Combinatorics of Chord Progressions

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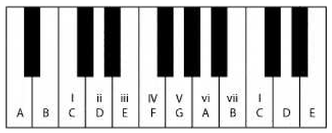
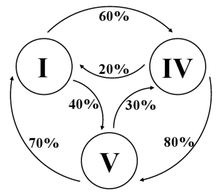
1. Music Background

- One of the first people to make a connection between mathematics and music was Pythagoras of Samos, nearly 2500 years ago. Since then, many mathematicians have explored areas of overlap between mathematics and music. In this project, we explored an overlap between combinatorics and music. Our goal is to show chord progressions that are common to a specific collection of music, composer, or era.
- To understand chord progressions, however, we must first understand the musical scale and its respective chords. A musical scale consists of seven notes and repeats at the octave, e.g., the C Major Scale is C-D-E-F-G-A-B-(C)-.... The notes of a musical scale are numbered by their steps from the root of the scale. In a C Major Scale, the root is C, the second D, the third E, and so on to the seventh B, at which point the scale starts over again one octave higher.
- A chord is three or more notes played simultaneously. In our work, we consider seven chords defined on a musical scale. The C Major scale, for example, consists of the I chord (C-E-G), ii (D-F-A), iii (E-G-B), IV (F-A-C), V (G-B-D), vi (A-C-E), and vii° (B-D-F). An upper case Roman numeral indicates a major chord where as a lower case Roman numeral indicates a minor chord.
- A chord progression, then, is the movement from one chord to another in a musical scale. For example, I-IV-V-I and I-vi-IV-V are two very common chord progressions used in pop music.



2. Mathematics and Music

- For example, suppose a piece of music has the following probabilities changing from one chord to another,



- We would represent this situation mathematically with the following Initial State matrix:

$$\begin{array}{c} \\ \text{I} \\ \text{IV} \\ \text{V} \end{array} \begin{array}{c} \begin{array}{ccc} \text{I} & \text{IV} & \text{V} \end{array} \\ \begin{pmatrix} 0 & .60 & .40 \\ .20 & 0 & .80 \\ .70 & .30 & 0 \end{pmatrix} \end{array}$$

- Using Markov Chain Analysis, we find that the steady state is

$$\begin{array}{ccc} \text{I} & \text{IV} & \text{V} \\ \begin{pmatrix} .322 & .305 & .373 \\ \vdots & \vdots & \vdots \end{pmatrix} \end{array}$$

which tells us that we expect 32.2% of chord changes to progress to I chords, 30.5% to IV chords, and 37.3% to V chords.

3. Chord Analysis - Initial State Matrices

- Each entry of the matrices below represents the probability that the chord in the column immediately follows the chord in the row. For example, in the first row of the Bach Minor matrix, the .41 means that there is a 41% chance that any I chord is followed by a V chord. Note that each row sums to 1, or 100%.

Bach Minor

	I	*ii*	*iii*	*IV*	*V*	*vi*	*vii°*
I	0	.18	.01	.20	.41	.09	.12
ii	.01	0	.03	0	.89	0	.07
iii	.06	.06	0	.25	.19	.31	.13
IV	.22	.14	0	0	.48	0	.15
V	.80	0	.02	.06	0	.10	.02
vi	.03	.54	.03	.14	.19	0	.08
vii°	.81	0	.01	.03	.15	0	0

Bach Major

	I	*ii*	*iii*	*IV*	*V*	*vi*	*vii°*
I	0	.15	.01	.28	.41	.09	.06
ii	.01	0	0	0	.71	.01	.25
iii	.03	.03	0	.52	.06	.32	.03
IV	.22	.13	0	0	.39	.02	.23
V	.82	.01	0	.07	0	.09	0
vi	.15	.29	.05	.11	.32	0	.09
vii°	.91	0	.01	.02	.04	.03	0

Mozart Minor

	I	*ii*	*iii*	*IV*	*V*	*vi*	*vii°*
I	0	.08	0	.07	.68	.06	.11
ii	.37	0	0	0	.46	0	.17
iii	0	0	0	1	0	0	0
IV	.42	.10	0	0	.39	0	.09
V	.82	0	0	.05	0	.07	.05
vi	.14	.51	0	.16	.05	0	.14
vii°	.76	.01	0	0	.23	0	0

Mozart Major

	I	*ii*	*iii*	*IV*	*V*	*vi*	*vii°*
I	0	.13	0	.15	.62	.05	.05
ii	.49	0	.01	0	.40	.01	.09
iii	.67	0	0	0	0	.33	0
IV	.64	.14	0	0	.15	0	.07
V	.94	0	0	.01	0	.04	.01
vi	.11	.51	0	.14	.20	0	.04
vii°	.82	0	.01	.01	.16	0	0

Palestrina

	I	*ii*	*iii*	*IV*	*V*	*vi*	*vii°*
I	0	.15	.13	.28	.14	.22	.08
ii	.08	0	.15	.13	.28	.14	.22
iii	.22	.08	0	.15	.13	.28	.14
IV	.14	.22	.08	0	.15	.13	.28
V	.28	.14	.22	.08	0	.15	.13
vi	.13	.28	.14	.22	.08	0	.15
vii°	.15	.13	.28	.14	.22	.08	0

Beethoven

	I	*ii*	*iii*	*IV*	*V*	*vi*	*vii°*
I	0	.10	.01	.13	.52	.02	.22
ii	.06	0	.02	0	.87	0	.05
iii	0	0	0	0	.67	.33	0
IV	.33	.03	.07	0	.40	.03	.13
V	.56	.22	.01	.04	0	.07	.11
vi	.06	.44	0	.06	.11	0	.33
vii°	.80	0	0	.03	.17	0	0

4. Markov Chain Analysis - Steady State Matrices

Bach Minor

I	ii	iii	IV	V	vi	vii°
.333	.109	.015	.010	.306	.063	.075
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Bach Major

I	ii	iii	IV	V	vi	vii°
.344	.089	.016	.131	.273	.067	.080
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Mozart Minor

I	ii	iii	IV	V	vi	vii°
.400	.066	0	.056	.345	.049	.084
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Mozart Major

I	ii	iii	IV	V	vi	vii°
.435	.086	.001	.073	.330	.037	.038
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Palestrina

I	ii	iii	IV	V	vi	vii°
.143	.143	.143	.143	.143	.143	.143
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Beethoven

I	ii	iii	IV	V	vi	vii°
.317	.120	.012	.060	.328	.034	.130
⋮	⋮	⋮	⋮	⋮	⋮	⋮

- For example, the results for Bach Minor show that there is a 33.3% probability of progressing to the I chord from the previous chord, 10.9% to ii, 1.5% to iii, 1.0% to IV, 30.6% to V, 6.3% to vi, and 7.5% to vii°.

5. Mathematical Analysis

- By analysing chord sequences in compositions by Bach, Mozart, Palestrina, and Beethoven, we used Markov Chains to obtain the probability of any chord progression in that composer's work and show the most probable chord progression. Our analysis shows that that progressions to the I and V chords are most probable in all compositions (except Palestrina), whereas progressions to the iii chord are the least probable.

Noteworthy Results

Composer	Most Probable Chord(s)	Least Probable Chord(s)
Palestrina	Equal Probability (14.3%)	Equal Probability (14.3%)
Bach	**I** (33.9%) and **V** (29.0%)	**iii** (1.5%)
Mozart	**I** (41.8%) and **V** (33.8%)	**iii** (0%)
Beethoven	**V** (32.8%) and **I** (31.7%)	**iii** (1.2%)

- Note the time line of the composers analysed and the musical eras they lived in:
  - Palestrina (1525 - 1594) - Renaissance (1400 - 1600)
  - Bach (1685 - 1750) - Baroque (1600 - 1760)
  - Mozart (1756 - 1791) - Classical (1750 - 1830)
  - Beethoven (1770 - 1827) - Classical, Romantic (1815 - 1910)

6. Music Analysis

- Music from the Renaissance Period emphasized the self, and introduced the concept of chord progressions. The Church, which emphasized the divine, disapproved of the Renaissance Period because of its contrasting view. Palestrina, a composer of sacred music, composed at a level the Church approved of - a level prior to the Renaissance Period when chord progressions did not exist. Our results on Palestrina's work correlate with this musical history because there is an equal probability of progressing to any chord; i.e., it suggests that Palestrina did not follow any chord progression model.
- Our results for Bach and Mozart agree with the Baroque and Classical eras, respectively, because they show the dominance of the I and V chord and the lack of the iii chord. Beethoven's *Piano Quartet No. 1*, which was composed in 1783, during the Classical Period, also conforms to this trend.
- Romantic music broke down formal structures, such as chord progression models, from the Classical Period. We would expect an analysis of chord progressions of Romantic compositions to show no trend. In particular, we would expect the dominance of the I and V chord and the lack of the iii chord from the Classical Period to disappear almost entirely.
- Our research can serve as a launching pad for chord and Markov Chain Analysis for any number of compositions and composers. With enough data, it would be possible to show the changes in musical eras, say from the Classical Period to the Romantic Era, and therefore defining features of each musical time period.

References

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Acknowledgments

- Office of Research and Sponsored Programs, UW-Eau Claire
- Professor James Walker, UW-Eau Claire
- Professor Dmitri Tymoczko, Princeton University