Power indices taking into account agents' preferences

Fuad Aleskerov

No 898

WARWICK ECONOMIC RESEARCH PAPERS

DEPARTMENT OF ECONOMICS

THE UNIVERSITY OF WARWICK

Power indices taking into account agents' preferences

Fuad Aleskerov

State University 'Higher School of Economics'
and
Institute of Control Sciences
Russian Academy of Sciences

Moscow 101000 Russia

alesk@hse.ru, alesk@ipu.ru

A set of new power indices is introduced extending Banzhaf power index and allowing to take into account agents' preferences to coalesce. An axiomatic characterization of intensity functions representing a desire of agents to coalesce is given. A set of axioms for new power indices is presented and discussed. An example of use of these indices for Russian parliament is given.

1. Introduction

Power indices have become a very powerful instrument for study of electoral bodies and an institutional balance of power in these bodies [5-8, 11].

One of the main shortcomings mentioned almost in all publications on power indices is the fact that known indices do not take into account the preferences of agents [6,10]. Indeed, in construction of those indices, e.g., Shapley-Shubik or Banzhaf power indices [4,12], all agents are assumed to be able to coalesce. Moreover, none of those indices evaluates to which extent the agents are free in their wishes to create coalition, how intensive are the connections inside one or another coalition.

Consider an example. Let three parties A, B and C with 50, 49 and 1 sets, respectively, are presented in a parliament, and the voting rule is simple majority, i.e., 51 votes for.

Then winning coalitions are $A+B$, $A+C$, $A+B+C$ and A is pivotal in all coalitions, B is pivotal in the first coalition and C is pivotal in the second one. Banzhaf power index β for these parties is equal to[1]

[1] Banzhaf power index is evaluated as

$$\beta_i = \frac{b_i}{\sum_j b_j},$$

$$\beta(A) = 3/5, \quad \beta(B) = \beta(C) = 1/5 .$$

Assume now that parties *A* and *B* never coalesce in pairwise coalition, i.e., coalition *A*+*B* is impossible. Let us, however, assume that the coalition *A*+*B*+*C* can be implemented, i.e. in the presence of `moderator' *C* parties *A* and *B* can coalesce. Then the winning coalitions are *A*+*C* and *A*+*B*+*C*, and *A* is pivotal in both coalitions while *C* is in one; *B* is pivotal in none of the winning coalitions. In this case $\beta(A) = 2/3$, $\beta(B) = 0$ and $\beta(C) = 1/3$, i.e., although *B* has almost half of the seats in the parliament, its power is equal to 0.

If *A* and *B* never coalesce even in the presence of a moderator *C*, then the only winning coalition is *A*+*C*, in which both parties are pivotal. Then $\beta(A) = \beta(C) = 1/2$.

Such situations are met in real political systems. For instance, Russian Communist Party in the second parliament (1997-2000) had had about 35% of seats, however, its power during that period was always almost equal to 0 [1].

We introduce here two new types of indices based on the idea similar to Banzhaf power index, however, taking into account agents' preferences to coalesce.

In the first type the information is used about agents' preferences over other agents. These preferences are assumed to be linear orders. Since these preferences may not be symmetric, the desire of agent 1 to coalesce with agent 2 can be different than the desire of agent 2 to coalesce with agent 1. These indices take into account in a different way such asymmetry of preferences.

In the second type of power index the information about the intensity of preferences is taken into account as well, i.e., we extend the former type of power index to cardinal information about agents' preferences.

The structure of the paper is as follows. Section 2 gives main notions. In Section 3 we define and discuss `ordinal' power indices. In Section 4 cardinal indices are introduced. In Section 5 we evaluate power distribution of groups and factions in the Russian Parlament in 2000-2003 using some of new indices. Section 6 and 7 provides some axioms for the indices introduced.

Acknowledgments. I am very thankful to the participants of the Workshop on Constitutional and Scientific Quandaries (International Center on Economic Research, Turin,

---

b_i is the number of winning coalitions in which agent *i* is pivotal, i.e., if agent *i* expels from the coalition it becomes a loosing one [4].

Italy, June 20-23, 2005) and Workshop on Voting Power and Procedures (Warwick University, UK, July 20-23, 2005), and particularly to Professors Dennis Leach, Norman Schofield, Kenneth Shepsle, John Ferejohn, Moshe Machover and Federico Valenciano for helpful comments.

Mrs. Natalya Andryshina and Ms. Anna Sokolova helped me a lot to prepare the text. Mr. Vyacheslav Yakuba prepared software to evaluate the indices.

This work was supported by the grant of the State University `Higher School of Economics' (grant #285/10-04). Professors Vladimir Avtonomov, Yaroslav Kuz'minov, Andrey Yakovlev and Evgeniy Yasin had supported the project from its very beginning.

Finally, this work is partially supported by the grant #05-01-00188 of Russian Foundation of Basic Research.

All these help and support are gratefully acknowledged.

2. Main notions

The set of agents is denoted as *N*, $N = \{1,\ldots,n\}$, *n*>1. A coalition ω is the subset of *N*, $\omega \subseteq N$.

We consider the situation when the decision of a body is made by voting procedure; agents who do not vote `yes' vote against it, i.e., the abstention is not allowed.

Each agent has a predefined number of votes, $v_i > 0,\ i = 1,\ldots,n$. It is assumed that a quota *q* is predetermined and as a decision making rule the voting with quota is used, i.e., the decision is made if the number of votes for it is not less than *q*,

$$\sum_i v_i \geq q .$$

The model describes a voting by simple and qualified majority, voting with veto (as in the Security Council of UNO), etc.

A coalition ω is called winning if the sum of votes in the coalition is not less than *q*. An agent *i* is called pivotal in the coalition ω if the coalition $\omega \setminus \{i\}$ is a loosing one.

For such voting rule the set of all winning coalitions Ω possesses the following properties:

$$\phi \notin \Omega ,$$

$$N \in \Omega ,$$

$$\omega \in \Omega,\ \omega' \supseteq \omega \Rightarrow \omega' \in \Omega .$$

Sometime, one additional condition is applied as well

$$\omega \in \Omega \Rightarrow N \setminus \omega \notin \Omega,$$

which implies that $q \geq \lceil n/2 \rceil$, where $\lceil x \rceil$ is the smallest integer greater or equal to *x*.

To solve the problem stated above, two types of indices, ordinal and cardinal, are introduced. Both types are constructed on the following basis: the intensity of connection $f(i,\omega)$ of the agent with other members of ω is defined. Then for such agent *i* the value χ_i is evaluated as

$$\chi_i = \sum_{\omega} f(i,\omega),$$

i.e. the sum of intensities of connections of i over those coalitions ω in which *i* is pivotal. Naturally, other functions instead of summation can be considered.

Then the power indices are constructed as

$$\alpha(i) = \frac{\chi_i}{\sum_j \chi_j}.$$

The very idea of $\alpha(i)$ is the same as for Banzhaf index, with the difference that in Banzhaf index we evaluate the number of coalitions in which *i* is pivotal.

The main question now is how to construct the intensity functions $f(i,\omega)$. Below we give two ways how to construct those functions.

Each agent *i* is assumed to have a linear order[2] P_i revealing her preferences over other agents in the sense that *i* prefers to coalesce with agent *j* than with agent *k* if P_i contains the pair (*j, k*). Obviously, P_i is defined on the cartesian product $(N \setminus \{i\}) \times (N \setminus \{i\})$.

Since P_i is a linear order, the rank p_{ij} of the agent j in P_i can be defined. We assume that $p_{ij} = |N|$ -1 for the most preferable agent *j* in P_i.

The value p_{ij} shows how many agents less preferable than *j* are in P_i. For instance, if *N*={*A,B,C,D*} and $P_A : B \succ C \succ D$, then $P_{AB} = 3$, $P_{AC} = 2$ and $P_{AD} = 1$.

Using these ranks, one can constract different intensity functions.

---

[2] i.e. irreflexive, transitive and connected binary relation. We often denote it as $\succ$.

A second way of construction of $f(i,\omega)$ is based on the idea that the values p_{ij} of connection of i with j are predetermined somehow. In general, it is not assumed $p_{ij} = p_{ji}$. Then the intensity function can be constructed as above.

Below we give four different ways how to construct $f(i,\omega)$ in ordinal case and sixteen ways of construction of cardinal function $f(i,\omega)$.

3. Ordinal indices

For each coalition ω and each agent i construct now an intensity $f(i,\omega)$ of connections in this coalition. In other words, f is a function which maps $N \times \Omega$ $(=2^N \setminus \{\text{Ø}\})$ into R^1, $f : N \times \Omega \to R^1$. This very value $f(i,\omega)$ is evaluated using the ranks of members of coalition. Several different ways to evaluate f using different information about agents' preferences are provided:

a) Intensity of i's preferences. In this form only preferences of i's agent over other agents are evaluated, i.e.,

$$f^+(i,\omega) = \sum_{j\in\omega} \frac{p_{ij}}{|\omega|}$$

b) Intensity of preferences for i. In this case consider the sum of ranks of i given by other members of coalition ω

$$f^-(i,\omega) = \sum_{j\in\omega} \frac{p_{ji}}{|\omega|}$$

c) Average intensity with respect to i's agent

$$f(i,\omega) = \frac{f^+(i,\omega) + f^-(i,\omega)}{2}$$

d) Total positive average intensity. Consider any coalition ω of size $k \le n$. Without loss of generality we can put $\omega = \{1,\ldots,k\}$. Then consider for each i $f^+(i,\omega)$ and constract

$$f^+(\omega) = \frac{\sum_{i\in\omega} f^+(i,\omega)}{|\omega|},$$

e) Total negative average intensity is defined similarly by the formula

$$f^{-}(\omega) = \frac{\sum_{i \in \omega} f^{-}(i,\omega)}{|\omega|}$$

f) Total average intensity is defined as

$$f(\omega) = \frac{\sum_{i \in \omega} f(i,\omega)}{|\omega|}$$

It is worth emphasizing here that the intensities d) - f) do not depend on agent *i*, i.e., for any agent *i* in the following calculation of power indices we will assume that for any *i* in the coalition ω the corresponding intensity is the same.

Consider now several examples.

Example 1. Let *n*=3, *N*={*A*, *B*, *C*}, $\nu(A) = 33$, $\nu(B) = \nu(C) = 33$, *q*=50. Consider two preference profiles given in Tables 1 and 2.

Table 1

P_A	P_B	P_C
C	C	A
B	A	B

Table 2

P_A	P_B	P_C
B	C	A
C	A	B

For both preference profiles there are three winning coalitions in which agents are pivotal. These coalitions are *A*+*B*, *A*+*C* and *B*+*C*.

Let us calculate the functions *f* as above for each agent in each winning coalition.

The preferences from Tables 1 and 2 can be re-written in the matrix form as

$$\left\| p_{ij} \right\| = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{A} & \mathrm{B} & C \end{array} \\ \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix} \end{array}$$

$$\left\| p'_{ij} \right\| = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} \mathrm{A} & \mathrm{B} & C \end{array} \\ \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix} \end{array}$$

Now, for the profile given in Table 1 one can calculate the values of intensities a)-f) obtained by each agent i in each winning coalition ω. These values for the first preference profile is given in Table 3 and for the second one – in Table 4.

	$f^+(i,\omega)$			$f^-(i,\omega)$			$f(i,\omega)$			$f^+(\omega)$			$f^-(\omega)$			$f(\omega)$		
	A	B	C	A	B	C	A	B	C	A	B	C	A	B	C	A	B	C
A+B	½	½	–	½	½	–	½	½	–	½	½	–	½	½	–	½	½	–
A+C	1	–	1	1	–	1	1	–	1	1	–	1	1	–	1	1	–	1
B+C	–	1	½	–	½	1	–	¾	¾	–	¾	¾	–	¾	¾	–	¾	¾

Table 3. Intensity values for the first preference profile

	$f^+(i,\omega)$			$f^-(i,\omega)$			$f(i,\omega)$			$f^+(\omega)$			$f^-(\omega)$			$f(\omega)$		
	A	B	C	A	B	C	A	B	C	A	B	C	A	B	C	A	B	C
A+B	1	½	–	½	1	–	¾	¾	–	¾	¾	–	¾	¾	–	¾	¾	–
A+C	½	–	1	1	–	½	¾	–	¾	¾	–	¾	¾	–	¾	¾	–	¾
B+C	–	1	½	–	½	1	–	¾	¾	–	¾	¾	–	¾	¾	–	¾	¾

Table 4. Intensity values for the second preference profile

Using these intensity functions one can define now the corresponding power indices $\alpha(i)$. Let i be a pivotal agent in a winning coalition ω. Denote as χ_i the number equal to the value of the intensity function for a given coalition ω and agent *i*. Then the power index is defined as follows

$$\alpha(i) = \frac{\sum\limits_{\substack{\omega, \\ i \text{ is pivotal in } \omega}} \chi_i}{\sum\limits_{j \in N} \sum\limits_{\substack{\omega, \\ j \text{ is pivotal in } \omega}} \chi_j}$$

As we already mentioned this index is similar to the Banzhaf index. The difference is that χ_i in the Banzhaf index is equal to 1, in the case under study χ_i represents some intensity value.

The indices $\alpha(i)$ will be denoted as $\alpha_1(i), \ldots, \alpha_6(i)$.

Let us evaluate now the values $\alpha_1(\cdot) - \alpha_6(\cdot)$ for all agents for the preference profile from Table 1.

The agent *A* (as well as agents *B* and *C*) is pivotal in two coalitions; the sum of the values $f^+(i,\omega)$ for each *i* is equal to 3/2. Then

$$\alpha_1(A) = \frac{3/2}{3/2 + 3/2 + 3/2} = \frac{1}{3} = \alpha_1(B) = \alpha_1(C).$$

The value $\alpha_2(\cdot)$ is evaluated differently. The sum of values $f^-(i,\omega)$ from Table 3 for all i and ω is equal to 9/2. However, for A $\sum\limits_{\omega} f(A,\omega) = 3/2$, $\sum\limits_{\omega} f(B,\omega) = 1$ and $\sum\limits_{\omega} f(C,\omega) = 2$. Then $\alpha_2(A) = \frac{3}{9} = \frac{1}{3}$; $\alpha_2(B) = \frac{2}{9}$ and $\alpha_2(C) = \frac{4}{9}$.

The values of the indices $\alpha_1 - \alpha_6$ for both preference profiles are given in Table 5 as well as the values of Banzhaf index β

Consider now another example.

Example 2. Let N={*A, B, C, D, E*}, each agent has one vote, *q*=3 and the preferences of agents are given in Table 6.

The values of indices $\alpha_2 - \alpha_4$ are given in Table 7.

Note that α_1 is equal to the Banzahalf index, which for this case gives $\forall i \in N \;\; \beta(i) = 1/5$.

	First profile (Table 1)			Second profile (Table 2)		
	A	B	C	A	B	C
α_1	1/3	1/3	1/3	1/3	1/3	1/3
α_2	1/3	2/9	4/9	1/3	1/3	1/3
α_3	1/3	5/18	7/18	1/3	1/3	1/3
α_4	1/3	5/18	7/18	1/3	1/3	1/3
α_5	1/3	5/18	7/18	1/3	1/3	1/3
α_6	1/3	5/18	7/18	1/3	1/3	1/3
β	1/3	1/3	1/3	1/3	1/3	1/3

Table 5. Power indices values

P_A	P_B	P_C	P_D	P_E	rank
B	A	D	A	B	4
C	C	A	B	A	3
D	D	B	C	D	2
E	E	E	E	C	1

Table 6. Preferences of agents for N={*A, B, C, D, E*}.

	A	B	C	D	E
α_2	0.28	0.26	0.18	0.2	0.08
α_3	0.24	0.23	0.19	0.2	0.14
α_4	0.22	0.21	0.2	0.2	0.17

Table 7. The values of the indices $\alpha_2 - \alpha_4$ for Example 2.

Example 3. Consider the case when 3 parties A, B and C have 50, 49 and 1 seats, respectively. Assume that decision making rule is simple majority, i.e. 51 votes. Then the winning coalitions are A+B, A+C and A+B+C. Note that A is pivotal in all three coalitions, B and C are pivotal in one coalition each. Then $\beta(A) = 3/5$, $\beta(B) = \beta(C) = 1/5$.

Consider now the case with the preferences of agents given below: $P_A : C \succ B$; $P_A : C \succ B$ and $P_C : A \succ B$.

Then the values of α_1 and α_2 (constructed by $f^+(i,\omega)$ and $f^-(i,\omega)$ are as follows

$$\alpha_1(A) = 5/8, \quad \alpha_1(B) = 1/5, \quad \alpha_1(C) = 1/8,$$

and the values of α_2 are equal to α_1.

Consider another preference profile: $P_A : C \succ B$, $P_A : C \succ B$; $P_C : A \succ B$, i.e., only agent C changes her preferences. Then one can easily evaluate that $\alpha_1(A) = 5/7, \quad \alpha_1(B) = \alpha_1(C) = 1/7$; $\alpha_2(A) = 10/19$, $\alpha_2(B) = 3/19$, $\alpha_2(C) = 6/19$.

4. Cardinal indices

Assume now that the desire of party i to coalesce with party j is given as real number p_{ij}, $\sum_j p_{ij} = 1,\ i, j = 1,\ldots,n$. In general, it is not assumed that $p_{ij} = p_{ji}$.

One can call the value p_{ij} as an intensity of connection of i with j. It may be interpreted as, for instance, a probability for i to form a coalition with j.

We define now several intensity functions

a) average intensity of i is connection with other members of coalition ω

$$f^{+}(i,\omega)=\frac{\sum_{j\in\omega}p_{ij}}{|\omega|};$$

b) average intensity of connection of other members of coalition ω with i

$$f^{-}(i,\omega)=\frac{\sum_{j\in\omega}p_{ji}}{|\omega|};$$

c) average intensity for i

$$f(i,\omega)=\frac{1}{2}(f^{+}(i,\omega)+f^{-}(i,\omega));$$

d) average positive intensity in ω

$$f^{+}(\omega)=\frac{\sum_{i\in\omega}f^{+}(i,\omega)}{|\omega|},$$

e) average negative intensity in ω

$$f^{-}(\omega)=\frac{\sum_{i\in\omega}f^{-}(i,\omega)}{|\omega|},$$

f) average intensity in ω

$$f(\omega)=\frac{\sum_{i\in\omega}f(i,\omega)}{|\omega|},$$

In contrast to ordinal case now we can introduce several new intensity functions:

g) minimal intensity of i's connections

$$f^{+}_{\min}(i,\omega)=\min_{j}p_{ij};$$

h) maximal intensity of i's connections

$$f^{+}_{\max}(i,\omega)=\max_{j}p_{ij};$$

i) maximal fluctuation of i's connections

$$f_{mf}(i,\omega)=\frac{1}{2}(\min_{j}p_{ij}+\max_{j}p_{ij})$$

j) minimal intensity of connections of other agents in ω with i

$$f^{-}_{\min}(i,\omega)=\min_{j}p_{ji};$$

k) maximal intensity of connections of other agents in ω with i

$$f^{-}_{\max}(i,\omega) = \max_{j} p_{ji}$$

l) s-mean intensity of i's connections with other agents in ω

$$f^{+}_{sm}(i,\omega) = \frac{1}{|\omega|}\sqrt[s]{\sum_{j} p^{s}_{ij}} \;;$$

m) s-mean intensity of connections of other agents in ω with i

$$f^{+}_{sm}(i,\omega) = \frac{1}{|\omega|}\sqrt[s]{\sum_{j} p^{s}_{ji}} \;;$$

n) max min intensity

$$f_{\max\min}(\omega) = \max_{i}\min_{j} p_{ij} \;;$$

o) min max intensity

$$f_{\min\max}(\omega) = \min_{i}\max_{j} p_{ji} \;;$$

p) maximal fluctuation

$$f_{mf}(\omega) = \frac{1}{2}(f_{\max\min}(\omega) + f_{\min\max}(\omega))$$

Note that the intensity functions in the cases d)–f), n)–p) do not depend on agent herself but only on coalition ω.

Now the corresponding power indices can be define as above, i.e.,

$$\alpha^{\text{card}}(i) = \frac{\sum\limits_{\substack{\omega \text{ is winning,} \\ i \text{ is pivotal in } \omega}} \chi_i}{\sum\limits_{j\in N} \sum\limits_{\substack{\omega \text{ is winning,} \\ j \text{ is pivotal in } \omega}} \chi_i(\omega)},$$

where χ is one of the above intensity functions.

Example 4. Let N={A,B,C,D}, each voter has only one vote, the quota is equal to q=3, and the matrix $\left\| p_{ij} \right\|$ is given in Table 8. In Table 9 the power indices are given for the cases a), b), e), h).

	A	B	C	D
A		0.7	0.2	0.1
B	0.3		0.5	0.2
C	0.1	0.7		0.2
D	0.7	0.2	0.1	

Table 8. Matrix $\|p_{ij}\|$ for Example 3

	A	B	C	D
$\alpha_{a)}$	0.25	0.25	0.25	0.25
$\alpha_{b)}$	0.27	0.40	0.20	0.13
$\alpha_{e)}$	0.25	0.27	0.24	0.23
$\alpha_{h)}$	0.25	0.25	0.25	0.25

Table 9. Some cardinal indices for Example 3

5. **Evaluation for Russian Parliament**

We will study now a distribution of power among factions in the third Russian Parliament (1999-2003) using these new indices.

The matrix $\|p_{ij}\|$ is constructed using the consistency index; the latter (the index of consistency of positions of two groups) is constructed as

$$c(q_1, q_2) = 1 - \frac{|q_1 - q_2|}{\max(q_1, 1 - q_1, q_2, 1 - q_2)}.$$

where q_1 and q_2 be the share of "ay" votes in two groups of MPs [1].

We consider the value of consistency index as the value of intensity of connections between agents *i* and *j*. Then we are in cardinal framework, and one can use one of the indices introduced in the previous section.

On Fig. 1 the values of $\alpha_{a)}$ index are given for the Russian Parliament from 2000 to 2003 on the monthly basis. It can be readily seen that index α gives lower values for Communist Party (sometimes up to 3%) and higher values for Edinstvo (up to 1%). It is interesting to note that Liberal-Democrats (Jirinovski's Party) had had almost equal values by both indices, which corresponds to the well-known flexibility of that party position.

Let us note that different ways to use the index α are possible. For instance, following the approach from [1], we may assume that if the consistency value for two factions is less than some threshold value δ, then parties do not coalesce, i.e., $p_{ij} = 0$.

5. Axiomatic construction of a cardinal intensity function

Now we will try to axiomatize a construction of cardinal intensity function.

First, we define an intensity function depending on intensities p_{ij} of connections of *i* with other members of coalition ω, i.e., if $\omega = \{1,\dots,m\}$, $m \le n$,

$$f(i,\omega) = f_i(p_{11},\dots,p_{1m},p_{21},\dots,p_{2m},\dots,p_{i1},\dots,p_{im},\dots,p_{mm})$$

As it is seen, the intensity function for *i* depends not only of *i*'s connections with other members of coalition, but depends also of connections of other members among themselves.

However, we will restrict this function in a way which is similar to independence of irrelevant alternatives [3]: $f(i,\omega)$ will depend on connections of agent *i* with other members of coalition ω only, i.e.,

$$f(i,\omega) = f_i(p_{i1},\dots,p_{im}).$$

For the sake of simplicity we put $p_{ij}^{\omega} \ge 0$ for all i, j and $\forall i$ $\sum_{j\in\omega} p_{ij}^{\omega} = 1$.

I would like to emphasize that in this formulation the sum of p_{ij}^{ω} is equal to 1 in each ω, i.e., now connections are defined by $2^N - 1$ matrices $\left\| p_{ij}^{\omega} \right\|$ for each coalition ω.

Consider several axioms which reasonable function $f(i,\omega)$ should satisfy to.

Axiom 1. For any m – tuple of values $(p_{i1},\ldots,p_{im})$ there exist a function $f(i,\omega)$ such that $0 \le f(i,\omega) \le 1$, f is continious differentiable function of each of its arguments.

Axiom 2. If $p_{ij} = 0$ for any j, then $f(i,\omega)$ =0.

Axiom 3. (Monotonicity). A value of $f(i,\omega)$ increases if any value p_{ij} increases, and a value of $f(i,\omega)$ decreases if p_{ij} decreases. Moreover, equal changes in intensities p_{ij} lead to equal changes of $f(i,\omega)$. This means that

$$\frac{\partial f_i}{\partial p_{ij}} = \mu_i \quad \text{for any} \quad j$$

and

$$\frac{\partial f_i}{\partial p_{lj}} = 0 \quad \text{for any} \quad l \neq i\,.$$

Then the following theorem holds

Theorem. An intensity function $f(i,\omega)$ satisfies Axioms 1–3 iff it is represented in the form

$$f(i,\omega) = \frac{\sum_j p_{ij}}{|\omega|}\,.$$

Proof is similar to the proof of the theorem from [9] given in the framework of probabilistic social choice and hence is omitted.

An axiomatic characterization of other types of intensity functions is still an open problem.

7. Axioms for power indices

We introduce several axioms, which any reasonable power index should satisfy to.

First, we call a voting situation a four-tuple $\{N,q,v,\vec{P}\}$, where N is a set of agents, $|N| = n, n > 1$, q is a quota, $v = (v_1,\ldots,v_n)$ is a set of votes which agents possess, $\vec{P}$ is a preference profile, where each agent $i \in N$ has a preference (linear order) P_i over $N \setminus \{i\}$ or preference matrix $\|p_{ij}\|$.

Axiom 1. Under a given quota rule for any agent $i \in N$ there exists an intensity profile $\vec{P}$ such that $\alpha(i) > 0$.

In words, for no agent it is known in advance, independently of agents' preferences, that her power is equal to 0.

Axiom 2. Consider two voting situations $\{N,q,v,\vec{P}\}$ and $\{N,q,v',\vec{P}\}$. Let $\exists A \in N$ s.t. $v'(A) \geq v(A)$, and $\forall B \in N$, $B \neq A$, $v'(B) = v(B)$. Then $\alpha'(A) \geq \alpha(A)$.

Assume that for a given distribution of votes and a given preference profile we evaluate power distribution among agents. Then we increase the number of votes for a given agent A, keeping the votes of other agents as before. Then Axiom 2 states that voting power of A in new situation should be not less than before.

Axiom 3. (Symmetry) Let η be a one-to-one correspondence of *N* to *N*. Then

$$\eta(\alpha_1,\ldots,\alpha_n) = (\alpha_{\eta(1)},\ldots,\alpha_{\eta(n)}) .$$

Axiom 3 states that power of agents does not depend of their names, i.e., the procedure of evaluation of power distribution must treat agents in a similar way.

Axiom 4. Let $i \in N$ be pivotal in no winning coalition ω. Then $\alpha(i) = 0$.

It is usual axiom in voting power models (in fact, in game – theoretic models – see [12]): a dummy player has power equal 1 to 0.

Axiom 5'. <u>First Monotonicity Axiom (FMA).</u> Consider two voting situations $[N,q,v,\vec{P}]$ and $[N,q,v,\vec{P}']$. Let for some *i* and any $k \neq i$ $P_k = P_k^{'}$ holds. Let additionally for some *j* $p_{ij}^{'} > p_{ij}$ holds. Then $\alpha'(j) \geq \alpha(j)$.

This axiom can be explained in a simple way: all preferences except *i*'s are the same in two profiles; in *i*'th preference the evaluation of *j* is higher in new profile than in the old one. Then the power of *j* should not be less in new voting situation (with $\vec{P}'$).

Axiom 5". <u>Second Monotonicity Axiom (SMA).</u> Consider two voting situations $[N,q,v,\vec{P}]$ and $[N,q,v,\vec{P}']$. Let for two agents *i* and *j* $\alpha(i) \geq \alpha(j)$ holds, where $\alpha(i)$ is the voting power of *i* in the first voting situation. Let $\vec{P}'$ is such that for any $k \neq l$ $P_k = P_k^{'}$ holds, and in the preferences of *l*'s agent

$$p_{li}^{'} - p_{lj}^{'} > p_{li} - p_{lj}$$

holds.

Then $\alpha'(i) \geq \alpha'(j)$ (weak version of SMA) or $\alpha'(i) > \alpha(j)$ (strong version of SMA), where $\alpha'(i)$ is the voting power of *i* with respect to second voting situation.

In words, let power of i is not less than the power of j with respect to first voting situation. Let $\vec{P}'$ is such that for any agent but ℓ new preferences of agents coincide with old preferences, and in l's preference the relative position of i with respect to j is higher in $p_l^{'}$ than in p_l. Then the voting power of i should be not less than that of j in new voting situation (weak version) or even must be greater than that of j (strong version).

Axiom 6. Let P' be an intensity matrix such that $p_{ij}^{'} = kp_{ij}$ for every $i, j = 1, \dots, n$. Then $\alpha'(i) = \alpha(i)$, where α' is the power vector obtained from P'.

Axiom 6 deals with cardinal power indices. It says that voting power of agents does not change under the transformation of scale of intensities in the form

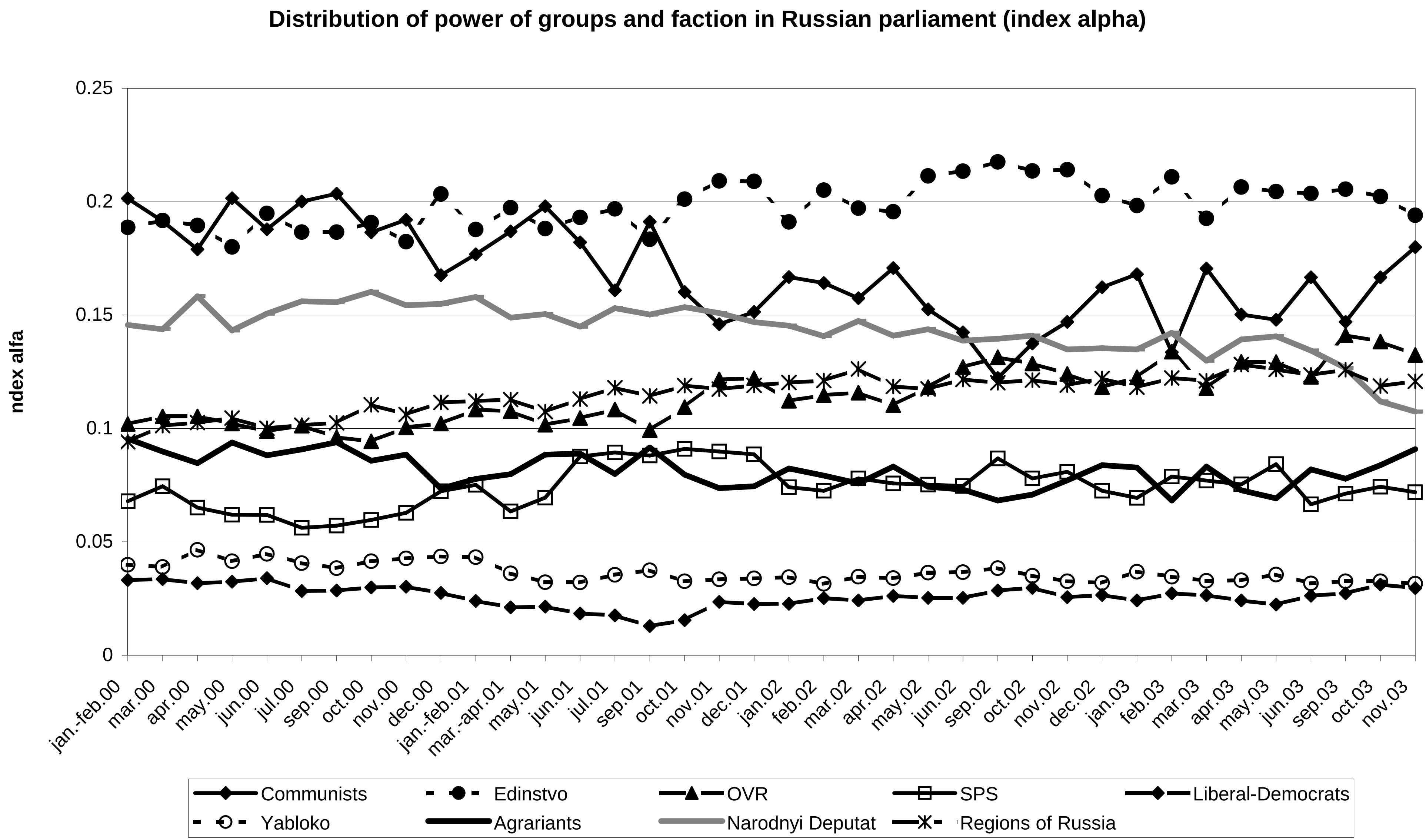
$$p_{ij}^{'} = kp_{ij,}$$

i.e., when intensities multiply to the same constant k.

It is possible to formulate axioms similar to those from Section 5 and prove the theorem similar to the given above but for α-indices. However, it will be interesting to analize how the axioms from this Section provide an axiomatic characterization of α indices.

References

1. Aleskerov F., Blagoveschenskiy N., Satarov G., Sokolova A., Yakuba V. "Evaluation of power of groups and factions in the Russian parliament (1994-2003)", WP7/2003/01, Moscow: State University "High School of Economics", 2003 (in Russian)
2. Aleskerov F., Blagoveschensky N., Konstantinov M., Satarov G., Yakuba V. “A balancedness of the 3d State Duma of Russian Federation evaluated by the use of claster-analysis”, WP7/2005/04, Moscow: State University "High School of Economics", 2005 (in Russian)
3. Arrow K.J. “Social Choice and Individual Values, – 2nd ed., New Haven: Yale University Press, 1963.
4. Banzhaf, J. F., “Weighted Voting Doesn't Work: A Mathematical Analysis”, Rutgers Law Review, 1965, v.19, 317-343.
5. Brams S. “Game Theory and Politics”, The Free Press, New York, 1975
6. Felsenthal D., Machover M "The Measurment of Voting Power: Theory and Practices, Problems and Paradoxes", Edgar Elgar Publishing House, 1998
7. Grofman, B. and H. Scarrow “Ianucci and Its Aftermath: The Application of Banzhaf index to Weighted Voting in the State of New York”, Brams, S., Schotter A. and G Schwodiatuer (eds.) Applied Game Theory, 1979
8. Heme, K. and H Nurmi “A Priori Distribution of Power in the EU Council of Ministers and the European Parliament”, Scandinavian Journal of Po1ilitical Studies, 1993, v.16, 269-284
9. Intriligator M.D. “A Probabilistic Model of Social Choice”, The Review of Economic Studies, v.XL(4), 1973, 553-560.
10. Laruelle A., Valenciano F. “Shapley-Shubik and Banzhaf indices revisited”, Mathematics of Operations Research, v.26, 2001, 89-104.
11. Leech, D. “Voting power in the governance of the International Monetary Fund”, Annals of Operations Research, 2002, v.109, 375-397.
12. Shapley, L.S., and M. Shubik “A method for Evaluting the Distribution of Power in a Committee System”, American Political Science Review, 1954, v. 48, 787-792.


Distribution of power of groups and faction in Russian parliament (index alpha)
ndex alfa
0.25
0.2
0.15
0.1
0.05
0
jan.-feb.00
mar.00
apr.00
may.00
jun.00
jul.00
sep.00
oct.00
nov.00
dec.00
jan.-feb.01
mar.-apr.01
may.01
jun.01
jul.01
sep.01
oct.01
nov.01
dec.01
jan.02
feb.02
mar.02
apr.02
may.02
jun.02
sep.02
oct.02
nov.02
dec.02
jan.03
feb.03
mar.03
apr.03
may.03
jun.03
sep.03
oct.03
nov.03
Communists
Edinstvo
OVR
SPS
Liberal-Democrats
Yabloko
Agrariants
Narodnyi Deputat
Regions of Russia