On Kaneko Congruences

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ON KANEKO CONGRUENCES

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ABSTRACT. We present a proof of certain congruences modulo powers of an odd prime for the coefficients of a series produced by repeated application of U-operator to a certain weakly holomorphic modular form. This kind of congruences were first observed by Kaneko as a result of numerical experiments, and later proved in a different (but similar) case by Guerzhoy [6]. It is interesting to note that, in our case, the congruences become different, both experimentally and theoretically, depending on whether the prime is congruent to 1 or 3 modulo 4.

1. Introduction

In this paper we prove congruences of q-series coefficients modulo powers of primes. These congruences may be described in an elementary way, and we begin with this description.

Consider the following formal power series in variable q

$$e_4(q) = 1+240\sum_{n>0}\left(\sum_{d|n} d^3\right)q^n = 1+240q+2160q^2+6720q^3+17520q^4+30240q^5+60480q^6+\dots$$

and

$$\eta^*(q) = \prod_{n\geq 1}(1-q^n) = 1-q-q^2+q^5+q^7-q^{12}-q^{15}+\dots$$

By construction, both series have integer coefficients, i.e. $e_4, \eta^* \in \mathbb{Z}[[q]]$. Moreover, it is easy to see that the series

$$\frac{e_4(q^4)}{\eta^*(q^4)^2\eta^*(q^8)^2} = 1+242q^4+2647q^8+12734q^{12}+49475q^{16}+151026q^{20}+429260q^{24}+\dots$$

also has integer coefficients. In order to observe the properties in question, we divide the series by q in order to shift the indexing by one, and introduce the notation $c(n)$ for the series coefficients

$$\sum_{n\geq -1} c(n)q^n = \frac{e_4(q^4)}{q\eta^*(q^4)^2\eta^*(q^8)^2} = q^{-1}+242q^3+2647q^7+12734q^{11}+49475q^{15}+\dots$$

Note that $c(n) = 0$ unless $n \equiv 3 \mod 4$.

The following division properties of the coefficients $c(n)$ were first observed by M. Kaneko on the basis of numerical experiments.

Conjecture 1. Let p be an odd prime, and let l, n be positive integers.

(1) If $p \equiv 1 \mod 4$, then $c(p^l n) \equiv 0 \mod p^l$.
(2) If $p \equiv 3 \mod 4$, then $c(p^l n) \equiv 0 \mod p^{[l/2]}$.

An important fact about the series $\sum c(n)q^n$ is that this series appears to be a q-expansion of a weakly holomorphic modular form. This form shows up in his investigation, and he had a reason to suspect interesting division properties of its coefficients. Despite of the simplicity of the conjecture above, it is natural to formulate it in the framework of the theory of modular forms. However, this reformulation provides little clue on how one can prove the conjecture. In a similar case [7], Kaneko and Honda proved the special case $l = 1$ using the modular forms techniques. Their proof is difficult, and, although it may be possible to repeat it in the case under consideration, there is no hope to generalize that proof to the case of arbitrary l. In this paper, we prove only slightly weaker congruences for arbitrary l (see Theorem 1 below) using the theory of weak harmonic Maass forms.

We conclude this introduction with some numerical evidence in favor of the above conjecture. The coefficients $c(n)$ themselves look pretty random; here are the first non-zero values:

n	3	7	11	15	19	23	27	31	35	39
$c(n)$	242	2647	12734	49475	151026	429260	1072092	2572325	5704200	12283752

We now pick $p = 5$. Here are the coefficients $c(3 \cdot 5^l)$ along with their 5-orders illustrating (i) of the conjecture above.

l	$c(3 \cdot 5^l)$	$\mathrm{ord}_5(c(3 \cdot 5^l))$
0	242	0
1	49475	2
2	3105466750	3
3	99882055543236545625	4
4	1858614781560119022805304517492111001545625 0	5

In contrast, in order to illustrate (ii) of the conjecture, we pick $p = 3$, and list $c(5 \cdot 3^l)$ along with their 3-orders.

l	$c(5 \cdot 3^l)$	$\mathrm{ord}_3(c(5 \cdot 3^l))$
1	49475	0
3	2588257009650	1
5	118804073284466436614083007230891575	2

For a prime p, denote U_p-operator by U. We say that a function ϕ with a Fourier expansion $\phi = \sum u(n)q^n$ is congruent to zero modulo a power of a prime p,

$$\phi = \sum u(n)q^n \equiv 0 \mod p^w$$

if all its Fourier expansion coefficients are divisible by this power of the prime, $u(n) \equiv 0 \mod p^w$ for all n.

In this paper we prove the following congruences

Theorem 1. (i) *Let p be a prime such that $p \equiv 1 \mod 4$. There exists an integer $A_p \geq 0$ such that for all integers $l \geq A_p$*

$$\left(\frac{E_4(4\tau)}{\eta(4\tau)^2\eta(8\tau)^2} \right) \bigg| U^l \equiv 0 \mod p^{l-A_p}.$$

(ii) *Let p be a prime such that $p \equiv 3 \mod 4$. There exists an integer $A_p \geq 0$ such that for all integers $[l/2] \geq A_p$*

$$\left(\frac{E_4(4\tau)}{\eta(4\tau)^2\eta(8\tau)^2} \right) \bigg| U^l \equiv 0 \mod p^{[l/2]-A_p}.$$

2. Notation and Preparation

We call $\mathrm{SL}_2(\mathbb{Z})$ modular group.

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \bigg| \; a, b, c, d \in \mathbb{Z}, \; ad - bc = 1 \right\}.$$

When N is a positive integer, we define $\Gamma_0(N)$ a level N congruence subgroup

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \bigg| \; c \equiv 0 \pmod{N} \right\}.$$

In particular $\Gamma_0(1) = \mathrm{SL}_2(\mathbb{Z})$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ act on $\mathscr{H}$, the upper half of the complex plane, by the linear fractional transformation

$$A\tau = \frac{a\tau + b}{c\tau + d},$$

where $\tau = x + iy \in \mathscr{H}$.

Definition 1. Suppose that Γ is a level N congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. A **cusp** of Γ is an equivalence class in $\mathbb{Q} \bigcup \{i\infty\}$ under the action of Γ.

Definition 2. Let $\gamma \in \Gamma_0(N)$. If $f(\tau)$ is a meromorphic function on $\mathscr{H}$ and k is an even integer, then define **the operator** $|_k$ by

$$(f|_k\gamma)(\tau) := (c\tau + d)^{-k} f(\gamma\tau).$$

It is easy to check that the operator $|_k$ is compatible with the group operation:

$$f|_k(\gamma_1\gamma_2) = (f|_k\gamma_1)|_k\gamma_2$$

for any two matrices $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$.

Definition 3. Suppose that $f(\tau)$ is a meromorphic function on $\mathscr{H}$, that k is an even integer, and that Γ is a congruence subgroup of level N. Then $f(\tau)$ is called **a meromorphic modular form with weight k on Γ** if the following hold:

(1) We have
$$f\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^k f(\tau),$$
for all $\tau\in\mathscr{H}$ and all $\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\ \Gamma$.

(2) If $\gamma\in \mathrm{SL}_2(\mathbb{Z})$, then $(f|_k\gamma)(\tau)$ has a Fourier expansion of the form
$$(f|_k\gamma)(\tau)=\sum_{n\geq n_\gamma} a_\gamma(n)q_N^n,$$
where $q_N:=e^{\frac{2\pi i\tau}{N}}$ and $a_\gamma(n_\gamma)\neq 0$.

Remark 1.

(1) Condition (2) of Definition 3 means that $f(\tau)$ is meromorphic at the cusps of Γ. If $n_\gamma\geq 0$ for each $\gamma\in\mathrm{SL}_2(\mathbb{Z})$, then we say that $f(\tau)$ is holomorphic at the cusps of Γ.
(2) Condition (1) may be rewritten as $(f|_k\gamma)=f$ for all $\gamma\in\Gamma$.
(3) By [13, III.3 Modular forms for congruence subgroups], the function $f|_k\gamma$ is periodic with period N for $\gamma\in\mathrm{SL}_2(\mathbb{Z})/\Gamma$.
(4) One uses only even integers k in the definition above since it is easy to check that there are no non-zero modular forms of odd weight.

Definition 4. Suppose that $f(\tau)$ is an even integer weight meromorphic modular form on a congruence subgroup $\Gamma_0(N)$.
We say that $f(\tau)$ is **a holomorphic modular form** if $f(\tau)$ is holomorphic on $\mathscr{H}$ and is holomorphic at the cusps of $\Gamma_0(N)$.
We say that $f(\tau)$ is **a holomorphic cusp form** if $f(\tau)$ is holomorphic on $\mathscr{H}$ and vanishes at the cusps of $\Gamma_0(N)$.
We say that $f(\tau)$ is **a weakly holomorphic modular form** if it is holomorphic on $\mathscr{H}$ with possible poles in cusps of $\Gamma_0(N)$.

When $\Gamma=\Gamma_0(N)$ is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$, we put
$$\begin{aligned} M_k(\Gamma) &= \{\, f\colon f \text{ is a holomorphic modular form of weight } k \text{ on } \Gamma\},\\ S_k(\Gamma) &= \{\, f\colon f \text{ is a cusp form of weight } k \text{ on } \Gamma\},\\ M_k^!(\Gamma) &= \{\, f\colon f \text{ is a weakly holomorphic modular form of weight } k \text{ on } \Gamma\}.\end{aligned}$$
Let $q=e^{2\pi i\tau}$ with $\tau\in\mathscr{H}$. One of the first classical examples of a cusp form is
$$\Delta=q\prod_{n\geq 1}(1-q^n)^{24}\in S_{12}(\Gamma_0(1)).$$

$M_k(\Gamma)$ and $S_k(\Gamma)$ are finite dimensional vector spaces. $M_k^!(\Gamma)$ is an infinite dimensional vector space. $S_k(\Gamma)$ has a basis $\{g_i\}$ such that $g_i=\sum_{n=1}^\infty c_i(n)q^n$ with $c_i(n)\in\mathbb{Z}$. The latter fact is rather difficult to prove, and we take it for granted. This fact has several implications. Let $f=\sum_{n\gg-\infty}a(n)q^n\in M_k^!(\Gamma)$ be a modular form such that all coefficients $a(n)$ are algebraic numbers. Then there is an algebraic number field K (i.e. a finite extension of $\mathbb{Q}$) such that $a(n)\in K$ for all n. Moreover, there exists $T\in K$ such that $Ta(n)\in\mathcal{O}_K$, the ring of integers of K, for all n. The latter statement is often called "bounded denominators principle". In order to derive these facts it suffices to notice that there exists a positive integer S such that $\Delta^S f$ is a cusp form.

For a non-positive weight $k \leq 0$ modular form $f = \sum_{n \gg -\infty} a(n)q^n \in M_k^!(\Gamma)$ there is a criterion which allows one to determine whether all its coefficients $a(n)$ are algebraic numbers. A polynomial $P_f := \sum_{n \leq 0} a(n)q^n \in \overline{\mathbb{Q}}[q^{-1}]$ is called the *principal part* of f. For that, it suffices that the principal part of $f|\gamma$ has algebraic Fourier coefficients for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. One abbreviates that saying that "the principal parts and constant terms at all cusps are algebraic" and considers as an instance of the"q-expansion principle". Note that this abbreviation is a bit ambiguous since the notion of q-expansion at a cusp different from infinity is not well-defined. Note also that such a quantity as the constant term at a cusp different from infinity is not well defined while one may speak without any ambiguity about zero or non-zero constant term at such a cusp.

The first examples of modular forms are Eisenstein series $E_k \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ with Fourier expansions

$$E_k = 1 - \frac{2k}{B_k} \sum_{n>0} \left(\sum_{d|n} d^{k-1} \right) q^n,$$

where B_k is the k-th Bernoulli number. We will be particularly interested in E_4. Note that

$$e_4(q) = E_4(\tau).$$

Of course, E_k is not a cusp form, and we now give an example of a cusp form. The Dedekind η-function is defined by

$$\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n).$$

A classical result of Dedekind states that η transforms as a modular form of weight 1/2. In particular, its 24-th power is known to be a cusp form of weight 12. In this paper, we will make use of

$$g = \eta(4\tau)^2 \eta(8\tau)^2 \in S_2(\Gamma_0(32)).$$

We will need the following operators which act on the functions on the upper half-plane. Let p be a prime. For a function f we let

$$(f|V)(\tau) = f(p\tau),$$

and

$$(f|U)(\tau) = \frac{1}{p} \sum_{j=0}^{p-1} f\left(\frac{\tau + j}{p}\right).$$

If f has a Fourier expansion $f = \sum a(n)q^n$, then it is easy to check that

$$f|V = \sum a(n)q^{pn},$$

and

$$f|U = \sum a(pn)q^n.$$

These operators may not preserve the spaces of modular forms. However, one can check that if $f \in M_k(\Gamma_0(N))$ (resp. $M_k^!(\Gamma_0(N))$, $S_k(\Gamma_0(N))$), then both $f|U, f|V \in M_k(\Gamma_0(pN))$ (resp. $M_k^!(\Gamma_0(pN))$, $S_k(\Gamma_0(pN))$).

Remark 2. Note that the operators U and V defined above may also be considered as operators acting on $K[[q]]$, the ring of formal power series in q with coefficients in arbitrary field K.

Our series of interest now can be rewritten as the Fourier expansion of a weakly holomorphic modular form of weight 2 on $\Gamma_0(32)$:

$$\frac{E_4(4\tau)}{g(\tau)} = \sum_{n\geq -1} c(n)q^n \in M_2^!(\Gamma_0(32))$$

The finite-dimensional spaces $S_k(\Gamma_0(N))$ are supplied with an inner product.

Definition 5. If $f(\tau)$ and $g(\tau)$ are cusp forms in $S_k(\Gamma_0(N))$, then their **Petersson product** is defined by

$$\langle f, g\rangle := \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \cdot \int_{D(N)} f(\tau)\overline{g(\tau)}y^{k-2}\,\mathrm{d}x\mathrm{d}y,$$

where D(N) denotes a connected fundamental domain for the action of $\Gamma_0(N)$ on $\mathscr{H}$, and where $\tau = x + iy$.

We denote the **Petersson norm** of g by $||g|| = \langle g, g\rangle^{\frac{1}{2}}$.

We now introduce a generalization of modular forms which are weak harmonic Maass forms.

Definition 6. Let $\tau = x + iy \in \mathscr{H}$, the upper-half of the complex plane, with $x, y \in \mathbb{R}$. Suppose that $k \in \mathbb{N}$. We define the weight k hyperbolic Laplacian by

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Then **a weak harmonic Maass form of weight k** on $\Gamma_0(N)$ is any smooth function on $\mathscr{H}$ satisfying:

(i) $M(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k M(\tau)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$;
(ii) $\Delta_k M = 0$;
(iii) There is a polynomial $P_M = \sum_{n\leq 0} c_M^+(n)q^n \in \mathbb{C}[q^{-1}]$ such that $M(\tau) - P_M(\tau) = O(e^{-\epsilon y})$ as $y \to \infty$ for some $\epsilon > 0$. Analogous conditions are required at all cusps.

Weakly holomorphic modular forms naturally sit in spaces of *weak harmonic Maass forms*. In this paper, only weight zero weak harmonic Maass forms show up. In the case of weight zero, the hyperbolic Laplacian simplifies to the classical Laplacian (multiplied by $-y^2$), and condition (ii) becomes

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) M = 0,$$

saying that M is a harmonic function on the upper half-plane.

By [4], "it is natural to investigate the arithmeticity of the Fourier coefficients of such Maass forms, and to also investigate their nontrivial interplay with holomorphic and weakly holomorphic modular forms. In the works above, the differential operator,

$$\xi_\omega := 2iy^\omega \cdot \overline{\frac{\partial}{\partial \bar{\tau}}}$$

(where ω is the weight and $\overline{\frac{\partial}{\partial \bar{\tau}}}$ represents differentiation with respect to $\bar{\tau}$ followed by conjugation), plays a central role. It is a nontrivial fact that

$$\xi_{2-k} : H_{2-k}(\Gamma_0(N)) \to S_k(\Gamma_0(N)).$$

Here $H_\omega(\Gamma_0(N))$ denotes the space of weight ω weak harmonic Maass forms on $\Gamma_0(N)$. We say that a weak harmonic Maass form $f \in H_{2-k}(\Gamma_0(N))$ is **good for g** if it satisfies the following properties:

(i) The principal part of f at the cusps $i\infty$ belongs to the number field obtained by adjoining the coefficients of g to $\mathbb{Q}$.
(ii) The principal parts of f at the other cusps of $\Gamma_0(N)$ are constant.
(iii) We have that $\xi_{2-k}(f) = ||g||^{-2}g$."

One has (see [4]) a natural decomposition

$$M = M^+ + M^-$$

of a weak harmonic Maass form into a sum of a holomorphic function M^+ and a non-holomorphic function M^-. In the case of weak harmonic Maass forms of weight zero, which are harmonic functions, this becomes the decomposition of a harmonic function into its holomorphic and anti-holomorphic parts, and

$$\xi(M) = \xi_0(M) = \xi_0(M^-) = 2i \cdot \overline{\frac{\partial M^-}{\partial \bar{\tau}}}.$$

We thus have that the anti-holomorphic differentiation $\xi = \xi_0$ takes weak harmonic Maass forms defined above to cusp forms. The map ξ is surjective, moreover; by [4, Proposition 5.1] for every cusp form g of weight k there exists a weak harmonic Maass form M of weight $2-k$ (same level) which is good for g.

Along with the anti-holomorphic differentiation, it is natural to look at the holomorphic differentiation

$$D := \frac{1}{2\pi i}\frac{d}{d\tau},$$

and its interplay with the matrix action. In general, a derivative of a modular form is not modular. However, classical Bol's identity tells us that

$$D^{k-1}(M|_{2-k}\gamma) = (D^{k-1}M)|_k\gamma \quad \text{for } k > 1$$

for any smooth function M. This implies, in particular, for a weak harmonic Maass form M of weight zero

$$D(M) = \frac{1}{2\pi i}\frac{dM^+}{d\tau}$$

is a weakly holomorphic modular form of weight 2 (and same level as M).

For a weak harmonic Maass form of weight zero, one can write the Fourier expansions (see [4]) of both holomorphic and anti-holomorphic parts of M as

$$M^+ = \sum_{n \gg -\infty} a(n)q^n, \quad M^- = \sum_{n<0} a^-(n)e^{2\pi i n\bar{\tau}}.$$

Thus

$$D(M) = \sum_{n\gg-\infty} na(n)q^n \quad \xi(M) = -4\pi \sum_{n>0} na^-(-n)q^n.$$

We will make use of one more operation which intertwines well with matrix action, that is character twist. Let r be a positive integer. A quadratic Dirichlet character χ modulo r is a r-periodic function $\mathbb{Z} \to 0, -1, 1$ which factors through a multiplicative group homomorphism (we denote the latter by the same letter χ)

$$\chi \ : \ (\mathbb{Z}/r\mathbb{Z})^* \to \pm 1,$$

where ± 1 is the multiplicative group of two elements. It is assumed that $\chi(a) = 0$ if $gcd(a, r) \neq 1$. A quadratic Dirichlet character is called primitive if r is the exact period of χ.

For a weak harmonic Maass form $M = M^+ + M^-$ with a Fourier expansion as above, we let $M \otimes \chi = M^+ \otimes \chi + M^- \otimes \chi$, where

$$M^+ \otimes \chi = \sum_{n\gg-\infty} a(n)\chi(n)q^n, \quad M^- \otimes \chi = \sum_{n<0} \chi(n)a^-(n)e^{2\pi i n\overline{\tau}}.$$

Similarly, if $f = \sum a(n)q^n$ is a modular form, then $f \otimes \chi = \sum \chi(n)a(n)q^n$.

It is known ([16, Proposition 3.64]) that if f (resp. M) is modular (resp. weak harmonic Maass) form then so is $f \otimes \chi$ (resp. $M \otimes \chi$). While the weight of the twist stays the same, the level may not, and becomes a divisor of Nr^2.

Let p be a prime. The operators $U = U_p$ and $V = V_p$ defined above take modular forms (resp. weak harmonic Maass forms) to modular forms (resp. weak harmonic Maass forms) of the same weight, but they may not (and typically do not) preserve the level. We now introduce their linear combination which preserves everything (including the level). The Hecke operator $T(p)$ acting on modular (resp. weak harmonic Maass) forms of weight k and level N is defined by

$$T(p) = \begin{cases} U + p^{k-1}V & \text{if } p \text{ does not divide } N \\ U & \text{if } p \text{ divides } N. \end{cases}$$

In fact, every space $M_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$ admits a basis which consists of common eigenforms of all Hecke operators. We will need only the special case of the space $S_2(\Gamma_0(32))$. Since the dimension of this space is one, clearly $g|T(p)$ is a multiple of g for every p. More precisely, if $f = \sum_{n\geq 0} a(n)q^n \in M_k(\Gamma_0(N))$ is a common eigenform of all Hecke operators $T(p)$, then $a(1) \neq 0$, and assuming $a(1) = 1$, we have

$$f|T(p) = a(p)f.$$

In other words, the Fourier coefficients of a Hecke eigenform are, at the same time, the eigenvalues of the Hecke operators. More generally, one defines Hecke operators $T(m)$ for a positive integer $m \geq 1$ such that this statement still holds true. Namely, $T(1)$ is the identity operator, and for powers of a prime p, put

$$T\left(p^{l+1}\right) := \begin{cases} U^{l+1} & \text{if } p|N \\ T(p^l)T(p) - p^{k-1}T(p^{l-1}) & \text{otherwise.} \end{cases}$$

$$T(mn) := T(m)T(n) \quad \text{if } gcd(m,n) = 1.$$

An inductive argument shows that a common eigenform $f = \sum_{n\geq 0} a(n)q^n \in M_k(\Gamma_0(N))$ of all $T(p)$ is also a common eigenform of all $T(m)$. Moreover, assuming the normalization $a(1) = 1$ we have that

$$f|T(m) = a(m)f$$

for every positive integer m.

Remark 3. Since (see remark 2 above) the operators U and V make sense as operators acting on the the ring of formal power series $K[[q]]$ in variable q, the Hecke operators $T(m)$ constructed above also can be viewed as operators acting on $K[[q]]$. Strictly speaking, one should specify weight and level, and write $|_{k,N}T(m)$. We suppress this data from our notations for the sake of brevity.

The modular form $g = \sum_{n\geq 1} b(n)q^n$ will play a significant role in this paper, and we now record some of its properties. It is known that g is a CM-form which comes from a certain Hecke character for $\mathbb{Q}(\sqrt{-1})$. That fact is used in the proof of Proposition 4 below. This proof in turn makes use of Theorem 1.3 from [4]. However, an inspection of the proof of this theorem reveals that the only facts about the Fourier coefficients of g which are used $b(n) \in \mathbb{Q}$ for all n, and

$$b(n) = 0 \quad \text{if} \quad n \equiv 3 \mod 4 \text{ or } n \equiv 0 \mod 2.$$

Although these facts indeed follow from the theory of CM modular forms, they also follow more easily from the definition $g = \eta(4\tau)^2\eta(8\tau)^2$.

Another fact which we will make use of is an estimate

$$|b(p)| < 2\sqrt{p},$$

which may be derived either from the theory of complex multiplication or, alternatively, from the theory of elliptic curves. We take this estimate for granted. Note that this estimate implies that either $b(p) = 0$ (this happens for $p \equiv 3 \mod 4$ and for $p = 2$), or $b(p)$ is not divisible by p (this happens for $p \equiv 1 \mod 4$).

We will also need the some notation and facts pertaining to p-adic analysis. Let p be a prime. For a non-zero integer a we write $a = p^m a'$ with an integer a' which is not divisible by p, and let $\operatorname{ord}_p a := m$. We also let $\operatorname{ord}_p(0) = \infty$. For a rational number $x = a/b \in \mathbb{Q}$ we let $\operatorname{ord}_p(x) := \operatorname{ord}_p(a) - \operatorname{ord}_p(b)$. Further define a metric $|\ |_p$ on $\mathbb{Q}$ as follows:

$$|x|_p = \begin{cases} \frac{1}{p^{\operatorname{ord}_p x}}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

The field of p-adic numbers $\mathbb{Q}_p \supset \mathbb{Q}$ is the completion of $\mathbb{Q}$ with respect to this metric, and $\mathbb{Z}_p := \{x \in \mathbb{Q}_p \mid |x|_p \leq 1\}$ is the ring of p-adic integers.

We further introduce the p-adic metric on the ring $\mathbb{Q}_p \otimes \mathbb{Z}_p[[q]]$ of formal power series in variable q. For a power series $z = \sum_{n\geq 0} a(n)q^n$ let

$$|z|_p := p^{-\inf_n(\operatorname{ord}_p(a(n)))}.$$

Thus, for a family of power series $z_l = \sum_{n\geq 0} a_l(n)q^n$,

$$\lim_{l\to\infty} z_l = z$$

means $a_l(n) \to a(n)$ as $l \to \infty$ uniformly in n.

Lemma 1. *(Hensel's Lemma). Let $F(x) = c_0 + c_1x + \cdots + c_nx^n$ be a ploynomial whose coefficients are p-adic integers. Let $F'(x) = c_1 + 2c_2x + 3c_3x^2 + \cdots + nc_nx^{n-1}$ be the derivative of $F(x)$. Let a_0 be a p-adic integer such that $F(a_0) \equiv 0(mod\ p)$ and $F'(a_0) \not\equiv 0(mod\ p)$. Then there exists a unique p-adic integer a such that $F(a) = 0$ and $a \equiv a_0(mod\ p)$.*

Proposition 1. *Suppose $f(\tau) = \frac{E_4(4\tau)}{\eta(4\tau)^2\eta(8\tau)^2}$. Then $f|\gamma$ has zero constant terms and algebraic principal parts at all cusps for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.*

Proof. Let $\phi(\tau) = \frac{E_4(\tau)}{\eta(\tau)^2\eta(2\tau)^2}$. Then, $f(\tau) = \phi(4\tau)$. We shall show the following:

(i) If $\phi|\gamma$ has zero constant terms and algebraic principal parts at all cusps for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, then so does $f|\gamma$ for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

(ii) $\phi|\gamma$ has zero constant terms and algebraic principal parts at all cusps for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

For (i), suppose $\phi|\gamma$ has zero constant terms and algebraic principal parts at all cusps for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$

$$f|\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (c\tau + d)^{-k}\phi\left(4\ \frac{a\tau + b}{c\tau + d}\right).$$

It suffices to check that for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ there exists $\sigma = \begin{pmatrix} \alpha & \beta \\ \rho & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}\gamma = \sigma^{-1}\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \textit{with } \frac{A}{D} > 0 \textit{ and } A, B, D \in \mathbb{Q}.$$

We want to find $\sigma = \begin{pmatrix} \alpha & \beta \\ \rho & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ such that

$$\begin{pmatrix} \alpha & \beta \\ \rho & \delta \end{pmatrix}\begin{pmatrix} 4a & 4b \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \text{.i.e. } 4a\rho + \delta c = 0.$$

Note: If $gcd(\rho, \delta) = 1$, then there exists α, β such that $\alpha\delta - \beta\rho = 1$.
In order to find such σ, it suffices to show the existence of ρ, δ such that $4a\rho + \delta c = 0$ and $gcd(\rho, \delta) = 1$. So, we show the following three cases.
Case(1), suppose $gcd(4a, c) = 1$, then let $\rho = -c$ and $\delta = 4a$. Then, $gcd(\rho, \delta) = 1$. Therefore, there exists α, β such that $\alpha\delta - \beta\rho = 1$.
Case(2), suppose $2|c$ and $4 \nmid c$, then $2a\rho + \frac{c}{2}\delta = 0$. Since $gcd(a, c) = 1$, $gcd(2a, \frac{c}{2}) = 1$. Let $\rho = -\frac{c}{2}, \delta = 2a$. Then, $gcd(\rho, \delta) = 1$. Therefore, there exists α, β such that $\alpha\delta - \beta\rho = 1$.
Case (3), suppose $4|c$, then $a\rho + \frac{c}{4}\delta = 0$. Since $gcd(a, c) = 1$, clearly $gcd(a, \frac{c}{4}) = 1$.
Let $\rho = -\frac{c}{4}, \delta = a$. Then, $gcd(\rho, \delta) = 1$. Therefore, there exists α, β such that $\alpha\delta - \beta\rho = 1$.

For (ii), by [13, Proposition 20.] we have

$$\eta(\tau)^8\eta(2\tau)^8 \in S_8(\Gamma_0(2)).$$

We conclude that for every $\gamma \in \Gamma_0(2)$

$$\eta(\tau)^2\eta(2\tau)^2|_2\ \gamma = u_\gamma(\eta(\tau)^2\eta(2\tau)^2)$$

with a fourth root of unity u_γ (which depends on γ.)
Thus,

$$\phi(\tau)|_2\, \gamma = u_\gamma^{-1}\phi(\tau) \tag{1}$$

for every $\gamma \in \Gamma_0(2)$ since $E_4(\tau)|\gamma = E_4(\tau)$.
Note the coset decomposition

$$\mathrm{SL}_2(\mathbb{Z}) = \Gamma_0(2) \cup \Gamma_0(2)S \cup \Gamma_0(2)ST$$

with $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.
Taking into the account (1), it thus suffices to show that $\phi|S$ (and, therefore, $\phi|ST$) has zero constant term and algebraic principal part. By [13, Proposition 14.], indeed,

$$\phi|S = \frac{E_4(-\frac{1}{\tau})}{\eta(-\frac{1}{\tau})^2\eta(-\frac{2}{\tau})^2} \cdot \tau^{-2} = (-2)\frac{E_4(\tau)}{\eta(\tau)^2\eta(\frac{\tau}{2})^2}.$$

□

Remark 4.
(i) $f(\tau) = \frac{E_4(4\tau)}{\eta(4\tau)^2\eta(8\tau)^2}$ has integer $q-$expansion coefficients.
(ii) $f(\tau) = \frac{E_4(4\tau)}{\eta(4\tau)^2\eta(8\tau)^2} = \sum_{n=0}^{\infty} c(n)q^{4n-1}$, for all $\tau \in \mathscr{H}$.

For a prime p, denote U_p-operator by U. We say that a function ϕ with a Fourier expansion $\phi = \sum u(n)q^n$ is congruent to zero modulo a power of a prime p,

$$\phi = \sum u(n)q^n \equiv 0 \mod p^w$$

if all its Fourier expansion coefficients are divisible by this power of the prime, $u(n) \equiv 0 \mod p^w$ for all n.

3. Proof of the theorem

In this paper we prove the following congruences

Theorem 2. (i) *Let p be a prime such that $p \equiv 1 \mod 4$. There exists an integer $A_p \geq 0$ such that for all integers $l \geq A_p$*

$$\left(\frac{E_4(4\tau)}{\eta(4\tau)^2\eta(8\tau)^2}\right)\Bigg| U^l \equiv 0 \mod p^{l-A_p}.$$

(ii) *Let p be a prime such that $p \equiv 3 \mod 4$. There exists an integer $A_p \geq 0$ such that for all integers $[l/2] \geq A_p$*

$$\left(\frac{E_4(4\tau)}{\eta(4\tau)^2\eta(8\tau)^2}\right)\Bigg| U^l \equiv 0 \mod p^{[l/2]-A_p}.$$

The conditions $p \equiv 1$ or $3 \mod 4$ indeed make a difference, and are related to the theory of complex multiplication. Similar congruences may exist for all weakly holomorphic modular forms dual to complex multiplication cusp forms. In particular, let $g = \sum b(n)q^n$ be the weight two normalized cusp form. Incidentally, $g = \eta(4\tau)^2\eta(8\tau)^2$. If a prime p is inert in the CM field (in this case the CM field is $\mathbb{Q}(\sqrt{-1})$, and inert primes are the odd primes $p \equiv 3$

mod 4) then $b(p)=0$. The congruences of Theorem 2 (ii) are, in a sense, dual to this fact. In this paper, however, we only concentrate on this particular case. A general theory which indicates how to produce similar congruences, is developed in [1].

Denote by D the differentiation

$$D := \frac{1}{2\pi i}\frac{d}{d\tau} = q\,\frac{d}{dq}.$$

Recall $M_k^! = M_k^!(N)$ for the space of weakly holomorphic (i.e. holomorphic in the upper half plane with possible poles in cusps) modular forms of weight k on $\Gamma_0(N)$. Recall that Bol's identity

$$D^{k-1}(f|_{2-k}\gamma) = (D^{k-1}f)|_k\gamma \quad \text{for } k>1$$

implies that for an even positive integer k

$$D^{k-1} : M_{2-k}^! \to M_k^!. \tag{2}$$

Recall that for a weakly holomorphic modular form $f \in M_{2-k}^!$, which has rational q-expansion coefficients, the bounded denominators principle allows us to claim the existence of an integer T such that all q-expansion coefficients of Tf are integers. The following fact is obvious.

Proposition 2. *Let p be a prime. If $f \in M_{2-k}^!$ has rational q-expansion coefficients, then there exists an integer $A \geq 0$ such that for all integers $l \geq 0$*

$$(D^{k-1}f)|U^l \equiv 0 \mod p^{l(k-1)-A}.$$

In particular, if $f \in M_{2-k}^!$ has p-integral q-expansion coefficients, then $A=0$.

Proof. Let $f \in M_{2-k}^!$ have rational q-expansion coefficients. Then $f(\tau) = \sum_{n\gg-\infty} r(n)q^n$ where $r(n)\in\mathbb{Q}$. $Df = \sum_{n\gg-\infty} r(n)nq^n$. So, $D^{k-1}f = \sum_{n\gg-\infty} r(n)n^{k-1}q^n$. Thus,

$$(D^{k-1}f)|U^l = \sum_{n\gg-\infty} r(p^l n)p^{l(k-1)}n^{k-1}q^n.$$

Since the rational coefficients $r(p^l n)$ have bounded demominators, it is clear that

$$(D^{k-1}f)|U^l \equiv 0 \mod p^{l(k-1)-A}.$$

In particular, suppose $f \in M_{2-k}^!$ has p-integral q-expansion coefficients. Then, the denominators of $r(p^l n)$ are not divisble by p for each n. So, $(D^{k-1}f)|U^l \equiv 0 \mod p^{l(k-1)}$. ☐

We obtain our results as an application of the theory of *weak harmonic Maass forms*. We refer to [2, 4] for definitions and detailed discussion of their properties. The extension of (2) to the space $H_{2-k} \supset M_{2-k}^!$ of weak harmonic Maass forms is $D^{k-1} : H_{2-k} \to M_k^!$.

Proposition 3. *There exists a weak harmonic Maass form M of weight zero on $\Gamma_0(32)$ such that*

$$\frac{E_4(4\tau)}{\eta(4\tau)^2\eta(8\tau)^2} = D(M) + \gamma\eta(4\tau)^2\eta(8\tau)^2$$

with some quantity $\gamma \in \mathbb{C}$.

Proof. The existence of M with any given principal parts at cusps, in particular, such that the principal parts of $D(M)$ and $E_4(4\tau)/\eta(4\tau)^2\eta(8\tau)^2$ at all cusps coincide follows from [2, Proposition 3.11]. Since the constant terms of the Fourier expansion of $E_4(4\tau)/\eta(4\tau)^2\eta(8\tau)^2$ at all cusps vanish, the difference $D(M) - E_4(4\tau)/\eta(4\tau)^2\eta(8\tau)^2 \in S_2(\Gamma_0(32))$ is a cusp form. However, $\dim S_2(\Gamma_0(32)) = 1$, and this space is generated by the unique normalized cusp form $\eta(4\tau)^2\eta(8\tau)^2$. □

We will later show that in fact $\gamma = 0$. First, we need to investigate some properties of M. Recall (see [4, 2] for the details) that a weight zero weak harmonic Maass form M naturally decomposes into two summands,

$$M = M^+ + M^-,$$

with a holomorphic function M^+ and an anti-holomorphic function M^-. The former has a Fourier expansion

$$M^+ = \sum_{n \gg -\infty} a(n)q^n.$$

Since M is of weight zero, we may and will assume from now on that $a(0) = 0$.

Remark 5. Proposition 3, $\frac{E_4(4\tau)}{\eta(4\tau)^2\eta(8\tau)^2} = D(M) + \gamma\eta(4\tau)^2\eta(8\tau)^2$ allows us to further assume that the coefficients of the principal part of M^+ are algebraic numbers. Moreover, the coefficients of the principal part of M are algebraic numbers.

Proposition 4. *The Fourier coefficients $a(n)$ of M^+ are algebraic numbers at $i\infty$. More specifically there is an algebraic number field K, which is a finite extension of $\mathbb{Q}$ such that $a(n) \in K$ for all n.*

Proof. Let $g := \eta(4\tau)^2\eta(8\tau)^2 \in S_2(\Gamma_0(32))$ be the unique normalized cusp form of weight 2 and level 32. Let $\tau = x + iy$. The differential operator $\xi := 2i\overline{\frac{\partial}{\partial \bar{\tau}}}$ takes weight zero weak harmonic Maass forms to cusp forms since $\xi_{2-k} : H_{2-k}(\Gamma_0(N)) \to S_k(\Gamma_0(N))$. In particular, since $\dim S_2(\Gamma_0(32)) = 1$, we conclude that

$$\xi(M) = tg$$

for some quantity $t \in \mathbb{C}$. Since M^+ has algebraic principal parts of its Fourier expansion at all cusps and $g := \eta(4\tau)^2\eta(8\tau)^2 \in S_2(\Gamma_0(32))$, we derive from [2, Proposition 3.5] that $t||g||^2$ is algebraic, where $||g||^2$ denotes the Petersson norm of g. At the same time, it follows from [4, Proposition 5.1] that there exists a weak harmonic Maass form M_g which is *good for g*. In particular, $\xi(M_g) = ||g||^{-2}g$. Since the linear combination $M - t||g||^2M_g$ obviously satisfies

$$\xi(M - t||g||^2M_g) = 0,$$

we conclude that it is a weight zero weakly holomorphic modular form

$$M - t||g||^2M_g \in M_0^!(\Gamma_0(32)).$$

Since M_g is *good for g*, M_g has its principal part at $i\infty$ in $\overline{\mathbb{Q}}[q^{-1}]$, and is constant at all other cusps. It follows that the modular form $M - t||g||^2M_g$ has algebraic principal parts of its Fourier expansion at all cusps, and, therefore, algebraic Fourier coefficients at $i\infty$. Proposition 4 now follows from a theorem of Bruinier, Ono, and Rhoades [4, Theorem 1.3] which tells us that the Fourier expansion coefficients of the holomorphic part M_g^+ belong to K, because g is a CM-form.

□

We now prove that in Proposition 3, in fact, $\gamma = 0$.

Proposition 5. *There exists a weak harmonic Maass form M of weight zero on $\Gamma_0(32)$ such that*

$$\frac{E_4(4\tau)}{\eta(4\tau)^2\eta(8\tau)^2} = D(M).$$

Proof. We write the Fourier expansion of $M = M^+ + M^-$ as

$$M^+ = \sum_{n \gg -\infty} a(n)q^n, \quad M^- = \sum_{n<0} a^-(n)e^{2\pi i n\overline{\tau}},$$

We thus have that

$$tg = \xi(M) = -4\pi \sum_{n\geq 1} a^-(-n)nq^n.$$

Since $g = \eta(4\tau)^2\eta(8\tau)^2$, we conclude that $a^-(n) = 0$ if $n \equiv 0 \mod 2$ or $n \equiv 1 \mod 4$. As in [4, p.12], we conclude that the weak harmonic Maass form

$$u := M + M \otimes \chi$$

has the property $\xi(u) = 0$, and is therefore a weakly holomorphic weight zero modular form. It follows that the denominators of its Fourier coefficients are bounded. Namely, there exists a non-zero $T \in K$ such that the coefficients of

$$Tu = T(M + M \otimes \chi) = T(M^+ + M^+ \otimes \chi) = \sum_{n \gg -\infty} T\left(a(n) + \chi(n)a(n)\right)q^n$$

all belong to the ring of integers $\mathcal{O}_K \subset K$. In particular, for a prime $p \equiv 1 \mod 4$ we conclude that the p-adic limit of the coefficients $p^m a(p^m)$ of q^{p^m} of $D(M) = D(M^+)$ as $m \to \infty$ is zero. Since all coefficients of q^{p^m} in $E_4(4\tau)/\eta(4\tau)^2\eta(8\tau)^2$ are zeros, and the coefficients of q^{p^m} in g are not divisible by p, we conclude that $\gamma = 0$.

$\square$

Proposition 5 allows us to further assume that the Fourier coefficients of M^+ are rational numbers.

We now need the Hecke operators' action on M. For a prime p, let $T(p) := U + p^{k-1}V$ be the p-th Hecke operator at weight k. Let

$$g = \sum_{n\geq 1} b(n)q^n.$$

The form g is a Hecke eigenform.

Using the same argument as in [3, Lemma 7.4], we have that

$$M|_0T(p) = p^{-1}b(p)M + R_p,$$

where $R_p \in M_0^!(32)$ is a weakly holomorphic modular form with coefficients in $\mathbb{Q}$. We apply the differential operator D to this identity and use the (obvious) commutation relation

$$pD\left(H|_0T(p)\right) = \left(D(H)\right)|_2T(p)$$

valid for any function H with a Fourier expansion in q. We obtain that

$$(D(M))\,|_2T(p) = b(p)D(M) + pD(R_p). \tag{3}$$

Let β, β' be the roots of equation

$$X^2 - b(p)X + p = 0,$$

such that $\mathrm{ord}_p(\beta) \leq \mathrm{ord}_p(\beta')$. Note that g is a complex multiplication cusp form, and the complex multiplication field is $\mathbb{Q}(\sqrt{-1})$. In particular, if $p \equiv 1 \mod 4$ then $\beta, \beta' \in \mathbb{Q}_p$ by Hensel's lemma, and $\mathrm{ord}_p(\beta) = 0$ while $\mathrm{ord}_p(\beta') = 1$. If $p \equiv 3 \mod 4$, then $b(p) = 0$, and we have that $\beta = -\beta'$. Thus $\beta, \beta' \in \mathcal{F} = \mathbb{Q}_p(\sqrt{-p})$ and $\mathrm{ord}_p(\beta) = \mathrm{ord}_p(\beta') = 1/2$ in this case.

Our next proposition, let $\mathcal{R} \subset \mathcal{F}$ be the ring of p-integers. We consider the topology on $\mathcal{F} \otimes \mathcal{R}[[q]] \supset \mathbb{Q}_p \otimes \mathbb{Z}_p[[q]]$ determined by the metric

$$\left| \sum_{n\geq 0} u(n)q^n \right|_p = p^{-\inf_n(\mathrm{ord}_p(u(n)))}.$$

Proposition 6.

(i) *If* $p \equiv 1 \mod 4$ *is a prime, then in* $\mathbb{Q}_p \otimes \mathbb{Z}_p[[q]]$

$$\lim_{l\to\infty} \beta^{-l}(D(M))|U^l = 0.$$

(ii) *If* $p \equiv 3 \mod 4$ *is a prime, then in* $\mathcal{F} \otimes \mathcal{R}[[q]]$ *the limits*

$$\lim_{l\to\infty} \beta^{-2l}(D(M)|U^{2l}$$

and

$$\lim_{l\to\infty} \beta^{-2l-1}(D(M)|U^{2l+1}$$

exist.

Proof. Abbreviate

$$F := D(M), \qquad r_p = pD(R_p),$$

and note that it follows from (3) that the Fourier coefficients of r_p are rational integers since those of F are rational integers. We first prove that all limits exist.

Set

$$G(\tau) = F(\tau) - \beta' F(p\tau) \quad \text{and} \quad G'(\tau) = F(\tau) - \beta F(p\tau),$$

and rewrite equation (3) as

$$(F|U)(\tau) + \beta\beta' F(p\tau) = (\beta + \beta')F(\tau) + r_p.$$

We obtain that

$$G|U = \beta G + r_p, \quad G'|U = \beta' G' + r_p,$$

and

$$F|U = \frac{\beta}{\beta - \beta'}(\beta G + r_p) - \frac{\beta'}{\beta - \beta'}(\beta' G' + r_p).$$

It follows that

$$\begin{aligned} (\beta - \beta')\beta^{-l} F|U^l = \; & \left(\beta G + r_p + \tfrac{1}{\beta} r_p|U + \ldots + \tfrac{1}{\beta^{l-1}} r_p|U^{l-1}\right) \\ & -(\beta'/\beta)^l \left(\beta' G' + r_p + \tfrac{1}{\beta'} r_p|U + \ldots + \tfrac{1}{\beta'^{l-1}} r_p|U^{l-1}\right). \end{aligned} \tag{4}$$

The existence of the limit in part (i) follows from (4) since $(\beta'/\beta)^l \to 0$ and the second expression in parenthesis has bounded denominators by Proposition 2, while $\beta^{1-l} r_p|U^{l-1} \to 0$

as $l \to \infty$ again by Proposition 2. In order to prove the existence of the limits in part (ii), we rewrite (4) in this case, taking into the account that $\beta = -\beta'$, as

$$2\beta^{-2l+1} F|U^{2l} = \beta G + \beta G' + 2\frac{1}{\beta} r_p|U + 2\frac{1}{\beta^3} r_p|U^3 + \ldots + 2\frac{1}{\beta^{2l-1}} r_p|U^{2l-1}$$

and

$$2\beta^{-2l} F|U^{2l+1} = \beta G - \beta G' + 2r_p + 2\frac{1}{\beta^2} r_p|U^2 + 2\frac{1}{\beta^4} r_p|U^4 + \ldots + 2\frac{1}{\beta^{2l}} r_p|U^{2l},$$

and Proposition 6 (ii) follows since we still have that $\beta^{-m} r_p|U^m \to 0$ as $m \to \infty$ by Proposition 2.

We now prove that the limit in Proposition 6 (i) is actually zero. Write

$$\lim_{l\to\infty} \beta^{-l}(D(M))|U^l = \sum_{n>0} c(n)q^n.$$

Obviously

$$\left(\sum_{n>0} c(n)q^n\right)|U = \beta \left(\sum_{n>0} c(n)q^n\right),$$

and we derive from (3), Proposition 2, and the fact that the operators U and $T(m)$ commute for any prime $m \neq p$, that

$$\left(\sum_{n>0} c(n)q^n\right)|T(m) = b(m)\left(\sum_{n>0} c(n)q^n\right).$$

We can conclude that for every positive integer m not divisible by p

$$\left(\sum_{n>0} c(n)q^n\right)|T(m) = b^*(m)\left(\sum_{n>0} c(n)q^n\right),$$

where $b^*(m)$ are defined by $g(\tau) - \beta' g(p\tau) = \sum_{n\geq 1} b^*(n)q^n$. It follows that $\sum c(n)q^n$ must be a multiple of $g(\tau) - \beta' g(p\tau)$. However, $c(1) = 0$ (because $F = \frac{E_4(4\tau)}{\eta(4\tau)^2\eta(8\tau)^2}$, and therefore $c(p^l) = 0$ for all l). Thus the series $\sum_{n>0} c(n)q^n$ must be a zero multiple of $g(\tau) - \beta' g(p\tau)$. $\square$

We are now ready to prove Theorem 2 (i),(ii).

Proof of Theorem 2 (i),(ii). Recall that $F = D(M) = \frac{E_4(4\tau)}{\eta(4\tau)^2\eta(8\tau)^2}$. Theorem 2 (ii) follows immediately from Proposition 6 (ii) since $\mathrm{ord}_p(\beta) = 1/2$ for $p \equiv 3 \mod 4$.

Assume that $p \equiv 1 \mod 4$. Proposition 2 allows us to pick $A_p \geq 0$ such that

$$\mathrm{ord}_p\left(p^{A_p} r_p|U^m\right) \geq m \tag{5}$$

for all $m \geq 0$. Since F has p-integral Fourier coefficients, so does G', and in view of (5) and the fact that $\mathrm{ord}_p(\beta') = 1$, it now follows from (4) that

$$(\beta - \beta')\beta^{-l} p^{A_p} F|U^l \equiv p^{A_p}\beta G + p^{A_p} r_p + \frac{p^{A_p}}{\beta} r_p|U + \ldots + \frac{p^{A_p}}{\beta^{l-1}} r_p|U^{l-1} \mod p^l.$$

Let $s \geq 1$ be an integer. Pick $l > s$ large enough such that $F|U^l \equiv 0 \mod p^s$, take into the account that both $(\beta - \beta')$ and β are p-adic units, and rewrite the previous congruence as

$$0 \equiv p^{A_p} \frac{\beta - \beta'}{\beta^s} F|U^s + \frac{p^{A_p}}{\beta^s} r_p|U^s + \ldots + \frac{p^{A_p}}{\beta^{l-1}} r_p|U^{l-1} \mod p^s.$$

It now follows from (5) that all terms on the right in this congruence, except possibly the first one, vanish modulo p^s, and we conclude that $p^{A_p} F|U^s \equiv 0 \mod p^s$ as required. $\square$

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