Appendix for
RedAlert: Determinacy Inference for Prolog

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Abstract

Proofs of the mathematical foundations and propositions and theorems stated and used in (Kriener and King, 2011).

1 Introduction

In (Kriener and King, 2011) we present a determinacy inference for Prolog including *cut*. This inference is developed in as a static analysis, comprising of a number of concrete semantics and abstractions thereof. Three theorems and a number of propositions formalise the connections that hold between these. We present here formal justification for the mathematical backdrop used in (Kriener and King, 2011), as well as proofs of the propositions and theorems stated there.

2 Appendix - Proofs

2.1 $Con^{\downarrow}_{seq}$ is a complete lattice

2.1.1 Relation on $Con^{\downarrow}_{seq}$ is a partial order

The relation is reflexive: $\vec{\Theta} \sqsubseteq \vec{\Theta}$

$Observe\ that: \ \forall \vec{\Theta} \in Con^{\downarrow}_{seq}(\vec{\Theta} \subseteq_{pw} \vec{\Theta} \wedge \vec{\Theta} \in Sub_{|\vec{\Theta}|})$

$hence\ \forall \vec{\Theta} \in Con^{\downarrow}_{seq}(\vec{\Theta} \sqsubseteq \vec{\Theta})$

$by\ selecting\ \Phi = \Theta$

The relation is transitive: $\vec{\Theta_1} \sqsubseteq \vec{\Theta_2} \wedge \vec{\Theta_2} \sqsubseteq \vec{\Theta_3} \rightarrow \vec{\Theta_1} \sqsubseteq \vec{\Theta_3}$

$\forall \vec{\Theta}_1, \vec{\Theta}_2, \vec{\Theta}_3 \in Con^{\downarrow}_{seq}((\vec{\Theta}_1 \sqsubseteq \vec{\Theta}_2 \wedge \vec{\Theta}_2 \sqsubseteq \vec{\Theta}_3) \rightarrow (\vec{\Theta}_1 \sqsubseteq \vec{\Theta}_3))$

$let\ |\vec{\Theta}_1| = l,\ |\vec{\Theta}_2| = m,\ |\vec{\Theta}_3| = n,$

$l \leq m \leq n$

$(\vec{\Theta}_1 \sqsubseteq \vec{\Theta}_2) \rightarrow \exists \vec{\Phi}_1 \in Sub_l(\vec{\Theta}_2).(\vec{\Theta}_1 \subseteq_{pw} \vec{\Phi}_1)$

$(\vec{\Theta}_2 \sqsubseteq \vec{\Theta}_3) \rightarrow \exists \vec{\Phi}_2 \in Sub_m(\vec{\Theta}_3).(\vec{\Theta}_2 \subseteq_{pw} \vec{\Phi}_2)$

$since\ \vec{\Theta}_2 \subseteq_{pw} \vec{\Phi}_2\ and\ \exists \vec{\Phi}_1 \in Sub_l(\vec{\Theta}_2).(\vec{\Theta}_1 \subseteq_{pw} \vec{\Phi}_1):\ \exists \vec{\Phi}_3 \in Sub_l(\vec{\Phi}_2).(\vec{\Theta}_1 \subseteq_{pw} \vec{\Phi}_3)$

$Sub_l(\vec{\Phi}_2) \subseteq Sub_l(\vec{\Theta}_3)$

$hence\ \exists \vec{\Phi}_3 \in Sub_l(\vec{\Theta}_3).(\vec{\Theta}_1 \subseteq_{pw} \vec{\Phi}_3)$

$therefore\ \vec{\Theta}_1 \sqsubseteq \vec{\Theta}_3$

The relation is anti-symmetric: $\forall \vec{\Theta}_1, \vec{\Theta}_2 \in Con^{\downarrow}_{seq}(\vec{\Theta}_1 \sqsubseteq \vec{\Theta}_2 \wedge \vec{\Theta}_2 \sqsubseteq \vec{\Theta}_1 \rightarrow \vec{\Theta}_1 = \vec{\Theta}_2)$

$let\ |\vec{\Theta}_1| = m,\ |\vec{\Theta}_2| = n$

$(\vec{\Theta}_1 \sqsubseteq \vec{\Theta}_2) \rightarrow \exists \vec{\Phi}_1 \in Sub_m(\vec{\Theta}_2)\ such\ that\ \vec{\Theta}_1 \subseteq_{pw} \vec{\Phi}_1$

$(\vec{\Theta}_2 \sqsubseteq \vec{\Theta}_1) \rightarrow \exists \vec{\Phi}_2 \in Sub_n(\vec{\Theta}_1)\ such\ that\ \vec{\Theta}_2 \subseteq_{pw} \vec{\Phi}_2$

$|\vec{\Phi}_1| = m\ and\ |\vec{\Phi}_1| \leq n\ hence\ m \leq n$

$|\vec{\Phi}_2| = n\ and\ |\vec{\Phi}_2| \leq m\ hence\ n \leq m$

$hence\ m = n\ (by\ anti-symmetry\ of\ \ \leq)$

$hence\ \vec{\Phi}_1 = \vec{\Theta}_2\ and\ \vec{\Phi}_2 = \vec{\Theta}_1$

$hence\ \vec{\Theta}_1 \subseteq_{pw} \vec{\Theta}_2\ and\ \vec{\Theta}_2 \subseteq_{pw} \vec{\Theta}_1$

$therefore:$

$\vec{\Theta}_1 = \vec{\Theta}_2\ (by\ anti-symmetry\ of\ \ \subseteq_{pw})$

2.1.2 The meet of two sequences is unique and therefore well defined:

First note that by the definition of $\sqcap$, $\vec{\Theta} \sqcap \vec{\Psi} \sqsubseteq \vec{\Theta}$ and $\vec{\Theta} \sqcap \vec{\Psi} \sqsubseteq \vec{\Psi}$.
Then show: $\forall \vec{\Theta}, \vec{\Psi}, \vec{\Gamma} \in Con^{\downarrow}_{seq} : \vec{\Gamma} \sqsubseteq \vec{\Theta} \wedge \vec{\Gamma} \sqsubseteq \vec{\Psi} \rightarrow \vec{\Gamma} \sqsubseteq (\vec{\Theta} \sqcap \vec{\Psi})$

$|\vec{\Theta}| = n,\ |\vec{\Psi}| = m,\ |\vec{\Gamma}| = k$
$\vec{\Gamma} \sqsubseteq \vec{\Theta} \rightarrow \exists \vec{\Theta}_1\ \in Sub_k(\vec{\Theta}).(\vec{\Gamma} \subseteq_{pw} \vec{\Theta}_1)$
$\vec{\Gamma} \sqsubseteq \vec{\Psi} \rightarrow \exists \vec{\Psi}_1\ \in Sub_k(\vec{\Psi}).(\vec{\Gamma} \subseteq_{pw} \vec{\Psi}_1)$
$|\vec{\Theta}_1| = k,\ |\vec{\Psi}_1| = k$
assume (without loss of generality): $n \geq m$, then: $|\vec{\Theta} \sqcap \vec{\Psi}| = l,\ l \leq m$
since $\vec{\Gamma} \sqsubseteq \vec{\Theta}$ and $\vec{\Gamma} \sqsubseteq \vec{\Psi}$, $k \leq m$ $(and\ k \leq n)$
since $\vec{\Gamma} \subseteq_{pw} \vec{\Theta}_1$ *and* $\vec{\Gamma} \subseteq_{pw} \vec{\Psi}_1$, $\vec{\Gamma} \subseteq_{pw} (\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1)$
hence $\vec{\Gamma} \sqsubseteq (\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1)$
$(\vec{\Psi}_1 \in Sub_k(\vec{\Psi})) \rightarrow (\vec{\Psi}_1 \sqsubseteq \vec{\Psi})$
$(\vec{\Theta}_1 \in Sub_k(\vec{\Theta})) \rightarrow (\vec{\Theta}_1 \sqsubseteq \vec{\Theta})$
$(\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1) \in \{\vec{X} \cap_{pw} \vec{\Psi}_1 \mid \vec{X} \in Sub_k(\vec{\Theta})\}$ $(since\ \vec{\Theta}_1 \in Sub_k(\vec{\Theta}))$
$(\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1) \subseteq_{pw} \bigcup_{pw} \{\vec{X} \cap_{pw} \vec{\Psi}_1 \mid \vec{X} \in Sub_k(\vec{\Theta})\}$
$(note\ that\ since\ \vec{\Gamma} \in Con^{\downarrow}_{seq},\ \vec{\Gamma}\ does\ not\ contain\ \{false\}$
$and\ since\ \vec{\Gamma} \subseteq_{pw} (\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1),\ \vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1\ does\ not\ contain\ \{false\}$
$hence\ (\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1) = trim(\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1))$
$(\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1) \sqsubseteq (\vec{\Theta} \sqcap \vec{\Psi}_1)$
$(\vec{\Theta} \sqcap \vec{\Psi}_1) \sqsubseteq (\vec{\Theta} \sqcap \vec{\Psi})$ $(since\ \vec{\Psi}_1 \sqsubseteq \vec{\Psi}\ and\ \sqcap\ is\ monotonic)$
$therefore\ \vec{\Gamma} \sqsubseteq (\vec{\Theta} \sqcap \vec{\Psi})$

2.2 Cut-normal form

We transform Prolog predicates that are defined by any number of clauses, none of which contains a disjunction, into this form by constructing G_1, G_2, G_3 and G_4 as follows:

G_1:	If no clause precedes the clause containing the first *cut*, set G_1 to post(*false*). Else, if a single clause precedes the clause containing the first *cut*, set G_1 to the body of this clause. Otherwise, define an auxiliary predicate to wrap up all clauses preceding the clause containing the first *cut* and set G_1 to a call to that predicate.
G_2:	If there is no *cut* in the predicate, set G_2 to post(*false*). Else, if no atom precedes the first *cut*, set G_2 to post(*true*). Otherwise, set G_2 to the compound goal before the first *cut*.

G_3:	If there is no *cut* in the predicate, set G_3 to any goal, e.g. $\mathsf{post}(true)$. Else, if no goal follows the first *cut*, set G_3 to $\mathsf{post}(true)$. Else, if the compound goal following the first *cut* does not contain another *cut*, set G_3 to that goal. Otherwise, define an auxiliary predicate to wrap up the compound goal following the first *cut* and set G_3 to a call to that predicate.
G_4:	If no clause follows the clause containing the first *cut*, set G_4 to $\mathsf{post}(false)$. Else, if a single, *cut*-free clause follows the clause containing the first *cut*, set G_4 to the body of this clause. Otherwise, define an auxiliary predicate to wrap up all clauses following the clause containing the first *cut* and set G_4 to a call to that predicate.

2.3 Theorem 1: $\bigcup(\mathcal{F}_G[\![G]\!]\mu_P\vec{\Theta}) \subseteq \bigcup(\vec{\Theta}) \cap S_G[\![G]\!]$

Notice first that the following things hold:
$\bigcup(\vec{\Psi}) \subseteq \bigcup(trim(\vec{\Psi}))$
$\Downarrow(\Theta \cup \Phi) = \Downarrow\Theta \cup \Downarrow\Phi$
$\Downarrow(\Theta \cap \Phi) = \Downarrow\Theta \cap \Downarrow\Phi$
$\overline{\exists}_{\vec{y}}(\Theta \cup \Phi) = \overline{\exists}_{\vec{y}}(\Theta) \cup \overline{\exists}_{\vec{y}}(\Phi)$
$\overline{\exists}_{\vec{y}}(\Theta \cap \Phi) \subseteq \overline{\exists}_{\vec{y}}(\Theta) \cap \overline{\exists}_{\vec{y}}(\Phi)$
$\rho_{\vec{x},\vec{y}}(\Theta \cup \Phi) = \rho_{\vec{x},\vec{y}}\Theta \cup \rho_{\vec{x},\vec{y}}\Phi$
$\rho_{\vec{x},\vec{y}}(\Theta \cap \Phi) \subseteq \rho_{\vec{x},\vec{y}}\Theta \cap \rho_{\vec{x},\vec{y}}\Phi$
$\overline{\exists}_{\vec{y}}(\Downarrow\overline{\exists}_{\vec{y}}(\Theta)) = \overline{\exists}_{\vec{y}}(\Theta)$

Proof by induction on length of $\vec{\Theta}$:
Base Case: $\vec{\Theta} = []$
$\bigcup(\mathcal{F}_G[\![G]\!]\mu_P[]) = \bigcup([]) = \emptyset$
$\bigcup([]) \cap S_G[\![G]\!] = \emptyset \cap S_G[\![G]\!] = \emptyset$
$\emptyset \subseteq \emptyset$
therefore: $\bigcup(\mathcal{F}_G[\![G]\!]\mu_P[]) \subseteq \bigcup([]) \cap S_G[\![G]\!]$

Induction Step:
Assume: $\bigcup(\mathcal{F}_G[\![G]\!]\mu_P\vec{\Theta}) \subseteq \bigcup(\vec{\Theta}) \cap S_G[\![G]\!]$
Show: $\bigcup(\mathcal{F}_G[\![G]\!]\mu_P(\Theta : \vec{\Theta})) \subseteq \bigcup(\Theta : \vec{\Theta}) \cap S_G[\![G]\!]$

Induction on structure of G:
Two base cases: (1) $G = \mathsf{post}(\phi)$, (2) $G = p(\vec{x})$

(1) $G = \mathsf{post}(\phi)$
Assume: $\bigcup(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_P\vec{\Theta}) \subseteq \bigcup(\vec{\Theta}) \cap S_G[\![\mathsf{post}(\phi)]\!]$
Show: $\bigcup(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_P(\Theta : \vec{\Theta})) \subseteq \bigcup(\Theta : \vec{\Theta}) \cap S_G[\![\mathsf{post}(\phi)]\!]$
$\bigcup(\Theta : \vec{\Theta}) \cap S_G[\![\mathsf{post})(\phi)]\!]$
$= (\Theta \cap S_G[\![\mathsf{post}(\phi)]\!]) \cup (\bigcup(\vec{\Theta}) \cap S_G[\![\mathsf{post}(\phi)]\!])$
$= (\Theta \cap \Downarrow\{\phi\}) \cup (\bigcup(\vec{\Theta}) \cap S_G[\![\mathsf{post}(\phi)]\!])$

$\bigcup(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_P(\Theta:\vec{\Theta}))$
$= \bigcup(trim(\downarrow\!\!\{\phi\} \cap \Theta : \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_P\vec{\Theta}))$
$\subseteq \bigcup(\downarrow\!\!\{\phi\} \cap \Theta : \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_P\vec{\Theta})$
$= (\downarrow\!\!\{\phi\} \cap \Theta) \cup \bigcup(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_P\vec{\Theta})$
$\subseteq (\downarrow\!\!\{\phi\} \cap \Theta) \cup (\bigcup(\vec{\Theta}) \cap S_G[\![\mathsf{post}(\phi)]\!])$
therefore: $\bigcup(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_P(\Theta:\vec{\Theta})) \subseteq \bigcup(\Theta:\vec{\Theta}) \cap S_G[\![\mathsf{post}(\phi)]\!]$

(2) $G = p(\vec{x})$
Assume (without loss of generality): $p(\vec{y}) \leftarrow G_1; G_2, !, G_3; G_4 \in P$
Assume: $\bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_P\vec{\Theta}) \subseteq \bigcup(\vec{\Theta}) \cap S_G[\![p(\vec{x})]\!]$
Show: $\bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_P(\Theta:\vec{\Theta})) \subseteq \bigcup(\Theta:\vec{\Theta}) \cap S_G[\![p(\vec{x})]\!]$

$\bigcup(\Theta:\vec{\Theta}) \cap S_G[\![p(\vec{x})]\!]$
$= (\Theta \cap S_G[\![p(\vec{x})]\!]) \cup (\bigcup(\vec{\Theta}) \cap S_G[\![p(\vec{x})]\!])$
$= (\Theta \cap \downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_H[\![p(\vec{y})]\!])) \cup (\bigcup(\vec{\Theta}) \cap S_G[\![p(\vec{x})]\!])$
$= (\Theta \cap \downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\downarrow\!\overline{\exists}_{\vec{y}}(S_G[\![G_1]\!] \cup S_G[\![G_2, G_3]\!] \cup S_G[\![G_4]\!]))) \cup (\bigcup(\vec{\Theta}) \cap S_G[\![p(\vec{x})]\!])$
$= (\Theta \cap \downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_1]\!] \cup S_G[\![G_2, G_3]\!] \cup S_G[\![G_4]\!])) \cup (\bigcup(\vec{\Theta}) \cap S_G[\![p(\vec{x})]\!])$
$= (\Theta \cap (\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_1]\!]) \cup \downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_2, G_3]\!]) \cup \downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_4]\!]))$
$\quad \cup (\bigcup(\vec{\Theta}) \cap S_G[\![p(\vec{x})]\!])$

$\bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_P(\Theta:\vec{\Theta}))$
$= \bigcup(\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mu\ (p(\vec{y}))\ \downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])) \cap \Theta : \mathcal{F}_G[\![p(\vec{x})]\!]\mu\vec{\Theta})$
$= (\bigcup(\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mu\ (p(\vec{y}))\ \downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]))) \cap \Theta) \cup \bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu\vec{\Theta})$
$= (\bigcup(\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\downarrow\!\overline{\exists}_{y}(\mathcal{F}_G[\![G_1]\!]\mu[\Theta'] : \vec{\Psi}))) \cap \Theta) \cup \bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu\vec{\Theta})$
$$\textit{where } \vec{\Psi} = \begin{cases} \mathcal{F}_G[\![G_3]\!]\mu[\Phi] & \textit{if } \mathcal{F}_G[\![G_2]\!]\mu[\Theta'] = \Phi : \vec{\Phi} \\ \mathcal{F}_G[\![G_4]\!]\mu[\Theta'] & \textit{if } \mathcal{F}_G[\![G_2]\!]\mu[\Theta'] = [] \end{cases}$$
and $\Theta' = \downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta)$

To be on the safe side, consider the sequence resulting from appending *both* possibilities for $\vec{\Psi}$, the union of which is certainly a superset of the above:
$\subseteq (\bigcup(\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\downarrow\!\overline{\exists}_{y}(\mathcal{F}_G[\![G_1]\!]\mu[\Theta'] : \mathcal{F}_G[\![G_3]\!]\mu[\Phi] : \mathcal{F}_G[\![G_4]\!]\mu[\Theta']))) \cap \Theta) \cup \bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu\vec{\Theta})$
where $\Phi : \vec{\Phi} = \mathcal{F}_G[\![G_2]\!]\mu[\Theta']$

Again, changing this to include all, rather than only the first, possibilities for $\mathcal{F}_G[\![G_2]\!]\mu[\Theta']$ will result in a safe over-approximation, i.e. a superset of the above: $\subseteq (\bigcup(\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\downarrow\!\overline{\exists}_{y}(\mathcal{F}_G[\![G_1]\!]\mu[\Theta'] :$ $\mathcal{F}_G[\![G_3]\!]\mu(\mathcal{F}_G[\![G_2]\!]\mu[\Theta']) : \mathcal{F}_G[\![G_4]\!]\mu[\Theta']))) \cap \Theta)$
$\quad \cup \bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu\vec{\Theta})$
$= (\bigcup(\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu[\Theta'] : \mathcal{F}_G[\![G_3]\!]\mu(\mathcal{F}_G[\![G_2]\!]\mu[\Theta']) : \mathcal{F}_G[\![G_4]\!]\mu[\Theta'])) \cap \Theta) \cup \bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu\vec{\Theta})$
$= (\bigcup(\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu[\Theta']) : \downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_3]\!]\mu(\mathcal{F}_G[\![G_2]\!]\mu[\Theta'])) : \downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_4]\!]\mu[\Theta']))$
$\quad \cap \Theta) \cup \bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu\vec{\Theta})$
$= ((\bigcup(\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu[\Theta'])) \cup \bigcup(\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_3]\!]\mu(\mathcal{F}_G[\![G_2]\!]\mu[\Theta']))) \cup \bigcup(\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_4]\!]\mu[\Theta'])))$
$\quad \cap \Theta) \cup \bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu\vec{\Theta})$
$= (\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\bigcup(\mathcal{F}_G[\![G_1]\!]\mu[\Theta'])) \cup \downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\bigcup(\mathcal{F}_G[\![G_3]\!]\mu(\mathcal{F}_G[\![G_2]\!]\mu[\Theta']))) \cup \downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\bigcup(\mathcal{F}_G[\![G_4]\!]\mu[\Theta']))$
$\quad \cap \Theta) \cup \bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu\vec{\Theta})$
since: $\bigcup(\mathcal{F}_G[\![G_1]\!]\mu[\Theta']) \subseteq S_G[\![G_1]\!] \cap \Theta'$
and: $\bigcup(\mathcal{F}_G[\![G_2]\!]\mu[\Theta']) \subseteq S_G[\![G_2]\!] \cap \Theta'$
hence: $\bigcup(\mathcal{F}_G[\![G_3]\!]\mu(\mathcal{F}_G[\![G_2]\!]\mu[\Theta'])) \subseteq S_G[\![G_3]\!] \cap (S_G[\![G_2]\!] \cap \Theta')$
hence: $\bigcup(\mathcal{F}_G[\![G_3]\!]\mu(\mathcal{F}_G[\![G_2]\!]\mu[\Theta'])) \subseteq (S_G[\![G_3]\!] \cap S_G[\![G_2]\!]) \cap \Theta'$

hence: $\bigcup(\mathcal{F}_G[\![G_3]\!]\mu(\mathcal{F}_G[\![G_2]\!]\mu[\Theta'])) \subseteq S_G[\![G_2, G_3]\!] \cap \Theta'$
and: $\bigcup(\mathcal{F}_G[\![G_4]\!]\mu[\Theta']) \subseteq S_G[\![G_4]\!] \cap \Theta'$
using these, therefore, the above superset of $\bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_P(\Theta : \vec{\Theta}))$ is a subset of:
$\subseteq ((\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_1]\!] \cap \Theta') \cup \downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_2, G_3]\!] \cap \Theta') \cup \downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_4]\!] \cap \Theta')) \cap \Theta)$
$\quad \cup \bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu\vec{\Theta})$
since: $\Theta' = \downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta)$, the following holds: $\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\Theta') = \downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta))$
$\quad = \downarrow\!\overline{\exists}_{\vec{x}}(\Theta) \supseteq \Theta$
intersecting this with Θ therefore gives Θ itself: $\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\Theta') \cap \Theta = \Theta$ distributing the projections and collecting and intersection the occurrences of $\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\Theta')$ and Θ above therefore gives:
$\subseteq ((\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_1]\!]) \cup \downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_2, G_3]\!]) \cup \downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_4]\!])) \cap \Theta) \cup \bigcup(\mathcal{F}_G[\![p(\vec{x})]\!]\mu\vec{\Theta})$
$\subseteq ((\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_1]\!]) \cup \downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_2, G_3]\!]) \cup \downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_4]\!])) \cap \Theta)$
$\quad \cup (\bigcup(\vec{\Theta}) \cap S_G[\![p(\vec{x})]\!])$
$= \bigcup(\Theta : \vec{\Theta}) \cap S_G[\![p(\vec{x})]\!]$

Induction Step: $G = G_1, G_2$
Assume: $\bigcup(\mathcal{F}_G[\![G_1, G_2]\!]\mu_P\vec{\Theta}) \subseteq \bigcup(\vec{\Theta}) \cap S_G[\![G_1, G_2]\!]$
And: $\bigcup(\mathcal{F}_G[\![G_1]\!]\mu_P\vec{\Phi}) \subseteq \bigcup(\vec{\Phi}) \cap S_G[\![G_1]\!]$
And: $\bigcup(\mathcal{F}_G[\![G_2]\!]\mu_P\vec{\Phi}) \subseteq \bigcup(\vec{\Phi}) \cap S_G[\![G_2]\!]$
Show: $\bigcup(\mathcal{F}_G[\![G_1, G_2]\!]\mu_P(\Theta : \vec{\Theta})) \subseteq \bigcup(\Theta : \vec{\Theta}) \cap S_G[\![G_1, G_2]\!]$

$\quad \bigcup(\Theta : \vec{\Theta}) \cap S_G[\![G_1, G_2]\!]$
$= (\Theta \cap S_G[\![G_1, G_2]\!]) \cup (\bigcup(\vec{\Theta}) \cap S_G[\![G_1, G_2]\!])$
$= (\Theta \cap S_G[\![G_1]\!] \cap S_G[\![G_2]\!]) \cup (\bigcup(\vec{\Theta}) \cap S_G[\![G_1, G_2]\!])$

$\quad \bigcup(\mathcal{F}_G[\![G_1, G_2]\!]\mu_P(\Theta : \vec{\Theta}))$
$= \bigcup(\mathcal{F}_G[\![G_2]\!]\mu(\mathcal{F}_G[\![G_1]\!]\mu(\Theta : \vec{\Theta})))$
$\subseteq \bigcup(\mathcal{F}_G[\![G_1]\!]\mu(\Theta : \vec{\Theta}) \cap S_G[\![G_2]\!]$
$\subseteq \bigcup(\Theta : \vec{\Theta}) \cap S_G[\![G_1]\!] \cap S_G[\![G_2]\!]$
$= \bigcup(\Theta : \vec{\Theta}) \cap S_G[\![G_1, G_2]\!]$
QED

2.4 Theorem 2: For $\Theta \in Con^{\downarrow}$ and stratified $P = P_0 \cup \ldots \cup P_n$: $\Theta \subseteq \mathcal{D}_G[\![G]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G]\!]\mu_P[\Theta]| \leq 1$.

2.4.1 Lemma 1: $(\mathcal{F}_G[\![G]\!]\mu\vec{\Theta}) \cap \Psi = \mathcal{F}_G[\![G]\!]\mu(\vec{\Theta} \cap \Psi)$

Proof by nested induction on:
1. μ,
2. $|\vec{\Theta}|$,
3. structure of G

1 Base Case: $\mu = \mu_\perp$
Show: $(\mathcal{F}_G[\![G]\!]\mu_\perp\vec{\Theta}) \cap \Psi = \mathcal{F}_G[\![G]\!]\mu_\perp(\vec{\Theta} \cap \Psi)$

1.1 Base Case: $\vec{\Theta} = []$
Show: $(\mathcal{F}_G[\![G]\!]\mu_\perp[]) \cap \Psi = \mathcal{F}_G[\![G]\!]\mu_\perp([] \cap \Psi)$
$([]) \cap \Psi = \mathcal{F}_G[\![G]\!]\mu_\perp([])$
$[] = []$

1.2 Induction Step: $(\Theta : \vec{\Theta})$
Assume: $(\mathcal{F}_G[\![H]\!]\mu_\perp\vec{\Theta}) \cap \Psi = \mathcal{F}_G[\![H]\!]\mu_\perp(\vec{\Theta} \cap \Psi)$
Show: $(\mathcal{F}_G[\![G]\!]\mu_\perp(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![G]\!]\mu_\perp((\Theta : \vec{\Theta}) \cap \Psi)$

1.2.1 Two Base Cases: (1) $G = \mathsf{post}(\phi)$, (2) $G = p(\vec{x})$

(1) $G = \mathsf{post}(\phi)$
Show: $(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp((\Theta : \vec{\Theta}) \cap \Psi)$

$(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp((\Theta : \vec{\Theta}) \cap \Psi)$
$\mathit{trim}(\downarrow\!\!\{\phi\} \cap \Theta : \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp\vec{\Theta}) \cap \Psi = \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp((\Theta \cap \Psi) : (\vec{\Theta} \cap \Psi))$
$(\mathit{trim}[\downarrow\!\!\{\phi\} \cap \Theta] : \mathit{trim}(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp\vec{\Theta})) \cap \Psi$
$\quad = \mathit{trim}(\downarrow\!\!\{\phi\} \cap \Theta \cap \Psi : \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp(\vec{\Theta} \cap \Psi))$
$(\mathit{trim}[\downarrow\!\!\{\phi\} \cap \Theta] \cap \Psi) : (\mathit{trim}(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp\vec{\Theta}) \cap \Psi)$
$\quad = \mathit{trim}[\downarrow\!\!\{\phi\} \cap \Theta \cap \Psi] : \mathit{trim}(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp(\vec{\Theta} \cap \Psi))$
by assumption:$(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp\vec{\Theta}) \cap \Psi = \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp(\vec{\Theta} \cap \Psi)$
$\mathit{trim}(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp\vec{\Theta}) \cap \Psi = \mathit{trim}(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp(\vec{\Theta} \cap \Psi))$
$\mathit{trim}(\downarrow\!\!\{\phi\} \cap \Theta) \cap \Psi = \mathit{trim}(\downarrow\!\!\{\phi\} \cap \Theta \cap \Psi)$

(2) $G = p(\vec{x})$
Show: $(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp((\Theta : \vec{\Theta}) \cap \Psi)$
$(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp(\Theta : \vec{\Theta})) \cap \Psi$
$= (\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mu_\perp(p(\vec{y}))\ \downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])) \cap \Theta : \mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp\vec{\Theta}) \cap \Psi$
$= (\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}([]) \cap \Theta : \mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp\vec{\Theta}) \cap \Psi$
$= ([]) : (\mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp\vec{\Theta}) \cap \Psi$
by assumption: $(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp\vec{\Theta}) \cap \Psi = \mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp(\vec{\Theta} \cap \Psi)$

$\mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp((\Theta : \vec{\Theta}) \cap \Psi)$
$= \mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp((\Theta \cap \Psi) : (\vec{\Theta} \cap \Psi))$
$= \downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mu_\perp(p(\vec{y}))\ \downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap \Psi])) \cap \Theta \cap \Psi : \mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp(\vec{\Theta} \cap \Psi)$
$= \downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}([]) \cap \Theta \cap \Psi : \mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp(\vec{\Theta} \cap \Psi)$
$= ([]) : \mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp(\vec{\Theta} \cap \Psi)$
hence: $(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp((\Theta : \vec{\Theta}) \cap \Psi)$

1.2.2 Induction Step: $G = G_1, G_2$
Assume: $(\mathcal{F}_G[\![G_1]\!]\mu_\perp(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![G_1]\!]\mu_\perp((\Theta : \vec{\Theta}) \cap \Psi)$
And: $(\mathcal{F}_G[\![G_2]\!]\mu_\perp(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![G_2]\!]\mu_\perp((\Theta : \vec{\Theta}) \cap \Psi)$
Show: $(\mathcal{F}_G[\![G_1, G_2]\!]\mu_\perp(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![G_1, G_2]\!]\mu_\perp((\Theta : \vec{\Theta}) \cap \Psi)$
$(\mathcal{F}_G[\![G_1, G_2]\!]\mu_\perp(\Theta : \vec{\Theta})) \cap \Psi$
$= (\mathcal{F}_G[\![G_2]\!]\mu_\perp(\mathcal{F}_G[\![G_1]\!]\mu_\perp(\Theta : \vec{\Theta}))) \cap \Psi$
$= (\mathcal{F}_G[\![G_2]\!]\mu_\perp(\mathcal{F}_G[\![G_1]\!]\mu_\perp(\Theta : \vec{\Theta})) \cap \Psi)$
$= \mathcal{F}_G[\![G_2]\!]\mu_\perp(\mathcal{F}_G[\![G_1]\!]\mu_\perp((\Theta : \vec{\Theta}) \cap \Psi))$
$= \mathcal{F}_G[\![G_1, G_2]\!]\mu_\perp((\Theta : \vec{\Theta}) \cap \Psi)$

2 Induction Step: $\mu = \mu_{k+1}$
Assume: $(\mathcal{F}_G[\![H]\!]\mu_k\vec{\Delta}) \cap \Lambda = \mathcal{F}_G[\![H]\!]\mu_k(\vec{\Delta} \cap \Lambda)$

Show: $(\mathcal{F}_G[\![G]\!]\mu_{k+1}\vec{\Theta}) \cap \Psi = \mathcal{F}_G[\![G]\!]\mu_{k+1}(\vec{\Theta} \cap \Psi)$
where $\mu_{k+1} = \mathcal{F}_P[\![P]\!]\mu_k$

2.1 Base Case: $\vec{\Theta} = []$
Show: $(\mathcal{F}_G[\![G]\!]\mu_{k+1}[]) \cap \Psi = \mathcal{F}_G[\![G]\!]\mu_{k+1}([] \cap \Psi)$
$([]) \cap \Psi = \mathcal{F}_G[\![G]\!]\mu_{k+1}([])$
$([]) = ([])$

2.2 Induction Step: $\vec{\Theta} = (\Theta : \vec{\Theta})$
Assume: $(\mathcal{F}_G[\![G]\!]\mu_{k+1}\vec{\Theta}) \cap \Psi = \mathcal{F}_G[\![G]\!]\mu_{k+1}(\vec{\Theta} \cap \Psi)$
Show: $(\mathcal{F}_G[\![G]\!]\mu_{k+1}(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![G]\!]\mu_{k+1}((\Theta : \vec{\Theta}) \cap \Psi)$

2.2.1 Two Base Cases: (1) $G = \mathsf{post}(\phi)$, (2) $G = p(\vec{x})$
(1) $G = \mathsf{post}(\phi)$
Show: $(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}((\Theta : \vec{\Theta}) \cap \Psi)$

$(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}((\Theta : \vec{\Theta}) \cap \Psi)$
$\mathit{trim}(\Downarrow\{\phi\} \cap \Theta : \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}\vec{\Theta}) \cap \Psi = \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}((\Theta \cap \Psi) : (\vec{\Theta} \cap \Psi))$
$(\mathit{trim}[\Downarrow\{\phi\} \cap \Theta] : \mathit{trim}(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}\vec{\Theta})) \cap \Psi$
$= \mathit{trim}(\Downarrow\{\phi\} \cap \Theta \cap \Psi : \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}(\vec{\Theta} \cap \Psi))$
$(\mathit{trim}(\Downarrow\{\phi\} \cap \Theta) \cap \Psi) : (\mathit{trim}(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}\vec{\Theta}) \cap \Psi)$
$= \mathit{trim}(\Downarrow\{\phi\} \cap \Theta \cap \Psi) : \mathit{trim}(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}(\vec{\Theta} \cap \Psi))$
by assumption: $\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}\vec{\Theta}) \cap \Psi = \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}(\vec{\Theta} \cap \Psi)$
hence: $\mathit{trim}(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}\vec{\Theta}) \cap \Psi = \mathit{trim}(\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}(\vec{\Theta} \cap \Psi))$
$\mathit{trim}(\Downarrow\{\phi\} \cap \Theta) \cap \Psi = \mathit{trim}(\Downarrow\{\phi\} \cap \Theta \cap \Psi)$

(2) $G = p(\vec{x})$
Assume (without loss of generality): $p(\vec{y}) \leftarrow G_1; G_2, !, G_3; G_4 \in P$
Show: $(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}((\Theta : \vec{\Theta}) \cap \Psi)$

$(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}(\Theta : \vec{\Theta})) \cap \Psi$
$= (\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y}))\ \downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])) \cap \Theta : \mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}\vec{\Theta}) \cap \Psi$
$= (\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y}))\ \downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])) \cap \Theta) \cap \Psi : (\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}\vec{\Theta}) \cap \Psi$

$\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}((\Theta : \vec{\Theta}) \cap \Psi)$
$= \mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}((\Theta \cap \Psi) : (\vec{\Theta} \cap \Psi))$
$= (\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y}))\ \downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap \Psi])) \cap \Theta \cap \Psi) : (\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}(\vec{\Theta} \cap \Psi))$
by assumption: $(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}\vec{\Theta}) \cap \Psi = (\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}(\vec{\Theta} \cap \Psi))$
hence the question is whether the following holds:
$(\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y}))\ \downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])) \cap \Theta) \cap \Psi$
$\quad = (\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y}))\ \downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap \Psi])) \cap \Theta \cap \Psi)$

$(\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y}))\ \downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap \Psi])) \cap \Theta) \cap \Psi$
$= \downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\downarrow\!\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu_k\ \downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap \Psi])) : \vec{\Delta})) \cap \Theta \cap \Psi$

$$\mathit{where}\ \vec{\Delta} = \begin{cases} \mathcal{F}_G[\![G_3]\!]\mu_k[\Lambda] & \mathit{if}\ \mathcal{F}_G[\![G_2]\!]\mu_k\ \downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap \Psi]) = \Lambda : \vec{\Lambda} \\ \mathcal{F}_G[\![G_4]\!]\mu_k\ \downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap \Psi]) & \mathit{if}\ \mathcal{F}_G[\![G_2]\!]\mu_k\ \downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap \Psi]) = [] \end{cases}$$

$= (\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu_k\ \downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap \Psi]))) \cap \Theta \cap \Psi) : (\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\vec{\Delta}) \cap \Theta \cap \Psi)$

Observe that for any F: $\downarrow\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![F]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])) = \downarrow\overline{\exists}_{\vec{x}}(\mathcal{F}_G[\![F]\!]\mu_k \downarrow\overline{\exists}_{\vec{x}}([\Theta]))$
hence: $(\downarrow\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap \Psi]))) \cap \Theta \cap \Psi$
$= (\downarrow\overline{\exists}_{\vec{x}}(\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\overline{\exists}_{\vec{x}}([\Theta \cap \Psi]))) \cap \Theta \cap \Psi$
$= (\downarrow\overline{\exists}_{\vec{x}}(\mathcal{F}_G[\![G_1]\!]\mu_k\ [\downarrow\overline{\exists}_{\vec{x}}(\Theta \cap \Psi) \cap \downarrow\overline{\exists}_{\vec{x}}(\Theta)])) \cap \Theta \cap \Psi$
(since $\downarrow\overline{\exists}_{\vec{x}}(\Theta \cap \Psi) \subseteq \downarrow\overline{\exists}_{\vec{x}}(\Theta)$)
which by assumption is equal to:
$(\downarrow\overline{\exists}_{\vec{x}}(\mathcal{F}_G[\![G_1]\!]\mu_k\ [\downarrow\overline{\exists}_{\vec{x}}(\Theta)]) \cap \downarrow\overline{\exists}_{\vec{x}}(\Theta \cap \Psi)) \cap \Theta \cap \Psi$
$= (\downarrow\overline{\exists}_{\vec{x}}(\downarrow\overline{\exists}_{\vec{x}}(\mathcal{F}_G[\![G_1]\!]\mu_k\ [\downarrow\overline{\exists}_{\vec{x}}(\Theta)]) \cap \downarrow\overline{\exists}_{\vec{x}}(\Theta \cap \Psi)) \cap \Theta \cap \Psi$
(since $\downarrow\overline{\exists}_{\vec{x}}(A \cap B) = \downarrow\overline{\exists}_{\vec{x}}(\downarrow\overline{\exists}_{\vec{x}}(A) \cap B)$)
$= (\downarrow\overline{\exists}_{\vec{x}}(\mathcal{F}_G[\![G_1]\!]\mu_k\ [\downarrow\overline{\exists}_{\vec{x}}(\Theta)]) \cap \downarrow\overline{\exists}_{\vec{x}}(\Theta \cap \Psi) \cap \Theta \cap \Psi$
(since $\downarrow\overline{\exists}_{\vec{x}}(\downarrow\overline{\exists}_{\vec{x}}(A) \cap \downarrow\overline{\exists}_{\vec{x}}(B)) = \downarrow\overline{\exists}_{\vec{x}}(A) \cap \downarrow\overline{\exists}_{\vec{x}}(B)$)
$= (\downarrow\overline{\exists}_{\vec{x}}(\mathcal{F}_G[\![G_1]\!]\mu_k\ [\downarrow\overline{\exists}_{\vec{x}}(\Theta)]) \cap \Theta \cap \Psi$
(since $\Theta \cap \Psi \subseteq \downarrow\overline{\exists}_{\vec{x}}(\Theta \cap \Psi)$)
$= (\downarrow\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]))) \cap \Theta \cap \Psi$
by parallel reasoning:
$(\downarrow\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_4]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap \Psi]))) \cap \Theta \cap \Psi$
$= (\downarrow\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_4]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]))) \cap \Theta \cap \Psi$
also by parallel reasoning:
$(\downarrow\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_2]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap \Psi]))) \cap \Theta \cap \Psi$
$= (\downarrow\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_2]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]))) \cap \Theta \cap \Psi$

hence if $(\downarrow\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_2]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap \Psi]))) \cap \Theta \cap \Psi = \Lambda : \vec{\Lambda}$
and $(\downarrow\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_2]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]))) \cap \Theta \cap \Psi = \Phi : \vec{\Phi}$
then $\Lambda = \Phi$
hence $(\downarrow\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_3]\!]\mu_k\ [\Lambda]))) \cap \Theta \cap \Psi = (\downarrow\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_3]\!]\mu_k\ [\Phi]))) \cap \Theta \cap \Psi$

now say $\vec{\Gamma} = \begin{cases} \mathcal{F}_G[\![G_3]\!]\mu_k[\Phi] & \textit{if } \mathcal{F}_G[\![G_2]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) = \Phi : \vec{\Phi} \\ \mathcal{F}_G[\![G_4]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) & \textit{if } \mathcal{F}_G[\![G_2]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) = [] \end{cases}$
then $\vec{\Gamma} = \vec{\Delta}$
hence: $\downarrow\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\downarrow\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap \Psi])) : \vec{\Delta})) \cap \Theta \cap \Psi$
$= \downarrow\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\downarrow\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])) : \vec{\Gamma})) \cap \Theta \cap \Psi$
hence: $(\downarrow\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y})) \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])) \cap \Theta) \cap \Psi$
$= (\downarrow\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y})) \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap \Psi])) \cap \Theta \cap \Psi)$
therefore: $(\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}((\Theta : \vec{\Theta}) \cap \Psi)$

2.2.2 Induction Step $G = G_1, G_2$
Assume: $(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![G_1]\!]\mu_{k+1}((\Theta : \vec{\Theta}) \cap \Psi)$
And: $(\mathcal{F}_G[\![G_2]\!]\mu_{k+1}(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![G_2]\!]\mu_{k+1}((\Theta : \vec{\Theta}) \cap \Psi)$
Show: $(\mathcal{F}_G[\![G_1, G_2]\!]\mu_{k+1}(\Theta : \vec{\Theta})) \cap \Psi = \mathcal{F}_G[\![G_1, G_2]\!]\mu_{k+1}((\Theta : \vec{\Theta}) \cap \Psi)$

$(\mathcal{F}_G[\![G_1, G_2]\!]\mu_{k+1}(\Theta : \vec{\Theta})) \cap \Psi$
$= (\mathcal{F}_G[\![G_2]\!]\mu_{k+1}(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}(\Theta : \vec{\Theta}))) \cap \Psi$
$= (\mathcal{F}_G[\![G_2]\!]\mu_{k+1}(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}(\Theta : \vec{\Theta})) \cap \Psi)$
$= \mathcal{F}_G[\![G_2]\!]\mu_{k+1}(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}((\Theta : \vec{\Theta}) \cap \Psi))$
$= \mathcal{F}_G[\![G_1, G_2]\!]\mu_{k+1}((\Theta : \vec{\Theta}) \cap \Psi)$
QED

2.4.2 Lemma 2: $\mathcal{F}_G[\![G]\!]\mu[\Theta] = \mathcal{F}_G[\![G]\!]\mu[\Theta \cap S_G[\![G]\!]]$

Proof in two stages:
(a) $\mathcal{F}_G[\![G]\!]\mu[\Theta \cap S_G[\![G]\!]] \sqsubseteq \mathcal{F}_G[\![G]\!]\mu[\Theta]$
(b) $\mathcal{F}_G[\![G]\!]\mu[\Theta] \sqsubseteq \mathcal{F}_G[\![G]\!]\mu[\Theta \cap S_G[\![G]\!]]$

(a) by monotonicity of $\mathcal{F}_G$:
$[\Theta \cap S_G[\![G]\!]] \sqsubseteq [\Theta] \Rightarrow \mathcal{F}_G[\![G]\!]\mu[\Theta \cap S_G[\![G]\!]] \sqsubseteq \mathcal{F}_G[\![G]\!]\mu[\Theta]$

(b) $\mathcal{F}_G[\![G]\!]\mu[\Theta] \sqsubseteq \mathcal{F}_G[\![G]\!]\mu[\Theta \cap S_G[\![G]\!]]$
Proof by nested induction on:
1. μ,
2. structure of G:

1 Base Case: $\mathcal{F}_G[\![G]\!]\mu_\perp[\Theta] \sqsubseteq \mathcal{F}_G[\![G]\!]\mu_\perp[\Theta \cap S_G[\![G]\!]]$

induction on structure of G:
1.1 Two Base Cases: (1) $G = \mathsf{post}(\phi)$, (2) $G = p(\vec{x})$
(1) $G = \mathsf{post}(\phi)$
Show: $\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp[\Theta] \sqsubseteq \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp[\Theta \cap S_G[\![\mathsf{post}(\phi)]\!]]$

$\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp[\Theta] = \mathit{trim}([\Theta \cap \Downarrow\{\phi\}])$
$\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp[\Theta \cap S_G[\![\mathsf{post}(\phi)]\!]]$
$\quad = \mathit{trim}([\Theta \cap S_G[\![\mathsf{post}(\phi)]\!] \cap \Downarrow\{\phi\}])$
$\quad = \mathit{trim}([\Theta \cap \Downarrow\{\phi\} \cap \Downarrow\{\phi\}])$
$\quad = \mathit{trim}([\Theta \cap \Downarrow\{\phi\}])$

(2) $G = p(\vec{x})$
Show: $\mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp[\Theta] \sqsubseteq \mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp[\Theta \cap S_G[\![p(\vec{x})]\!]]$

$\mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp[\Theta] = []$
$\mathcal{F}_G[\![p(\vec{x})]\!]\mu_\perp[\Theta \cap S_G[\![p(\vec{x})]\!]] = []$

1.2 Induction Step: $G = G_1, G_2$
Assume: $\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta_1] \sqsubseteq \mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta_1 \cap S_G[\![G_1]\!]]$
And: $\mathcal{F}_G[\![G_2]\!]\mu_\perp[\Theta_2] \sqsubseteq \mathcal{F}_G[\![G_2]\!]\mu_\perp[\Theta_2 \cap S_G[\![G_2]\!]]$
Show: $\mathcal{F}_G[\![G_1, G_2]\!]\mu_\perp[\Theta] \sqsubseteq \mathcal{F}_G[\![G_1, G_2]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]]$

$\mathcal{F}_G[\![G_1, G_2]\!]\mu_\perp[\Theta] = \mathcal{F}_G[\![G_2]\!]\mu_\perp(\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta])$
by assumption: $\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta] \sqsubseteq \mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1]\!]]$
hence: $\mathcal{F}_G[\![G_2]\!]\mu_\perp(\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta]) \sqsubseteq \mathcal{F}_G[\![G_2]\!]\mu_\perp(\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1]\!]])$
by assumption:
$\mathcal{F}_G[\![G_2]\!]\mu_\perp(\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1]\!]])$
$\quad \sqsubseteq \mathcal{F}_G[\![G_2]\!]\mu_\perp((\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1]\!]]) \cap S_G[\![G_2]\!])$
by Lemma 1: $(\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1]\!]]) \cap S_G[\![G_2]\!]$
$\quad = \mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1]\!] \cap S_G[\![G_2]\!]]$
hence: $\mathcal{F}_G[\![G_2]\!]\mu_\perp((\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1]\!]]) \cap S_G[\![G_2]\!])$
$\quad = \mathcal{F}_G[\![G_2]\!]\mu_\perp(\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1]\!] \cap S_G[\![G_2]\!]])$

$= \mathcal{F}_G[\![G_1, G_2]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]]$
therefore: $\mathcal{F}_G[\![G_1, G_2]\!]\mu_\perp[\Theta] \sqsubseteq \mathcal{F}_G[\![G_1, G_2]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]]$

2 Induction Step:
Assume: $\mathcal{F}_G[\![H]\!]\mu_k[\Theta'] \sqsubseteq \mathcal{F}_G[\![H]\!]\mu_k[\Theta' \cap S_G[\![H]\!]]$
Show: $\mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta] \sqsubseteq \mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta \cap S_G[\![G]\!]]$
where $\mu_{k+1} = \mathcal{F}_P[\![P]\!]\mu_k$

induction on structure of G:
2.1 Two Base Cases: (1) $G = \mathsf{post}(\phi)$, (2) $G = p(\vec{x})$
(1) $G = \mathsf{post}(\phi)$
Show: $\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}[\Theta] \sqsubseteq \mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}[\Theta \cap S_G[\![\mathsf{post}(\phi)]\!]]$
where $\mu_{k+1} = \mathcal{F}_P[\![P]\!]\mu_k$

$\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}[\Theta] = \mathit{trim}([\Theta \cap \downarrow\!\!\downarrow\{\phi\}])$
$\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}[\Theta \cap S_G[\![\mathsf{post}(\phi)]\!]]$
$= \mathit{trim}([\Theta \cap S_G[\![\mathsf{post}(\phi)]\!] \cap \downarrow\!\!\downarrow\{\phi\}])$
$= \mathit{trim}([\Theta \cap \downarrow\!\!\downarrow\{\phi\} \cap \downarrow\!\!\downarrow\{\phi\}])$
$= \mathit{trim}([\Theta \cap \downarrow\!\!\downarrow\{\phi\}])$

(2) $G = p(\vec{x})$
Assume (without loss of generality): $p(\vec{y}) \leftarrow G_1; G_2, !, G_3; G_4 \in P$
Show: $\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}[\Theta] \sqsubseteq \mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}[\Theta \cap S_G[\![p(\vec{x})]\!]]$
where: $\mu_{k+1} = \mathcal{F}_P[\![P]\!]\mu_k$
$\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}[\Theta \cap S_G[\![p(\vec{x})]\!]]$
$= \downarrow\!\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y}))\ \downarrow\!\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]])) \cap \Theta \cap S_G[\![p(\vec{x})]\!]$
$\mu_{k+1}(p(\vec{y}))\ \downarrow\!\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]]) = \downarrow\!\!\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu_k\ \downarrow\!\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]]) : \vec{\Psi})$
where
$$\vec{\Psi} = \begin{cases} \mathcal{F}_G[\![G_3]\!]\mu_k[\Phi] & \mathit{if}\ \mathcal{F}_G[\![G_2]\!]\mu_k\ \downarrow\!\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]]) = \Phi : \vec{\Phi} \\ \mathcal{F}_G[\![G_4]\!]\mu_k\ \downarrow\!\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]]) & \mathit{if}\ \mathcal{F}_G[\![G_2]\!]\mu_k\ \downarrow\!\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap S_G[\![p(\vec{x})]\!]]) = [] \end{cases}$$
now: $S_G[\![p(\vec{x})]\!] = \downarrow\!\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_H[\![p(\vec{y})]\!])$
$= \downarrow\!\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\downarrow\!\!\overline{\exists}_{\vec{y}}(S_G[\![G_1]\!] \cup S_G[\![G_2, G_3]\!] \cup S_G[\![G_4]\!]))$
$= \downarrow\!\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_1]\!] \cup S_G[\![G_2, G_3]\!] \cup S_G[\![G_4]\!])$
$= \downarrow\!\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_1]\!]) \cup \downarrow\!\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_2, G_3]\!]) \cup \downarrow\!\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_4]\!])$
(because $\overline{\exists}$ distributes over $\cup$)
Since $S_G[\![p(\vec{x})]\!]$ is the union of these three components, it is a superset of each of them, hence: $S_G[\![p(\vec{x})]\!] \supseteq \downarrow\!\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_1]\!])$
and: $S_G[\![p(\vec{x})]\!] \supseteq \downarrow\!\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_2, G_3]\!])$
and: $S_G[\![p(\vec{x})]\!] \supseteq \downarrow\!\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_4]\!])$
Intersecting each side with Θ preserves the order, hence: $\Theta \cap S_G[\![p(\vec{x})]\!] \supseteq \Theta \cap \downarrow\!\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_1]\!])$
and: $\Theta \cap S_G[\![p(\vec{x})]\!] \supseteq \Theta \cap \downarrow\!\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_2, G_3]\!])$
and: $\Theta \cap S_G[\![p(\vec{x})]\!] \supseteq \Theta \cap \downarrow\!\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_4]\!])$
Again, projecting and renaming both sides in the same way preserves the order, hence: $\downarrow\!\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta \cap S_G[\![p(\vec{x})]\!]) \supseteq \downarrow\!\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta \cap \downarrow\!\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_1]\!]))$
and: $\downarrow\!\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta \cap S_G[\![p(\vec{x})]\!]) \supseteq \downarrow\!\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta \cap \downarrow\!\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_2, G_3]\!]))$
and: $\downarrow\!\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta \cap S_G[\![p(\vec{x})]\!]) \supseteq \downarrow\!\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta \cap \downarrow\!\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(S_G[\![G_4]\!]))$
Now, since the following holds in general:

$$\downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Gamma_1 \cap \downharpoonleft\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\Gamma_2)) \supseteq \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Gamma_1) \cap \Gamma_2,$$

performing the same transformation on the above still preserves the order,
hence: $\downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta \cap S_G\llbracket p(\vec{x})\rrbracket) \supseteq \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G\llbracket G_1\rrbracket$
and: $\downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta \cap S_G\llbracket p(\vec{x})\rrbracket) \supseteq \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G\llbracket G_2, G_3\rrbracket$
$= \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G\llbracket G_2\rrbracket \cap S_G\llbracket G_3\rrbracket$
and: $\downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta \cap S_G\llbracket p(\vec{x})\rrbracket) \supseteq \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G\llbracket G_4\rrbracket$
by monotonicity of $\mathcal{F}_G$, therefore:
$\mathcal{F}_G\llbracket G_1\rrbracket\mu_k \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap S_G\llbracket p(\vec{x})\rrbracket]) \sqsupseteq \mathcal{F}_G\llbracket G_1\rrbracket\mu_k[\downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G\llbracket G_1\rrbracket]$
by assumption: $\mathcal{F}_G\llbracket G_1\rrbracket\mu_k[\downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G\llbracket G_1\rrbracket] \sqsupseteq \mathcal{F}_G\llbracket G_1\rrbracket\mu_k[\downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta)]$
hence the following holds of the first part of the sequence:
$\mathcal{F}_G\llbracket G_1\rrbracket\mu_k \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap S_G\llbracket p(\vec{x})\rrbracket]) \sqsupseteq \mathcal{F}_G\llbracket G_1\rrbracket\mu_k \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}[\Theta]$
and similarly: $\mathcal{F}_G\llbracket G_4\rrbracket\mu_k \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap S_G\llbracket p(\vec{x})\rrbracket]) \sqsupseteq \mathcal{F}_G\llbracket G_4\rrbracket\mu_k[\downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G\llbracket G_4\rrbracket]$
by assumption: $\mathcal{F}_G\llbracket G_4\rrbracket\mu_k[\downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G\llbracket G_4\rrbracket] \sqsupseteq \mathcal{F}_G\llbracket G_4\rrbracket\mu_k[\downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta)]$
hence the parallel thing holds for the second possibility of the second part of the sequence:
$\mathcal{F}_G\llbracket G_4\rrbracket\mu_k \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap S_G\llbracket p(\vec{x})\rrbracket]) \sqsupseteq \mathcal{F}_G\llbracket G_4\rrbracket\mu_k \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}[\Theta]$
As for the first possibility for the second part of the sequence, consider this:
by monotonicity of $\mathcal{F}_G$:
$\Phi : \vec{\Phi} = \mathcal{F}_G\llbracket G_2\rrbracket\mu_k \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap S_G\llbracket p(\vec{x})\rrbracket])$
$\sqsupseteq \mathcal{F}_G\llbracket G_2\rrbracket\mu_k[\downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G\llbracket G_2\rrbracket \cap S_G\llbracket G3\rrbracket]$
by assumption:
$\mathcal{F}_G\llbracket G_2\rrbracket\mu_k[\downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G\llbracket G_2\rrbracket \cap S_G\llbracket G3\rrbracket] \sqsupseteq \mathcal{F}_G\llbracket G_2\rrbracket\mu_k[\downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G\llbracket G3\rrbracket]$
by Lemma 1: $\mathcal{F}_G\llbracket G_2\rrbracket\mu_k[\downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G\llbracket G3\rrbracket] = \mathcal{F}_G\llbracket G_2\rrbracket\mu_k[\downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}[\Theta] \cap S_G\llbracket G3\rrbracket$
hence: $\Phi : \vec{\Phi} \sqsupseteq \mathcal{F}_G\llbracket G_2\rrbracket\mu_k[\downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}[\Theta] \cap S_G\llbracket G3\rrbracket$
now call the part of the sequence we are aiming for here $\Lambda : \vec{\Lambda} = \mathcal{F}_G\llbracket G_2\rrbracket\mu_k \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])$
then: $\Phi : \vec{\Phi} \sqsupseteq (\Lambda : \vec{\Lambda}) \cap S_G\llbracket G_3\rrbracket$
hence: $[\Phi] \sqsupseteq [\Lambda \cap S_G\llbracket G_3\rrbracket]$
hence: $\mathcal{F}_G\llbracket G_3\rrbracket\mu_k[\Phi] \sqsupseteq \mathcal{F}_G\llbracket G_3\rrbracket\mu_k[\Lambda \cap \llbracket G_3\rrbracket]$
by assumption: $\mathcal{F}_G\llbracket G_3\rrbracket\mu_k[\Lambda \cap \llbracket G_3\rrbracket] \sqsupseteq \mathcal{F}_G\llbracket G_3\rrbracket\mu_k[\Lambda]$
hence: $\mathcal{F}_G\llbracket G_3\rrbracket\mu_k[\Phi] \sqsupseteq \mathcal{F}_G\llbracket G_3\rrbracket\mu_k[\Lambda]$

These last few lines show that each part of the sequence we are considering is greater than the sequence we are aiming for. Pulling these together, we arrive at:
$\downharpoonleft\overline{\exists}_{\vec{y}}(\mathcal{F}_G\llbracket G_1\rrbracket\mu_k \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap S_G\llbracket p(\vec{x})\rrbracket]) : \vec{\Psi}) \sqsupseteq \downharpoonleft\overline{\exists}_{\vec{y}}(\mathcal{F}_G\llbracket G_1\rrbracket\mu_k \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) : \vec{\Delta})$
where

$$\vec{\Psi} = \begin{cases} \mathcal{F}_G\llbracket G_3\rrbracket\mu_k[\Phi] & \textit{if } \mathcal{F}_G\llbracket G_2\rrbracket\mu_k \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap S_G\llbracket p(\vec{x})\rrbracket]) = \Phi : \vec{\Phi} \\ \mathcal{F}_G\llbracket G_4\rrbracket\mu_k \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap S_G\llbracket p(\vec{x})\rrbracket]) & \textit{if } \mathcal{F}_G\llbracket G_2\rrbracket\mu_k \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap S_G\llbracket p(\vec{x})\rrbracket]) = [] \end{cases}$$

and $$\vec{\Delta} = \begin{cases} \mathcal{F}_G\llbracket G_3\rrbracket\mu_k[\Lambda] & \textit{if } \mathcal{F}_G\llbracket G_2\rrbracket\mu_k \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) = \Lambda : \vec{\Lambda} \\ \mathcal{F}_G\llbracket G_4\rrbracket\mu_k \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) & \textit{if } \mathcal{F}_G\llbracket G_2\rrbracket\mu_k \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) = [] \end{cases}$$

therefore: $\mu_{k+1}(p(\vec{y})) \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap S_G\llbracket p(\vec{x})\rrbracket]) \sqsupseteq \mu_{k+1}(p(\vec{y})) \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])$
applying the same renaming and projection to both sides preserves the order:
$\downharpoonleft\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y})) \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap S_G\llbracket p(\vec{x})\rrbracket])) \sqsupseteq \downharpoonleft\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y})) \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]))$
now name these two sequences:
$\downharpoonleft\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y})) \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta \cap S_G\llbracket p(\vec{x})\rrbracket])) = \vec{\Psi}'$
and: $\downharpoonleft\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mu_{k+1}(p(\vec{y})) \downharpoonleft\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])) = \vec{\Delta}'$
and notice the following two facts:
(1) $\vec{\Delta}' \cap \Theta = \mathcal{F}_G\llbracket p(\vec{x})\rrbracket\mu_{k+1}[\Theta]$
(2) $\vec{\Psi}' \cap S_G\llbracket p(\vec{x})\rrbracket \cap \Theta = \mathcal{F}_G\llbracket p(\vec{x})\rrbracket\mu_{k+1}[\Theta \cap S_G\llbracket p(\vec{x})\rrbracket]$

then from above we have: $\vec{\Delta}' \sqsubseteq \vec{\Psi}'$
hence: $\vec{\Delta}' \cap \Theta \sqsubseteq \vec{\Psi}'$
by (1) and Theorem 1, therefore: $\bigcup(\vec{\Delta}') \cap \Theta = \bigcup(\vec{\Delta}' \cap \Theta) \subseteq \Theta \cap S_G[\![p(\vec{x})]\!]$
hence: $\bigcup(\vec{\Delta}') \subseteq S_G[\![p(\vec{x})]\!]$
therefore for each Δ' in $\vec{\Delta}'$: $\Delta' \subseteq S_G[\![p(\vec{x})]\!]$
hence for each Δ' in $\vec{\Delta}'$: $\Delta' \cap S_G[\![p(\vec{x})]\!] = \Delta'$
hence: $\vec{\Delta}' \cap S_G[\![p(\vec{x})]\!] = \vec{\Delta}'$
hence: $\vec{\Delta}' \cap \Theta = \vec{\Delta}' \cap S_G[\![p(\vec{x})]\!] \cap \Theta \sqsubseteq \vec{\Psi}' \cap S_G[\![p(\vec{x})]\!] \cap \Theta$
substituting using (2), we therefore arrive at:
$\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}[\Theta] \sqsubseteq \mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}[\Theta \cap S_G[\![p(\vec{x})]\!]]$

2.2 Induction Step: $G = G_1, G_2$
Assume: $\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta_1] \sqsubseteq \mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta_1 \cap S_G[\![G_1]\!]]$
And: $\mathcal{F}_G[\![G_2]\!]\mu_{k+1}[\Theta_2] \sqsubseteq \mathcal{F}_G[\![G_2]\!]\mu_{k+1}[\Theta_2 \cap S_G[\![G_2]\!]]$
Show: $\mathcal{F}_G[\![G_1, G_2]\!]\mu_{k+1}[\Theta] \sqsubseteq \mathcal{F}_G[\![G_1, G_2]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]]$

$\mathcal{F}_G[\![G_1, G_2]\!]\mu_{k+1}[\Theta] = \mathcal{F}_G[\![G_2]\!]\mu_{k+1}(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta])$
by assumption: $\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta] \sqsubseteq \mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1]\!]]$
hence: $\mathcal{F}_G[\![G_2]\!]\mu_{k+1}(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta]) \sqsubseteq \mathcal{F}_G[\![G_2]\!]\mu_{k+1}(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1]\!]])$
by assumption:
$\mathcal{F}_G[\![G_2]\!]\mu_{k+1}(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1]\!]]) \sqsubseteq \mathcal{F}_G[\![G_2]\!]\mu_{k+1}((\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1]\!]]) \cap S_G[\![G_2]\!])$
by Lemma 1: $(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1]\!]]) \cap S_G[\![G_2]\!] = \mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1]\!] \cap S_G[\![G_2]\!]]$
hence: $\mathcal{F}_G[\![G_2]\!]\mu_{k+1}((\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1]\!]]) \cap S_G[\![G_2]\!])$
$= \mathcal{F}_G[\![G_2]\!]\mu_{k+1}(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1]\!] \cap S_G[\![G_2]\!]])$
$= \mathcal{F}_G[\![G_1, G_2]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]]$
therefore: $\mathcal{F}_G[\![G_1, G_2]\!]\mu_{k+1}[\Theta] \sqsubseteq \mathcal{F}_G[\![G_1, G_2]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]]$

Therefore since: (a) $\mathcal{F}_G[\![G]\!]\mu[\Theta \cap S_G[\![G]\!]] \sqsubseteq \mathcal{F}_G[\![G]\!]\mu[\Theta]$
and (b) $\mathcal{F}_G[\![G]\!]\mu[\Theta] \sqsubseteq \mathcal{F}_G[\![G]\!]\mu[\Theta \cap S_G[\![G]\!]]$,
it follows that: $\mathcal{F}_G[\![G]\!]\mu[\Theta] = \mathcal{F}_G[\![G]\!]\mu[\Theta \cap S_G[\![G]\!]]$
QED

2.4.3 Proof of Theorem 2: For $\Theta \in Con^{\downarrow}$ and stratified $P = P_0 \cup \ldots \cup P_n$: $\Theta \subseteq \mathcal{D}_G[\![G]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G]\!]\mu_P[\Theta]| \leq 1$.

First notice that the following things hold:
(1) $\Theta \subseteq (\Phi \to \Psi) \Rightarrow \Theta \cap \Phi \subseteq \Psi$
(2) $\Theta \subseteq mux(\Phi, \Psi) \Rightarrow (\Theta \cap \Phi = \{false\}) \vee (\Theta \cap \Psi = \{false\})$
(3) $\mathcal{F}_G[\![G]\!]\mu\vec{\Theta} \subseteq \bigcup \vec{\Theta} \cap S_G[\![G]\!]$
for any μ constructed by application of $\mathcal{F}_P[\![P]\!]$ to $\mu_\perp$
(4) $\overline{\forall}_{\vec{y}}(\Theta \cap \Phi) = \overline{\forall}_{\vec{y}}(\Theta) \cap \overline{\forall}_{\vec{y}}(\Phi)$
(5) $\Theta \subseteq \downarrow\!\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\Phi) \Rightarrow \downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \subseteq \Phi$
This holds due to the following few lines of reasoning:
$\Theta \subseteq \overline{\exists}_{\vec{x}}(\Theta)$ (since $\overline{\exists}$ is extensive)
if $\Theta \subseteq \downarrow\!\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\Phi)$
then $\overline{\exists}_{\vec{x}}(\Theta) \subseteq \overline{\exists}_{\vec{x}}(\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\Phi))$
(by monotonicity of $\overline{\exists}$)
then $\downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \subseteq \downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\Phi)) = \downarrow\!\rho_{\vec{x},\vec{y}}(\downarrow\!\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\Phi))$

(by monotonicity of $\downarrow$, ρ)

$\downarrow\!\rho_{\vec{x},\vec{y}}\downarrow\!\rho_{\vec{y},\vec{x}}$ cancel out and $\overline{\forall}$ is reductive, hence:

$\downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \subseteq \Phi$

(6) $\mathcal{F}_G[\![G]\!]\mu[\Theta] \sqsubseteq \mathcal{F}_G[\![G]\!]\mu(\Theta : \vec{\Theta})$

again for any μ constructed by application of $\mathcal{F}_P[\![P]\!]$ to $\mu_\perp$

(7) $\vec{\Theta}_1 \sqsubseteq \vec{\Theta}_2 \Rightarrow |\vec{\Theta}_1| \leq |\vec{\Theta}_2|$

Proof by nested induction on:
1. μ,
2. structure of G:

1 Base Case: $\mu = \mu_\perp$

show: $\Theta \subseteq \mathcal{D}_G[\![G]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G]\!]\mu_\perp[\Theta]| \leq 1$

Induction on structure of G:

1.1 Two Base Cases: (1) $G = \mathsf{post}(\phi)$, (2) $G = p(\vec{x})$

(1) $G = \mathsf{post}(\phi)$:

Show: $\Theta \subseteq \mathcal{D}_G[\![\mathsf{post}(\phi)]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp[\Theta]| \leq 1$

$\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp[\Theta] = \mathit{trim}([\downarrow\!\{\phi\} \cap \Theta])$

hence: $|\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_\perp[\Theta]| = |\mathit{trim}([\downarrow\!\{\phi\} \cap \Theta])| \leq 1$

(2) $G = p(\vec{x})$

Show: $\Theta \subseteq \mathcal{D}_G[\![p(\vec{y})]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![p(\vec{y})]\!]\mu_\perp[\Theta]| \leq 1$

$\mathcal{F}_G[\![p(\vec{y})]\!]\mu_\perp[\Theta] = \downarrow\!\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mu_\perp\ (p(\vec{y}))\ \downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])) \cap \Theta : []$

$\mu_\perp\ (p(\vec{y}))\ \downarrow\!\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) = []$

hence: $\mathcal{F}_G[\![p(\vec{y})]\!]\mu_\perp[\Theta] = []$

hence: $|\mathcal{F}_G[\![p(\vec{y})]\!]\mu_\perp[\Theta]| = |[]| = 0$

1.2 Induction Step:

$G = G_1, G_2$:

Assume: $\Theta_1 \subseteq \mathcal{D}_G[\![G_1]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta_1]| \leq 1$

And: $\Theta_2 \subseteq \mathcal{D}_G[\![G_2]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G_2]\!]\mu_\perp[\Theta_2]| \leq 1$

Show: $\Theta \subseteq \mathcal{D}_G[\![G]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G]\!]\mu_\perp[\Theta]| \leq 1$

$\mathcal{D}_G[\![G]\!]\delta_P = (S_G[\![G_2]\!] \to \mathcal{D}_G[\![G_1]\!]\delta_P) \cap (S_G[\![G_1]\!] \to \mathcal{D}_G[\![G_2]\!]\delta_P)$

$\Theta \subseteq \mathcal{D}_G[\![G]\!]\delta_P \Rightarrow \Theta \subseteq (S_G[\![G_1]\!] \to \mathcal{D}_G[\![G_2]\!]\delta_P) \Rightarrow \Theta \cap S_G[\![G_1]\!] \subseteq \mathcal{D}_G[\![G_2]\!]\delta_P$

$\Theta \subseteq \mathcal{D}_G[\![G]\!]\delta_P \Rightarrow \Theta \subseteq (S_G[\![G_2]\!] \to \mathcal{D}_G[\![G_1]\!]\delta_P) \Rightarrow \Theta \cap S_G[\![G_2]\!] \subseteq \mathcal{D}_G[\![G_1]\!]\delta_P$

$\mathcal{F}_G[\![G]\!]\mu_\perp[\Theta] = \mathcal{F}_G[\![G]\!]\mu_\perp[\Theta \cap S_G[\![G]\!]]$ (*by Lemma* 2)

$\mathcal{F}_G[\![G]\!]\mu_\perp[\Theta \cap S_G[\![G]\!]] = \mathcal{F}_G[\![G_2]\!]\mu_\perp(\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]])$

$\Theta \cap S_G[\![G_1, G_2]\!] = \Theta \cap S_G[\![G_1]\!] \cap S_G[\![G_2]\!] \subseteq \Theta \cap S_G[\![G_2]\!] \subseteq \mathcal{D}_G[\![G_1]\!]\delta_P$

hence by assumption: $|\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]]| \leq 1$

distinguish two cases:

(a) $|\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]]| = 0$,

(b) $|\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]]| = 1$

(a) $|\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]]| = 0$
$\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]] = []$
$\mathcal{F}_G[\![G]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]] = \mathcal{F}_G[\![G_2]\!]\mu_\perp[] = []$
hence: $|\mathcal{F}_G[\![G]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]]| \leq 1$
by Lemma 2 (remembering $G = G_1, G_2$): $|\mathcal{F}_G[\![G]\!]\mu_\perp[\Theta]| \leq 1$

(b) $|\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]]| = 1$
$\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]] = [\Psi]$
by Theorem 1: $\bigcup(\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]])$
$\subseteq \Theta \cap S_G[\![G_1, G_2]\!] \cap S_G[\![G_1]\!]$
$\subseteq \Theta \cap S_G[\![G_1]\!]$
hence: $\Psi \subseteq \Theta \cap S_G[\![G_1]\!]$
hence: $\Psi \subseteq \mathcal{D}_G[\![G_2]\!]\delta_P$
hence by assumption: $|\mathcal{F}_G[\![G_2]\!]\mu_\perp[\Psi]| \leq 1$
hence (again by Lemma 2): $|\mathcal{F}_G[\![G_2]\!]\mu_\perp(\mathcal{F}_G[\![G_1]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]])|$
$= |\mathcal{F}_G[\![G]\!]\mu_\perp[\Theta \cap S_G[\![G_1, G_2]\!]]|$
$= |\mathcal{F}_G[\![G]\!]\mu_\perp[\Theta]| \leq 1$

2 Induction Step:
Assume: $X \subseteq \mathcal{D}_G[\![H]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![H]\!]\mu_k[X]| \leq 1$
Show: $\Theta \subseteq \mathcal{D}_G[\![G]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta]| \leq 1$
where $\mu_{k+1} = \mathcal{F}_P[\![P]\!]\mu_k$

Induction on structure of G:
2.1 Two base cases: (1) $G = \mathsf{post}(\phi)$, (2) $G = p(\vec{x})$

(1) $G = \mathsf{post}(\phi)$:
Show: $\Theta \subseteq \mathcal{D}_G[\![\mathsf{post}(\phi)]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}[\Theta]| \leq 1$

$\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}[\Theta] = \mathit{trim}([\downarrow\{\phi\} \cap \Theta])$
hence: $|\mathcal{F}_G[\![\mathsf{post}(\phi)]\!]\mu_{k+1}[\Theta]| = |\mathit{trim}([\downarrow\{\phi\} \cap \Theta])| \leq 1$

(2) $G = p(\vec{x})$
Assume (without loss of generality): $p(\vec{y}) \leftarrow G_1; G_2, !, G_3; G_4 \in P$
Show: $\Theta \subseteq \mathcal{D}_G[\![p(\vec{x})]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}[\Theta]| \leq 1$

$\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}[\Theta] = \downarrow\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mu_{k+1}\ (p(\vec{y}))\ \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])) \cap \Theta$
hence: $|\mathcal{F}_G[\![p(\vec{x})]\!]\mu_{k+1}[\Theta]| = |\downarrow\rho_{\vec{y},\vec{x}}\overline{\exists}_{\vec{y}}(\mu_{k+1}\ (p(\vec{y}))\ \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])) \cap \Theta| = |\mu_{k+1}\ (p(\vec{y}))\ \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])|$
and: $\mu_{k+1}\ (p(\vec{y}))\ \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) = \downarrow\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu_k\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) : \vec{\Psi})$
where $\vec{\Psi} = \begin{cases} \mathcal{F}_G[\![G_3]\!]\mu_k[\Phi] & \mathit{if}\ \mathcal{F}_G[\![G_2]\!]\mu_k\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) = \Phi : \vec{\Phi} \\ \mathcal{F}_G[\![G_4]\!]\mu_k\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) & \mathit{if}\ \mathcal{F}_G[\![G_2]\!]\mu_k\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) = [] \end{cases}$

$|\downarrow\overline{\exists}_{\vec{y}}(\mathcal{F}_G[\![G_1]\!]\mu_k\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) : \vec{\Psi})| = |\mathcal{F}_G[\![G_1]\!]\mu_k\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) : \vec{\Psi}|$

Show $\Theta \subseteq \mathcal{D}_G[\![p(\vec{x})]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G_1]\!]\mu_k\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) : \vec{\Psi}| \leq 1$ in two steps:

1 Show that each component cannot be longer than 1:
1a Show: $\Theta \subseteq \mathcal{D}_G[\![p(\vec{x})]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G_1]\!]\mu_k\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \leq 1$

1b Show: $\Theta \subseteq \mathcal{D}_G\llbracket p(\vec{x})\rrbracket\delta_P \Rightarrow |\mathcal{F}_G\llbracket G_4\rrbracket\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \leq 1$
1c Show: $\Theta \subseteq \mathcal{D}_G\llbracket p(\vec{x})\rrbracket\delta_P \Rightarrow |\mathcal{F}_G\llbracket G_3\rrbracket\mu_k[\Phi]| \leq 1$
where $\mathcal{F}_G\llbracket G_2\rrbracket\mu_k\ {\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) = \Phi : \vec{\Phi}$

2 Show that only one component can be longer than 0:
$\Theta \subseteq \mathcal{D}_G\llbracket p(\vec{x})\rrbracket\delta_P \Rightarrow \neg(|\mathcal{F}_G\llbracket G_1\rrbracket\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \neq 0 \wedge |\vec{\Psi}| \neq 0)$
This is done thus:
2a Show:
$\Theta \subseteq \mathcal{D}_G\llbracket p(\vec{x})\rrbracket\delta_P \Rightarrow \neg(|\mathcal{F}_G\llbracket G_1\rrbracket\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \neq 0 \wedge |\mathcal{F}_G\llbracket G_4\rrbracket\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \neq 0)$
2b Show:
$\Theta \subseteq \mathcal{D}_G\llbracket p(\vec{x})\rrbracket\delta_P \Rightarrow \neg(|\mathcal{F}_G\llbracket G_1\rrbracket\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \neq 0 \wedge |\mathcal{F}_G\llbracket G_3\rrbracket\mu_k[\Phi]| \neq 0)$
where $\mathcal{F}_G\llbracket G_2\rrbracket\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) = \Phi : \vec{\Phi}$

$\mathcal{D}_G\llbracket p(\vec{x})\rrbracket\delta_P = {\downarrow}\rho_{\vec{y},\vec{x}}(\overline{\forall}_{\vec{y}}(\delta_P(p(\vec{y}))))$
$= {\downarrow}\rho_{\vec{y},\vec{x}}(\overline{\forall}_{\vec{y}}({\downarrow}\overline{\forall}_{\vec{y}}(\mathcal{D}_G\llbracket G_1\rrbracket\delta_P \cap (S_G\llbracket G_2\rrbracket \rightarrow \mathcal{D}_G\llbracket G_3\rrbracket\delta_P) \cap \mathcal{D}_G\llbracket G_4\rrbracket\delta_P \cap \Theta_1 \cap \Theta_2)))$
$= {\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\mathcal{D}_G\llbracket G_1\rrbracket\delta_P \cap (S_G\llbracket G_2\rrbracket \rightarrow \mathcal{D}_G\llbracket G_3\rrbracket\delta_P) \cap \mathcal{D}_G\llbracket G_4\rrbracket\delta_P \cap \Theta_1 \cap \Theta_2)$
where $\Theta_1 = mux(S_G\llbracket G_1\rrbracket, S_G\llbracket G_4\rrbracket)$
and $\Theta_2 = mux(S_G\llbracket G_1\rrbracket, S_G\llbracket G_2, G_3\rrbracket)$
$= {\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\mathcal{D}_G\llbracket G_1\rrbracket\delta_P)$
$\cap\, {\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(S_G\llbracket G_2\rrbracket \rightarrow \mathcal{D}_G\llbracket G_3\rrbracket\delta_P)$
$\cap\, {\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\mathcal{D}_G\llbracket G_4\rrbracket\delta_P)$
$\cap\, {\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(mux(S_G\llbracket G_1\rrbracket, S_G\llbracket G_4\rrbracket))$
$\cap\, {\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(mux(S_G\llbracket G_1\rrbracket, S_G\llbracket G_2, G_3\rrbracket))$

1a Show: $\Theta \subseteq \mathcal{D}_G\llbracket p(\vec{x})\rrbracket\delta_P \Rightarrow |\mathcal{F}_G\llbracket G_1\rrbracket\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \leq 1$
$\Theta \subseteq \mathcal{D}_G\llbracket p(\vec{x})\rrbracket\delta_P \Rightarrow \Theta \subseteq {\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\mathcal{D}_G\llbracket G_1\rrbracket)\delta_P$
hence (by (5) stated above): ${\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \subseteq \mathcal{D}_G\llbracket G_1\rrbracket\delta_P$
hence by assumption: $|\mathcal{F}_G\llbracket G_1\rrbracket\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \leq 1$

1b Show: $\Theta \subseteq \mathcal{D}_G\llbracket p(\vec{x})\rrbracket\delta_P \Rightarrow |\mathcal{F}_G\llbracket G_4\rrbracket\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \leq 1$
$\Theta \subseteq \mathcal{D}_G\llbracket p(\vec{x})\rrbracket\delta_P \Rightarrow \Theta \subseteq {\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\mathcal{D}_G\llbracket G_4\rrbracket\delta_P)$
hence (again by (5) above): ${\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \subseteq \mathcal{D}_G\llbracket G_4\rrbracket\delta_P$
hence by assumption: $|\mathcal{F}_G\llbracket G_4\rrbracket\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \leq 1$

1c Show: $\Theta \subseteq \mathcal{D}_G\llbracket p(\vec{x})\rrbracket\delta_P \Rightarrow |\mathcal{F}_G\llbracket G_3\rrbracket\mu_k[\Phi])| \leq 1$
where $\mathcal{F}_G\llbracket G_2\rrbracket\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) = \Phi : \vec{\Phi}$
$\Theta \subseteq \mathcal{D}_G\llbracket p(\vec{x})\rrbracket\delta_P \Rightarrow \Theta \subseteq {\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(S_G\llbracket G_2\rrbracket \rightarrow \mathcal{D}_G\llbracket G_3\rrbracket\delta_P)$
hence (again by (5) above): ${\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \subseteq (S_G\llbracket G_2\rrbracket \rightarrow \mathcal{D}_G\llbracket G_3\rrbracket\delta_P)$
hence (by (1) stated above): ${\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G\llbracket G_2\rrbracket \subseteq \mathcal{D}_G\llbracket G_3\rrbracket\delta_P$
by Theorem 1: $\bigcup(\Phi : \vec{\Phi}) = \bigcup(\mathcal{F}_G\llbracket G_2\rrbracket\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])) \subseteq {\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) \cap S_G\llbracket G_2\rrbracket$
therefore (since $\Phi \subseteq \bigcup(\Phi : \vec{\Phi})$): $\Phi \subseteq {\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) \cap S_G\llbracket G_2\rrbracket \subseteq \mathcal{D}_G\llbracket G_3\rrbracket\delta_P$
by assumption: $|\mathcal{F}_G\llbracket G_3\rrbracket\mu_k[\Phi]| \leq 1$

2a Show:
$\Theta \subseteq \mathcal{D}_G\llbracket p(\vec{x})\rrbracket\delta_P \Rightarrow \neg(|\mathcal{F}_G\llbracket G_1\rrbracket\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \neq 0 \wedge |\mathcal{F}_G\llbracket G_4\rrbracket\mu_k{\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \neq 0)$
$\Theta \subseteq \mathcal{D}_G\llbracket p(\vec{x})\rrbracket\delta_P \Rightarrow \Theta \subseteq {\downarrow}\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(mux(S_G\llbracket G_1\rrbracket, S_G\llbracket G_4\rrbracket))$
hence (by (5) stated above): ${\downarrow}\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \subseteq mux(S_G\llbracket G_1\rrbracket, S_G\llbracket G_4\rrbracket)$

hence (by (2) stated above):
$(\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_1]\!] = \{false\}) \vee (\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_4]\!] = \{false\})$
by Theorem 1: $\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_1]\!] = \{false\} \Rightarrow \mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) = []$
hence: $\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_1]\!] = \{false\} \Rightarrow |\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| = 0$
similarly: $\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_4]\!] = \{false\} \Rightarrow \mathcal{F}_G[\![G_4]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) = []$
hence: $\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_4]\!] = \{false\} \Rightarrow |\mathcal{F}_G[\![G_4]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| = 0$
therefore: $(|\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| = 0) \vee (|\mathcal{F}_G[\![G_4]\!]\mu_k\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| = 0)$
hence: $\neg((|\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \neq 0) \wedge (|\mathcal{F}_G[\![G_4]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \neq 0))$

2b Show: $\Theta \subseteq \mathcal{D}_G[\![p(\vec{x})]\!]\delta_P \Rightarrow \neg(|\mathcal{F}_G[\![G_1]\!]\mu_k\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \neq 0 \wedge |\mathcal{F}_G[\![G_3]\!]\mu_k[\Phi]| \neq 0)$
where $\mathcal{F}_G[\![G_2]\!]\mu_k\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) = \Phi : \vec{\Phi}$
$\Theta \subseteq \mathcal{D}_G[\![p(\vec{x})]\!]\delta_P \Rightarrow \Theta \subseteq \downarrow\rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(mux(S_G[\![G_1]\!], S_G[\![G_2, G_3]\!]))$
hence (again by (5) above): $\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \subseteq mux(S_G[\![G_1]\!], S_G[\![G_2, G_3]\!])$
hence (again by (2) above):
$(\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_1]\!] = \{false\}) \vee (\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_2, G_3]\!] = \{false\})$
by Theorem 1: $\Phi \subseteq \bigcup(\mathcal{F}_G[\![G_2]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])) \subseteq \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_2]\!]$
by Theorem 1: $\bigcup(\mathcal{F}_G[\![G_3]\!]\mu_k\ [\Phi]) \subseteq \Phi \cap S_G[\![G_3]\!] \subseteq \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_2]\!] \cap S_G[\![G_3]\!]$
hence: $\bigcup(\mathcal{F}_G[\![G_3]\!]\mu_k[\Phi]) \subseteq \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_2, G_3]\!]$
hence: $\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_2, G_3]\!] = \{false\} \Rightarrow \mathcal{F}_G[\![G_3]\!]\mu_k[\Phi] = []$
hence: $\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_2, G_3]\!] = \{false\} \Rightarrow |\mathcal{F}_G[\![G_3]\!]\mu_k[\Phi]| = 0$
also (by Theorem 1): $\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_1]\!] = \{false\} \Rightarrow \mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta]) = []$
hence: $\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}(\Theta) \cap S_G[\![G_1]\!] = \{false\} \Rightarrow |\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| = 0$
hence:$(|\mathcal{F}_G[\![G_1]\!]\mu_k\downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| = 0) \vee (|\mathcal{F}_G[\![G_3]\!]\mu_k\ [\Phi]| = 0)$
hence: $\neg((|\mathcal{F}_G[\![G_1]\!]\mu_k \downarrow\rho_{\vec{x},\vec{y}}\overline{\exists}_{\vec{x}}([\Theta])| \neq 0) \vee (|\mathcal{F}_G[\![G_3]\!]\mu_k[\Phi]| \neq 0))$

2.2 Induction Step:
$G = G1, G2$:
Assume: $\Theta_1 \subseteq \mathcal{D}_G[\![G_1]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta_1]| \leq 1$
And: $\Theta_2 \subseteq \mathcal{D}_G[\![G_2]\!]\delta_P \Rightarrow |\mathcal{F}_G[\![G_2]\!]\mu_{k+1}[\Theta_2]| \leq 1$
Show: $\Theta \subseteq \mathcal{D}_G[\![G]\!] \Rightarrow |\mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta]| \leq 1$
where $\mu_{k+1} = \mathcal{F}_P[\![P]\!]\mu_k$

$\mathcal{D}_G[\![G]\!]\delta_P = (S_G[\![G_2]\!] \to \mathcal{D}_G[\![G_1]\!]\delta_P) \cap (S_G[\![G_1]\!] \to \mathcal{D}_G[\![G_2]\!]\delta_P)$
therefore if $\Theta \subseteq \mathcal{D}_G[\![G]\!]\delta_P$
  then $\Theta \subseteq (S_G[\![G_1]\!] \to \mathcal{D}_G[\![G_2]\!]\delta_P)$
  and hence $\Theta \cap S_G[\![G_1]\!] \subseteq \mathcal{D}_G[\![G_2]\!]\delta_P$
similarly if $\Theta \subseteq \mathcal{D}_G[\![G]\!]\delta_P$
  then $\Theta \subseteq (S_G[\![G_2]\!] \to \mathcal{D}_G[\![G_1]\!]\delta_P)$
  and hence $\Theta \cap S_G[\![G_2]\!] \subseteq \mathcal{D}_G[\![G_1]\!]\delta_P$
by Lemma 2: $\mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta] = \mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta \cap S_G[\![G]\!]]$
applying the definition of $\mathcal{F}_G$:
$\mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta \cap S_G[\![G]\!]] = \mathcal{F}_G[\![G_2]\!]\mu_{k+1}(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]])$
now notice that: $\Theta \cap S_G[\![G_1, G_2]\!]$
$= \Theta \cap S_G[\![G_1]\!] \cap S_G[\![G_2]\!]$
$\subseteq \Theta \cap S_G[\![G_2]\!]$
$\subseteq \mathcal{D}_G[\![G_1]\!]\delta_P$
hence by assumption: $|\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]]| \leq 1$

distinguish two cases:
(a) $|\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]]| = 0$,
(b) $|\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]]| = 1$

(a) $|\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]]| = 0$
$\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]] = []$
$\mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]] = \mathcal{F}_G[\![G_2]\!]\mu_{k+1}[] = []$
hence: $|\mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]]| \leq 1$
hence by Lemma 2 (remembering $G = G_1, G_2$): $|\mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta]| \leq 1$

(b) $|\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]]| = 1$
$\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]] = [\Psi]$
therefore: $\bigcup(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]]) = \Psi$
by Theorem 1: $\Psi \subseteq \Theta \cap S_G[\![G_1, G_2]\!] \cap S_G[\![G_1]\!] \subseteq \Theta \cap S_G[\![G_1]\!]$
hence since $\Theta \cap S_G[\![G_1]\!] \subseteq \mathcal{D}_G[\![G_2]\!]\delta_P$ (see above): $\Psi \subseteq \mathcal{D}_G[\![G_2]\!]\delta_P$
hence by assumption: $|\mathcal{F}_G[\![G_2]\!]\mu_{k+1}[\Psi]| \leq 1$
hence (again using Lemma 2): $|\mathcal{F}_G[\![G_2]\!]\mu_{k+1}(\mathcal{F}_G[\![G_1]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]])|$
$= |\mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta \cap S_G[\![G_1, G_2]\!]]|$
$= |\mathcal{F}_G[\![G]\!]\mu_{k+1}[\Theta]| \leq 1$
QED

2.5 Abstraction Proofs

2.5.1 Proposition 1: If $\Theta_1 \subseteq \gamma_{\vec{x}}(f_1)$ and $\gamma_{\vec{x}}(f_1) \subseteq \Theta_2$ then $\gamma_{\vec{x}}(f_1 \Rightarrow f_2) \subseteq \Theta_1 \rightarrow \Theta_2$

$\gamma_{\vec{x}}(f_1 \Rightarrow f_2)$
$= \bigcup\{\gamma_{\vec{x}}(f) \mid f \models f_1 \Rightarrow f_2\}$
$= \bigcup\{\Theta \mid \alpha_{\vec{x}}(\Theta) \models f_1 \Rightarrow f_2\}$
$= \bigcup\{\Theta \mid (\alpha_{\vec{x}}(\Theta) \models f_1) \Rightarrow (\alpha_{\vec{x}}(\Theta) \models f_2)\}$
$= \bigcup\{\Theta \mid (\Theta \subseteq \gamma_{\vec{x}}(f_1)) \Rightarrow (\Theta \subseteq \gamma_{\vec{x}}(f_2))\}$
$\subseteq \bigcup\{\Theta \mid (\Theta \subseteq \Theta_1) \Rightarrow (\Theta \subseteq \Theta_2)\}$
$= \bigcup\{\Theta \mid (\Theta \subseteq \Theta_1 \cap \Theta_2) \vee (\Theta \not\subseteq \Theta_1)\}$
$= \bigcup\{\Theta \mid \Theta \subseteq (\Theta_1 \cap \Theta_2) \cup (Con \setminus \Theta_1)\}$
$= \bigcup\{\Theta \mid \Theta \cap \Theta_1 \subseteq \Theta_2\}$
$= \Theta_1 \rightarrow \Theta_2$

2.5.2 Proposition 2: $\gamma_{\vec{x}}(mux^{\alpha}_{\vec{x}}(\Theta_1^{DK}, \Theta_2^{DK})) \subseteq mux(\Theta_1, \Theta_2)$

Proof:
First notice that by the definition of the Galois connection (i.e. of $\gamma()$ and $\alpha()$ the following:
$\gamma_{\vec{x}}(mux^{\alpha}_{\vec{x}}(\Theta_1^{DK}, \Theta_2^{DK})) \subseteq mux(\Theta_1, \Theta_2)$
is equivalent to: $\alpha_{\vec{x}}(\Psi) \models mux^{\alpha}_{\vec{x}}(\Theta_1^{DK}, \Theta_2^{DK}) \rightarrow \Psi \subseteq mux(\Theta_1, \Theta_2)$

Now: $\alpha_{\vec{x}}(\Psi) \models mux^{\alpha}_{\vec{x}}(\Theta_1^{DK}, \Theta_2^{DK})$ iff for each clause in $\alpha_{\vec{x}}(\Psi)$ there is a clause in $mux^{\alpha}_{\vec{x}}(\Theta_1^{DK}, \Theta_2^{DK})$ that is entailed by it, ie:
$\forall \psi \in \Psi . \exists Y \subseteq vars(\vec{x}).(\forall \theta_1 \in \Theta_1^{DK}.\forall \theta_2 \in \Theta_2^{DK}.$

$(\overline{\exists}_Y(\theta_1) \wedge \overline{\exists}_Y(\theta_2) = \mathit{false}) \wedge \alpha_{\vec{x}}(\psi) \models \bigwedge Y)$

Since $mux^{\alpha}_{\vec{x}}(\Theta_1^{DK}, \Theta_2^{DK})$ contains only positive (ie non-negated) literals, only the positive literals entailed by $\alpha_{\vec{x}}(\psi)$ are relevant.

Now, the positive literals entailed by $\alpha_{\vec{x}}(\psi)$ are exactly $vars(\vec{x}) \cap \mathit{fix}(\psi)$.

Therefore: $\psi \in \gamma_{\vec{x}}(mux^{\alpha}_{\vec{x}}(\Theta_1^{DK}, \Theta_2^{DK}))$

iff $\exists Y \subseteq (vars(\vec{x}) \cap \mathit{fix}(\psi)).(\forall \theta_1 \in \Theta_1^{DK}.\forall \theta_2 \in \Theta_2^{DK}.(\overline{\exists}_Y(\theta_1) \wedge \overline{\exists}_Y(\theta_2) = \mathit{false}))$

Now observe that the following three things hold:

(1) $\forall \phi \in \Phi.\exists \phi' \in \Phi^{DK}.(\phi \models \phi')$

(2) $((f_1 \models f_1') \wedge (f_2 \models f_2') \wedge (f_1' \wedge f_2' = \mathit{false})) \rightarrow f_1 \wedge f_2 = \mathit{false}$

(3) $\phi \models \phi' \rightarrow \overline{\exists}_Y(\phi) \models \overline{\exists}_Y(\phi')$

Therefore from $\forall \theta_1' \in \Theta_1^{DK}.\forall \theta_2' \in \Theta_2^{DK}.(\overline{\exists}_Y(\theta_1') \wedge \overline{\exists}_Y(\theta_2') = \mathit{false})$

it follows: $\forall \theta_1 \in \Theta_1.\forall \theta_2 \in \Theta_2.(\overline{\exists}_Y(\theta_1) \wedge \overline{\exists}_Y(\theta_2) = \mathit{false})$

And thus: $\overline{\exists}_Y(\Theta_1) \cap \overline{\exists}_Y(\Theta_2) = \{\mathit{false}\}$

Hence the following entailment holds:

$\forall \phi.(\exists Y \subseteq (vars(\vec{x}) \cap \mathit{fix}(\psi)).(\forall \theta_1 \in \Theta_1^{DK}.\forall \theta_2 \in \Theta_2^{DK}.(\overline{\exists}_Y(\theta_1) \wedge \overline{\exists}_Y(\theta_2) = \mathit{false}))$

$\models$

$\exists Y \subseteq \mathit{fix}(\phi).(\overline{\exists}_Y(\Theta_1) \cap \overline{\exists}_Y(\Theta_2) = \{\mathit{false}\}))$

Therefore: $\forall \phi.(\phi \in \gamma_{\vec{x}}(mux^{\alpha}_{\vec{x}}(\Theta_1^{DK}, \Theta_2^{DK})) \rightarrow \phi \in mux(\Theta_1, \Theta_2))$

From which it follows: $\gamma_{\vec{x}}(mux^{\alpha}_{\vec{x}}(\Theta_1^{DK}, \Theta_2^{DK})) \subseteq mux(\Theta_1, \Theta_2)$

2.6 Theorem 3: $\forall i \in \mathbb{N} : \gamma_{vars(G)}(\mathcal{D}^{\alpha}_G[\![G]\!]\delta^{\alpha}_i) \subseteq \mathcal{D}_G[\![G]\!]\delta_i$ where $\delta^{\alpha}_i/\delta_i$ are the results of i applications of $\mathcal{D}^{\alpha}_P[\![P]\!]/\mathcal{D}_P[\![P]\!]$ to $\delta^{\alpha}_{\top}/\delta_{\top}$ respectively.

Proof by nested induction on:

1. i,

2. the structure of G:

notice first that: $\gamma_{vars(\vec{x})}(\rho^{\alpha}_{\vec{y},\vec{x}}\overline{\forall}^{\alpha}_{\vec{y}}(f)) \subseteq \rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\gamma_{vars(\vec{y})}(f))$

1 Base Case: $i = 0$

$\delta^{\alpha}_0 = \delta^{\alpha}_{\top}$

$\delta_0 = \delta_{\top}$

Show: $\gamma_{vars(G)}(\mathcal{D}^{\alpha}_G[\![G]\!]\delta^{\alpha}_{\top}) \subseteq \mathcal{D}_G[\![G]\!]\delta_{\top}$

Induction on structure of G:

1.1 Two base cases: (1) $G = post(\phi)$, (2) $G = p(\vec{x})$

(1) $G = post(\phi)$

$\gamma_{vars(\phi)}(\mathcal{D}^{\alpha}_G[\![post(\phi)]\!]\delta^{\alpha}_{\top})$

$= \gamma_{vars(\phi)}(true)$

$= \Downarrow\{true\}$

$= \mathcal{D}_G[\![post(\phi)]\!]\delta_{\top}$

hence: $\gamma_{vars(\phi)}(\mathcal{D}^{\alpha}_G[\![post(\phi)]\!]\delta^{\alpha}_{\top}) \subseteq \mathcal{D}_G[\![post(\phi)]\!]\delta_{\top}$

(2) $G = p(\vec{x})$

$\gamma_{vars(\vec{x})}(\mathcal{D}^{\alpha}_G[\![p(\vec{x})]\!]\delta^{\alpha}_{\top})$

$= \gamma_{vars(\vec{x})}(\rho^{\alpha}_{\vec{y},\vec{x}}\overline{\forall}^{\alpha}_{\vec{y}}(true))$

$\subseteq \rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\gamma_{vars(\vec{y})}(true))$

$= \rho_{\vec{y},\vec{x}} \overline{\forall}_{\vec{y}}(\Downarrow\{true\})$
$= \mathcal{D}_G[\![p(\vec{x})]\!]\delta_\top$

1.2 Induction step: $G = G_1, G_2$
Assume: $\gamma_{vars(G_{1/2})}(\mathcal{D}^\alpha_G[\![G_{1/2}]\!]\delta^\alpha_\top) \subseteq \mathcal{D}_G[\![G_{1/2}]\!]\delta_\top$

$$\begin{aligned}
&\gamma_{vars(G_1,G_2)}(\mathcal{D}^\alpha_G[\![G_1, G_2]\!]\delta^\alpha_\top)\\
&= \gamma_{vars(G_1,G_2)}((S^\alpha_G[\![G_2]\!] \Rightarrow \mathcal{D}^\alpha_G[\![G_1]\!]\delta^\alpha_\top) \wedge (S^\alpha_G[\![G_1]\!] \Rightarrow \mathcal{D}^\alpha_G[\![G_2]\!]\delta^\alpha_\top))\\
&\subseteq \gamma_{vars(G_1,G_2)}(S^\alpha_G[\![G_2]\!] \Rightarrow \mathcal{D}^\alpha_G[\![G_1]\!]\delta^\alpha_\top) \cap \gamma_{vars(G_1,G_2)}(S^\alpha_G[\![G_1]\!] \Rightarrow \mathcal{D}^\alpha_G[\![G_2]\!]\delta^\alpha_\top)\\
&\quad \text{(by monotonicity i.e. } \gamma_{vars(G_1,G_2)}(f_1 \wedge f_2) \subseteq \gamma_{vars(G_1,G_2)}(f_i))\\
&\subseteq (S_G[\![G_2]\!] \rightarrow \mathcal{D}_G[\![G_1]\!]\delta_\top) \cap (S_G[\![G_1]\!] \rightarrow \mathcal{D}_G[\![G_2]\!]\delta_\top)\\
&\quad \text{(by Proposition 1 and Proposition 3 and the induction assumption)}\\
&= \mathcal{D}_G[\![G_1, G_2]\!]\delta_\top
\end{aligned}$$

2 Induction step: $i = k + 1$
Assume: $\gamma_{vars(G)}(\mathcal{D}^\alpha_G[\![G]\!]\delta^\alpha_k) \subseteq \mathcal{D}_G[\![G]\!]\delta_k$
Show: $\gamma_{vars(G)}(\mathcal{D}^\alpha_G[\![G]\!]\delta^\alpha_{k+1}) \subseteq \mathcal{D}_G[\![G]\!]\delta_{k+1}$
where $\delta_{k+1} = \mathcal{D}_P[\![P]\!]\delta_k$ and $\delta^\alpha_{k+1} = \mathcal{D}^\alpha_P[\![P]\!]\delta^\alpha_k$

Induction on structure of G:
2.1 Two base cases: (1) $G = post(\phi)$, (2) $G = p(\vec{x})$
(1) $G = post(\phi)$
$\gamma_{vars(\phi)}(\mathcal{D}^\alpha_G[\![post(\phi)]\!]\delta^\alpha_{k+1})$
$= \gamma_{vars(\phi)}(true)$
$= \Downarrow\{true\}$
$= \mathcal{D}_G[\![post(\phi)]\!]\delta_{k+1}$
hence: $\gamma_{vars(\phi)}(\mathcal{D}^\alpha_G[\![post(\phi)]\!]\delta^\alpha_\top) \subseteq \mathcal{D}_G[\![post(\phi)]\!]\delta_\top$

(2) $G = p(\vec{x})$
Assume (without loss of generality): $p(\vec{y}) \leftarrow G_1; G_2, !, G_3; G_4 \in P$

$$\begin{aligned}
&\gamma_{vars(\vec{x})}(\mathcal{D}^\alpha_G[\![p(\vec{x})]\!]\delta^\alpha_{k+1})\\
&= \gamma_{vars(\vec{x})}(\rho^\alpha_{\vec{y},\vec{x}}(\overline{\forall}^\alpha_{\vec{y}}(\mathcal{D}^\alpha_G[\![p(\vec{y})]\!]\delta^\alpha_{k+1})))\\
&= \gamma_{vars(\vec{x})}(\rho^\alpha_{\vec{y},\vec{x}}(\overline{\forall}^\alpha_{\vec{y}}(\overline{\forall}^\alpha_{\vec{y}}(\mathcal{D}^\alpha_G[\![G_1]\!]\delta^\alpha_k \wedge (S^\alpha_G[\![G_2]\!] \Rightarrow \mathcal{D}^\alpha_G[\![G_3]\!]\delta^\alpha_k) \wedge \mathcal{D}^\alpha_G[\![G_4]\!]\delta^\alpha_k\\
&\quad \wedge mux^\alpha_{vars(\vec{y})}(S^{DK}_G[\![G_1]\!], S^{DK}_G[\![G_4]\!])\\
&\quad \wedge mux^\alpha_{vars(\vec{y})}(S^{DK}_G[\![G_1]\!], S^{DK}_G[\![G_2, G_3]\!])))))\\
&= \gamma_{vars(\vec{x})}(\rho^\alpha_{\vec{y},\vec{x}}(\overline{\forall}^\alpha_{\vec{y}}(\mathcal{D}^\alpha_G[\![G_1]\!]\delta^\alpha_k \wedge (S^\alpha_G[\![G_2]\!] \Rightarrow \mathcal{D}^\alpha_G[\![G_3]\!]\delta^\alpha_k) \wedge \mathcal{D}^\alpha_G[\![G_4]\!]\delta^\alpha_k\\
&\quad \wedge mux^\alpha_{vars(\vec{y})}(S^{DK}_G[\![G_1]\!], S^{DK}_G[\![G_4]\!])\\
&\quad \wedge mux^\alpha_{vars(\vec{y})}(S^{DK}_G[\![G_1]\!], S^{DK}_G[\![G_2, G_3]\!]))))\\
&\subseteq \rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\gamma_{vars(\vec{y})}(\mathcal{D}^\alpha_G[\![G_1]\!]\delta^\alpha_k \wedge (S^\alpha_G[\![G_2]\!] \Rightarrow \mathcal{D}^\alpha_G[\![G_3]\!]\delta^\alpha_k) \wedge \mathcal{D}^\alpha_G[\![G_4]\!]\delta^\alpha_k\\
&\quad \wedge mux^\alpha_{vars(\vec{y})}(S^{DK}_G[\![G_1]\!], S^{DK}_G[\![G_4]\!])\\
&\quad \wedge mux^\alpha_{vars(\vec{y})}(S^{DK}_G[\![G_1]\!], S^{DK}_G[\![G_2, G_3]\!])))\\
&\subseteq \rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\gamma_{vars(\vec{y})}(\mathcal{D}^\alpha_G[\![G_1]\!]\delta^\alpha_k) \cap (\gamma_{vars(\vec{y})}(S^\alpha_G[\![G_2]\!]) \rightarrow \gamma_{vars(\vec{y})}(\mathcal{D}^\alpha_G[\![G_3]\!]\delta^\alpha_k)) \cap \gamma_{vars(\vec{y})}(\mathcal{D}^\alpha_G[\![G_4]\!]\delta^\alpha_k)\\
&\quad \cap \gamma_{vars(\vec{y})}(mux^\alpha_{vars(\vec{y})}(S^{DK}_G[\![G_1]\!], S^{DK}_G[\![G_4]\!]))\\
&\quad \cap \gamma_{vars(\vec{y})}(mux^\alpha_{vars(\vec{y})}(S^{DK}_G[\![G_1]\!], S^{DK}_G[\![G_2, G_3]\!])))
\end{aligned}$$

$\subseteq \rho_{\vec{y},\vec{x}}\overline{\forall}_{\vec{y}}(\mathcal{D}_G[\![G_1]\!]\delta_k \cap (S_G[\![G_2]\!] \rightarrow \mathcal{D}_G[\![G_3]\!]\delta_k) \cap \mathcal{D}_G[\![G_4]\!]\delta_k$
$\quad \cap\, mux(S_G[\![G_1]\!], S_G[\![G_4]\!])$
$\quad \cap\, mux(S_G[\![G_1]\!], S_G[\![G_2, G_3]\!]))$
$= \mathcal{D}_G[\![p(\vec{x})]\!]\delta_{k+1}$

2.2 Induction step: $G = G_1, G_2$
Assume: $\gamma_{vars(G_{1/2})}(\mathcal{D}^{\alpha}_G[\![G_{1/2}]\!]) \subseteq \mathcal{D}_G[\![G_{1/2}]\!]$
again, notice that: (1) $A \subseteq B \Rightarrow \gamma_B(f) \subseteq \gamma_A(f)$
and: $vars(G_1, G_2) = vars(G_1) \cup vars(G_2)$
and hence: (2^1) $vars(G_1) \subseteq vars(G_1, G_2)$
and similarly: (2^2) $vars(G_2) \subseteq vars(G_1, G_2)$
$\quad \gamma_{vars(G_1,G_2)}(\mathcal{D}^{\alpha}_G[\![G_1, G_2]\!]\delta^{\alpha}_{k+1})$
$= \gamma_{vars(G_1,G_2)}((S^{\alpha}_G[\![G_2]\!] \Rightarrow \mathcal{D}^{\alpha}_G[\![G_1]\!]\delta^{\alpha}_{k+1}) \wedge (S^{\alpha}_G[\![G_1]\!] \Rightarrow \mathcal{D}^{\alpha}_G[\![G_2]\!]\delta^{\alpha}_{k+1}))$
$\subseteq \gamma_{vars(G_1,G_2)}((S^{\alpha}_G[\![G_2]\!] \Rightarrow \mathcal{D}^{\alpha}_G[\![G_1]\!]\delta^{\alpha}_{k+1})) \cap \gamma_{vars(G_1,G_2)}((S^{\alpha}_G[\![G_1]\!] \Rightarrow \mathcal{D}^{\alpha}_G[\![G_2]\!]\delta^{\alpha}_{k+1}))$
(by monotonicity i.e. $\gamma_{vars(G_1,G_2)}(f_1 \wedge f_2) \subseteq \gamma_{vars(G_1,G_2)}(f_i)$)
$\subseteq \gamma_{vars(G_1,G_2)}(S^{\alpha}_G[\![G_2]\!]) \rightarrow \gamma_{vars(G_1,G_2)}(\mathcal{D}^{\alpha}_G[\![G_1]\!]\delta^{\alpha}_{k+1}) \cap \gamma_{vars(G_1,G_2)}(S^{\alpha}_G[\![G_1]\!]) \rightarrow \gamma_{vars(G_1,G_2)}(\mathcal{D}^{\alpha}_G[\![G_2]\!]\delta^{\alpha}_{k+1})$
(by Proposition 1 and Proposition 3 and the induction assumption)
$\subseteq \gamma_{vars(G_2)}(S^{\alpha}_G[\![G_2]\!]) \rightarrow \gamma_{vars(G_1)}(\mathcal{D}^{\alpha}_G[\![G_1]\!]\delta^{\alpha}_{k+1}) \cap \gamma_{vars(G_1)}(S^{\alpha}_G[\![G_1]\!]) \rightarrow \gamma_{vars(G_2)}(\mathcal{D}^{\alpha}_G[\![G_2]\!]\delta^{\alpha}_{k+1})$
(by (1), (2^1) and (2^2) above)
$\subseteq S_G[\![G_2]\!] \rightarrow \mathcal{D}_G[\![G_1]\!]\delta_{k+1} \cap S_G[\![G_1]\!] \rightarrow \mathcal{D}_G[\![G_2]\!]\delta_{k+1}$
$= \mathcal{D}_G[\![G_1, G_2]\!]\delta_{k+1}$
QED

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