Fragments of Spider Diagrams of Order and their Relative Expressiveness

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Abstract. Investigating the expressiveness of a diagrammatic logic provides insight into how its syntactic elements interact at the semantic level. Moreover, it allows for comparisons with other notations. Various expressiveness results for diagrammatic logics are known, such as the theorem that Shin's Venn-II system is equivalent to monadic first order logic. The techniques employed by Shin for Venn-II were adapted to allow the expressiveness of Euler diagrams to be investigated. We consider the expressiveness of spider diagrams of order (SDoO), which extend spider diagrams by including syntax that provides ordering information between elements. Fragments of SDoO are created by systematically removing each aspect of the syntax. We establish the relative expressiveness of the various fragments. In particular, one result establishes that spiders are syntactic sugar in any fragment that contains order, negation and shading. We also show that shading is syntactic sugar in any fragment containing negation and spiders. The existence of syntactic redundancy within the spider diagram of order logic is unsurprising however, we find it interesting that spiders or shading are redundant in fragments of the logic. Further expressiveness results are presented throughout the paper. The techniques we employ may well extend to related notations, such as the Euler/Venn logic of Swoboda et al. and Kent's constraint diagrams.

1 Introduction

Recent years have seen the development of a number of diagrammatic logics, including constraint diagrams [1], existential graphs [2], Euler diagrams [3], Euler/Venn [4], spider diagrams [5], and Venn-II [6]. Each of these logics, except constraint diagrams, have sound and complete reasoning systems; for constraint diagrams, complete fragments exist, such as that in [7]. Recently, an extension of spider diagrams has been proposed that permits the specification of ordering information on the universal set [8]; this extension is called spider diagrams of order and is the primary focus of this paper.

By contrast to the relatively large body of work on reasoning with these logics, relatively little exploration has been conducted into their expressive power. To our knowledge, the first expressiveness result for formal diagrammatic logics was due to Shin, who proved that her Venn-II system is equivalent to Monadic First Order Logic (MFOL) [6]; recall, in MFOL all predicate symbols are one place. Her proof strategy used syntactic manipulations of sentences in MFOL, turning them into a normal form that could easily be translated into a Venn-II diagram. Shin's strategy was adapted to establish that the expressiveness of Euler diagrams with shading was also that of MFOL [9]. Thus, the general techniques used to investigate and evaluate expressiveness in one notation may be helpful in other domains.

It has also been shown that spider diagrams are equivalent to MFOL with equality [10]; MFOL[=] extends MFOL by including =, allowing one to assert the distinctness of elements. To establish the expressiveness of spider diagrams, a different approach to that of Shin's for Venn-II was utilized. The proof strategy involved a model theoretic analysis of the closure properties of the model sets for the formulas of the language. In the case of spider diagrams of order, so-called becasue they provide ordering constraints on elements, it has been shown that they are equivalent to MFOL of Order [11]; MFOL[$<$] extends MFOL by including $<$, which is interpreted as a strict total order. MFOL[$<$] is strictly more expressive than MFOL[=] which, in turn, is strictly more expressive than MFOL. For this expressiveness result, spider diagrams of order were not directly compared MFOL[$<$]. Instead, it was shown that spider diagrams of order could define precisely the star-free regular languages. It is known that these languages are also precisely those definable by MFOL[$<$] [12].

In this paper, we establish the relative expressiveness of fragments of spider diagrams of order. If two distinct fragments are equivalent in expressive power then this gives insight into what may be expressed by syntactically different but semantically equivalent fragments. Such insight allows one to consider the manner in which any particular semantic concept may be defined syntactically, possibly leading to more helpful or more appropriate diagrams. If two fragments have differing expressive power then this allows us to identify when certain syntactic elements are necessary for formulating particular semantic concepts. This can allow for more effective diagrams to be chosen when defining concepts. In section 2, we define the syntax and semantics of spider diagrams of order. In section 3, we identify natural fragments of spider diagrams of order and our novel expressiveness results concerning their relative expressiveness.

2 Spider Diagrams of Order

This section provides a brief overview of spider diagrams of order (SDoO), slightly modified from [8]. Diagram d_1 in figure 1 contains two labelled closed curves, called *contours*. The diagram d_1 contains four minimal regions, called *zones*: one zone is inside just P, another inside just Q, and another is outside both P and Q. The zone inside both P and Q of d_1 is *shaded*. This diagram also contains two *spiders*, s and r. The diagram $d_2 \lhd d_3$ is a compound spider diagram of order.

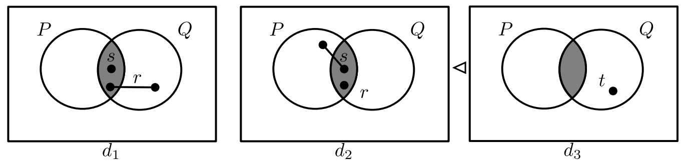

Fig. 1. A unitary diagram and a compound spider diagram.

First, we formally define the syntax, before proceeding to the semantics. The contour labels in spider diagrams are selected from a set $\mathcal{C}$. A *zone* is defined to be a pair, (in, out), of finite, disjoint subsets of $\mathcal{C}$. The set in contains the labels of the contours that the zone is inside whereas out contains the labels of the contours that the zone is outside. The set of all zones is denoted $\mathcal{Z}$. A *region* is a set of zones. To describe the spiders in a diagram, it is sufficient to say how many spiders are placed in each region. Thus, the abstract definition of a spider diagram will specify the labels used, the zones, the shaded zones and use a set of spider identifiers to describe the spiders.

Definition 1 (Delaney et. al. [11]). *A* ***unitary spider diagram of order****, d, is a quadruple $\langle C, Z, ShZ, SI \rangle$ where:*

1. *$C = C(d) \subseteq \mathcal{C}$ is a finite set of contour labels,*
2. *$Z = Z(d) \subseteq \{(in, C - in) : in \subseteq C\}$ is a set of zones,*
3. *$ShZ \subseteq Z(d)$ is a set of shaded zones, and*
4. *$SI = SI(d) \subsetneq \mathbb{N}^+ \times \mathbb{P}Z$ is a finite set of spider identifiers such that for all $(n, r), (m, s) \in SI(d)$ if $r = s$ then $n = m$.*

The set of spiders in d is defined to be $S(d) = \{(i, r) : (n, r) \in SI(d) \wedge 1 \leq i \leq n\}$. The symbol $\perp$ is also a unitary spider diagram of order. If d_1 and d_2 are spider diagrams of order then $(d_1 \vee d_2)$, $(d_1 \wedge d_2)$, $(d_1 \lhd d_2)$ and $\neg d_1$ are ***spider diagrams of order****. Any diagram that is not a unitary diagram is a* ***compound*** *diagram.*

The abstract syntax of the diagram d_1 in figure 1 is

$$\begin{aligned}
C(d_1) &= \{P, Q\}, \\
Z(d_1) &= \{(\{\}, \{P, Q\}), (\{P\}, \{Q\}), (\{Q\}, \{P\}), (\{P, Q\}, \{\})\}, \\
ShZ(d_1) &= \{(\{P, Q\}, \{\})\}, \\
SI(d_1) &= \{(1, \{(\{P, Q\}, \{\})\}), (1, \{(\{Q\}, \{P\}), (\{P, Q\}), \{\}\})\}.
\end{aligned}$$

By convention, we employ a lower-case d to denote a spider diagram. An upper case D will denote an arbitrary diagram. It is also useful to identify which zones could be present in a unitary diagram, given the label set, but are not present; semantically, *missing* zones provide information.

Definition 2 (Howse et. al. [5]). *Given a unitary diagram, d, a zone* (in, out) *is* ***missing*** *from d if it is in the set* $\{(in, C(d) - in) : in \subseteq C(d)\} - Z(d)$ *with the set of such zones denoted* $MZ(d)$*. If* $MZ(d) = \emptyset$ *then d is in* ***Venn form****.*

Unitary diagrams make statements about sets (represented by contours) and their cardinalities (by using spiders and shading). The spiders in d_1, figure 1, represent distinct elements in the sets represented by the regions in which they are placed; spiders provide lower bounds on set cardinality. The spider r provides disjunctive information: the element it represents is in one of the sets represented by the zones in which it is placed. Shading places an upper bound on set cardinality: in a shaded region, all elements must be represented by spiders. Taken together, the spiders s and r allow for the set represented by the shaded zone to contain between 1 and 2 elements. The semantics of spider diagrams are model-based: a model is an assignment of sets to contour labels that agrees with the intended meaning of the diagram.

Definition 3 (Delaney et. al. [11]). *An* ***interpretation*** *is a triple* $I = (U, <, \Psi)$ *where U is called the universal set and* $\Psi : \mathcal{C} \rightarrow \mathbb{P}U$ *is a function that assigns a subset of U to each contour label and < is a strict total order on U. The function* Ψ *can be extended to interpret zones and regions as follows:*

1. *each zone,* $(a, b) \in \mathcal{Z}$*, represents the set* $\bigcap_{l \in a} \Psi(l) \cap \bigcap_{l \in b} (U - \Psi(l))$ *and*
2. *each region,* $r \in \mathbb{P}\mathcal{Z}$*, represents the set which is the union of the sets represented by r's constituent zones.*

If $U = \emptyset$ *then I is the* ***empty*** *interpretation.*

Definition 4 (Delaney et. al. [11]). *Let* $I = (U, <, \Psi)$ *be an interpretation and let d* $(\neq \perp)$ *be a unitary spider diagram. Then I is a* ***model*** *for d, denoted* $m \models d$*, if and only if the following conditions hold.*

1. ***The missing zones condition*** *All of the missing zones represent the empty set, that is,* $\bigcup_{z \in MZ(d)} \Psi(z) = \emptyset$.
2. ***The function extension condition*** *There exists an extension of* Ψ *to spiders,* $\Psi : \mathcal{C} \cup S(d) \rightarrow \mathbb{P}U$ *for which the following hold.*
   (a) ***The spiders' locations condition*** *All spiders represent elements (strictly, singleton sets) in the sets represented by the regions in which they are placed:* $\forall (i, r) \in S(d)\, (\Psi(i, r) \subseteq \Psi(r) \wedge |\Psi(i, r)| = 1)$.
   (b) ***The distinct spiders condition*** *Distinct spiders denote distinct elements:* $\forall s_1, s_2 \in S(d) (\Psi(s_1) = \Psi(s_2) \Rightarrow s_1 = s_2)$.
   (c) ***The shading condition*** *Shaded regions represent a subset of elements denoted by spiders:* $\Psi(ShZ(d)) \subseteq \bigcup_{s \in S(d)} \Psi(s)$.

If $d = \perp$ *then no interpretation is a model for d.*

The interpretation $m = (U, <, \Psi)$ where $U = \{1, 2, 3, 4\}$, $<$ is the natural order over U, $\Psi(P) = \{2\}$ and $\Psi(Q) = \{2, 3\}$ is a model for the diagram d_1 in figure 1, but not for d_2 or d_3. For compound diagrams, the definition of a model extends inductively. In the case of $\neg D_1$, $D_1 \vee D_2$ and $D_1 \wedge D_2$ the extension is obvious. A diagram of the form $D_1 \lhd D_2$ provides a constraint on the interpretation of $<$. For instance, $d_2 \lhd d_3$ in figure 1 asserts, in part, that no elements in $P \cap Q$ can be ordered after the elements represented by s and r in d_2. We need to ensure that the ordering information provided by an interpretation respects the intended meaning of the diagram.

Definition 5 (adapted from Ebbinghaus & Flum [13]). *Let $I_1 = (U_1, <_1, \Psi_1)$ and $I_2 = (U_2, <_2, \Psi_2)$ be interpretations where U_1 and U_2 are disjoint. The* ***ordered sum*** *of I_1 and I_2, denoted $I_1 + I_2$, is defined to be the interpretation $I_3 = (U_3, <_3, \Psi_3)$ such that*

1. *$U_3 = U_1 \cup U_2$*
2. *$<_3 = <_1 \cup <_2 \cup \{(u_1, u_2) : u_1 \in U_1 \wedge u_2 \in U_2\}$, and*
3. *for each $c \in \mathcal{C}$, $\Psi_3(c) = \Psi_1(c) \cup \Psi_2(c)$.*

Definition 6. *Let $I = (U, <, \Psi)$ be an interpretation and let D be a compound diagram. Then I is a* ***model*** *for D provided:*

1. *if $D = D_1 \vee D_2$ then I models D whenever I models D_1 or I models D_2,*
2. *if $D = D_1 \wedge D_2$ then I models D whenever I models D_1 and I models D_2,*
3. *if $D = \neg D_1$ then I models D whenever I does not model D_1, and*
4. *if $D = D_1 \lhd D_2$ then I models D whenever there exist interpretations I_1 and I_2 such that $I = I_1 + I_2$ and I_1 models D_1 and I_2 models D_2.*

For the purpose of establishing relative expressiveness, we need the notion of *satisfiability* and to know when two diagrams are *semantically equivalent*.

Definition 7 (Delaney et. al. [11]). *Spider diagrams of order, D_1 and D_2, are* ***semantically equivalent*** *provided they have exactly the same models. If D_1 has a model then we say that D_1 is* ***satisfiable****.*

3 Expressiveness

We will now establish the relative expressiveness of various fragments of spider diagrams of order. In subsection 3.1 we define our notation for discussing fragments of SDoO and in subsection 3.2 we summarize previously known expressiveness results. Then in subsections 3.3 and 3.4 we provide definitions and results that are helpful for our analysis. The remainder of this section provides new expressiveness results.

	$[CDNOSpSh]$	$-C$	$-D$	$-N$	$-O$	$-Sp$	$-Sh$	
$-C$	$=$	$-$			$=$	$=$	$=$	$-C$
$-D$	$=$		$-$		$=$	$=$	$=$	$-D$
$-N$				$-$	$=$			$-N$
$-O$	$<$	$<$	$<$	$<$	$-$	$<$	$<$	$-O$
$-Sp$					$<$	$-$		$-Sp$
$-Sh$							$-$	$-Sh$

Table 1. Summary of known relative expressiveness results.

3.1 Fragments of Spider Diagrams of Order

We observe that spider diagrams of order can be thought of as being built from Euler diagrams, with various syntactic additions. We view (unitary) Euler diagrams as the basic building blocks and this motivates our method of defining natural fragments of SDoO. To these basic building blocks we can add connectives ($\wedge$, $\vee$, $\lhd$), the negation operation ($\neg$), spiders, and shading. Using any set of these additions to Euler diagrams gives rise to a fragment of SDoO.

We denote the unitary Euler diagrams fragment by ED. Using notation similar to that seen in description logics, $ED[C]$ is taken to be the class of Euler diagrams formed by joining them with the conjunction, $\wedge$, operator. Equivalently, this is the fragment of SDoO in which there are no spiders, no shading, no negation, and the only logical connective is $\wedge$. If we wanted to include spiders, Sp and conjunction, C, but no other operators and no shading then this fragment would be denoted by $ED[C, Sp]$. Given some list, $IncSyn$, that is a sublist of $[C, D, N, O, Sp, Sh]$, the fragment $ED[IncSyn]$ is then defined in the obvious manner, where C=conjunction, D=disjunction, N=negation, O=order ($\lhd$), Sp=spiders, and Sh=shading. Thus, 'full' SDoO is $ED[C, D, N, O, Sp, Sh]$. Importantly, we define it to be the case that fragments with no shading also do not include unitary diagrams with missing zones, since such zones can be replaced by shaded zones. We will frequently omit ED from the fragment description and write, for example, $[CSp]$ rather than $ED[C, Sp]$.

3.2 Known expressiveness results

Known results for relative expressive power are summarized in table 1; all of these results were presented in the introduction, follow immediately from them, or appear elsewhere in the literature (primarily in [5]). The column headings give a fragment of SDoO, with the second column considering SDoO: $[CDNOSpSh]$. The third through eight columns define the fragment of $[CDNOShSp]$ without the syntax indicated by the heading i.e. the $-C$ column is the fragment $[DNOSpSh]$. Similarly, each row removes a (second) piece of syntax from the fragment, giving another fragment. Thus, column 3 in row 5 identifies that

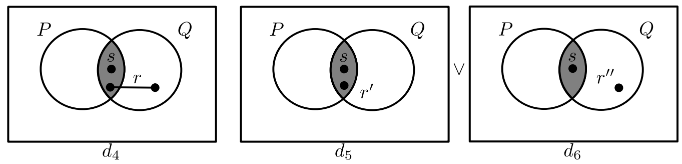

Fig. 2. Creating an α-diagram.

$[DNOSpSh]$ has greater expressiveness than $[DNSpSh]$. In this paper, we complete most of the missing entries in table 1.

3.3 The α-diagram fragments

Spiders whose habitats comprise more than one zone make disjunctive statements within a unitary diagram. However, it has been observed that this disjunctive information can also be made using a compound diagram. For example, d_4 in figure 2 is semantically equivalent to $d_5 \vee d_6$. One approach to investigating expressiveness is to consider only diagrams whose spiders are placed in single zones. Such diagrams are called **α-diagrams** [5].

Theorem 1 (Howse et al. [5]). *Every unitary diagram is semantically equivalent to a disjunction of unitary α-diagrams.*

Theorem 2. *Let D_1 be drawn from a fragment, F, of SDoO that contains*

1. *disjunction (D), or*
2. *conjunction (C) and negation (N).*

Then there exists an α-diagram, D_2, also in F, such that D_1 is semantically equivalent to D_2.

Proof (Sketch). The proof follows by induction on the depth of D_1 in the inductive construction of diagrams, with the base case provided by theorem 1.

3.4 Literals

As well as reducing expressiveness questions to those for α-diagrams, it is also helpful to consider unitary diagrams that contain information in, at most, one zone. For example, the unitary α-diagram, d_7, in figure 3 contains exactly two zones which provide semantic information, and is semantically equivalent to $d_8 \wedge d_9$. The diagrams d_8 and d_9 are called *literals*, since they give information about exactly one zone; we say that they are *literal parts* of d_7. All diagrams in this example are in Venn form; missing zones would provide semantic information and we are seeking diagrams that provide information about a single zone. Our definition of a literal extends that of an Euler diagram literal [9].

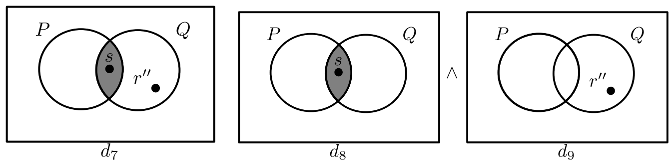

Fig. 3. Creating a diagram in literal form.

Definition 8. *Let d be a unitary α-diagram in Venn form that contains at most one zone which contains spiders or shading. Then d and $\neg d$ are called **literals**. The diagram d is a **positive** literal, whereas $\neg d$ is a **negative** literal.*

Definition 9. *Let D be an SDoO. If each unitary part of D is a literal or $\perp$ then D is in **literal form**.*

Definition 10. *Let d_1 ($\neq\perp$) be a unitary α-diagram. A **literal part** of d_1 is a positive literal, d_2, that is formed from d_1 by adding all missing zones to the zone set and shaded zone set and, subsequently, deleting the spiders and shading from all except at most one zone.*

Theorem 3. *Let d ($\neq\perp$) be a unitary α-diagram. Then d is semantically equivalent to the conjunction of its literal parts.*

Proof (Sketch). Since d is a unitary α-diagram it contains no disjunctive information, and so the semantics of the whole diagram is equivalent to the conjunction of the constraints in each zone.

Theorem 4. *Let D_1 be drawn from a fragment, F, of SDoO that contains either*

1. *disjunction (D) and conjunction (C), or*
2. *disjunction and negation (N), or*
3. *conjunction and negation (N).*

Then there exists D_2, also in F, in literal form such that D_1 is semantically equivalent to D_2.

Proof (Sketch). Noting that conversion of D_1 to an α-diagram requires either disjunction or both conjunction and negation, we can use theorem 2 to reduce D_1 to an α-diagram. The proof then follows by induction on the depth of D_1 in the inductive construction of diagrams, with the base case provided by theorem 3.

3.5 Removing spiders

Some of our fragments do not contain spiders so we need to know whether their absence impacts expressiveness. Intuitively, one might expect their removal to decrease expressiveness, but this is not always so. Figure 4 demonstrates that

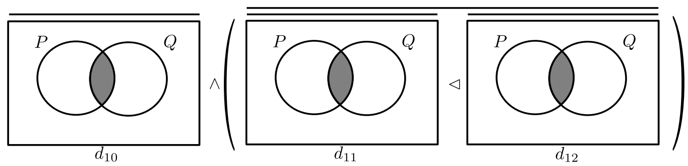

Fig. 4. Removing spiders from literal d_8 in figure 3.

it is possible to remove spiders from a positive literal without altering expressiveness provided we have access to negation, order, and shading: d_8 (figure 3) is semantically equivalent to $\neg d_{10} \wedge \neg(\neg d_{11} \lhd \neg d_{12})$, in figure 4.

Definition 11. *Let d_1 be a positive literal with a zone z containing spiders. Then the* ***spider-free*** *diagram associated with d_1 is a copy of d_1 except that z contains no spiders and z is shaded.*

We adopt the notation d^n to mean $d \lhd d \lhd ... \lhd d$ (n times).

Theorem 5. *Let d_1 be a positive literal, with the zone z containing exactly n spiders. Let d_2 be the spider-free diagram associated with d_1. If z is not shaded then d_1 is semantically equivalent to $(\neg d_2)^n$. If z is shaded then d_1 is semantically equivalent to $(\neg d_2)^n \wedge \neg(\neg d_2)^{n+1}$.*

Proof (Sketch). The models of $\neg d_2$ are those interpretations containing at least one element in $\Psi(z)$. The models of $\neg d_2 \lhd \neg d_2$ are, therefore, those interpretations which contain at least two elements in $\Psi(z)$. The result follows.

Theorem 6. *Let D_1 be be a diagram drawn from any fragment, F, of spider diagrams of order, provided that F contains negation (N), order (O), and shading (Sh), and at least one of conjunction (C) and disjunction (D). Then D_1 is semantically equivalent to some diagram, D_2, also in F, such that D_2 contains no spiders.*

Proof. Since we have negation, having one or both of conjunction and disjunction does not alter expressiveness. Thus, without loss of generality, our proof assumes we have access to both C and D. Theorem 2 allows us to replace D_1 by an α-diagram, whilst remaining within F. Theorem 4 allows us to reduce the α-diagram to literal form, again whilst remaining within F (this replacement uses C). The result then essentially follows by theorem 5 (which uses N and C).

Theorem 6 allows us to complete some of row concerning removal of spiders in table 1; see table 2 (new results shown in bold typeface).

Theorem 7. *Euler diagrams of order are equivalent in expressiveness to SDoO.*

Proof (Sketch). This theorem is a restatement of theorem 6 with respect to the specific fragment $ED[C, D, N, O, Sh]$.

	$[CDNOSpSh]$	$-C$	$-D$	$-N$	$-O$	$-Sp$	$-Sh$	
$-Sp$	$=$	$=$	$=$		$<$	–		$-Sp$

Table 2. Expressiveness results when removing spiders.

There are two entries to be completed in table 2. Concerning the first, spiders are removed from $ED[C, D, O, Sp, Sh]$ to give $ED[C, D, O, Sh]$. The following theorem allows us to deduce that this reduces expressiveness.

Theorem 8. *Let $D \in ED[C, D, O, Sh]$. If D is satisfiable then $I = (\emptyset, <, \Psi)$ models D.*

Proof (Sketch). The proof proceeds by induction on the depth of D in the inductive construction. Assume D has a model. Trivially, if D is a unitary diagram then D contains no spiders and I models D. If $D = D_1 \wedge D_2$ or $D = D_1 \vee D_2$ then the result follows trivially. Consider $D = D_1 \lhd D_2$. D is satisfiable if and only if both D_1 and D_2 are satisfiable. Should this be the case, I models D_1 and I models D_2, by assumption. Now, $I = I + I$, so I models D. Hence D is modelled by the empty interpretation.

Corollary 1. *$ED[C, D, O, Sp, Sh]$ is more expressive than $ED[C, D, O, Sh]$.*

Theorem 9. *Any diagram drawn from $ED[C, D, N, O]$ is modelled by every interpretation or is not modelled by any interpretation.*

Proof (Sketch). The property of being equivalent to true or equivalent to false holds for the unitary diagrams in this language, and the property is preserved when formulas are conjoined, disjoined, negated or connected with product, (O).

Thus, the $ED[C, D, N, O]$ fragment is not terribly interesting: it can only make statements that are either valid or contradictory. We immediately have the following corollary and are now able complete the 'remove spiders' row.

Corollary 2. *$ED[C, D, N, O, Sp]$ is more expressive than $ED[C, D, N, O]$.*

3.6 Removing shading

We now proceed to show that, under some circumstances, shading is syntactic sugar. For example, the diagram $d_{13} \wedge d_{14}$ presented in figure 3 is semantically equivalent to d_8 in figure 5. Intuitively, d_8 tells us that the shaded zone represents a set containing exactly 1 element, which is equivalent to saying there are at least 1 element (d_{13}) and not at least 2 elements ($\neg d_{14}$).

Theorem 10. *Let d be a positive literal with a shaded zone, z. Let d_1 be a copy of d, except that z contains no shading. Let d_2 be a copy of d except that z contains no shading and exactly one more spider than in d. Then d is semantically equivalent to $d_1 \wedge \neg d_2$.*

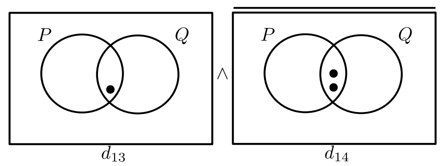

Fig. 5. Removing shading from literal d_8 in figure 3.

	$[CDNOSpSh]$	$-C$	$-D$	$-N$	$-O$	$-Sp$	$-Sh$	
$-Sh$	=	=	=		=		–	$-Sh$

Table 3. Expressiveness results when removing shading.

Proof (Sketch). Let $I = (U, <, \Psi)$ be an interpretation that models d. It follows that $|\Psi(z)| = n$, where n is the number of spiders in z, since z is shaded and d is an α-diagram. Clearly, I is a model for d_1, since z contains n spiders in d_1. In d_2, z contains $n+1$ spiders, so any model for d_2 has $|\Psi(z)| \geq n+1$. Thus, I does not model d_2, so I models $\neg d_2$. Hence, I models $d_1 \wedge \neg d_2$. Conversely, suppose that I models $d_1 \wedge \neg d_2$. Then $|\Psi(z)| \geq n$ (from d_1) and $|\Psi(z)| < n+1$ (from d_2). Since d is a literal, I models d. Hence d is semantically equivalent to $d_1 \wedge \neg d_2$.

Theorem 11. *Let D_1 be a diagram drawn from any fragment, F, of spider diagrams of order, provided that F contains negation (N), spiders, and either conjunction (C) or disjunction (D) (or both). Then D_1 is semantically equivalent to some diagram, D_2, also in F, such that D_2 contains no shading.*

Proof (Sketch). The proof is similar to that of theorem 6.

This theorem allows us to complete some of row concerning removal of shading in table 1; see table 3 (all entries are new results). There are two entries left to be completed in the removal of shading row in table 3. Concerning the first, shading is removed from $ED[C, D, O, Sp, Sh]$ to give $ED[C, D, O, Sp]$. We observe that an entirely shaded unitary diagram is satisfiable, but does not have models of arbitrarily large cardinality. Without shading and negation, we cannot provide upper bounds on set cardinality, captured by the following theorem.

Theorem 12. *Any diagram drawn from $ED[C, D, O, Sp]$ that is satisfiable has models of arbitrarily large cardinality.*

Proof (Sketch). This property holds of unitary diagrams which contain only spiders, and it is preserved when formulas are combined using conjunction, disjunction and product.

Corollary 3. *$ED[C, D, O, Sp, Sh]$ is more expressive than $ED[C, D, O, Sp]$.*

For the final entry in this row, we have a further corollary to theorem 9:

Corollary 4. *$ED[C, D, N, O, Sh]$ is more expressive than $ED[C, D, N, O]$.*

3.7 Removing Logical Operators

We now give a further four results concerning relative expressiveness, where we consider the removal of a logical operator from a fragment. The proofs of these results all use model theoretic arguments. First, we observe that any diagram, D, drawn from $ED[C, O, Sp, Sh]$ that is satisfied by the empty interpretation does not contain spiders. Thus, D can make assertions such as a particular zone represents the empty set, or that elements in the set represented by one zone cannot be ordered before elements in another such set. Therefore, given a non-empty model for D, we can remove elements from the universal set (updating the interpretations of $<$ and the contour labels appropriately) and obtain another model for D. To make this insight precise, we first define a *sub-interpretation* of an interpretation.

Definition 12 (adapted from Manzano [14]). *A* ***sub-interpretation*** *of an interpretation $I = (U, <, \Psi)$ is an interpretation, $I_r = (U_r, <_r, \Psi_r)$ where*

1. *$U_r \subseteq U$*
2. *$<_r = < \cap (U_r \times U_r)$, and*
3. *$\Psi_r(c) = \Psi(c) \cap U_r$, for each $c \in \mathcal{C}$.*

Lemma 1. *Let $I = (U, <, \Psi)$ be an interpretation. with sub-interpretations $I_s = (U_s, <_s, \Psi_s)$. If $I = I_1 + I_2$ for some interpretations $I_1 = (U_1, <_1, \Psi_1)$ and $I_2 = (U_2, <_2, \Psi_2)$ then*

$$I_s = I_{1,s} + I_{2,s}$$

where $I_{1,s}$ and $I_{2,s}$ are sub-interpretations of I_1 and I_2 with universal sets $U_1 \cap U_s$ and $U_2 \cap U_s$ respectively.

Theorem 13. *Let D be a diagram in $ED[C, O, Sp, Sh]$ such that $(\emptyset, <, \Psi)$ models D. Let $I = (U, <, \Psi)$ be a model for D. Then any sub-interpretation of I also models D.*

Proof. Again, the result proceeds by induction where the interesting case is $D = D_1 \lhd D_2$. Given that the empty interpretation models D, it also models both D_1 and D_2. $I = I_1 + I_2$ where I_1 and I_2 are models for D_1 and D_2 respectively. Given a sub-interpretation, I_s, of I, by lemma 1 $I_s = I_{1,s} + I_{2,s}$ and by assumption $I_{1,s}$ and $I_{2,s}$ model D_1 and D_2 respectively. Hence I_s models D_1.

Corollary 5. *$ED[C, D, O, Sp, Sh]$ is more expressive than $ED[C, O, Sp, Sh]$.*

To justify corollary 5, construct $D = d_{15} \vee d_{16}$ (figure 6) in $ED[C, D, O, Sp, Sh]$ where d_{15} is unitary, containing no spiders and fully shaded, and d_{16} is unitary, containing exactly two spiders and fully shaded. D is satisfied by the empty interpretation (this satisfies d_{15}) and is satisfied by any model with exactly two elements in the universal set (this satisfies d_{16}). Take any model for D with two elements and create a sub-interpretation with one element. This is not a model for D, so $ED[C, D, O, Sp, Sh]$ can axiomatise more classes of interpretations than $ED[C, O, Sp, Sh]$. For corollary 6, observe that $D = \neg(\neg d_{15} \wedge \neg d_{16})$ (d_1 and d_2 as above) has same models as $d_{15} \vee d_{16}$, so the proof is similar.

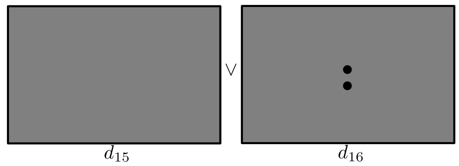

Fig. 6. An example to justify corollary 5

Corollary 6. *$ED[C, N, O, Sp, Sh]$ is more expressive than $ED[C, O, Sp, Sh]$.*

Corollary 7. *$ED[C, D, N, O, Sh]$ is more expressive than $ED[C, D, O, Sh]$.*

Finally, consider diagrams in $ED[C, D, O, Sp]$.

Theorem 14. *Let $D \in ED[C, D, O, Sp]$. If D is modelled by $(\emptyset, <, \Psi)$ then every interpretation models D.*

Proof (Sketch). Any unitary diagram in this fragment satisfied by the empty interpretation does not contain any spiders. Since there is no shading (and, therefore, no missing zones), such a diagram is satisfied by every interpretation. It can be shown, by induction, that any compound diagram, D, satisfied by the empty interpretation is also satisfied by every interpretation:

Corollary 8. *$ED[C, D, N, O, Sp]$ is more expressive than $ED[C, D, O, Sp]$*

To justify corollary 8, observe that the unitary diagram that contains no contours and exactly one spider is modelled by every interpretation except the empty interpretation. Therefore its negation is modelled by the empty interpretation, but has no other models.

3.8 Summary

Table 4a summarises the relative expressive power of fragments of the spider diagram of order logic, including the results in this paper (presented in bold typeface) and previously known results. We can use table 4a to deduce the expressive power of some fragments, due to the previously known expressiveness results. These results and deductions are presented in table 4b. Each entry identifies the expressiveness of the fragment obtained by removing syntax as indicated by the row and column heading. For example, the last row of the third column gives the expressiveness of $ED[D, N, O, Sp]$.

A different way to view the expressive power of a logic is to identify which regular languages it is capable of defining. It is known that MFOL[<] (equivalently, SDoO) is capable of defining precisely the star-free regular languages [12] and we have recently shown that MFOL[=] (equivalently SD) is capable of defining precisely the commutative star-free regular languages [15] (a language is commutative if it is closed under permutation). Thus, the table 4b can be rewritten in terms of expressiveness as compared to regular languages, where $\top$ defines Σ^* and $\bot$ defines $\emptyset$ (the empty language).

	[CDNOSpSh]	−C	−D	−N	−O	−Sp	−Sh	
−C	=	–			=	=	=	−C
−D	=		–	<	=	=	=	−D
−N			<	–	=	<	<	−N
−O	<	<	<	<	–	<	<	−O
−Sp	=	=	=	<	<	–	<	−Sp
−Sh	=	=	=	<	=		–	−Sh

(a) Summary of results shown in this section.

	[CDNOSpSh]	−C	−D	−N	−O	−Sp	−Sh	
−C	MFOL[<]	–			MFOL[=]	**MFOL[<]**	**MFOL[<]**	−C
−D	MFOL[<]		–		MFOL[=]	**MFOL[<]**	**MFOL[<]**	−D
−N				–	MFOL[=]		**MFOL[<]**	−N
−O	MFOL[=]	MFOL[=]	MFOL[=]	MFOL[=]	–	MFOL	**MFOL[=]**	−O
−Sp	**MFOL[<]**	**MFOL[<]**	**MFOL[<]**	**MFOL[<]**	MFOL	–	⊤,⊥	−Sp
−Sh	**MFOL[<]**	**MFOL[<]**	**MFOL[<]**		**MFOL[=]**	⊤,⊥	–	−Sh

(b) Expressiveness in terms of classes of symbolic logic.

Table 4. A summary of the presented results.

4 Conclusion

The key results in this paper concern the relative expressiveness of fragments of spider diagrams of order. Perhaps surprisingly, we have shown that spiders and shading can each be removed from certain fragments whilst maintaining expressiveness. The model theoretic analysis we have provided for some of the fragments also provide insight into the kinds of statements that the diagrams in these fragments can make. Whilst we completed 14 of the 36 entries in table 1, 5 gaps remain. We conjecture that the two missing entries in the $-N$ row will be $<$, but this is not clear. A difficultly with analysing these two cases stems from the fact that there is no analogy to De Morgan's Laws for negation and $\lhd$. Thus, in fragments containing N and O , there are no obvious normal forms that explicitly reflect the semantics of the diagrams.

The proof strategies used throughout the paper are likely to adapt to other systems, such as the Euler/Venn logic. Whilst this logic is less expressive than spider diagrams, its strong syntactic similarity justifies our claim. Moreover, the kinds of results we have provided concerning when the removal of syntax impacts expressiveness may well provide a basis for similar conjectures in Euler/Venn and other related notations. We expect to use these results when developing more expressive notations based on SDoO: they will inform us about what syntax it is necessary to include. Our immediate plans involve extending SDoO to a monadic second order logic, since MSOL is capable of defining precisely the regular languages.

As well as providing insight into what can be expressed with the presence or absence of certain pieces of syntax, there are further benefits to this work concerning, for instance, the development of reasoning systems. For example, theorem 5 can be restated as a (syntactic) inference rule. Now consider a fragment, F_1, from which we can remove spiders using this inference rule. We can, therefore, immediately obtain a sound and complete inference system for F_1 provided F_2 is sound and complete, where F_2 is F_1 without spiders; SDoO is an example of such an F_1. Currently, there is no sound and complete set of inference rules for SDoO, so the results in this paper may aid in their development.

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