p^2} \\
&= \tfrac{-a^2+2a-1}{2p^2}
\end{aligned}$$

Then $M_Q(v) - \gamma((a,-2)) = \frac{2p^2-8}{p^2} - \frac{-a^2+2a-1}{2p^2} = \frac{a^2-2a-15}{2p^2} + 2.$

As before, we must determine for which p, a does there exist k such that

$$\frac{a^2-2a-15}{2p^2} + 2 = 2k \iff (a-5)(a+3) = 4kp^2$$

First note that $a = 5$ is a solution to the equation regardless of the prime p. If $a \neq 5$, then either $p^2|(a-5)$ or $p^2|(a+3)$. Otherwise there exists m, $l \in \mathbb{Z}$ such that $a-5 = lp$ and $a+3 = mp$. As in case 1, this implies $p|8$, a contradiction. Finally neither $p^2|(a-5)$ nor $p^2|(a+3)$ is possible for $a \neq 5$ and $1 \leq a \leq \frac{p^2+1}{2}$, as in case 1.

Table 4.6: $k = 7$

i	$-7+2i$	j	$[v]-[w(i)] (\text{mod } k^2)$	$m'_Q(i)$	w maximizing $(-k^2)w^TQ^{-1}w$
1	-5	-3	$24 \equiv -25$	3	$(0,-2,0,0,0,0,-2,-2,-5)$
3	-1	-1	$6 \equiv -43$	4	$(0,0,0,0,-2,-2,-2,-2,-1)$
5	3	1	-12	1	$(0,0,0,0,0,-2,0,0,3)$
7	7	3	-30	3	$(0,-2,0,0,0,-2,-2,0,7)$

- If $j = 1$, then $(-k^2)w^TQ^{-1}w = 393$
- If $j = 3$, then $(-k^2)w^TQ^{-1}w = 1961$
- If $j = -3$, then $(-k^2)w^TQ^{-1}w = 2353$
- If $j = -1$, then $(-k^2)w^TQ^{-1}w = 4705$

Table 4.7: $k = 11$

i	$-11+2i$	j	$[v]-[w(i)] \pmod{k^2}$	$m'_Q(i)$	w maximizing $(-k^2)w^TQ^{-1}w$
0	-11	-5	$60 \equiv -61$	4	$(0,0,0,-2,0,0,0,0,0,-2,-2,-2,-11)$
2	-7	-3	$34 \equiv -89$	6	$(-2,0,0,0,0,0,0,-2,-2,-2,-2,-2,-7)$
4	-3	-1	$8 \equiv 113$	1	$(0,0,0,0,0,-2,-2,-2,-2,-2,-2,-2,-3)$
6	1	1	-18	1	$(0,0,0,0,0,0,0,0,-2,0,0,0,1)$
8	5	3	-44	3	$(-2,0,0,0,0,0,0,0,0,-2,-2,0,5)$
10	9	5	-70	5	$(0,0,0,0,-2,0,0,0,-2,-2,-2,0,9)$

- If $j = 1$, then $(-k^2)w^TQ^{-1}w = 1381$
- If $j = 3$, then $(-k^2)w^TQ^{-1}w = 4285$
- If $j = 5$, then $(-k^2)w^TQ^{-1}w = 12997$
- If $j = -5$, then $(-k^2)w^TQ^{-1}w = 13965$
- If $j = -3$, then $(-k^2)w^TQ^{-1}w = 25373$
- If $j = -1$, then $(-k^2)w^TQ^{-1}w = 43973$

Table 4.8: $k = 15$

i	$-15+2i$	j	$[v]-[w(i)] (\mathrm{mod}\ k^2)$	$m'_Q(i)$	w maximizing $(-k^2)w^TQ^{-1}w$
1	-13	-5	$78 \equiv -147$	4	$(-2, 0, \ldots, 0, -2, -2, -2, -2, -2, -2, -13)$
3	-9	-3	$44 \equiv -181$	6	$(0, 0, 0, 0, 0, 0, 0, -2, 0, -2, \ldots, -2, -9)$
5	-5	-1	$10 \equiv -215$	1	$(-2, 0, 0, 0, 0, 0, -2, \ldots, -2, -5)$
7	-1	1	-24	1	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -2, 0, 0, 0, 0, -1)$
9	3	3	-58	2	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -2, -2, 0, 3)$
11	7	5	-92	4	$(0, 0, 0, -2, 0, 0, 0, 0, 0, 0, 0, 0, -2, -2, -2, 0, 7)$
13	11	7	-126	5	$(0, 0, 0, 0, 0, 0, 0, 0, -2, 0, 0, -2, -2, -2, -2, 0, 11)$
15	15	9	-160	7	$(0, 0, 0, 0, -2, 0, 0, 0, 0, -2, \ldots, -2, 0, 15)$

- If $j = 1$, then $(-k^2)w^TQ^{-1}w = 3425$

  $\vdots$

- If $j = 9$, then $(-k^2)w^TQ^{-1}w = 93425$

- If $j = -5$, then $(-k^2)w^TQ^{-1}w = 77225$

  $\vdots$

- If $j = -1$, then $(-k^2)w^TQ^{-1}w = 208625$

Table 4.9: $k = 19$

i	$-19+2i$	j	$[v]-[w(i)](k^2)$	$m'_Q(i)$	w maximizing $(-k^2)w^TQ^{-1}w$
0	-19	-7	$138 \equiv -223$	8	$(0,0,-2,0,0,0,0,0,0,0,0,0,-2,\ldots,-2,-19)$
2	-15	-5	$96 \equiv -265$	9	$(0,0,0,0,0,0,0,0,0,0,-2,0,-2,\ldots,-2,-15)$
4	-11	-3	$54 \equiv -307$	11	$(0,0,0,0,0,0,0,0,-2,0,-2,\ldots,-2,-11)$
6	-7	-1	$12 \equiv -349$	14	$(0,0,-2,0,0,0,0,-2,\ldots,-2,-2,-7)$
8	-3	1	-30	1	$(0,0,0,0,0,0,0,0,0,0,0,0,0,-2,0,0,0,0,0,0,-3)$
10	1	3	-72	2	$(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-2,0-2,0,1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
18	17	11	-240	8	$(0,0,0,0,0,0,0,-2,0,0,0,0,-2,\ldots,-2,0,17)$

- If $j = 1$, then $(-k^2)w^TQ^{-1}w = 6909$

$\vdots$

- If $j = 11$, then $(-k^2)w^TQ^{-1}w = 235061$

- If $j = -7$, then $(-k^2)w^TQ^{-1}w = 197517$

$\vdots$

- If $j = -1$, then $(-k^2)w^TQ^{-1}w = 688477$

Table 4.10: $k = 23$

i	$-23+2i$	j	$[v]-[w(i)] \pmod{k^2}$	$m'_Q(i)$	w maximizing $(-k^2)w^TQ^{-1}w$
1	-21	-7	$164 \equiv -365$	10	$(0,\ldots,0,-2,-2,-2,-2,\ldots,-2,-21)$
3	-17	-5	$114 \equiv -415$	12	$(0,0,0,0,0,0,0,0,0,0,-2,0,0,-2,\ldots,-2,-17)$
5	-13	-3	$64 \equiv -465$	14	$(0,0,0,0,0,0,0,0,0,0,-2,\ldots,-2,-13)$
7	-9	-1	$14 \equiv -515$	18	$(0,0,0,0,0,-2,0,0,-2,\ldots,-2,-9)$
9	-5	1	-36	1	$(0,0,\ldots,0,0,0,0,-2,0,0,0,0,0,0,-5)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
23	23	15	-386	11	$(0,0,0,0,0,0,0,-2,0,0,0,0,0,-2,\ldots,-2,0,23)$

- If $j = 1$, then $(-k^2)w^TQ^{-1}w = 12217$

$\vdots$

- If $j = -7$, then $(-k^2)w^TQ^{-1}w = 621625$

- If $j = 15$, then $(-k^2)w^TQ^{-1}w = 769695$

$\vdots$

- If $j = -1$, then $(-k^2)w^TQ^{-1}w = 1821257$

Table 4.11: $k = 27$

i	$-27+2i$	j	$[v]-[w(i)](k^2)$	$m'_Q(i)$	w maximizing $(-k^2)w^TQ^{-1}w$
0	-27	-9	$248 \equiv -481$	12	$(0,-2,0,\ldots,0,-2,-2,\ldots,-2,-2,-27)$
2	-23	-7	$190 \equiv -539$	13	$(0,0,0,0,0,0,0,0,0,0,0,0,0,0,-2,0,0,-2,\ldots,-2,-23)$
4	-19	-5	$132 \equiv -597$	15	$(0,0,0,0,0,0,0,0,0,0,0,0,-2,0,0,-2,\ldots,-2,-19)$
6	-15	-3	$74 \equiv -655$	18	$(0,-2,0,0,0,0,0,0,0,0,0,-2,\ldots,-2,-15)$
8	-11	-1	$16 \equiv -713$	21	$(-2,0,0,0,0,0,0,0,-2,-2,\ldots,-2,-11)$
10	-7	1	-42	1	$(0,0,\ldots,0,0,0,-2,0,0,0,0,0,0,0,-7)$
12	-3	3	-100	2	$(0,0,\ldots,0,0,-2,0,0,0,-2,0,-3)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
26	25	17	-506	12	$(0,0,0,0,0,0,0,0,0,0,-2,0,0,0,0,0,-2,\ldots,-2,-2,0,25)$

- If $j = 1$, then $(-k^2)w^TQ^{-1}w = 19733$

$\vdots$

- If $j = 9$, then $(-k^2)w^TQ^{-1}w = 264677$

- If $j = -9$, then $(-k^2)w^TQ^{-1}w = 1352981$

$\vdots$

- If $j = -1$, then $(-k^2)w^TQ^{-1}w = 4084637$

Table 4.12: $k = 7$

i	$-7+2i$	j	$[v]-[w(i)] (\text{mod } k^2)$	$m'_Q(i)$	w maximizing $(-k^2)w^TQ^{-1}w$
0	-7	-4	$33 \equiv -16$	2	$(-2,0,0,0,0,0,-2,0,-7)$
2	-3	-2	$15 \equiv -34$	3	$(0,0,0,-2,0,-2,-2,0,-3)$
4	1	0	$-3 \equiv -52$	6	$(-2,0,-2,-2,-2,-2,-2,0,1)$
6	5	2	-21	2	$(0,0,0,0,0,0,-2,-2,5)$

- If $j = 2$, then $(-k^2)w^TQ^{-1}w = 785$
- If $j = -4$, then $(-k^2)w^TQ^{-1}w = 1569$
- If $j = -2$, then $(-k^2)w^TQ^{-1}w = 3529$
- If $j = 0$, then $(-k^2)w^TQ^{-1}w = 9017$

Table 4.13: $k = 11$

i	$-11+2i$	j	$[v]-[w(i)] \pmod{k^2}$	$m'_Q(i)$	w maximizing $(-k^2)w^TQ^{-1}w$
1	-9	-4	$47 \equiv -74$	4	$(0,0,0,0,0,0,-2,0,-2,-2,-2,0,-9)$
3	-5	-2	$21 \equiv -100$	6	$(0,0,0,0,-2,0,-2,-2,-2,-2,-2,0,-5)$
5	-1	0	$-5 \equiv 126$	9	$(0,0,-2,-2,-2,-2,-2,-2,-2,-2,-2,0,-1)$
7	3	2	-31	2	$(0,0,0,0,0,0,0,0,0,-2,0,-2,3)$
9	7	4	-57	4	$(0,-2,0,0,0,0,0,0,0,-2,-2,-2,7)$
11	11	6	-83	5	$(0,0,0,0,0,-2,0,0,-2,-2,-2,-2,11)$

- If $j = 2$, then $(-k^2)w^TQ^{-1}w = 2349$
- If $j = 4$, then $(-k^2)w^TQ^{-1}w = 7189$
- If $j = 6$, then $(-k^2)w^TQ^{-1}w = 16869$
- If $j = -4$, then $(-k^2)w^TQ^{-1}w = 19773$
- If $j = -2$, then $(-k^2)w^TQ^{-1}w = 38165$
- If $j = 0$, then $(-k^2)w^TQ^{-1}w =$73981

Table 4.14: $k = 15$

i	$-15+2i$	j	$[v]-[w(i)] \pmod{k^2}$	$m'_Q(i)$	w maximizing $(-k^2)w^TQ^{-1}w$
0	-15	-6	$95 \equiv -130$	5	$(0,0,0,0,0,0,0,0,0,0,-2,-2,-2,-2,-2,0,-15)$
2	-11	-4	$61 \equiv -164$	7	$(0,0,0,0,0,0,-2,0,0,-2,\ldots,-2,0,-11)$
4	-7	-2	$27 \equiv -198$	9	$(0,0,0,0,0,0,-2,\ldots,-2,0,-7)$
6	-3	0	$-7 \equiv -232$	13	$(0,-2,0,-2,\ldots,-2,-2,0,-3)$
8	1	2	-41	2	$(0,0,0,0,0,0,0,0,0,0,0,0,-2,0,0,-2,1)$
10	5	4	-75	4	$(-2,0,0,0,0,0,0,0,0,0,0,0,0,-2,-2,-2,5)$
12	9	6	-109	5	$(0,0,0,0,-2,0,0,0,0,0,0,0,-2,-2,-2,-2,9)$
14	13	8	-143	6	$(0,0,0,0,0,0,0,0,0,-2,0,-2,-2,-2,-2,-2,13)$

- If $j = 2$, then $(-k^2)w^TQ^{-1}w = 5225$

  $\vdots$
- If $j = 8$, then $(-k^2)w^TQ^{-1}w = 57425$
- If $j = -6$, then $(-k^2)w^TQ^{-1}w = 66425$

  $\vdots$
- If $j = 0$, then $(-k^2)w^TQ^{-1}w = 322025$

Table 4.15: $k = 19$

i	$-19+2i$	j	$[v]-[w(i)](k^2)$	$m'_Q(i)$	w maximizing $(-k^2)w^TQ^{-1}w$
1	-17	-6	$117 \equiv -244$	8	$(0,\dots,0,-2,0,0,-2,-2,-2,-2,-2,-2,-2,0,-17)$
3	-13	-4	$75 \equiv -286$	10	$(0,0,0,0,0,0,0,-2,0,0,-2,\dots,-2,0,-13)$
5	-9	-2	$33 \equiv -328$	13	$(0,-2,0,0,0,0,0,-2,\dots,-2,0,-9)$
7	-5	0	$-9 \equiv -370$	17	$(-2,0,0,-2,-2,-2,\dots,-2,0,-5)$
9	-1	2	-51	2	$(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-2,0,0,0,-2,-1)$
11	3	4	-93	3	$(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-2,-2,-2,3)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
19	19	12	-261	9	$(0,0,0,0,0,0,0,0,-2,0,0,0,-2,\dots,-2,19)$

- If $j = 2$, then $(-k^2)w^TQ^{-1}w = 9797$
- If $j = -6$, then $(-k^2)w^TQ^{-1}w = 278381$

$\vdots$

- If $j = 12$, then $(-k^2)w^TQ^{-1}w = 258165$
- If $j = 0$, then $(-k^2)w^TQ^{-1}w = 1004578$

Table 4.16: $k = 23$

i	$-23+2i$	j	$[v]-[w(i)] (\text{mod } k^2)$	$m'_Q(i)$	w maximizing $(-k^2)w^TQ^{-1}w$
0	-23	-8	$189 \equiv -340$	9	$(0,\dots,0,-2,0,-2,-2,-2,-2,-2,-2,-2,-2,0,-23)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
8	-7	0	$-11 \equiv -540$	19	$(0,0,0,0,-2,-2,-2,\dots,-2,0,-7)$
10	-3	2	-61	2	$(0,\dots,0,0,0,0,-2,0,0,0,0,-2,-3)$
12	1	4	-111	3	$(0,0,\dots,0,0,-2,0,-2,-2,1)$
14	5	6	-161	5	$(0,0,-2,0,\dots,0,-2,-2,-2,-2,5)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
22	21	14	-361	10	$(0,0,0,0,0,0,0,0,0,0,0,0,-2,0,0,-2,\dots,-2,21)$

- If $j = 1$, then $(-k^2)w^TQ^{-1}w = 3425$

$\vdots$

- If $j = 9$, then $(-k^2)w^TQ^{-1}w = 93425$
- If $j = -5$, then $(-k^2)w^TQ^{-1}w = 77225$

$\vdots$

- If $j = -1$, then $(-k^2)w^TQ^{-1}w = 208625$

Table 4.17: $k = 27$

i	$-27+2i$	j	$[v]-[w(i)](k^2)$	$m'_Q(i)$	w maximizing $(-k^2)w^TQ^{-1}w$
1	-25	-8	$219 \equiv -510$	12	$(0,0,0,0,0,0,0,0,0,0,0,0,-2,0,0,0,-2,\dots,-2,0,-25)$
3	-21	-6	$161 \equiv -568$	14	$(0,0,0,0,0,0,0,0,0,0,-2,0,0,0,-2,\dots,-2,0,-21)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
9	-9	0	$-13 \equiv 742$	24	$(0,0,-2,0,-2,-2,\dots,-2,-2,0,-9)$
11	-5	2	-71	2	$(0,0,0,\dots,0,0,-2,0,0,0,0,0,-2,-5)$
13	-1	4	-129	3	$(0,0,\dots,0,-2,0,0,-2,-2,-1)$
15	3	6	-187	5	$(0,-2,0,0,\dots,0,0,-2,-2,-2,-2,3)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
27	27	18	-535	13	$(0,0,0,0,0,0,0,0,0,0,0,-2,0,0,0,0,-2,\dots,-2,27)$

- If $j = 2$, then $(-k^2)w^TQ^{-1}w = 25565$

$\vdots$

- If $j = 18$, then $(-k^2)w^TQ^{-1}w = 1495229$

- If $j = -8$, then $(-k^2)w^TQ^{-1}w = 1559381$

$\vdots$

- If $j = 0$, then $(-k^2)w^TQ^{-1}w = 5455157$

Table 4.18: $k = 31$

i	$-27+2i$	j	$[v]-[w(i)](k^2)$	$m'_Q(i)$	w maximizing $(-k^2)w^TQ^{-1}w$
0	−31	−10	$315 \equiv -646$	12	$(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-2,0,0,-2,\ldots,-2,0,-31)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
10	−11	0	$-15 \equiv 976$	2	$(0,-2,0,0,-2,-2,\ldots,-2,0,-11)$
12	−7	2	−81	3	$(0,0,\ldots,0,0,-2,0,0,0,0,0,0,-2,-7)$
14	−3	4	−147	5	$(0,0,\ldots,0,0,-2,0,0,0,-2,-2,-3)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
28	25	18	−609	12	$(0,0,-2,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-2,\ldots,-2,25)$
30	29	20	−675	13	$(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-2,0,0,0,-2,\ldots,-2,29)$

- If $j = 2$, then $(-k^2)w^TQ^{-1}w = 37529$

$\vdots$

- If $j = 20$, then $(-k^2)w^TQ^{-1}w = 2559193$

- If $j = -10$, then $(-k^2)w^TQ^{-1}w = 2659137$

$\vdots$

- If $j = 0$, then $(-k^2)w^TQ^{-1}w = 10693097$

Chapter 5

The Remaining Cases

In this section we explain why we are not able to solve the problem in the remaining cases.

5.1 The Case $P(2m, k, -k+2)$, $\det(G(K)) < 0$

These knots have signature equal to -2. If we were to use Greene's strengthening of Donaldson's Theorem, we must consider knots $K = P(-2m, -k, k-2)$. Unfortunately, the double branched cover of these knots do not bound a negative definite manifold.

We could use the method described in Chapter 4, however $\Sigma(K)$ is not always an L-Space.

5.2 The Case $P(2m, k, -k+2)$, $\det(G(K)) > 0$

These knots have signature equal to 0. Therefore Greene's strengthening of Donaldson's Theorem would only give us a statement about when the knot can be unknotted by a positive crossing.

We could use the method described in Chapter 4, however $\Sigma(K)$ is never an L-Space. Furthermore, some knots have determinant one, and therefore there would be no symmetry obstruction. An example of this is $P(8,5,-3)$.

5.3 The Case $P(2m, k, -k+4)$, $\det(G(K)) > 0$

These knots have signature equal to -2. Because of this, the difficulties in determining unknotting number one for these knots are the same as those in Section 5.1

Bibliography

[1] Steven A. Bleiler. A note on unknotting number. *Math. Proc. Cambridge Philos. Soc.*, 96(3):469–471, 1984.

[2] Abhijit Champanerkar and Ilya Kofman. Twisting quasi-alternating links. *Proc. Amer. Math. Soc.*, 137(7):2451–2458, 2009.

[3] T. D. Cochran and W. B. R. Lickorish. Unknotting information from 4-manifolds. *Trans. Amer. Math. Soc.*, 297(1):125–142, 1986.

[4] S. K. Donaldson. The orientation of Yang-Mills moduli spaces and 4-manifold topology. *J. Differential Geom.*, 26(3):397–428, 1987.

[5] C. McA. Gordon and R. A. Litherland. On the signature of a link. *Invent. Math.*, 47(1):53–69, 1978.

[6] C. McA. Gordon and John Luecke. Knots with unknotting number 1 and essential Conway spheres. *Algebr. Geom. Topol.*, 6:2051–2116 (electronic), 2006.

[7] Joshua Greene. On closed 3-braids with unknotting number one. `arXiv:0902.1573v1`, 2009.

[8] Joshua Greene and Stanislav Jabuka. The slice-ribbon conjecture for 3-stranded pretzel knots. *Amer. J. Math.*, 133(3):555–580, 2011.

[9] Taizo Kanenobu and Hitoshi Murakami. Two-bridge knots with unknotting number one. *Proc. Amer. Math. Soc.*, 98(3):499–502, 1986.

[10] Tsuyoshi Kobayashi. Minimal genus Seifert surfaces for unknotting number 1 knots. *Kobe J. Math.*, 6(1):53–62, 1989.

[11] Peter Kohn. Two-bridge links with unlinking number one. *Proc. Amer. Math. Soc.*, 113(4):1135–1147, 1991.

[12] W. B. Raymond Lickorish. The unknotting number of a classical knot. In *Combinatorial methods in topology and algebraic geometry (Rochester, N.Y., 1982)*, volume 44 of *Contemp. Math.*, pages 117–121. Amer. Math. Soc., Providence, RI, 1985.

[13] Katura Miyazaki and Kimihiko Motegi. Seifert fibred manifolds and Dehn surgery. *Topology*, 36(2):579–603, 1997.

[14] José M. Montesinos. Surgery on links and double branched covers of S^3. In *Knots, groups, and 3-manifolds (Papers dedicated to the memory of R. H. Fox)*, pages 227–259. Ann. of Math. Studies, No. 84. Princeton Univ. Press, Princeton, N.J., 1975.

[15] K. Motegi. A note on unlinking numbers of Montesinos links. *Rev. Mat. Univ. Complut. Madrid*, 9(1):151–164, 1996.

[16] Kunio Murasugi. On a certain numerical invariant of link types. *Trans. Amer. Math. Soc.*, 117:387–422, 1965.

[17] Yasutaka Nakanishi. A note on unknotting number. *Math. Sem. Notes Kobe Univ.*, 9(1):99–108, 1981.

[18] Yasutaka Nakanishi. Unknotting numbers and knot diagrams with the minimum crossings. *Math. Sem. Notes Kobe Univ.*, 11(2):257–258, 1983.

[19] Walter D. Neumann and Frank Raymond. Seifert manifolds, plumbing, μ-invariant and orientation reversing maps. In *Algebraic and geometric topology (Proc. Sympos., Univ. California, Santa Barbara, Calif., 1977)*, volume 664 of *Lecture Notes in Math.*, pages 163–196. Springer, Berlin, 1978.

[20] Peter Ozsváth and Zoltán Szabó. Absolutely graded Floer homologies and intersection forms for four-manifolds with boundary. *Adv. Math.*, 173(2):179–261, 2003.

[21] Peter Ozsváth and Zoltán Szabó. On the Floer homology of plumbed three-manifolds. *Geom. Topol.*, 7:185–224 (electronic), 2003.

[22] Peter Ozsváth and Zoltán Szabó. Knots with unknotting number one and Heegaard Floer homology. *Topology*, 44(4):705–745, 2005.

[23] Jacob Rasmussen. Khovanov homology and the slice genus. *Invent. Math.*, 182(2):419–447, 2010.

[24] Martin Scharlemann and Abigail Thompson. Link genus and the Conway moves. *Comment. Math. Helv.*, 64(4):527–535, 1989.

[25] Martin G. Scharlemann. Unknotting number one knots are prime. *Invent. Math.*, 82(1):37–55, 1985.

[26] A. Stoimenow. Some examples related to 4-genera, unknotting numbers and knot polynomials. *J. London Math. Soc. (2)*, 63(2):487–500, 2001.

[27] Morwen B. Thistlethwaite. On the algebraic part of an alternating link. *Pacific J. Math.*, 151(2):317–333, 1991.

[28] Ichiro Torisu. A note on Montesinos links with unlinking number one (conjectures and partial solutions). *Kobe J. Math.*, 13(2):167–175, 1996.

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