Four Dimensional Tops

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Introduction

String theory predicts that the universe has extra dimensions, which have the structure of *Calabi-Yau varieties*; the universes defined by these varieties are conjectured to occur in physically indistinguishable pairs. The mathematical field of *mirror symmetry* seeks to understand the geometric correspondences between paired Calabi-Yau varieties.

For any polytope $\diamond$ with vertices in an integer lattice $\mathbb{Z}^n$, we define its polar dual within a dual space by $\diamond^* = \{\overline{x} \in \mathbb{R}^n | \langle v, \overline{x} \rangle \geq -1, \forall v \in \diamond\}$. A lattice polytope is defined to be reflexive if its polar dual is also a lattice polytope. It has already been shown that reflexive polytopes can be used to describe Calabi-Yau hypersurfaces, so by studying reflexive polytopes we may gain important insight into the nature of hidden dimensions in space. Reflexive polytopes have been classified in 3D and 4D, with 4,319 and 473,800,776 classes of equivalent polytopes respectively.

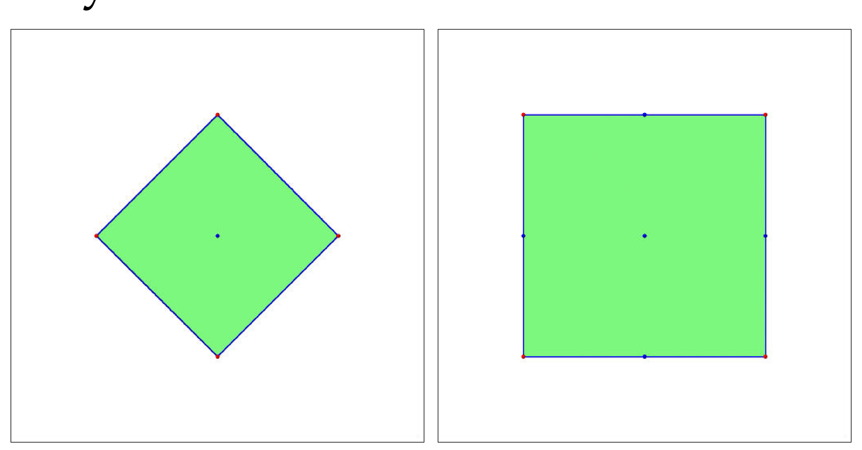

Figure 1: Dual pair of reflexive polytopes

A top Δ is defined to be a lattice polytope, for which one defining inequality is $\langle \overline{p_1}, x \rangle \geq 0$, and $\langle \overline{p_i}, x \rangle \geq -1$ for all other $\overline{p_i}$ in the dual space. The polar duality transformation can also be applied to tops and the structure changes to that of a dual top, with one facet infinitely extended.

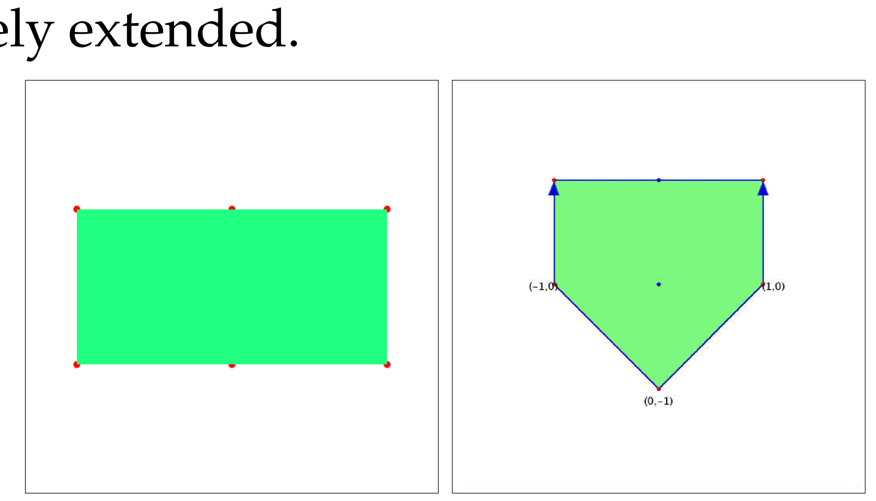

Figure 2: 2D top (left) and associated dual top

Bouchard and Skarke classified the 3D tops corresponding to each class of 2D reflexive polytope, based on the value of the point beneath the origin [B-S]. Through their classification, it was shown that there are infinitely many families of 3D tops. By extending the method of Bouchard and Skarke, we classify families of 4D tops.

Algorithm

A figure is convex if a line drawn between any two points in the figure is contained within the figure. For a polytope, this is equivalent to the simplex formed by every pair of adjacent facets being contained within the respective pair of hyperplanes. The containment of the facet can be established by computing a determinant involving the vertices, which must be positive. We follow the procedure shown below to construct a convex dual top and determine its dual:

1. Select the base polytope
2. Choose an appropriate triangulation
3. Choose an additional coordinate to concatenate to each point in the polytope
4. For each facet in the triangulation and each facet adjacent to it, check that the facet is locally convex as follows:
   (a) Add the point below the origin to a matrix
   (b) Add each point in the facet to the matrix
   (c) Add each point in the adjacent facet to the matrix
   (d) Compute the determinant of this matrix
   (e) Impose the requirement that this determinant be positive
5. Compute the polar dual of the constructed top

3D Fano Varieties

Of the 4,319 reflexive polytopes in three dimensions, five are smooth Fano polytopes with five or fewer vertices. These polytopes have a reasonably simple structure, and provide a good test situation for our computational setup. Our algorithm classifies all 4D tops with the point below the origin at a lattice point. As an initial application of the method, we implement our algorithm for these restricted smooth Fano polytopes using Sage mathematical software.

The 3D Simplex

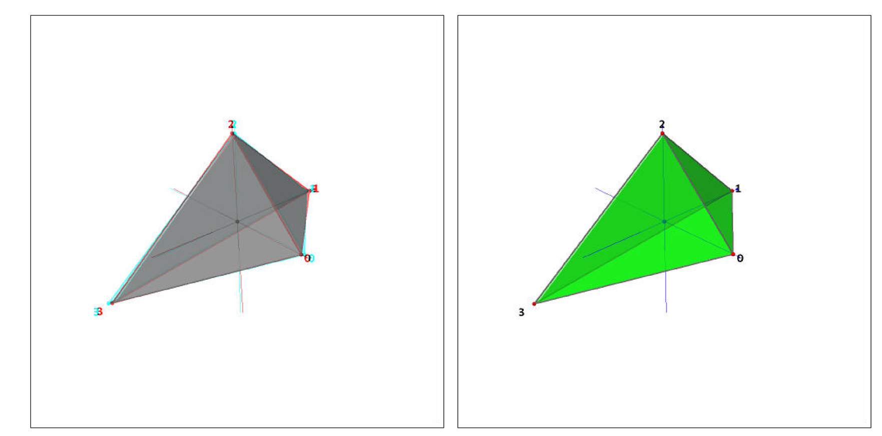

Figure 3: The 3D Simplex

The 3D simplex is defined by vertices at $(1\,0\,0), (0\,1\,0), (0\,0\,1)$, and $(-1\,-1\,-1)$. This dual top has one free parameter, and is convex when $a \in \mathbb{Z}$, with $a \geq -1$.

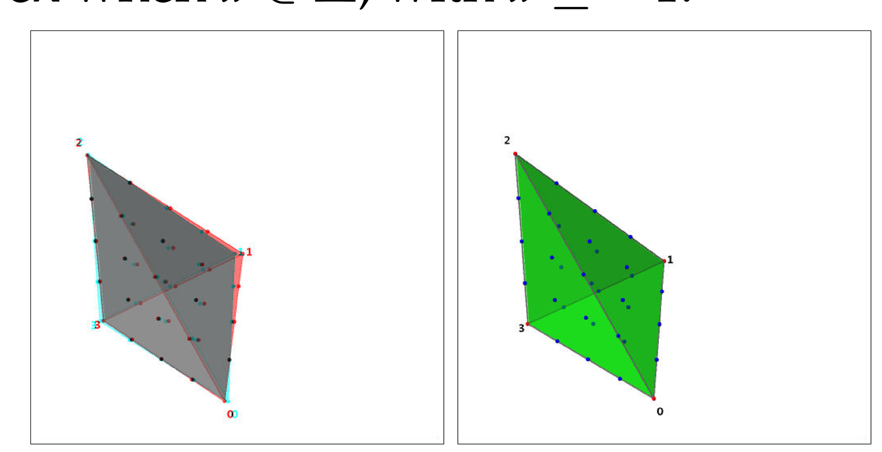

Figure 4: $x_4 = 0$ slice of the dual of the constructed top

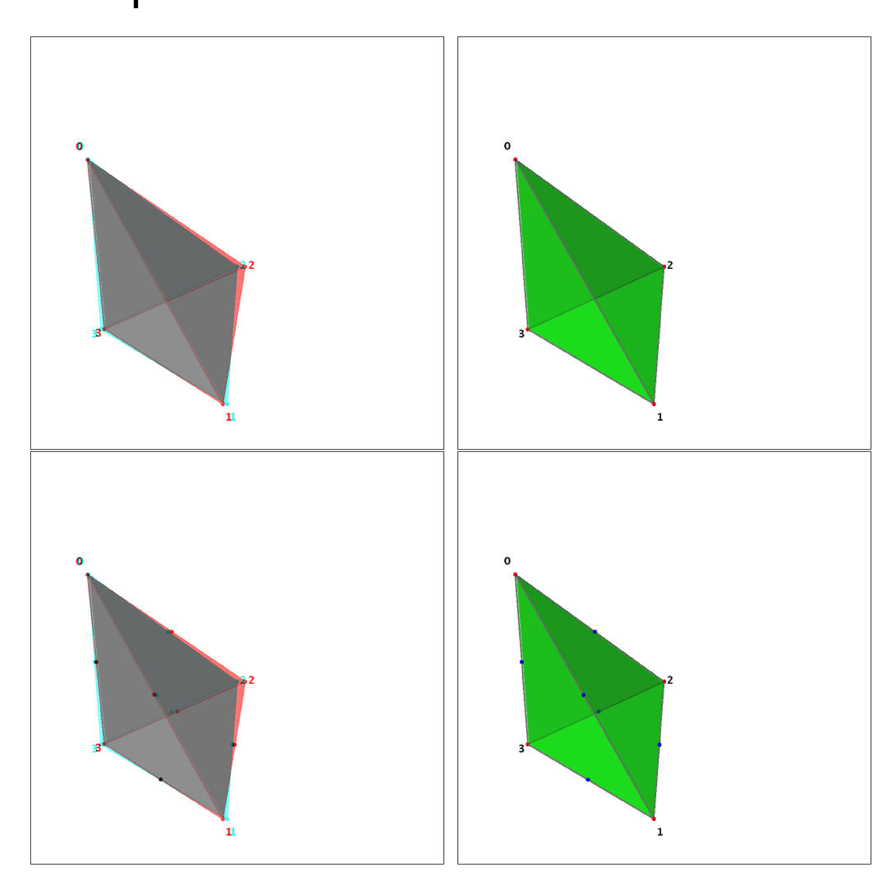

Figure 5: $x_4 = 0$ slice of the top corresponding to the constructed dual top, for $a = 0$ (top) and $a = 1$ (bottom)

Reflexive Polytope 6

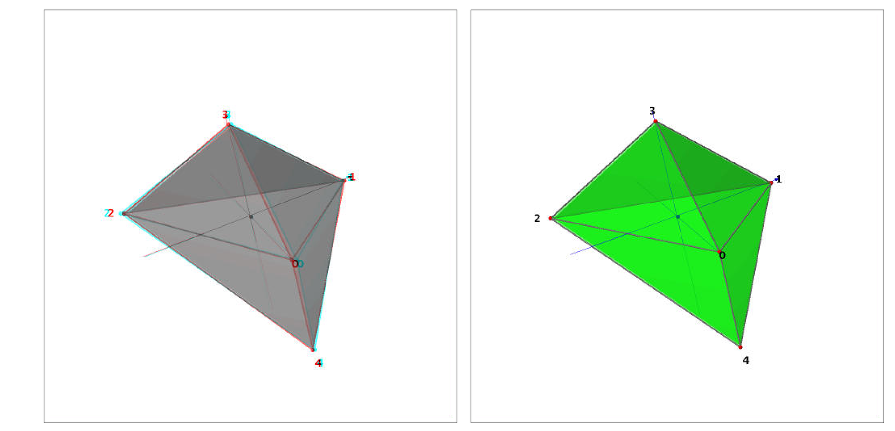

Figure 6: Reflexive polytope 6

Reflexive polytope 6 has vertices at $(1\,0\,0), (0\,1\,0), (0\,0\,1), (-1\,0\,1)$, and $(0\,-1\,-1)$. This dual top has two free parameters, and is convex when $a_2, a_4 \in \mathbb{Z}$, $a_2 \geq -1$ and $a_4 \geq -1$.

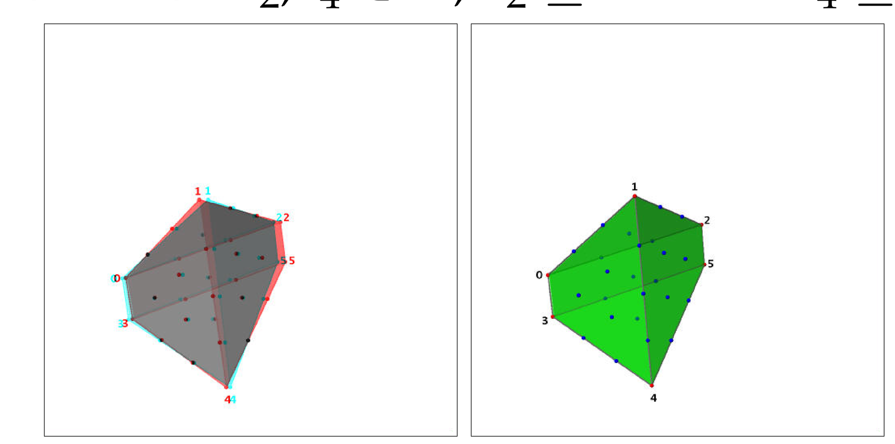

Figure 7: $x_4 = 0$ slice of the top corresponding to the constructed dual top

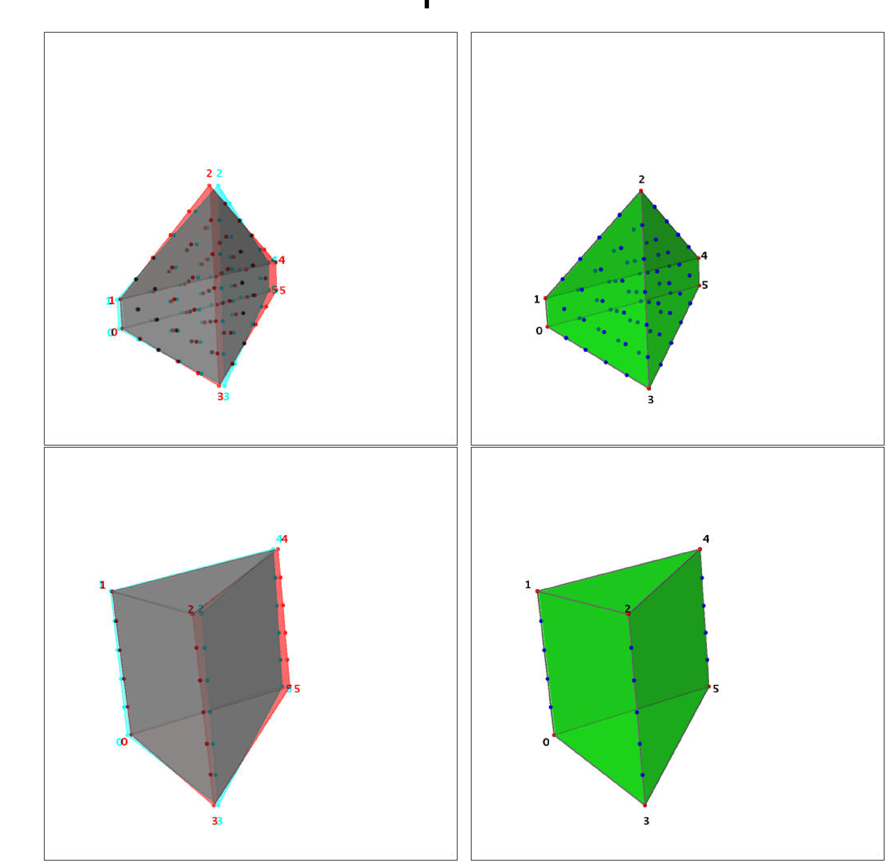

Figure 8: $x_4 = 1$ slice of the top corresponding to the constructed dual top, for $a_2 = 4, a_4 = 0$ (top) and $a_2 = 0, a_4 = 4$ (bottom)

References

[B-S] V. Bouchard, H. Skarke. 2003. ”Affine Kac-Moody algebras, CHL strings and the classification of tops”. *Adv. Theor. Math. Phys.* 7(2):205-232.

Undergraduate work performed at the University of Wisconsin- Eau Claire.

Support for this research provided by the Blugold Commitment Differential Tuition Grants Program.