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Author(s): Andres Carvajal and John K. H Quah
Article Title: A Nonparametric Analysis of the Cournot Model
Year of publication: 2009
Link to published article:
http://www2.warwick.ac.uk/fac/soc/economics/research/papers_2009/twerp_922.pdf
Publisher statement: None

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No 922

WARWICK ECONOMIC RESEARCH PAPERS

DEPARTMENT OF ECONOMICS

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A Nonparametric Analysis of the Cournot Model[*]

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November 26, 2009

Abstract

An observer makes a number of observations of an industry producing a homogeneous good. Each observation consists of the market price, the output of individual firms and perhaps information on each firm's production cost. We provide various tests (typically, linear programs) with which the observer can determine if the data set is consistent with the hypothesis that firms in this industry are playing a Cournot game at each observation. When cost information is wholly or partially unavailable, these tests could potentially be used to derive cost information on the firms. This paper is a contribution to the literature that aims to characterize (in various contexts) the restrictions that a data set must satisfy for it to be consistent with Nash outcomes in a game. It is also inspired by the seminal result of Afriat (and the subsequent literature) which addresses similar issues in the context of consumer demand, though one important technical difference from most of these results is that the objective functions of firms in a Cournot game are not necessarily quasiconcave.

Keywords: revealed preference, observable restrictions, linear programming, Cournot game, increasing marginal costs.

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1. INTRODUCTION

Consider an industry with I firms producing a homogeneous good. We make T observations of this industry over time. If we assume that the cost functions of firms in the industry do not vary across observations, so that the data is generated by fluctuations in the market demand function, how could we test the hypothesis that the firms in this industry are playing a Cournot game at each observation?

Suppose that the observation at t consists of the market price P_t and the output and the production costs of each individual firm, $Q_{i,t}$ and $C_{i,t}$ for every i. We wish to examine the conditions under which such a data set can be rationalized as a Cournot game. By this, we mean that we can find increasing, positive, and differentiable cost functions for each firm (fixed across all observations) and differentiable and downward sloping demand functions (one at each observation t) that give the observed outcome at each t as an equilibrium of the Cournot game that results from the firms having these cost functions and facing the demand function corresponding to that observation. It turns out that there is (in essence) just one property on the data set, which we call the *marginal property* (M), that is both necessary and sufficient for rationalizability. Property M requires that the data set not reveal instances of *over*-production by firms. Specifically, suppose that at time t firm i is producing more than at some other time t', i.e., $Q_{i,t} > Q_{i,t'}$. Then the data must not reveal that the firm is better off at time t by reducing its output to $Q_{i,t'}$; i.e.,

$$P_t Q_{i,t'} - C_{i,t'} < P_t Q_{i,t} - C_{i,t}.$$

The right hand side of this inequality is firm i's profit at time t if it produces $Q_{i,t}$. This is larger than the left hand side, which is an *under*-estimate of its profit at time t should it reduce output to $Q_{i,t'}$. Note that the left hand side is an under-estimate because the good's clearing price will be higher than P_t if firm i reduces its output (assuming, of course, that the market demand curve is downward sloping).

This result shows that there *are* observable restrictions on the Cournot model in this context. However, property M is a rather weak restriction. Indeed if the firms are colluding rather than playing a Cournot game, one can show that they will generate data that also satisfies M, so that collusion does not lead to any observable behavior that is inconsistent with the Cournot model. The main reason for this is that M is just a test that firms are not over-producing – it does not test for under-production.

There are two reasons why the predictions of the Cournot model are so weak in this context. Firstly, we assume that demand varies from one observation to the next and we

do not restrict the manner in which they vary. This means that, in principle, the demand curves at different observations may be completely unrelated to each other and, at each observation, we observe just one point on the demand curve. This gives the observer a great deal of freedom in choosing the demand curve to rationalize the data and thus, loosely speaking, more sets of observations are rationalizable.

The other reason is related to the cost curves constructed for rationalizing the data. We do not wish to limit the cost curves to a particular shape, for example, to those with constant, increasing, or decreasing marginal costs, because this detracts from a test of the Cournot model as such and, indeed, may well be directly contradicted by cost observations itself. However, this absence of restrictions can introduce a troubling dichotomy between observed and infinitesimal marginal costs, which we shall explain.

Suppose that for some firm i, the output below and closest to $Q_{i,t}$ was observed at t' and the output above and closest to $Q_{i,t}$ was observed at t''. The data allow the observer to calculate the average marginal cost of increasing output from $Q_{i,t'}$ to $Q_{i,t}$ (call it M') and from $Q_{i,t}$ to $Q_{i,t''}$ (call it M''). When rationalizing the observations, a cost function $\bar{C}_i$ must be constructed for this firm, that, amongst other things, will generate average marginal costs of M' and M'' over the respective output ranges. On the other hand, the observed output shares at t will impose restrictions on $\bar{C}_i'(Q_{i,t})$, i.e., the infinitesimal marginal cost at $Q_{i,t}$, through the first-order conditions. Since $\bar{C}_i'$ can vary freely (apart from being continuous), there need be no relation between $\bar{C}_i'(Q_{i,t})$, M', and M''. Indeed, there are data sets that can *only* be rationalized via a cost function for which the infinitesimal marginal cost at a point differs significantly from the observed marginal costs over intervals adjacent to that point. The 'oddness' of such a rationalization is most obvious when the observed average marginal costs (for discrete output changes) are monotonically increasing or decreasing for some firm i. In these cases, the rationalizing cost function for firm i may be such that its derivative (the marginal cost) does not display the same monotonicity.

This suggests that it may be fruitful to refine the concept of rationalizability. We say that a data set admits a *convincing rationalization* if it is rationalizable and the constructed cost functions have the property that the infinitesimal marginal cost at each observed output must lie between the observed marginal costs on either side of that point; formally, we require $\bar{C}_i'(Q_{i,t})$ to lie between M' and M''.[1] This is really an assumption

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[1] Strictly speaking this is just one of two conditions imposed in our definition of convincing rationalizability in Section 2, but it is this condition which leads to interesting implications. The other condition is, in some sense, always satisfied.

about the quality of the set of observations: we are assuming that it is rich enough for the observed marginal costs to convey some information on the infinitesimal marginal costs. We provide necessary and sufficient conditions under which a convincing rationalization exists. We also show, with an example, that the data set obtained from two colluding firms need not admit a *convincing rationalization* as a Cournot game.[2]

The paper also examines the case where cost information is not observed, so that each observation consists of just the market price and the output of each firm. In this case, we show that *any* set of observations can be rationalized as a Cournot game. Whether this result is surprising depends on one's perspective. It does not seem surprising given the meagerness of the information available to the observer; on the other hand, there are some well-known facts about the Cournot game that may suggest otherwise.

Specifically, it is known (and trivial to check via the first order conditions) that firms' market shares are inversely related to their marginal costs at any Cournot equilibrium. This means that if every firm has constant marginal cost (so its cost function is linear in output), then the ranking of firms according to market share does not vary with the demand function. In other words, if at observation t, some firm i produces more than firm j and at another observation t', firm j produces more than firm i, then the modeler knows that the observations are not consistent with a Cournot game with constant marginal costs, even though costs are not directly observed. Of course, constant marginal costs is a strong condition to impose on the cost functions, but there are similar observable restrictions if all firms have nondecreasing marginal costs.[3] In this paper, we elaborate on these remarks by providing an example of a data set (of price and outputs) that is not *convincingly* rationalizable; we also identify the necessary and sufficient conditions for such a data set to admit a convincing rationalization.

Related literature. This paper is a contribution to the literature that tries to identify the precise observable implications of various canonical economic models. Perhaps the most influential paper in this approach is that of Afriat (1967). Afriat showed that a finite

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[2] It is clear that we could, if we wish, be more stringent or permissive when restricting the relationship between observed and infinitesimal marginal costs. So, a weaker criterion is to require $\bar{C}_i'(Q_{i,t})$ to be in the interval $[k \min\{M', M''\}, K \max\{M', M''\}]$, where $0 < k < 1 < K < \infty$, with k and K chosen by the modeler. This does not affect the qualitative nature of our results on convincing rationalizability.

[3] Consider an oligopoly where at observation t, firm i produces 20 and firm j produces 15. At another observation t', firm i produces 15 and firm j produces 16. Observation t tells us that $C_i'(20) < C_j'(15)$, while observation t' tells us that $C_i'(15) > C_j'(16)$. Clearly, either firm i or firm j cannot have nondecreasing marginal costs. Note that this observable restriction is imposed solely on output levels. We do not rely on cost – or even price – observations.

data set of price and demand observations is compatible with the utility-maximization hypothesis if, and only if, there is solution to a linear program. This result has been extended in various ways – for example, to production theory; see Varian (1982) – and has also generated a very large empirical literature. Afriat's result has also been extended to the case of nonlinear, and possible nonconvex, budget sets (see Forges and Minelli (2008) and Matzkin (1991)), where rationalization may require utility functions that are not quasiconcave. By applying Afriat's result, Brown and Matzkin (1996) show that there are non-tautological conditions that are necessary and sufficient for a data set to be consistent with Walrasian outcomes, in the context of an exchange economy; this work has in turn been extended in a number of ways (see, for example, Kubler (2003), Carvajal (2004, 2009) and, for a survey, Carvajal et al. (2004)). Our paper bears some resemblance to these more recent contributions in its emphasis on nonconcave objective functions and multi-agent interaction.

Perhaps the work that is most closely related to ours are those that address similar issues in the context of games. Sprumont (2000) considers this question in the context of a static game, and asks when observed actions can be rationalized as Nash equilbria. Ray and Zhou (2001) address similar questions, but in the context of a dynamic game. Carvajal (2005) shows how weak the testable implications of Nash equilibrium are in games with continuous domains. In these papers, the variability in the data arises from changes to the strategy set across observations.[4] In our paper, we have chosen to focus on a specific and familiar game. In our study of the Cournot game, the set of strategies do not vary across observations. Instead, the observations are generated by changes to the payoff functions, which in turn arises from changes in the demand function. Partly motivated by earlier versions of this paper, Routledge (2009) has provided a nonparametric analysis of the Bertrand game. It is clear that there are many extensions and variations on this theme are potentially possible and worth studying, and also empirical work that can be done based on the nonparametric approach.

Lastly, our paper is of course related to the very large empirical IO literature (surveyed in Bresnahan (1989)) that, amongst other things, aims to determine the level of market power (for example, Genesove and Mullin (1998)) or to derive cost information from observed behavior under various game-theoretic assumptions. Our approach differs from most of this literature in a number of ways. Many of these models are parametric and ours

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[4] Another related paper is Zhou (2005) which gives the precise restrictions on subsets of the strategy space that could be rationalized as Nash equilibria of a two-player game with quasi-concave payoff functions.

is not. Secondly, our aim is to develop an *exhaustive* list of implications of a model, and not just *some* testable implications. Lastly, we make no assumptions at all about the evolution of demand across observations; we do this, not because we consider such assumptions always undesirable, but because we think it is worthwhile asking if the Cournot hypothesis yields any restrictions even without those assumptions.

Organization of paper. In Section 2, we consider the rationalizability problem for a monopolist. This is a useful exercise because it helps to develop intuition for the results in the Cournot case, and also because its results differ from that case in instructive ways. From Section 3 onwards we consider the Cournot model. Section 3 studies rationalizability and convincing rationalizability in a context where cost information is available to the observer, while Section 4 studies the same issues but in a context where cost information is (in part or in whole) unavailable. The final section discusses how the assumption that cost functions are unchanged across observations can be relaxed; in particular, we extend our results to accommodate linear perturbations to the cost function. Throughout the paper we also develop results for rationalizability in the cases where marginal costs are monotonic.

2. The Rationalizability Problem for a Monopoly

Consider an experiment in which we make T observations of a monopolist. The observations are indexed by t in the set $\mathcal{T} = \{1, 2, \ldots, T\}$; observation t consists of a triple (P_t, Q_t, C_t), respectively the price charged by the monopolist, the quantity it sells, and the cost it incurs.[5] We require $P_t > 0$ and $Q_t > 0$ for all t and $C_{t'} > C_t$ whenever $Q_{t'} > Q_t$ (so observed cost is higher if observed output is higher).

2.1. Rationalizable Observations

We say that the set of observations $\{(P_t, Q_t, C_t)\}_{t \in \mathcal{T}}$ is *rationalizable* if they are consistent with a profit-maximizing monopolist having a stable cost structure, with each observation corresponding to a different demand condition. Formally, we require that there be a $\mathbb{C}^1$ function $\bar{C} : \mathbb{R}_+ \to \mathbb{R}$ and $\mathbb{C}^1$ functions $\bar{P}_t : \mathbb{R}_+ \to \mathbb{R}$, for each t in $\mathcal{T}$, such that

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[5] For simplicity of presentation, we will assume that the firm incurs no fixed costs, or that the cost we observe is the variable cost. This assumption restricts the datasets that can be rationalized, since it implies that the observed cost can never be larger than the observed revenue. If we assume that the observed cost is a mix of fixed and variable cost, the analysis will not be substantially different, but the exposition of some results becomes more cumbersome.

(i) $\bar{C}(q) \geq 0$ for $q \geq 0$, with $\bar{C}(0) = 0$, and $\bar{C}'(q) > 0$;

(ii) $\bar{P}_t(q) \geq 0$ and $\bar{P}'_t(q) \leq 0$, with the the latter inequality being strict if $\bar{P}_t(q) > 0$;

(iii) $\bar{C}(Q_t) = C_t$ and $\bar{P}_t(Q_t) = P_t$; and

(iv) $\operatorname{argmax}_{q \geq 0}[\bar{P}_t(q)q - \bar{C}(q)] = Q_t$.

Function $\bar{C}$ is the monopolist's *cost function*; condition (i) says that it is positive and strictly increasing. Function $\bar{P}_t$ is the *inverse demand function* at observation t; (ii) says that more output can only be sold at a strictly lower price, until the price reaches zero. Henceforth, we shall refer to any $\mathbb{C}^1$ cost function satisfying (i) as a *regular* cost function; similarly, a *regular* inverse demand function is a $\mathbb{C}^1$ inverse demand function that obeys (ii). Condition (iii) requires the inverse demand and cost functions to coincide with their observed values at each t. Lastly, (iv) requires the observations to be consistent with profit maximization. It is clear that (iii) and (iv) together guarantee that the observed profit is the largest possible, i.e., $P_tQ_t - C_t = \max_{q \geq 0}[\bar{P}_t(q)q - \bar{C}(q)]$. Since producing nothing (and so incurring no cost) is always possible, rationality requires $P_tQ_t - C_t > 0$.

We say that the observations are *generic* if $Q_t \neq Q_{t'}$ whenever $t \neq t'$.[6] Let the set $\{(P_t, Q_t, C_t)\}_{t \in \mathcal{T}}$ be a generic set of observations. For each t, we define the set

$$L(t) = \{t' \in \mathcal{T} : Q_{t'} < Q_t\} \cup \{0\}.$$

This means that $L(t)$ consists of those observations with output levels lower than Q_t, as well as a fictitious observation 0, for which $Q_0 = 0$. Note that for the observation with the lowest output, we denote this observation by t^*, we have $L(t^*) = \{0\}$ whilst for any $t \neq t^*$, $L(t)$ will contain Q_{t^*}, 0, and possibly other elements. We denote $l(t) = \operatorname{argmax}_{t' \in L(t)} Q_{t'}$; that is, $l(t)$ is the observation corresponding to the highest output level below Q_t.[7] In a similar fashion, we denote the observation with the highest output level by t^{**}. For $t \neq t^{**}$, the set of observations with outputs higher than t is denoted by $U(t)$, with $u(t) = \operatorname{argmin}_{t' \in U(t)} Q_{t'}$, so $u(t)$ is the observation with the lowest output level above Q_t.

For any t in $\mathcal{T}$, define $\mathrm{d}Q_t = Q_t - Q_{l(t)}$ and $\mathrm{d}C_t = C_t - C_{l(t)}$. In words, $\mathrm{d}C_t$ is the extra cost incurred by the monopoly when it increases its output from $Q_{l(t)}$ to Q_t. We denote the *average marginal cost* over that output range by $M_t = \mathrm{d}C_t/\mathrm{d}Q_t$. The generic set of observations $\{(P_t, Q_t, C_t)\}_{t \in \mathcal{T}}$ is said to obey the *Marginal Property* (henceforth, property M) if for every t in $\mathcal{T}$,

$$P_tQ_{t'} - C_{t'} < P_tQ_t - C_t \text{ for all } t' \in L(t). \tag{1}$$

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[6] This assumption simply makes the notation and exposition simpler. It has no analytical significance and can be completely removed.

[7] In particular, $l(t^*) = 0$.

We may re-arrange this inequality to obtain

$$C_t - C_{t'} = \sum_{s \in (L(t) \cup \{t\}) \setminus (L(t') \cup \{t'\})} \mathrm{d}C_s < P_t(Q_t - Q_{t'}) \text{ for all } t' \in L(t). \qquad (2)$$

Note that the observer does not know the exact market price of the product at time t should the monopolist choose to produce $Q_{t'} < Q_t$, but he knows that it must be at least P_t. If Q_t is optimal, the cost saving in producing at $Q_{t'}$ rather than Q_t must be dominated by the revenue lost in producing at $Q_{t'}$ rather than Q_t; the latter does not exceed $P_t(Q_t - Q_{t'})$, so we obtain (2).[8] In short, property M requires that the monopolist is not *over*-producing given the data. We state this result formally in the following proposition.

PROPOSITION 1. *The generic set of observations* $\{(P_t, Q_t, C_t)\}_{t \in \mathcal{T}}$ *is rationalizable only if it obeys M.*

The next result says that M is also sufficient for rationalizability.

PROPOSITION 2. *Suppose that the generic set of observations* $\{(P_t, Q_t, C_t)\}_{t \in \mathcal{T}}$ *obeys M, and let* $\{\delta_t\}_{t \in \mathcal{T}}$ *be a set of numbers satisfying* $0 < \delta_t < P_t$. *Then, the set of observations are rationalizable and the cost function* $\bar{C}$ *can be chosen such that* $\bar{C}'(Q_t) = \delta_t$ *for all* $t \in \mathcal{T}$.

Note that the last condition in Proposition 2 says that we are free to choose the marginal cost at the optimal output level, subject to it being lower than the observed price. The extent to which we may freely choose the marginal cost at any observed output level turns out to be a crucial issue, as we shall see when we consider rationalizability in the Cournot model.

Since property M is a 'one-sided' condition – it requires that the monopoly is not *over*-producing given the data – Proposition 2 effectively says that the data does not permit the observer to check that the monopolist is not *under*-producing. The reason for this is that the monopolist decision at observation t to produce Q_t, *but not more*, can always be justified on the grounds that the price will fall (arbitrarily) sharply should it produce more. The fact that the demand curve changes from one observation to the next, and the fact that only one observation is made at each demand curve, mean that such a possibility cannot be excluded by the observer.

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[8] Note that the set $(L(t) \cup \{t\}) \setminus (L(t') \cup \{t'\})$ consists of those observations with output levels that are weakly below Q_t and strictly above $Q_{t'}$.

Proposition 2 follows from Lemmas 1 and 2 below. Loosely speaking, Lemma 1 provides us with the cost function needed to rationalize the set of observations, while Lemma 2 gives the demand functions corresponding to each observation.

LEMMA 1. *Suppose the generic set of observations $\{(P_t, Q_t, C_t)\}_{t \in \mathcal{T}}$ obeys M, and let $\{\delta_t\}_{t \in \mathcal{T}}$ be a set of numbers satisfying $0 < \delta_t < P_t$. Then, there is a regular cost function $\bar{C} : \mathbb{R}_+ \to \mathbb{R}$ such that, for all t in $\mathcal{T}$,*
(i) $\bar{C}(Q_t) = C_t$;
(ii) $\bar{C}'(Q_t) = \delta_t$; and
(iii) for all $q \in [0, Q_t)$,

$$P_t q - \bar{C}(q) < P_t Q_t - \bar{C}(Q_t). \tag{3}$$

Proof: Note that the inequality (3) may be re-written as

$$\bar{C}(q) > P_t(q - Q_t) + \bar{C}(Q_t). \tag{4}$$

The function $f_t(q) = P_t(q - Q_t) + C_t$, for q in $[0, Q_t)$, is represented by a line with slope P_t passing through the point (Q_t, C_t) – see Figure 1. Condition M guarantees that for t' in $L(t)$, $(Q_{t'}, C_{t'})$ lies *above* the line f_t. We require a cost function that satisfies (4). One such function is the one given by the linear interpolation of all the points (Q_t, C_t), since its graph stays above every one of the lines representing the functions f_t. This cost function can in turn be replaced by a smooth function where the derivative at Q_t is δ_t, since $\delta_t < P_t$ and the latter is the slope of f_t. $\hspace{2cm}$ Q.E.D.

Let $\bar{C}$ be a cost function that is *consistent with* $\{C_t\}_{t \in \mathcal{T}}$ in the sense that $\bar{C}(Q_t) = C_t$ for all t. We say that its marginal cost function $\bar{C}'$ *has minimal variation* (or that it *varies minimally*) if the following is true for all $t \in \mathcal{T}$:
(i) if M_t lies strictly between $\bar{C}'(Q_t)$ and $\bar{C}'(Q_{l(t)})$, then $\bar{C}'$ is either strictly increasing or strictly decreasing in the interval $[Q_{l(t)}, Q_t]$;
(ii) if $M_t = \bar{C}'(Q_t) = \bar{C}'(Q_{l(t)})$, then $\bar{C}'$ is constant in $[Q_{l(t)}, Q_t]$; and
(iii) if neither of the previous premises in (i) and (ii) are true, then $\bar{C}'$ has exactly one turning point in $[Q_{l(t)}, Q_t]$.[9]

Remark: It is clear form Figure 1 that $\bar{C}$ in Lemma Lemma 1 can be chosen with $\bar{C}'(0)$ taking any value. It is also clear that $\bar{C}'$ can be chosen to be of minimal variation, given

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[9] Note that in the last case, $\bar{C}'$ needs at least one turning point for it to be consistent with the data, i.e., for $\int_{Q_{l(t)}}^{Q(t)} \bar{C}'(q) \, dq$ to equal $M_t(Q_t - Q_{l(t)})$.
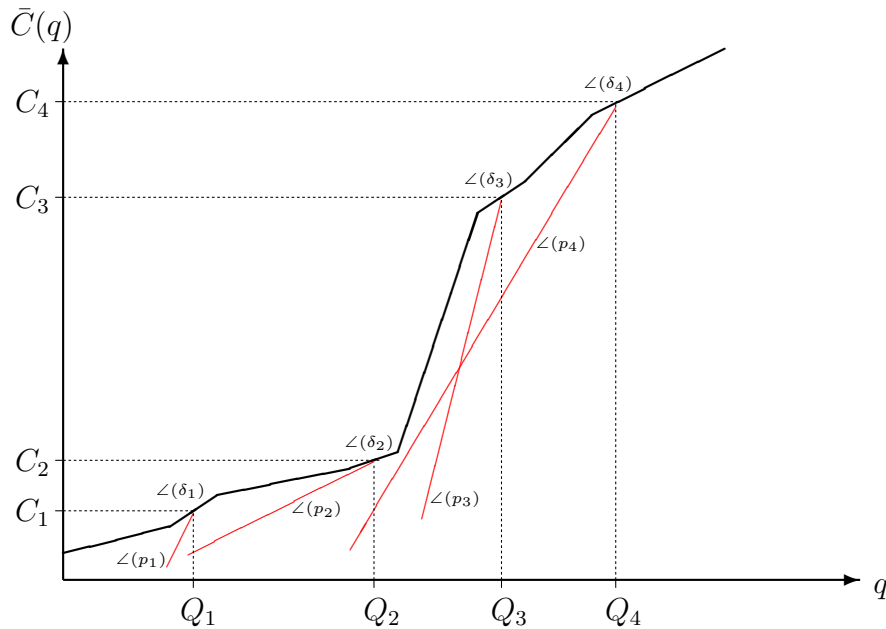
Figure 1: CONSTRUCTION OF A COST FUNCTION. The notation $\angle(\delta)$ is used to denote the slope (δ) at a point on the curve or of a line. The straight, thin lines represent the functions $f_t(q) = C_t + P_t(q - Q_t)$. Condition M guarantees that if $Q_{t'} < Q_t$, then $(Q_{t'}, C_{t'})$ lies above the graph of f_t.

the values of $\bar{C}'(Q_t)$ (for $t \in \mathcal{T} \cup \{0\}$).

Properties (i) and (ii) in Lemma 1 require the cost function to agree with the cost data at the observed output levels and to obey the specified marginal cost conditions. Property (iii) is a strengthening of condition M: M requires (3) to hold at discrete output levels, while (iii) requires it to hold at *all* output levels up to Q_t.

The next result says that, for the cost function guaranteed by Lemma 1, we could find a regular inverse demand function at each observation t such that the profit-maximizing output decision is Q_t. It is clear that Lemmas 1 and 2 together guarantee Proposition 2.

LEMMA 2. *Let $\{\delta_t\}_{t \in \mathcal{T}}$ be a set of numbers satisfying $0 < \delta_t < P_t$, and let $\bar{C} : \mathbb{R}_+ \to \mathbb{R}$ be a regular cost function satisfying the three properties of Lemma 1. Then, for any $t \in \mathcal{T}$, there is a regular inverse demand function $\bar{P}_t : \mathbb{R}_+ \to \mathbb{R}$ such that*
(i) $\bar{P}_t(Q_t) = P_t$; and
(ii) $\mathrm{argmax}_{q \geq 0}[\bar{P}_t(q)q - \bar{C}(q)] = Q_t$.

The proof of this lemma is in the Appendix, but the result is very intuitive. The lemma requires that we produce an inverse demand function. Property (iii) in Lemma 1 already provides us with one such function that obeys (i) and (ii) in Lemma 2: simply

assume that $\bar{P}_t(q) = P_t$ for all $q \leq Q_t$ and $\bar{P}_t(q) = 0$ for $q > Q_t$. This function is not regular, but we can always construct a regular demand function that is sufficiently close to it so that (i) and (ii) (in Lemma 2) remain valid.

2.2. The Convincing Criterion

The cost function $\bar{C}$ is consistent with the data in the sense that it agrees with the observed costs at the output levels Q_t (for all $t \in \mathcal{T}$). Note, however, that this restriction is very permissive: amongst other things, it does not restrict the relationship between infinitesimal marginal costs $\bar{C}'$ and the observed marginal costs. This problem is most obvious when observed marginal costs are monotonic. Formally, we say that the observed costs are *consistent with increasing marginal costs* if for all $t \neq t^*$, it is observed that $M_t > M_{l(t)}$. Similarly, observed costs are *consistent with decreasing marginal costs* if for all $t \neq t^*$, it is observed that $M_t < M_{l(t)}$, and they are *consistent with constant marginal costs* if $M_t = M_{t'}$ for all t and t'. If (say) we observe that marginal costs are consistent with increasing marginal costs, then it is reasonable to require that $\bar{C}$ also display increasing marginal costs at all output levels (i.e., that $\bar{C}'(q)$ be increasing in q). So, a rationalizing cost function with this property is *more convincing* than another rationalizing cost function that does not possess this property. More generally, it is desirable to have a rationalizing cost function such that $\bar{C}'$ adheres more closely to the pattern displayed by $\{M_t\}_{t \in \mathcal{T}}$.

This motivates our introduction of a *convincing criterion* on the rationalizing cost function $\bar{C}$. This criterion is satisfied if, in addition to all the conditions already imposed on $\bar{C}$ by rationalizability (and, in particular, that $\bar{C}(Q_t) = C_t$), the following conditions hold:

(C1) for all $t \neq t^{**}$, $\bar{C}'(Q_t)$ lies between M_t and $M_{u(t)}$;

(C2) the marginal cost $\bar{C}'$ varies minimally.

Condition (C1) says that the (infinitesimal) marginal cost at Q_t lies between the average marginal costs observed over the intervals $[l(t), t]$ and $[t, u(t)]$, while (C2) say that the marginal cost function does not 'wiggle' more than is necessary to be consistent with the data and to satisfy condition (C1). Figures 2 and 3 give examples of situations where the criterion is violated and where it is satisfied. It is easy to check that if the rationalizing function $\bar{C}$ has increasing marginal cost, then it must satisfy the convincing criterion (relative to the finite cost observations it rationalizes). Similarly, if $\bar{C}$ has constant or decreasing marginal cost, then it will also satisfy the convincing criterion.

In the case of a monopoly, the convincing criterion has no observable implications.
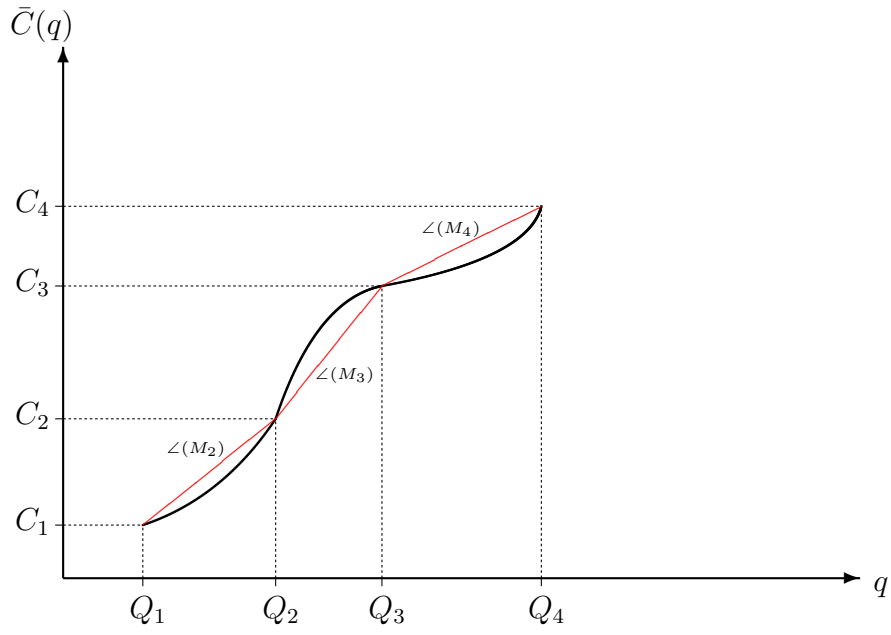
Figure 2: A NON-CONVINCING COST FUNCTION. The segments of line are the graph of the linear interpolation of the (Q_t, C_t) points, so their slopes are M_2, M_3 and M_4. Note that $\bar{C}'(Q_2) > M_3 > M_2$, while $\bar{C}'(Q_3) < M_4 < M_3$: this function violates condition (C1).
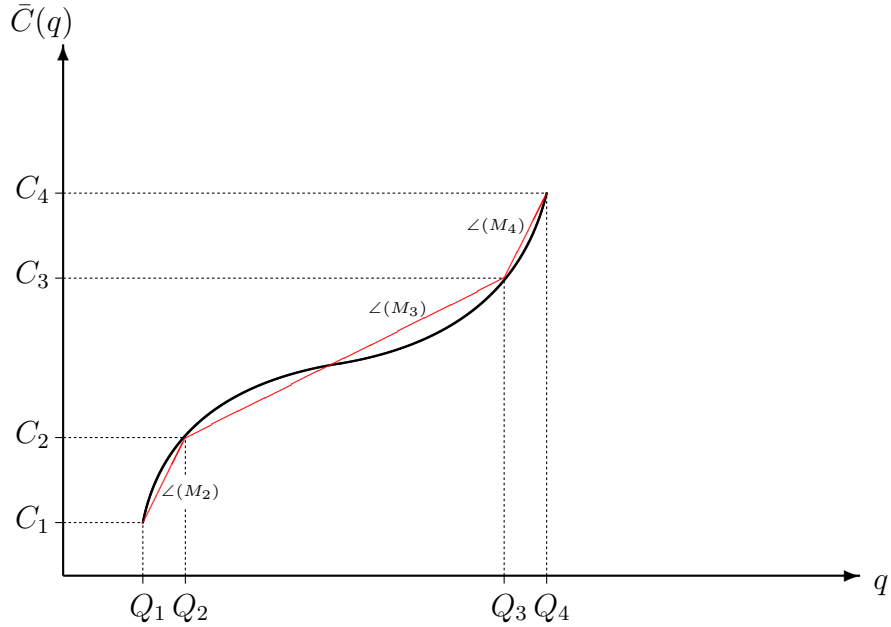

Figure 3: A CONVINCING COST FUNCTION. As in Figure 2, the segments of line are the graph of the linear interpolation of the (Q_t, C_t) points, but in this case $M_2 < \bar{C}'(Q_2) < M_3$ and $M_3 > \bar{C}'(Q_3) > M_4$.

By this we mean that any set of observations that is rationalizable in this context is also *convincingly rationalizable*, in the sense that the cost function can be chosen to satisfy the convincing criterion.

PROPOSITION 3. *Any generic set of observations $\{(P_t, Q_t, C_t)\}_{t \in \mathcal{T}}$ that obeys property M is rationalizable with a rationalizing cost function $\bar{C}$ that satisfies the convincing criterion. Furthermore, if the cost observations are consistent with increasing (resp. constant, resp. decreasing) marginal costs, then $\bar{C}$ can be chosen such that $\bar{C}'$ is increasing (resp. constant, resp. decreasing).*

Proof: Proposition 2 says that $\bar{C}$ can be chosen such that $\bar{C}'(Q_t) = \delta_t$, for any δ_t in $(0, P_t)$. Note, from (2), choosing $t' = l(t)$, that $M_t < P_t$, so we can certainly choose δ_t to be in $(0, P_t)$ and also to lie between M_t and $M_{u(t)}$ (for any $t \neq t^{**}$), thus ensuring that (C1) is satisfied. Furthermore, as we had pointed in the remark following Lemma 1, $\bar{C}$ may be chosen so that its marginal cost varies minimally, in addition to satisfying conditions (i)-(iii).

Consider now the case where the observations are consistent with increasing marginal cost.[10] Since $M_t > M_{l(t)}$ for all $t \neq t^*$, we can choose δ_t to lie in the interval $(M_t, M_{u(t)})$ for $t \neq t^{**}$. For $t = t^{**}$, choose $\delta_{t^{**}} > M_{t^{**}}$ and for $t = 0$ choose $\delta_0 < M_{t^*}$. The remark following Lemma 1 points out that the rationalizing cost function can be chosen such that $\bar{C}'$ varies minimally. Since $M_t \in (\delta_{l(t)}, \delta_t)$ for all t, it follows from the definition of minimal variation that $\bar{C}'$ is increasing.   *Q.E.D.*

As we shall see in the next sections, the equivalence between rationalizability and convincing rationalizability breaks down in a multi-firm context.

3.   COURNOT RATIONALIZABILITY

An industry consists of I firms producing a homogeneous good; we denote the set of firms by $\mathcal{I} = \{1, 2, \ldots, I\}$. Consider an experiment in which T observations are made of this industry. As in the previous section, we index the observations by t in $\mathcal{T} = \{1, 2, \ldots, T\}$. For each t, the industry price P_t, the output of each firm $(Q_{i,t})_{i \in \mathcal{I}}$ and the cost it incurs $(C_{i,t})_{i \in \mathcal{I}}$ are observed. We require $P_t > 0$ and $Q_{i,t} > 0$ for all t, $C_{i,t'} > C_{i,t}$ whenever $Q_{i,t'} > Q_{i,t}$, and also that $P_t Q_{i,t} - C_{i,t} > 0$. We let $Q_t = \sum_{i \in \mathcal{I}} Q_{i,t}$ denote the aggregate output of the industry at observation t.

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[10] The arguments for the decreasing and constant marginal cost cases are similar.

3.1. Rationalizable Observations

We say that the set of observations $\{[P_t, (Q_{i,t})_{i \in \mathcal{I}}, (C_{i,t})_{i \in \mathcal{I}}]\}_{t \in \mathcal{T}}$ is *Cournot rationalizable* if each observation can be explained as a Cournot equilibrium arising from a different market demand function, keeping the cost function of each firm fixed across observations. Formally, we require that there be a regular cost function $\bar{C}_i : \mathbb{R}_+ \to \mathbb{R}$ for each firm i and a regular demand function $\bar{P}_t : \mathbb{R}_+ \to \mathbb{R}$ for each t, such that

(i) $\bar{C}_i(Q_{i,t}) = C_{i,t}$ and $\bar{P}_t(Q_t) = P_t$; and

(ii) $\operatorname{argmax}_{q_i \geq 0}[q_i \bar{P}_t(q_i + \sum_{j \neq i} Q_{j,t}) - \bar{C}_i(q_i)] = Q_{i,t}$.

Condition (i) says that the inverse demand and cost functions must coincide with their observed values at each t. Condition (ii) says that, at each observation t, firm i's observed output level $Q_{i,t}$ maximizes its profit given the output of the other firms.

We say that the observations are *generic* if, for every firm i, we have $Q_{i,t} \neq Q_{i,t'}$ whenever $t \neq t'$. Let $\{[P_t, (Q_{i,t})_{i \in \mathcal{I}}, (C_{i,t})_{i \in \mathcal{I}}]\}_{t \in \mathcal{T}}$ be a generic set of observations. For each firm i, we may define L_i, l_i, u_i, t_i^*, t_i^{**}, and $M_{i,t}$, in a way analogous to our definitions in Section 2. We say that the observations obey the marginal property if, for every firm i, $\{(P_t, Q_{i,t}, C_{i,t})\}_{t \in T}$ obeys property M; formally, for every t in $\mathcal{T}$, we require

$$C_{i,t} - C_{i,t'} = \sum_{s \in (L_i(t) \cup \{t\}) \setminus (L_i(t') \cup \{t'\})} M_{i,s}(Q_{i,s} - Q_{i,l(s)}) < P_t(Q_t - Q_{t'}) \text{ for } t' \in L_i(t). \quad (5)$$

For exactly the same reason as the one given in the monopoly case, property M is *necessary* for a set of observations to be Cournot rationalizable; specifically, M guarantees that each firm is not better off by producing less than the observed output. The next result says that M is also sufficient for Cournot rationalizability.

THEOREM 1. *A generic set of observations* $\{[P_t, (Q_{i,t})_{i \in \mathcal{I}}, (C_{i,t})_{i \in \mathcal{I}}]\}_{t \in \mathcal{T}}$ *is Cournot rationalizable if, and only if, it obeys property M.*

Just as Proposition 2 follows from Lemmas 1 and 2, we can prove Theorem 1 with a similar two-step procedure. The next result is analogous to Lemma 1; the construction of the cost functions in this result is also identical to that in Lemma 1, so we shall omit its proof.

LEMMA 3. *Let* $\{[P_t, (Q_{i,t})_{i \in \mathcal{I}}, (C_{i,t})_{i \in \mathcal{I}}]\}_{t \in \mathcal{T}}$ *be a generic set of observations obeying M and suppose that the positive numbers* $\{\delta_{i,t}\}_{(i,t) \in \mathcal{I} \times \mathcal{T}}$ *satisfy* $0 < \delta_{i,t} < P_t$, *for all* (i,t). *Then, there are regular cost functions* $\bar{C}_i : \mathbb{R}_+ \to \mathbb{R}$ *such that*

(i) $\bar{C}_i(Q_{i,t}) = C_{i,t}$;

(ii) $\bar{C}'_i(Q_{i,t}) = \delta_{i,t}$; and

(iii) for all q_i in $[0, Q_{i,t})$,

$$P_t q_i - \bar{C}_i(q_i) < P_t Q_{i,t} - \bar{C}_i(Q_{i,t}). \tag{6}$$

Furthermore, $\bar{C}_i$ may be chosen such that $\bar{C}'_i(0) = \delta_{i,0}$ for any $\delta_{i,0} > 0$ and that $\bar{C}'_i$ varies minimally.

To motivate the next result, note that if firm i is playing its best response for demand function $\bar{P}_t$ and cost function $\bar{C}_i$, then it satisfies the first order condition

$$\bar{P}'_t(Q_t)Q_{i,t} + P_t = \bar{C}'_i(Q_{i,t}). \tag{7}$$

It follows that

$$-\bar{P}'_t(Q_t) = \frac{P_t - \bar{C}'_1(Q_{1,t})}{Q_{1,t}} = \frac{P_t - \bar{C}'_2(Q_{2,t})}{Q_{2,t}} = \ldots = \frac{P_t - \bar{C}'_I(Q_{I,t})}{Q_{I,t}}. \tag{8}$$

This accounts for the condition imposed on the cost functions in the next result, which is loosely analogous to Lemma 2.

LEMMA 4. *Let* $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ *be a set of positive numbers satisfying*

$$\frac{P_t - \delta_{1,t}}{Q_{1,t}} = \frac{P_t - \delta_{2,t}}{Q_{2,t}} = \ldots = \frac{P_t - \delta_{I,t}}{Q_{I,t}} > 0 \ \ \text{for each } t \in \mathcal{T} \tag{9}$$

and suppose that the cost functions $\bar{C}_i : \mathbb{R}_+ \to \mathbb{R}$ *satisfy properties (i)-(iii) in Lemma 3. Then, there are regular demand functions* $\bar{P}_t : \mathbb{R}_+ \to \mathbb{R}$ *such that,* $\bar{P}_t(\sum_{i\in\mathcal{I}} Q_{i,t}) = P_t$ *and, for every firm* i,

$$\operatorname{argmax}_{q_i \geq 0}[q_i \bar{P}_t(q_i + \sum_{j\neq i} Q_{j,t}) - \bar{C}_i(q_i)] = Q_{i,t}.$$

The proof of this lemma is deferred to the Appendix.

It is important to notice that for any P_t and $\{Q_{i,t}\}_{i\in\mathcal{I}}$ there *always* exist positive numbers $\{\delta_{i,t}\}_{i\in\mathcal{I}}$ such that equation (9) holds. Suppose that firm k produces more than any other firm at observation t, i.e., $Q_{k,t} \geq Q_{i,t}$ for all i in I. Let $\delta_{k,t}$ be any positive number smaller than P_t, and define $\beta = (P_t - \delta_{k,t})/Q_{k,t}$. Then,

$$\delta_{i,t} := P_t - \beta Q_{i,t} \geq P_t - \beta Q_{k,t} = \delta_{k,t} > 0.$$

It follows immediately from this observation that Lemmas 3 and 4 together establish Theorem 1.

Like Proposition 2, Theorem 1 has the feature that it only checks that each firm in the Cournot oligopoly is not over-producing, but it does not test that there is no

under-production. Indeed, property M is sufficiently weak that there are other reasonable scenarios of firm interaction under which it will also be satisfied. In particular, M holds if the firms are colluding to maximize total profits: this means that their collusion will never generate any evidence that is contrary to Cournot rationalizability.

To state this claim formally, we define a set of observations $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}, (C_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ as being *consistent with collusion* if there is a regular cost function $\tilde{C}_i : \mathbb{R}_+ \to \mathbb{R}$ for each firm i and a regular demand function $\tilde{P}_t : R_+ \to R$ for each t, such that

(i) $\tilde{C}_i(Q_{i,t}) = C_{i,t}$ and $\tilde{P}_t(Q_t) = P_t$; and

(ii) $\mathrm{argmax}_{(q_i)_{i\in\mathcal{I}}\geq 0} \left[(\sum_{i\in\mathcal{I}} q_i)\tilde{P}_t(\sum_{i\in\mathcal{I}} q_i) - \sum_{i\in\mathcal{I}} \tilde{C}_i(q_i) \right] = (Q_{i,t})_{i\in\mathcal{I}}$.

Condition (i) requires that the cost and inverse demand functions agree with the data while (ii) says that, at each observation t, the observed output distribution across firms must maximize total profit.

PROPOSITION 4. *The generic set of observations $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}, (C_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ is consistent with collusion only if it obeys M.*

Proof: We denote the profit of firm i at observation t by $\Pi_{i,t}$. Suppose that there is collusion but M is violated, so for some firm k, observation t, and $t' \in L_k(t)$,

$$P_t Q_{k,t'} - C_{k,t'} \geq P_t Q_{k,t} - C_{k,t} = \Pi_{k,t}.$$

By definition, $Q_{k,t'} < Q_{k,t}$, so $\tilde{P}_t(Q_{k,t'} + \sum_{j\neq k} Q_{j,t}) > P_t$. This implies that

$$\tilde{P}_t \left(Q_{k,t'} + \sum_{j\neq k} Q_{j,t} \right) Q_{k,t'} - \tilde{C}_k(Q_{k,t'}) > \Pi_{k,t}, \tag{10}$$

whereas for every $i \neq k$,

$$\tilde{P}_t \left(Q_{k,t'} + \sum_{j\neq k} Q_{j,t} \right) Q_{i,t} - \tilde{C}_i(Q_{i,t}) > \Pi_{i,t}. \tag{11}$$

In other words, both k and all other firms are strictly better off if k reduces its output (with the other firms benefitting from the higher market price). Clearly, the output vector $(Q_{i,t})_{i\in\mathcal{I}}$ does not maximize total industry profit at t, so collusion is excluded.   Q.E.D.

3.2. Convincingly Rationalizable Observations

The set of observations $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}, (C_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ is said to admit a *convincing Cournot rationalization* (or to be *convincingly Cournot rationalizable*) if it is Cournot rationalizable and, for each firm i, the rationalizing cost function $\bar{C}_i$ can be chosen to satisfy the convincing criterion. The next result gives necessary and sufficient conditions for a data set to admit a convincing rationalization.

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THEOREM 2. *A generic set of observations* $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}, (C_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ *admits a convincing Cournot rationalization if, and only if, it obeys property M and there exists positive scalars* $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ *that satisfy (9) and*

$$\delta_{i,t} \text{ lies between } M_{i,t} \text{ and } M_{i,u_i(t)} \text{ for all } t \neq t_i^{**}. \tag{12}$$

Proof: Suppose that $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}, (C_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ is convincingly Cournot rationalizable. Theorem 1 tells us that M holds. Choosing $\delta_{i,t} = \bar{C}'_i(Q_{i,t})$, (9) follows from (8) and (12) follows from condition (C1) of the convincing criterion.

Conversely, suppose M holds and there exist positive scalars $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ satisfying (9) and (12). Lemma 3 guarantees that there is $\bar{C}_i$ satisfying conditions (i)-(iii) in that lemma and condition (C2) of the convincing criterion. Furthermore, because of (12), $\bar{C}_i$ will also satisfy condition (C1) of the convincing criterion. Finally, Lemma 4 guarantees that a convincing rationalization exists. *Q.E.D.*

Remark: If $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ exists that satisfies the conditions stated in this theorem, then the rationalizing cost function for firm i may be chosen to satisfy $\bar{C}'_i(Q_{i,t}) = \delta_{i,t}$, with $\bar{C}'_i(0)$ taking on any positive value.

This result says that convincing Cournot rationalizability is equivalent to three conditions that (loosely speaking) perform three distinct roles: property M provides the justification for the global optimality of the observed output choices; condition (9) provides the justification for local optimality; and condition (12) guarantees that these justifications are convincingly related to each other. Note that condition M can be directly checked, while the existence of $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ obeying (9) and (12) is a linear programming problem, so the test proposed by Theorem 2 can be solved.[11]

In the single-firm setting considered in the last section, we concluded that a data set is rationalizable if and only if it is convincingly rationalizable. We claim that this is *not* the case in the multi-firm context of this section. Since it is always possible to find $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ that obey (9), we are effectively claiming that *there are data sets for which* (9) *and* (12) *cannot be simultaneously satisfied.* For such a data set, any rationalization will involve some firm j having a rationalizing cost function $\bar{C}_j$ that violates the convincing criterion. Indeed, because (12) is violated, we know that $\bar{C}_j$ violates condition (C1) at some observed output $Q_{j,t}$. We shall give an example of this phenomenon in the next section.

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[11]By this we mean that there is a procedure for determining, in a *finite* number of steps, whether this problem admits a solution.

In those situations where a firm has cost observations that are consistent with increasing, decreasing, or constant marginal cost, the modeler may wish to find a rationalizing cost function that displays the same behavior at all output levels. The next result, which is a slight variant of Theorem 2, provides the linear program for determining if such a rationalization is possible.

THEOREM 3. *Let $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}, (C_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ be a generic set of observations and suppose that all firms in $\mathcal{J} \subseteq \mathcal{I}$ have cost observations that are consistent with increasing marginal cost. Then the following conditions on the data set are equivalent:*
[A]. *It admits a Cournot rationalization such that the rationalizing cost function for every firm j in $\mathcal{J}$ has increasing marginal cost, i.e., $\bar{C}'_j(q)$ increases with q.*
[B]. *It obeys property M and there exists positive scalars $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ satisfying (9), with*

$$M_{i,t} < \delta_{i,t} < M_{i,u_i(t)} \ \ \text{for all } t \neq t_i^{**} \text{ and} \tag{13}$$

$$\delta_{i,t_i^{**}} > M_{i,t_i^{**}}. \tag{14}$$

Proof: That [A] implies [B] is easy to see. By Theorem 2, since the data set admits a Cournot rationalization, it must obey M and (9) is satisfied if $\delta_{i,t} = \bar{C}'_i(Q_{i,t})$. For firms in $\mathcal{J}$, $\bar{C}'_j$ is increasing, in which case (13) and (14) must hold.

Suppose that [B] holds; Lemma 3 guarantees that there is $\bar{C}_i$ satisfying conditions (i)-(iii) in that lemma. Furthermore the rationalizing cost functions $\bar{C}_j$ (for $j \in \mathcal{J}$) can be chosen to satisfy the convincing criterion (because of (13)). We may also require $\bar{C}'_j(0) = \delta_{j,0} < M_{j,t^*}$ for $j \in \mathcal{J}$; this condition, together with (13) and (14), imply that $M_{j,t}$ lies strictly between $\delta_{j,t}$ and $\delta_{j,l_j(t)}$ for all t. Therefore, condition (C2) of the convincing criterion says that $\bar{C}'_j$ is strictly increasing for all q. Finally, Lemma 4 guarantees that a rationalization exists. *Q.E.D.*

Note that there are some fairly obvious variations on Theorem 3. If the cost observations of the firms in $\mathcal{J}$ are consistent with decreasing marginal costs and we wish $\bar{C}_j$ to have the same property, then the necessary and sufficient conditions will involve (13) and (14), but with the inequalities reversed. If the cost observations of firms in $\mathcal{J}$ are consistent with constant marginal cost, then the necessary and sufficient conditions will involve replacing (13) and (14) with $\delta_{i,t} = M_{i,t}$ for all t. (Note that in this case $M_{i,t} = M_{i,t'}$ for any t and t'.) It is also possible to mix-and-match conditions. For example, if the cost observations of firms in $\mathcal{J}_1$ are consistent with increasing marginal cost and the cost observations of firms in $\mathcal{J}_2$ are consistent with decreasing marginal costs, with all other firms

displaying neither pattern, then by applying the relevant restrictions on $\delta_{i,t}$ for each firm, we could test whether there is a rationalization in which firms in $\mathcal{J}_1$ have cost functions with increasing marginal costs, firms in $\mathcal{J}_2$ have decreasing marginal costs, and firms in $\mathcal{I} \setminus (\mathcal{J}_1 \cup \mathcal{J}_2)$ have cost functions that satisfy the convincing criterion.

4. Cournot Rationalizability Without Observing Costs

In this section, we consider the problem of Cournot rationalizability under the assumption that the costs incurred by each firm are not observed. Formally, the set of observations reduces to $\{[P_t, (Q_{i,t})_{i \in \mathcal{I}}]\}_{t \in \mathcal{T}}$. This data set is said to be *generic* if $Q_{i,t'} \neq Q_{i,t}$ whenever $t \neq t'$; it is *Cournot rationalizable* if we can find a regular demand function, $\bar{P}_t$, for each observation t, and a regular cost function, $\bar{C}_i$, for each firm i, such that

(i) $\bar{P}_t(\sum_{i \in \mathcal{I}} Q_{i,t}) = P_t$; and

(ii) $\mathrm{argmax}_{q_i \geq 0}[q_i \bar{P}_t(q_i + \sum_{j \neq i} Q_{j,t}) - \bar{C}_i(Q_i)] = Q_{i,t}$.

In words, the t-th observation, $[P_t, (Q_{i,t})_{i \in I}]$, is the Cournot outcome when each firm i has cost function $\bar{C}_i$ and the market inverse demand function is $\bar{P}_t$.

4.1. Rationalizability and the Convincing Criterion

The following result says that Cournot competition imposes no restriction on the observations $\{[P_t, (Q_{i,t})_{i \in \mathcal{I}}]\}_{t \in \mathcal{T}}$.

COROLLARY 1. *Any generic set of observations $\{[P_t, (Q_{i,t})_{i \in \mathcal{I}}]\}_{t \in \mathcal{T}}$, is Cournot rationalizable.*

Proof: By Theorem 1, it suffices that we find an array of individual costs, $\{C_{i,t}\}_{(i,t) \in \mathcal{I} \times \mathcal{T}}$, that, when added to the observed data, gives a set of observations that obeys M. Equivalently, we need to find $\{M_{i,t}\}_{(i,t) \in \mathcal{I} \times \mathcal{T}}$ that obeys (5). But since the right end of that inequality is always positive and bounded away from zero for any t and t', it is clear that (5) holds if $M_{i,t}$ is sufficiently small. $\hspace{1cm}$ Q.E.D.

The next question we should ask is obvious: under what conditions is $\{[P_t, (Q_{i,t})_{i \in \mathcal{I}}]\}_{t \in \mathcal{T}}$ convincingly rationalizable? By this we mean that the cost functions $\bar{C}_i$ must satisfy the convincing criterion, i.e., conditions (C1) and (C2), where

$$M_{i,t} = \frac{\int_{Q_{i,l_i(t)}}^{Q_{i,t}} \bar{C}'(q_i) \, \mathrm{d}q_i}{Q_{i,t} - Q_{i,l_i(t)}}, \tag{15}$$

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in addition to the conditions imposed by rationalizability. The next result follows from Theorem 2.

THEOREM 4. *A generic set of observations $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ is convincingly Cournot rationalizable if, and only if, the following three conditions are satisfied:*
(a) there exists positive scalars $\{\Delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ such that, for each firm i and observation t,

$$\sum_{s\in(L_i(t)\cup\{t\})\backslash(L_i(t')\cup\{t'\})} \Delta_{i,s}\left(Q_{i,s} - Q_{i,l(s)}\right) < P_t(Q_t - Q_{t'}) \quad \text{for } t' \in L_i(t); \qquad (16)$$

(b) there exists positive scalars $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ that satisfy (9); and
(c) for all $t \neq t_i^{**}$,

$$\delta_{i,t} \text{ lies between } \Delta_{i,t} \text{ and } \Delta_{i,u_i(t)}. \qquad (17)$$

Proof: Suppose that $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ is convincingly Cournot rationalizable. Set $\delta_{i,t} = \bar{C}_i(Q_{i,t})$ and $\Delta_{i,t}$ equal to the right hand side of (15); then (a) and (b) holds because of (5) and (8) respectively. Finally, since the observations are convincingly rationalizable, (17) must also hold.

For the other direction, since (a) holds we may (fictitiously) assume that cost is observable, with

$$C_{i,t} = \sum_{s\in(L_i(t)\cup\{t\})\backslash\{0\}} \delta_{i,s}\left(Q_{i,s} - Q_{i,l_i(s)}\right). \qquad (18)$$

Then (16) is just property M (see Equation (5)). By Theorem 2, we know that (a), (b), and (c) guarantee that the set of observations is convincingly rationalizable.   *Q.E.D.*

Theorem 4 says that to determine if $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ is convincingly rationalizable we need to find $\Delta_{i,t}$ and $\delta_{i,t}$ (for $(i,t) \in I \times \mathcal{T}$) obeying conditions (a) to (c). Conditions (a) and (b) are linear conditions, but condition (c) is nonlinear, so this is no longer a linear programming problem.[12] However, it is clear that the problem *is* solvable, because it can always be broken up into a finite collection of linear programs. To see this, add the condition $\Delta_{i,t} \leq \Delta_{i,u(t)}$, so condition (c) can be written as $\Delta_{i,t} \leq \delta_{i,t} \leq \Delta_{i,u(t)}$, which is a linear condition. Alternatively, introduce the condition $\Delta_{i,t} \geq \Delta_{i,u(t)}$, so that (c) can be written as $\Delta_{i,t} \geq \delta_{i,t} \geq \Delta_{i,u(t)}$, which is another linear condition. Since there are only finitely many such (additional and linear) conditions on $\Delta_{i,t}$, we obtain a finite number of linear programs, each of which is solvable.[13]

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[12] Notice the difference between (12) and (17). In the former, the bounds on $\delta_{i,t}$ are obtained from the data while in the latter they are part of the solution to the program.

[13] Note that we are simply making an argument to establish the solvability of the problem. In an actual

Table 1: Data that cannot be convincingly rationalized.

t	P_t	Q_{1t}	Q_{2t}
1	2	40	90
2	10	50	100
3	1.5	60	110

Echoing something we said earlier in relation to Theorem 2, the conditions (a), (b), and (c) in Theorem 4 can be seen as ensuring three different aspects of a convincing rationalization. Condition (a) guarantees that $Q_{i,t}$ is superior to other output choices far away; condition (b) guarantees that $Q_{i,t}$ is locally optimal; and condition (c) guarantees that the values of $\Delta_{i,t}$ and $\delta_{i,t}$ obtained to satisfy (a) and (b) are related to each other in a convincing way.

We now turn to the issue that we raised in the previous section but did not resolve, i.e., that there *do* exist observations that are rationalizable but not convincingly rationalizable. If this is not the case, Theorem 2 and Theorem 4 will have no content. Consider the observations recorded in Table 1, which gives the prices and outputs for a duopoly. Corollary 1 says that any data set of prices and output is Cournot rationalizable, so this is certainly true of the observations in Table 1. However, we claim that these observations are not convincingly rationalizable.

Suppose instead that it is. By (8), $\bar{C}'_1(Q_{1,t}) = P_t - \left[P_t - \bar{C}'_2(Q_{2,t})\right] Q_{1,t}/Q_{2,t}$. Therefore,

$$\bar{C}'_1(Q_{1,t}) \geq P_t \left[1 - \frac{Q_{1,t}}{Q_{2,t}}\right].$$

From observation $t = 2$, we obtain $\bar{C}'_1(50) \geq 5$. By the convincing criterion (specifically, (C1)), the average marginal cost (for firm 1) of either increasing production from 40 to 50 or from 50 to 60 must be at least 5. The observed output at $t = 3$ is 60, so the cost of raising output from 40 to 60 is at least $5 \times 10 = 50$. On the other hand, the increased revenue that firm 1 earns by raising output from 40 to 60 is no more than $1.5 \times 20 = 30$, which means that the firm is better off producing at 40 rather than 60 at $t = 3$. We conclude that the data in Table 1 is not convincingly rationalizable.

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application with a large data set, it may be necessary to have an algorithm that is more computationally efficient. The programming problem we have here is an example of a problem in disjunctive programming, because we wish to determine if there is a solution to *any one* of a finite collection of linear programs. Mixed integer programming methods are potentially useful for this purpose.

It is worth emphasizing the role played by the convincing criterion in this argument: *it is used to extend the lower bound on $\bar{C}'_1(50)$ obtained through a first-order condition to some non-infinitesimal interval.* Without such a criterion, the value of $\bar{C}'_1(50)$) will have nothing to say about the marginal cost of any discrete change in output. Note also that if the modeler makes an *a priori* assumption that marginal cost is monotonic, that assumption will have a similar (indeed even more powerful) impact: if marginal cost is increasing, then the lower bound on marginal cost obtained at output 50 can be extended to all output levels above 50 and if marginal cost is decreasing, then the lower bound obtained at 50 can be extended to all output levels below 50.

4.2. Collusion

Note that while the observations in Table 1 are not convincingly rationalizable as Cournot outcomes, they *can* be rationalized as collusive outcomes. Indeed, any set of observations can be rationalized as a set of collusive outcomes in which all firms have constant and identical marginal costs. This is an easy consequence of our results in Section 2; in the result below, every firm should be interpreted as having the identical cost function $\bar{C}(q) = \epsilon q$.

COROLLARY 2. *For any generic set of observations $\{(P_t, (Q_{i,t})_{i\in\mathcal{I}})\}_{t\in\mathcal{T}}$, there is $\epsilon > 0$ and regular inverse demand functions $\tilde{P}_t : \mathbb{R}_+ \to \mathbb{R}$ for each t, such that, for every t,*

$$(Q_{i,t})_{i\in\mathcal{I}} \in \text{argmax}_{(q_i)_{i\in\mathcal{I}} \geq 0} \left[\left(\sum_{i\in\mathcal{I}} q_i \right) \tilde{P}_t \left(\sum_{i\in\mathcal{I}} q_i \right) - \epsilon \left(\sum_{i\in\mathcal{I}} q_i \right) \right].$$

Proof: Suppose that each firm has the cost function $\bar{C}(q) = \epsilon q$. Then every output allocation is cost efficient and if the firms are colluding, they must act like a monopoly with the same cost function $\bar{C}$. Choose ϵ sufficiently close to zero so that conditions (i), (ii), (iii) in Lemma 1 are satisfied, where $Q_t = \sum_{i\in\mathcal{I}} Q_{i,t}$. The existence of P_t is guaranteed by Lemma 2. *Q.E.D.*

4.3. Other Restrictions on Cost Functions

The convincing criterion is just one of many restrictions that a modeler may wish to impose on the rationalizing cost functions in the event that cost information is unavailable or incompletely available. These restrictions can be imposed, instead of or in addition to, the convincing criterion. We discuss some of these briefly. To keep the exposition simple, we assume that any restriction we consider is imposed on *all* firms in the industry,

though the reader should bear in mind that it is possible to mix-and-match by imposing different restrictions on different firms or multiple restrictions on any firm. Note also that all the restrictions are refutable in the sense that one can construct data sets which are not jointly consistent with these cost restrictions *and* Cournot rationalizability.

(1) Nondecreasing Marginal Costs. Suppose that some firm j has nondecreasing marginal costs. In this case, if a Cournot rationalization exists, it must be possible for $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ to be chosen such that, in addition to the conditions (9), we also have

$$\delta_{j,t'} \geq \delta_{j,t} \text{ whenever } Q_{j,t'} > Q_{j,t}. \tag{19}$$

This is clear since we know that $\delta_{j,t}$ can be chosen to equal $\bar{C}'_j(Q_{j,t})$. The next result gives necessary and sufficient conditions under which a data set of prices and quantities is rationalizable as Cournot outcomes and with all firms having increasing marginal costs. Note that, unlike Theorem 4, this result has no conditions explicitly involving $\Delta_{j,t}$. As will be clear from the proof, the existence of $\Delta_{j,t}$ obeying (16) is guaranteed by the (19). This is analogous to the fact that in concave optimization problems, local optimality implies global optimality.

THEOREM 5. *Let $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ be a generic set of observations. Then the following conditions on the data set are equivalent:*
[A] *It admits a Cournot rationalization such that the rationalizing cost functions have nondecreasing marginal costs, i.e., for every firm i, $\bar{C}'_i(q)$ is nondecreasing in q.*
[B]. *There exists positive scalars $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ that satisfy (9) and (19).*

Proof: Setting $\delta_{i,t} = \bar{C}'_i(Q_{i,t})$ it is clear that [A] implies [B]. Now suppose that [B] holds. Choose $\delta_{i,0}$ to be some positive number lower than δ_{i,t_i^*} and then $\Delta_{i,t}$ to be some number in the interval $(\delta_{i,l_i(t)}, \delta_{i,t})$ if $\delta_{i,l_i(t)} \neq \delta_{i,t}$ and $\Delta_{i,t} = \delta_{i,t}$ otherwise (for all $t \in \mathcal{T}$). Since $\delta_{i,t}$ is nondecreasing with $Q_{i,t}$, so is $\Delta_{i,t}$. Therefore, for s in $L_i(t)$,

$$\Delta_{i,s} \leq \Delta_{i,t} \leq \delta_{i,t} < P_t.$$

It follows that (16) is satisfied. Defining $C_{i,t}$ by (18), (16) is just property M (see (5)). Lemma 3 guarantees that, for every firm i, there is $\bar{C}_i$ that has minimal variation and satisfies conditions (i)-(iii) in that lemma. In this case, minimal variation means that $\bar{C}'_i$ is weakly increasing in q, with $\bar{C}'_i$ constant on any interval $[Q_{i,l_i(t)}, Q_{i,t}]$ where $\delta_{i,l_i(t)} = \delta_{i,t}$ and with $\bar{C}'_i$ increasing on any interval $[Q_{i,l_i(t)}, Q_{i,t}]$ where $\delta_{i,l_i(t)} < \delta_{i,t}$. Finally, Lemma 4 guarantees that a convincing rationalization exists. *Q.E.D.*

Note that [B] in Theorem 5 specifies a linear program, so it is solvable and computationally simpler than the nonlinear program specified in Theorem 4.

Instead of testing for Cournot rationalizability with nondecreasing marginal costs, the modeler may wish to restrict firms to have constant marginal costs. It is quite clear that Cournot rationalizability with this added requirement is equivalent to the following: *there exists positive scalars $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ that satisfy (9) and $\delta_{i,t} = \delta_{i,t'}$ for all t and t' in $\mathcal{T}$.* We leave the reader to verify this claim.

(2) Nonincreasing Marginal Costs. Suppose that we want the data set to be Cournot rationalizable, with all firms having nonincreasing marginal costs. We claim that this is possible if and only if there exists positive scalars $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$ such that (a) for each firm i and observation t, the condition (16) holds, with $\Delta_{i,t} = \delta_{i,t}$, (b) (9) holds for every $t \in \mathcal{T}$, and (c) $\delta_{j,t'} \leq \delta_{j,t}$ whenever $Q_{j,t'} > Q_{j,t}$.

It is not hard to see that conditions (a)-(c) are necessary. If such a rationalization exists, then we may choose $\delta_{i,t} = \bar{C}_i'(Q_{i,t})$, so (b) is satisfied because of (8) and (c) is satisfied because $\bar{C}_i'$ is nonincreasing. That $\bar{C}_i'$ is nonincreasing also guarantees that

$$\delta_{i,t} = \bar{C}_i'(Q_{i,t}) \leq \frac{\int_{Q_{l_i(t)}}^{Q_t} \bar{C}_i'(q)\,\mathrm{d}q}{Q_t - Q_{l_i(t)}},$$

so that (a) holds because of (5). We shall skip the proof that (a)-(c) are sufficient; this could be obtained by a straightforward modification of the proof of Theorems 4 or 5.

(3) A weaker version of the convincing criterion. An obvious weakening of the convincing criterion is to retain condition (C2), i.e., $\bar{C}_i'$ has minimal variation, but to modify (C1) by requiring that $\bar{C}_i'(Q_{i,t})$ be in the interval $[k\min\{M_{i,t}, M_{i,u_i(t)}\}, K\max\{M_{i,t}, M_{i,u_i(t)}\}]$, where $M_{i,t}$ is defined by (15) and k and K are chosen by the modeler, with $0 < k < 1 < K < \infty$. Such a generalization of the convincing criterion will not affect the qualitative nature of the results in this paper It is completely straightforward to check that the necessary sufficient conditions for this modified version of convincing rationalizability are given by (a) and (b) in Theorem 4, and the following:

$$\delta_{i,t} \in [k\min\{\Delta_{i,t}, \Delta_{i,u_i(t)}\}, K\max\{\Delta_{i,t}, \Delta_{i,u_i(t)}\}] \text{ for all } t \neq t_i^{**}.$$

Like Theorem 4, and for the same reason, this program is nonlinear but solvable.

(4) Bounds on Marginal Costs. Without knowing the precise costs incurred by the firm, the modeler may nonetheless know enough to impose some bounds on marginal

costs. For example, suppose that it knows that for firm i, marginal costs lies between b_i and B_i. In that case, it is clear that Cournot rationalizability, with $b_i \leq \bar{C}'_i(q) \leq B_i$ for all i and $q > 0$, is possible if and only if (a) and (b) in Theorem 4 are satisfied, and

$$b_i \leq \delta_{i,t}, \ \Delta_{i,t} \leq B_i.$$

Note that this specifies a linear program.

5. Cournot Rationalizability with Demand and Cost Fluctuations

Until now, we have assumed that the set of observations is generated by changes to the demand function and that firms' cost functions are unchanged across the entire set of observations. In this section, we show how observed fluctuations to costs of a particular type can be dealt with easily within the framework we have set up. To be specific, we shall assume that, in addition to P_t and $Q_{i,t}$ (for all $i \in \mathcal{I}$), the observer also knows that firm i's marginal cost has shifted by $A_{i,t}$ at observation t. We may normalize the fluctuations so $A_{i,1} = 0$ for all i; in other words, the change in marginal cost for each firm is measured against its value at $t = 1$. Note that $A_{i,t}$ may take positive or negative values. Cost changes of this kind are common; for example, they arise if the output is taxed and the tax rate changes across observations. Or they may arise from observed changes to the cost of a raw material that is used at a constant rate in the production process.[14] Formally, the set of observations is $\{[P_t, (Q_{i,t})_{i \in \mathcal{I}}, (A_{i,t})_{i \in \mathcal{I}}]\}_{t \in \mathcal{T}}$; this set is said to be *generic* if $Q_{i,t'} \neq Q_{i,t}$ whenever $t \neq t'$. It is *Cournot rationalizable* if we can find a regular demand function, $\bar{P}_t$, for each observation t, and a regular cost function, $\bar{C}_i$, for each firm i, such that

(i) $\bar{P}_t(\sum_{i \in \mathcal{I}} Q_{i,t}) = P_t$;

(ii) $\bar{C}'_i(q_i) + A_{i,t} > 0$ for all $q_i \geq 0$; and

(iii) $\operatorname{argmax}_{q_i \geq 0}[q_i \bar{P}_t(q_i + \sum_{j \neq i} Q_{j,t}) - \bar{C}_i(q_i) - A_{i,t} q_i] = Q_{i,t}$.

In short, $(P_t, (Q_{i,t})_{i \in I})$, is the Cournot outcome when each firm i has the increasing cost function $\hat{C}_{i,t}(q_i) = \bar{C}_i(q_i) + A_{i,t} q_i$ and the market inverse demand function is $\bar{P}_t$. A rationalization for $\{[P_t, (Q_{i,t})_{i \in \mathcal{I}}, (A_{i,t})_{i \in \mathcal{I}}]\}_{t \in \mathcal{T}}$ is convincing if the cost functions $\bar{C}_i$ satisfy the convincing criterion, i.e., the conditions (C1) and (C2), with $M_{i,t}$ given by (15). Note also that while linear cost shocks are observed, we are assuming here that the 'permanent' part of the cost function $\bar{C}$ remains completely unknown; it is not hard to modify the

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[14] For example, in the case of Genesove and Mullin (1998), this is the price of raw sugar.

exposition to deal with the case (like in Section 3) where costs at equilibrium outputs are observed and we shall leave that to the reader.

The next result should not be surprising given Theorem 4.

THEOREM 6. *A generic set of observations* $\{[P_t, (Q_{i,t})_{i\in\mathcal{I}}, (A_{i,t})_{i\in\mathcal{I}}]\}_{t\in\mathcal{T}}$ *(with* $A_{i,1} = 0$ *for* $i \in \mathcal{I}$*) is convincingly Cournot rationalizable if, and only if, the following three conditions are satisfied:*

(a) there exists scalars $\{\Delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$*, with* $\Delta_{i,t} > -\min_{s\in\mathcal{T}} A_{i,s}$*,[15] such that, for each firm* i *and observation* t*,*

$$\sum_{s\in(L_i(t)\cup\{t\})\backslash(L_i(t')\cup\{t'\})} (\Delta_{i,s} + A_{i,t}) \left(Q_{i,s} - Q_{i,l(s)}\right) < P_t(Q_{i,t} - Q_{i,t'}) \;\; for \; t' \in L_i(t); \quad (20)$$

(b) there exists scalars $\{\delta_{i,t}\}_{(i,t)\in\mathcal{I}\times\mathcal{T}}$*, with* $\delta_{i,t} > -\min_{s\in\mathcal{T}} A_{i,s}$*, that satisfy*

$$\frac{P_t - \delta_{1,t} - A_{1,t}}{Q_{1,t}} = \frac{P_t - \delta_{2,t} - A_{2,t}}{Q_{2,t}} = \ldots = \frac{P_t - \delta_{I,t} - A_{l,t}}{Q_{I,t}} > 0 \;\; for \; each \; t \in \mathcal{T}; \; and \quad (21)$$

(c)

$$\delta_{i,t} \; lies \; between \; \Delta_{i,t} \; and \; \Delta_{i,u_i(t)} \; for \; all \; t \neq t_i^{**}. \quad (22)$$

Proof: We shall only outline the argument since it is similar to those provided for Theorems 2 and 4.

If a convincing rationalization exists, set $\delta_{i,t} = \bar{C}_i(Q_{i,t})$ and

$$\Delta_{i,t} = \frac{\int_{Q_{l_i(t)}}^{Q_t} \bar{C}'_i(q)\mathrm{d}q}{Q_t - Q_{l_i(t)}}.$$

Both $\Delta_{i,t}$ and $\delta_{i,t}$ are strictly greater than $-\min_{s\in\mathcal{T}} A_{i,s}$ since $\bar{C}'_i + A_{i,t} > 0$ for all t. The condition (a) must hold since it is just a version of the marginal property (see (5) and (b) must hold since it follows from the first order condition (see the argument leading up to (8). Lastly, since the observations are convincingly rationalizable, (22) must hold.

For the other direction, since (a) holds we may (fictitiously) assume that cost at $Q_{i,t}$ is observable, with

$$C_{i,t} = \sum_{s\in(L_i(t)\cup\{t\})\backslash\{0\}} \Delta_{i,s} \left(Q_{i,s} - Q_{i,l_i(s)}\right). \quad (23)$$

Then (20) simply says that the marginal property holds (after taking into account the cost fluctuation). By adapting the argument of Lemma 3, we can conclude that there are

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[15] Since $A_{i,1} = 0$, we must have $\Delta_{i,t} > 0$.

regular cost functions $\bar{C}_i : R_+ \to R$ such that (i) $\bar{C}_i(Q_{i,t}) = C_{i,t}$; (ii) $\bar{C}_i'(Q_{i,t}) = \delta_{i,t}$; and (iii) for all q_i in $[0, Q_{i,t})$,

$$P_t q_i - \bar{C}_i(q_i) - A_{i,t} q_i < P_t Q_{i,t} - \bar{C}_i(Q_{i,t}) - A_{i,t} Q_{i,t}. \tag{24}$$

Furthermore, $\bar{C}_i$ may be chosen such that $\bar{C}_i'(0) = \delta_{i,0}$ for any $\delta_{i,0} > 0$ and $\bar{C}_i'$ varies minimally. If we choose $\delta_{i,0} > -\min_{s \in \mathcal{T}} A_{i,s}$, then the fact that $\delta_{i,t}$ and $\Delta_{i,t}$ are also greater than $-\min_{s \in \mathcal{T}} A_{i,s}$ by assumption means that $\bar{C}_i$ may be chosen such that $\bar{C}_i' + A_{i,t} > 0$ for all (i,t). Note that $\bar{C}_i$ satisfies the convincing criterion since it satisfies (C2) (minimal variation) and also (C1) (because of condition (c)). Since (b) holds and $\bar{C}_i$ obeys (i)-(iii), Lemma 4 guarantees the existence of regular inverse demand functions P_t that rationalizes the observed outputs at each observation. $Q.E.D.$

In Section 4, we considered various restrictions that could be imposed on the rationalizing cost functions, as an alternative (or in addition) to the convincing criterion, and developed tests for these restrictions. It is entirely straightforward (given Section 4) to carry out a similar exercise in this context; we leave the details to the reader.

Proof of Lemma 2: Define $g(q) = k(q - Q_t) + \delta_t$. The graph of g is a line, with positive slope k that passes through the point (Q_t, δ_t). Since $\alpha_t < P_t$, there is $k > 0$ and $\epsilon > 0$ such that, $P_t > g(Q_t - \epsilon)$ and for q in the interval $[Q_t - \epsilon, Q_t)$, we have

$$g(q) > \bar{C}'(q). \tag{25}$$

For q in $[0, Q_t - \epsilon]$, there exists $\zeta > 0$ such that

$$Pq - \bar{C}(q) < PQ_t - \bar{C}(Q_t) \text{ for } P_t < P < P_t + \zeta; \tag{26}$$

this follows from property (iii) in Lemma 1

Instead of constructing $\bar{P}_t$ directly, we shall specify the function $\bar{P}_t'$; $\bar{P}_t$ can then be obtained by integration. We denote the marginal revenue function induced by $\bar{P}_t$ by $\bar{m}_t$; i.e., $\bar{m}_t(q) = \bar{P}_t'(q)q + \bar{P}_t(q)$. We first consider the construction of $\bar{P}_t'$ in the interval $[0, Q_t]$. Choose $\bar{P}_t'$ with the following properties: (a) $\bar{P}_t'(Q_t) = (\delta_t - P_t)/Q_t$ (which is equivalent to the first order condition $\bar{m}_t(Q_t) = \bar{C}'(Q_t) = \delta_t$), (b) $\bar{P}_t'$ is negative, decreasing and concave in $[0, Q_t]$, and (c) $\int_0^{Q_t} \bar{P}_t'(q)dq = P_t - \bar{P}_t(0) > -\zeta$ and (d) $\bar{P}_t'(Q_t - \epsilon)$ is sufficiently close to zero so that $\bar{m}_t(Q_t - \epsilon) > g(Q_t - \epsilon)$. Property (b) guarantees that $\bar{m}_t$ is decreasing and concave. This fact, together with (a) and (d), ensures that $\bar{m}_t(q) > g(q)$ for q in $[Q_t - \epsilon, Q_t)$; together with (25), we obtain $\bar{m}_t(q) > \bar{C}'(q)$ for q in $[Q_t - \epsilon, Q_t)$. Therefore, in the interval $[Q_t - \epsilon, Q_t]$, profit is maximized at $q = Q_t$. Because of (c), $P_t < \bar{P}_t(q) < P_t + \zeta$, so by (26), $\bar{P}_t(q)q - \bar{C}(q) < P_t Q_t - \bar{C}(Q_t)$ for q in $[0, Q_t - \epsilon]$.

To recap, we have constructed $\bar{P}_t'$ (and hence $\bar{P}_t$) such that with this inverse demand function, profit st Q_t is higher than at any output below Q_t. We now need to specify $\bar{P}_t'$ for $q > Q_t$ such that profit at $q = Q_t$ is higher than at any output level above Q_t. It suffices to have $\bar{P}_t$ such that, for $q > Q_t$,

$$\bar{m}_t(q) = \bar{P}_t'(q)q + \bar{P}_t(q) < \bar{C}'(q),$$

so marginal cost always exceeds marginal revenue for $q > Q_t$. Provided $\bar{P}_t$ is decreasing, it suffices to have $\bar{P}_t'(q)q + P_t < \bar{C}'(q)$, which is equivalent to

$$-\bar{P}_t'(q) > \frac{P_t - \bar{C}'(q)}{q} \tag{27}$$

The right side of this inequality is a continuous function of q; clearly we can choose $\bar{P}_t' < 0$ such that (27) holds for $q > Q_t$.   Q.E.D.

Proof of Lemma 4: This is broadly similar to the proof of Lemma 2 except that we must now take into account the fact that there is more than one firm. For each firm i, define $g_i(q_i) = k_i(q_i - Q_{i,t}) + \delta_{i,t}$. The graph of g_i is a line, with positive slope k_i that passes through the point $(Q_{i,t}, \delta_{i,t})$. Since $\delta_{i,t} < P_t$, there is $\epsilon > 0$ and k_i (for $i \in calI$) such that, $P_t > g_i(Q_{i,t} - \epsilon)$ and for q_i in the interval $[Q_{i,t} - \epsilon, Q_{i,t})$, we have

$$g_i(q_i) > \bar{C}_i'(q_i). \tag{28}$$

For q_i in $[0, Q_{i,t} - \epsilon]$, there exists $\zeta > 0$ such that

$$Pq_i - \bar{C}_i(q_i) < PQ_{i,t} - \bar{C}_i(Q_{i,t}) \text{ for } P_t < P < P_t + \zeta; \tag{29}$$

t his follows from property (iii) in Lemma 3. Note that ζ is common across all firms.

We shall specify the function $\bar{P}'_t$, so $\bar{P}_t$ can be obtained by integration. Holding the output of firm j (for $j \neq i$) at $Q_{j,t}$, we denote the marginal revenue function for firm i by $\bar{m}_{i,t}$; i.e., $\bar{m}_{i,t}(q_i) = \bar{P}'_t(\sum_{j\neq i} Q_{j,t}+q_i)q_i+\bar{P}_t(\sum_{j\neq i} Q_{j,t}+q_i)$. We first consider the construction of $\bar{P}'_t$ in the interval $[0, Q_t]$, where $Q_t = \sum_{i\in\mathcal{I}} Q_{i,t}$. Choose $\bar{P}'_t$ with the following properties: (a) $\bar{P}'_t(Q_t) = (\delta_{i,t} - P_t)/Q_{i,t}$ (which is equivalent to the first order condition $\bar{m}_{i,t}(Q_{i,t}) = \bar{C}'_i(Q_{i,t}) = \delta_{i,t}$; note that there is no ambiguity here because of (9)), (b) $\bar{P}'_t$ is negative, decreasing and concave in $[0, Q_t]$, (c) $\int_0^{Q_t} \bar{P}'_t(q)dq = P_t - \bar{P}_t(0) > -\zeta$ and (d) $\bar{P}'_t(Q_t - \epsilon)$ is sufficiently close to zero so that $\bar{m}_{i,t}(Q_{i,t} - \epsilon) > g_i(Q_{i,t} - \epsilon)$. Property (b) guarantees that $\bar{m}_{i,t}$ is decreasing and concave (as a function of q_i). This fact, together with (a) and (d), ensures that $\bar{m}_{i,t}(q_i) > g_i(q_i)$ for all i and q_i in $[Q_{i,t} - \epsilon, Q_{i,t})$; combining with (28), we obtain $\bar{m}_{i,t}(q_i) > \bar{C}'_i(q)$. Therefore, in the interval $[Q_{i,t} - \epsilon, Q_{i,t}]$, firm i's profit is maximized at $q_i = Q_{i,t}$. Because of (c), $P_t < \bar{P}_t(q) < P_t + \zeta$, so by (29), $\bar{P}_t(\sum_{j\neq i} Q_{j,t} + q_i)q_i - \bar{C}_i(q_i) < P_t Q_{i,t} - \bar{C}_i(Q_{i,t})$ for q_i in $[0, Q_{i,t} - \epsilon]$.

To recap, we have constructed $\bar{P}'_t$ (and hence $\bar{P}_t$) such that, with this inverse demand function, firm i's profit at $Q_{i,t}$ is higher than at any output below $Q_{i,t}$, so long as other firms are producing $\sum_{j\neq i} Q_{j,t}$. Our next step is to show how to specify $\bar{P}'_t$ for $q > Q_t$ in such a way that firm i's profit at $q_i = Q_{i,t}$ is higher than at any output level above $Q_{i,t}$ (for every firm i). It suffices to have $\bar{P}_t$ such that, for $q_i > Q_{i,t}$,

$$\bar{m}_{i,t}(q_i) = \bar{P}'_t(\sum_{j\neq i} Q_{j,t} + q_i)q_i + \bar{P}_t(\sum_{j\neq i} Q_{j,t} + q_i) < \bar{C}'_i(q_i),$$

so firm i's marginal cost always exceeds its marginal revenue for $q_i > Q_{i,t}$. Provided $\bar{P}_t$ is decreasing, it suffices to have $\bar{P}'_t(\sum_{j\neq i} Q_{j,t} + q_i)q_i + P_t < \bar{C}'_i(q_i)$, which is equivalent to

$$-\bar{P}'_t(\sum_{j\neq i} Q_{j,t} + q_i) > \frac{P_t - \bar{C}'_i(q_i)}{q_i} \quad \text{for all firms } i.$$

This can be re-written as

$$-\bar{P}'_t(Q_t + x) > \frac{P_t - \bar{C}'_i(Q_{i,t} + x)}{Q_{i,t} + x} \quad \text{for } x > 0 \text{ and all firms } i \tag{30}$$

The right side of this inequality is a finite collection of continuous functions of x and at $x = 0$, the two sides are equal to each other (because of (9)). Clearly we can choose $\bar{P}'_t < 0$ such that (30) holds for $x > 0$. *Q.E.D.*

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