THE ILL-POSED LINEAR COMPLEMENTARITY PROBLEM

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Abstract. A regularization of the linear complementarity problem (LCP) is proposed that leads to an exact solution, if one exists, otherwise a minimizer of a natural residual of the problem is obtained. The regularized LCP (RLCP) turns out to be a linear program with equilibrium constraints (LPEC) that is always solvable. For the case when the underlying matrix M of the LCP is in the class Q_0 (LCP solvable if feasible), the RLCP can be solved by a quadratic program, which is convex if M is positive semidefinite. An explicitly exact penalty of the RLCP formulation is also given when $M \in Q_0$ and implicitly exact otherwise. Error bounds on the distance between an arbitrary point to the set of LCP residual minimizers follow from LCP error bound theory. Computational algorithms for solving the RLCP consist of solving a convex quadratic program for positive semidefinite M, otherwise a generally nonconvex quadratic program when $M \in Q_0$, for which a potentially finitely terminating Frank-Wolfe method is proposed. For a completely general M, a parametric method is proposed wherein for each value of the parameter a Frank-Wolfe algorithm is carried out.

Key words. Ill-posed linear complementarity, exact penalty, error bounds, parametric algorithm

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1. Introduction.

We consider the possibly unsolvable classical linear complementarity problem (LCP) [3, 9, 4]

$$0 \le x \perp Mx + q \ge 0, \tag{1}$$

for a given $n \times n$ real matrix M and a given $n \times 1$ vector q, where $\perp$ denotes orthogonality, that is $x^T(Mx + q) = 0$. In general problem (1) may not have a solution. This may be due to corruption of the problem data (M, q) or to other factors. We shall call instances of (1) when it has no solution, the *ill-posed linear complementarity problem* (ILCP). It is well known [10, 4] and easy to verify that (1) is equivalent to

$$x - (x - Mx - q)_+ = 0, \tag{2}$$

where $(\cdot)_+$ denotes $max\{\cdot, 0\}$ componentwise. Our regularization of the LCP (1) will consist of minimizing some norm of the left hand side of (2). We shall choose the 1-norm, because that will lead to a linear program with equilibrium constraints (LPEC), which in turn will enable us to obtain exact penalty formulations. We shall therefore solve the following problem:

$$\min_{x,y} \{\|y\|_1 \big| x + y - (x - Mx - q)_+ = 0\} \tag{3}$$

This is equivalent to the following LPEC [5], which we take as our *regularized LCP* (RLCP):

$$\min_{(x,y,z)\in S} e^T z, \quad \text{where } S := \{(x, y, z) \big| 0 \le x + y \perp Mx + y + q \ge 0, \ -z \le y \le z\}, \tag{4}$$

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where e is a vector of ones of appropriate dimension. It is easy to show that the RLCP (4) always has a solution and that its minimum value is zero if and only the LCP (1) is solvable (Theorem 2.1).

Another justification for the particular choice of the regularization (4) is that for $M \in Q_0$, the intuitive greedy sequential procedure of *first* generating a nonempty feasible region: $\{x \big| 0 \le x + \hat{y},\ Mx + \hat{y} + q \ge 0\}$ for some "minimal" $\hat{y}$, and then minimizing the complementarity condition $(x+\hat{y})^T(Mx+\hat{y}+q)$ on this nonempty feasible region, leads, in fact, to an exact solution of the RLCP (4) (Theorem 2.2). This result for Q_0 matrices for the LCP also leads to an exact penalty formulation of the RLCP (4) (Theorem 2.3a). When M is not in Q_0, an asymptotic penalty formulation of the RLCP (4) exists (Theorem 2.3b) from which an exact solution to the RLCP (4) can be extracted (Theorem 2.3c), and hence the terminology: "asymptotically exact". Section 3 of the paper extends classical error bounds of solvable LCPs [8, 7] to ILCPs. These results, summarized in Table 1, give bounds, in terms of residuals of arbitrary points, on the distance to the solution set of the RLCP (4). Section 4 of the paper gives computational algorithms for solving the RLCP (4). A quadratic programming algorithm (Algorithm 4.1) is given for the case when $M \in Q_0$, for which the quadratic program is convex when M is positive semidefinite, otherwise it is nonconvex, but solvable by a finite number of steps of the Frank-Wolfe algorithm, if the iterates accumulate to a solution of the RLCP (4). For a completely general M, a parametric algorithm (Algorithm 4.3) is proposed wherein for each value of the algorithm parameter a Frank-Wolfe algorithm is solved. The latter is finite if its iterates accumulate to a zero objective function value.

Although the 1-norm was used in (3) to formulate the RLCP (4), the 2-norm can be used as well. For the 2-norm formulation, most of the results of this paper go through with appropriate minor modifications, with the exception of Theorem 2.3 and Algorithm 4.3, which rely on the RLCP (4) being an LPEC. When the 2-norm is used the RLCP (4) is not an LPEC.

A word about our notation now. For a vector x in the n-dimensional real space R^n, x_+ will denote the vector in R^n with components $(x_+)_i := \max\{x_i, 0\}$, $i = 1, \ldots, n$. The notation $A \in R^{m \times n}$ will signify a real $m \times n$ matrix. For such a matrix, A^T will denote the transpose and A_i will denote row i. For two vectors x and y in R^n, $x^T y$ will denote the scalar product, and $x \perp y$ will denote $x^T y = 0$. A vector of ones in a real space of arbitrary dimension will be denoted by e. The notation $\arg\min\limits_{x \in S} f(x)$ will denote the set of minimizers of $f(x)$ on the set S, and the notation $\arg\operatorname{vertex}\min\limits_{x \in S} f(x)$ will denote the set of vertices of S that minimize f on S. An arbitrary norm on R^n will be denoted by $\|\cdot\|$, and the 1-norm will be denoted by $\|\cdot\|_1$. A matrix $M \in R^{n \times n}$ is in Q_0 if nonemptiness of the feasible region $\{0 \le x,\ Mx + q \ge 0\}$ implies solvability of the LCP (1); M is in R_0 if the LCP (1) with $q = 0$ is uniquely solvable by $0 \in R^n$; M is row sufficient if each Karush-Kuhn-Tucker point of the quadratic program

$$(5) \qquad \min_x \{x^T(Mx+q) \big| Mx + q \ge 0,\ x \ge 0\}$$

solves the LCP (1) [4]. A positive definite matrix is referred to as pd, and psd refers to a positive semidefinite matrix. The notations := and =: define a quantity on the colon side by a quantity on the equality side. For $f : R^n \to R$, $\nabla f(x)$ is the $1 \times n$ gradient vector.

2. The Regularized Linear Complementarity Problem. We begin by showing that the RLCP (4) always has a solution.

THEOREM 2.1. **(Existence of solution to the RLCP)** *The RLCP (4) has a solution for any matrix $M \in R^{n\times n}$ and any vector $q \in R^n$. The LCP (1) is solvable if and only if the RLCP (4) has a zero minimum.*

Proof. The feasible region S of the RLCP (4) is nonempty, because it contains the point $\big(x = 0,\ y = (-q)_+,\ z = (-q)_+\big)$. Since S is the union of a finite number of polyhedral sets on which the linear objective function is bounded below by zero, it follows that the minimum of the minima on each polyhedral set is the global minimum. The last statement of the theorem holds because $e^Tz = 0$ if and only if the LCP (1) is solvable. ▯

It will be convenient now to define a penalty problem (PLCP) associated with the RLCP (4) with a feasible region T containing S as follows:

$$\min_{(x,y,z)\in T} P(x,y,z,\alpha) := \min_{(x,y,z)\in T} e^Tz + \alpha(x+y)^T(Mx+y+q), \tag{6}$$

where

$$T := \left\{(x,y,z) \;\middle|\; \begin{array}{c} 0 \le x+y,\ Mx+y+q \ge 0, \\ -z \le y \le z \end{array} \right\} \tag{7}$$

Here α is some positive penalty parameter. With the help of this penalty parameter we establish the fact when $M \in Q_0$, the RLCP (4), which is an LPEC, can be solved by solving a quadratic program.

THEOREM 2.2. **(The RLCP as a quadratic program)** *Let $M \in Q_0$ (in particular M may be positive semidefinite or row sufficient), and let*

$$(\hat{x},\hat{y},\hat{z}) \in \arg\min_{(x,y,z)\in T} e^Tz := \arg\min_{(x,y,z)} \left\{e^Tz \;\middle|\; \begin{array}{c} 0 \le x+y,\ Mx+y+q \ge 0, \\ -z \le y \le z \end{array} \right\} \tag{8}$$

Then each solution of the quadratic program (QP)

$$\min_x \{(x+\hat{y})^T(Mx+\hat{y}+q) \big| 0 \le x+\hat{y},\ Mx+\hat{y}+q \ge 0\} \tag{9}$$

solves the RLCP (4).

Proof. Note first that the QP (9) is solvable, because it is feasible and its objective function is bounded below by zero. We next show that its minimum is zero as a consequence of $M \in Q_0$. Consider the transformed variable

$$s = x + \hat{y}, \tag{10}$$

where $\hat{y}$ is defined by (8). The QP (9) is related to the LCP

$$0 \le s \perp Ms + (I-M)\hat{y} + q \ge 0 \tag{11}$$

This LCP is feasible (take $s = \hat{x} + \hat{y}$) and hence solvable because $M \in Q_0$. Consequently, the QP (9) has a zero minimum, that is

$$(12)\qquad \begin{aligned}&0=(\bar{x}+\hat{y})^T(M\bar{x}+\hat{y}+q)\\&=\min_x\{(x+\hat{y})^T(Mx+\hat{y}+q)\mid 0\le x+\hat{y},\ Mx+\hat{y}+q\ge 0\}\end{aligned}$$

Let $\bar{x}$ be a solution of the QP (9) and let $\alpha>0$. Then

$$(13)\qquad \begin{aligned}&\min_{(x,y,z)\in T} P(x,y,z,\alpha)\\&\ge \min_{(x,y,z)\in T} e^Tz+\min_{(x,y,z)\in T}\alpha(x+y)^T(Mx+y+q)\\&=e^T\hat{z}\\&=e^T\hat{z}+\alpha(\bar{x}+\hat{y})^T(M\bar{x}+\hat{y}+q)=P(\bar{x},\hat{y},\hat{z},\alpha)\end{aligned}$$

The first equality above follows from (8) and the fact that by Theorem 2.1 $S\neq\emptyset$ and hence $(x+y)^T(Mx+y+q)=0$ for some $(x,y,z)\in T$. The second equality above follows from the fact that the QP (9) has a zero minimum for $M\in Q_0$. Consequently, $(\bar{x},\hat{y},\hat{z})\in S$, and $(\bar{x},\hat{y},\hat{z})$ is feasible for RLCP (4). It follows from (13) that

$$(14)\qquad e^T\hat{z}\le \min_{(x,y,z)\in T} P(x,y,z,\alpha)\le \min_{(x,y,z)\in S} P(x,y,z,\alpha)=\min_{(x,y,z)\in S} e^Tz$$

Hence $(\bar{x},\hat{y},\hat{z})$ solves the RLCP (4).□

We show now that when $M\in Q_0$ the penalty function formulation PLCP (6) for the RLCP (4) can be made exact. That is, a solution of the PLCP (6) for a finite value of the penalty parameter α, also solves the RLCP (4). For a general M, an exact solution to the RLCP (4) can also be obtained from the PLCP (6) for sufficiently large α as indicated in the following theorem.

Theorem 2.3. **(Exact penalty for the RLCP)**

a) For $M\in Q_0$, each solution of the PLCP (6) for $\alpha>0$, solves the RLCP (4).

b) For a general M, there exists $\bar{\alpha}>0$ such that for any fixed $\alpha\ge\bar{\alpha}$:

$$(15)\qquad \begin{bmatrix}x(\alpha)\\y(\alpha)\\z(\alpha)\end{bmatrix}=\begin{bmatrix}\frac{a^i}{\alpha}+x_0^i\\\frac{b^i}{\alpha}+y_0^i\\\frac{c^i}{\alpha}+z_0^i\end{bmatrix},\ \text{for some } i\in\{1,\ldots,\ell\}$$

where

$$(16)\qquad (x(\alpha),\,y(\alpha),\,z(\alpha))\in\arg\min_{(x,y,z)\in T} P(x,y,z,\alpha)$$

Here ℓ and the vectors a^i, b^i, c^i, x_0^i, y_0^i, z_0^i, $i=1,\ldots,\ell$, depend on the problem data (M,q) only. Furthermore, $(x_0^i,\,y_0^i,\,z_0^i)$, $i=1,\ldots,\ell$, solve the RLCP (4) and such that for any fixed $\alpha\ge\bar{\alpha}$, the following holds for some $i\in\{1,\ldots,\ell\}$:

$$(17)\qquad (x(\alpha)+y(\alpha))^T(Mx(\alpha)+y(\alpha)+q)=\frac{(a^i+b^i)^T(Ma^i+b^i)}{\alpha^2}$$

c) For a general M, let $\alpha_2 > \alpha_1 \geq \bar{\alpha}$, where $\bar{\alpha}$ is defined in part b) above. Let the two solutions $(x(\alpha_2),\, y(\alpha_2),\, z(\alpha_2))$ and $(x(\alpha_1),\, y(\alpha_1),\, z(\alpha_1))$ to the PLCP (6) have the same basis in the primal-dual space. Then $(x_0^i,\, y_0^i,\, z_0^i)$ solves the RLCP (4) where

$$\begin{aligned} x_0^i &= \tfrac{\alpha_2 x(\alpha_2) - \alpha_1 x(\alpha_1)}{\alpha_2 - \alpha_1} \\ y_0^i &= \tfrac{\alpha_2 y(\alpha_2) - \alpha_1 y(\alpha_1)}{\alpha_2 - \alpha_1} \\ z_0^i &= \tfrac{\alpha_2 z(\alpha_2) - \alpha_1 z(\alpha_1)}{\alpha_2 - \alpha_1} \end{aligned} \tag{18}$$

Proof.

a) From (12) and (13) it follows that for all $\alpha > 0$:

$$\begin{aligned} & e^T\hat{z} + \alpha \cdot 0 \\ & = P(\bar{x}, \hat{y}, \hat{z}, \alpha) \\ & = \min_{(x,y,z)\in T} P(x, y, z, \alpha) \\ & = e^T z(\alpha) + \alpha(x(\alpha) + y(\alpha))^T (Mx(\alpha) + y(\alpha) + q) \end{aligned} \tag{19}$$

Hence by (8) and (12) we have that:

$$e^T z(\alpha) = \min_{(x,y,z)\in T} e^T z \tag{20}$$

$$\begin{aligned} & 0 = (x(\alpha) + y(\alpha))^T (Mx(\alpha) + y(\alpha) + q) \\ & = \min_{(x,y,z)\in T} (x + y)^T (Mx + y + q) \end{aligned} \tag{21}$$

Consequently by Theorem 2.2, $x(\alpha)$ solves the RLCP (4).

b) This follows from the proof of [6, Theorem 3.2].

c) This follows from the proof of [6, Corollary 3.3].

□

We turn our attention to error bounds for the RLCP (4).

3. Error Bounds. In this section we show that standard error bounds for the linear complementarity problems given in [8, 7] extend to the RLCP (4) for the cases when $M \in Q_0$, $M \in Q_0 \cap R_0$, M positive semidefinite or positive definite. The key to this extension lies in the fact that when $M \in Q_0$, the LCP (11), where $\hat{y}$ is defined in (8) and s in (10), is solvable and any of its solutions $\bar{s}$ generates an $\bar{x} := \bar{s} - \hat{y}$ that solves the RLCP (4). Hence error bounds for (11) can be taken as error bounds for the RLCP (4). By taking into account the following relations between the variables s and x :

$$\begin{aligned} s &= x + \hat{y} \\ Ms + (I - M)\hat{y} + q &= Mx + \hat{y} + q, \end{aligned} \tag{22}$$

we can translate natural residuals in terms of s for the LCP (11) to residuals in terms of x. Indeed, with $\hat{y}$ defined as in (8) we have:

$$\begin{aligned} & \|s - (s - Ms - (I - M)\hat{y} - q)_+\| = \|x + \hat{y} - (x - Mx - q)_+\| =: r(x, \hat{y}) \\ & \|(-s,\, -Ms - (I - M)\hat{y} - q,\, s^T(Ms + (I - M)\hat{y} + q))_+\| \\ & \qquad = \|(-x - \hat{y},\, -Mx - \hat{y} - q,\, (x + \hat{y})^T(Mx + \hat{y} + q))_+\| =: s(x, \hat{y}) \end{aligned} \tag{23}$$

Thus, given an arbitrary point $x \in R^n$ we can find a point $\bar{x}(x,\hat{y})$ in the solution set of the RLCP (4) such that $\|x - \bar{x}(x,\hat{y})\|$ is bounded by some norm-dependent constant $\sigma(M,q)$ multiplied by the residuals $r(x,\hat{y})$, $s(x,\hat{y}) + s(x,\hat{y})^{\frac{1}{2}}$, or $r(x,\hat{y}) + s(x,\hat{y})$. If the bound is local, then x needs to be sufficiently close to a solution $\bar{x}(x,\hat{y})$ of the RLCP (4). These error bounds are summarized in Table 1 and are based on a similar table given in [8, Table 1].

Table 1

Residuals (23) as Error Bounds for RLCP (4)

Multiply Residual by Appropriate Norm-Dependent Constant $\sigma(M,q)$

	Residual		
M	$r(x,\hat{y})$	$s(x,\hat{y}) + s(x,\hat{y})^{\frac{1}{2}}$	$r(x,\hat{y}) + s(x,\hat{y})$
Q_0	local	?	local
psd	local	global	global
$Q_0 \cap R_0$	global	global	global
pd	global	global	global

We note that we cannot state, as was done in [8, Table 1] for an arbitrary matrix, that the residual $s(x,\hat{y}) + s(x,\hat{y})^{\frac{1}{2}}$ is not a local error bound for $M \in Q_0$, and hence the question mark in Table 1. This is so because the following matrix used to establish the non-local-error property of this residual for a general matrix [8, Remark 3.1] is *not* in Q_0:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{24}$$

We turn now to computational algorithms for solving the RLCP (4).

4. Computational Algorithms for the RLCP. For the case when M is positive semidefinite, solution of a single linear program (8) followed by the solution of a single convex quadratic program (9) leads to a solution of the LPEC that constitutes RLCP (4). We turn our attention now to the cases when M is *not* positive semidefinite.

For the case when $M \in Q_0$ we again propose solving the linear program (8) to obtain a $\hat{y}$ followed by the solution of the nonconvex quadratic program (9) by a Frank-Wolfe method, which terminates finitely if its iterates accumulate to a zero of the objective function. We state the algorithm and establish its convergence.

ALGORITHM 4.1. **(Finite Frank-Wolfe algorithm for $M \in Q_0$)**

(i) Solve the linear program (8) to determine $(\hat{x}, \hat{y}, \hat{z})$.

(ii) Use a Frank-Wolfe algorithm on the QP (9) transformed to the variable $s = x + \hat{y}$ *as follows:*

$$\min_{s} \{s^T(Ms + (I - M\hat{y} + q)) | 0 \le s,\ Ms + (I - M)\hat{y} + q \ge 0\} =: \min_{s \in X} f(s)$$

Compute s^{i+1} *from* s^i, *starting with* $s^0 = \hat{x} + \hat{y}$, *as follows:*

(a) $v^i \in \arg \operatorname{vertex} \min_{s \in X} \nabla f(s^i) s$

(b) Stop if $\nabla f(s^i)v^i = \nabla f(s^i)s^i$

(c) $s^{i+1} = (1-\lambda^i)s^i + \lambda^i v^i$ *where*

$$\lambda^i \in \arg\min_{0\le\lambda\le 1} f((1-\lambda)s^i + \lambda v^i)$$

THEOREM 4.2. **(Finite termination of Algorithm 4.1 for $M \in Q_0$)** *Let $M \in Q_0$. The sequence $\{s^i\}$ of the Algorithm 4.1 accumulates to an $\bar{s} \in X$. If $f(\bar{s}) = 0$, which must be the case if M is row sufficient, then one of the vertices $\{v^i\}$ of X generated by the algorithm is a solution. Else $\bar{x}$ satisfies the minimum principle necessary optimality condition:*

$$\nabla f(\bar{s})(s-\bar{s}) \ge 0 \qquad \forall s \in X \tag{25}$$

Proof. Follows from [2, Theorem A.2]. □

For an arbitrary M, we propose a parametric solution method for solving the RLCP (4), similar to that proposed in [6, Equations (40)-(41)], and successfully implemented in [1]. Define the parametric minimization problem based on the RLCP (4) as follows:

$$\begin{array}{l} \theta(\nu) := \min_{x,y,z} (x+y)^T(Mx+y+q) \\ \text{s.t. } 0 \le \mathrm{x}+\mathrm{y},\ \mathrm{Mx}+\mathrm{y}+\mathrm{q} \ge 0\ -\mathrm{z} \le \mathrm{y} \le \mathrm{z},\ \mathrm{e^T z} \le \nu \end{array} \tag{26}$$

The function $\theta(\nu)$ is a nonincreasing function of ν, for $\nu \ge 0$, and equals zero for $\nu \ge e^T(-q)_+$ (just take $x = 0,\ z = y = (-q)_+$). The problem then is to find the smallest nonnegative $\bar{\nu}$ such that $\theta(\bar{\nu}) = 0$, that is:

$$\bar{\nu} = \min_{\nu \ge 0} \{\nu | \theta(\nu) = 0\} \tag{27}$$

Obviously, if $\bar{\nu} = 0$, then the LCP (1) is solvable. We now state our parametric approach for approximately determining a sufficiently small interval containing $\bar{\nu}$.

ALGORITHM 4.3. **(Parametric algorithm for arbitrary M)** *Choose $\delta > 0$, an acceptably small interval length containing $\bar{\nu}$. (Note: $\bar{\nu} \in [\nu^1, \nu^2]$.)*

(i) Set $\nu^1 = 0,\ \nu^2 = e^T(-q)_+,\ j = 0$ *(Note:* $\theta(\nu^2) = 0$.*)*

(ii) Stop if $\nu^2 - \nu^1 \le \delta$

(iii) $\nu_j = \frac{\nu^1+\nu^2}{2}$

(iv) Determine $\theta(\nu_j)$ or an approximation thereof, by applying a Frank-Wolfe procedure to (26)

(v) $\nu^1 \leftarrow \nu_j$ *if* $\theta(\nu_j) > 0$

$\nu^2 \leftarrow \nu_j$ *if* $\theta(\nu_j) = 0$

(vi) $j \leftarrow j+1$. *Go to (ii)*

The "approximation" in step (iv) of Algorithm 4.3 refers to the fact that there is no guaranteed way for solving the nonconvex problem (26) short of total vertex enumeration. However, efficient solution of a very similar problem in [1] leads us to believe that this approach may be effective here as well.

5. Conclusion. We have proposed a regularization of the ill-posed linear complementarity problem that led to a linear program with equilibrium constraints. We have shown that the regularized problem always has a solution and have cast it as an explicitly or implicitly exact penalty problem or a quadratic program. We have also given error bounds on the distance from an arbitrary point to the solution set of the regularized problem. Computational algorithms have been proposed that are based on solving a quadratic program, which in general may be nonconvex. However, computational testing of these algorithms is warranted, as well as further theoretical and computational investigation of the proposed regularization and other possible reformulations of the ill-posed linear complementarity problem.

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