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Abstract

We give intuitive Samuelson conditions for a very general class of economies. Smoothness, monotonicity, transitivity and completeness are not required. We provide necessary and sufficient conditions for all Pareto efficient allocations, including those on the boundary. We also prove that if all agents have a cheaper point, the supporting prices fully decentralize the allocation. Finally, we show first and second welfare theorems as corollaries to the characterization of efficient allocations.

1. Introduction

Samuelson (1954, 1955) gave the first modern study of economies with public goods. One of his main results was calculus-based conditions for Pareto efficiency. These "Samuelson conditions" have since become one of the fundamental tools for understanding public goods economies. However, his work has several important limitations. In particular, he did not deal with the issue of corner allocations, in which at least one type of good is not consumed at all by at least one agent. Given that this is probably the typical rather than the exceptional case in real life, his omission is not trivial. Unless we can characterize corner allocations as well, we must doubt the practical relevance of the studies based on Samuelson conditions.

Later, economists assumed that the most obvious modification of Samuelson's efficiency conditions would be the correct ones for dealing with corners. However, as Campbell and Truchon (1988) point out in an important paper, there are cases where some efficient allocations violate the Samuelson conditions, even as modified. Campbell and Truchon conclude that the Samuelson conditions miss some efficient allocations, and they provide a different specification of the Samuelson conditions which they claim are necessary and sufficient for efficiency in economies with one private good and a finite number of public goods. They assume differentiability of the utility and cost functions, convexity of preferences and cost, and monotonicity of preferences.

Unfortunately, the analysis of Campbell and Truchon is limited by their assumption that there is only one private good and their need for differentiability. The first assumption reduces the relevance of their contribution to an essentially partial equilibrium domain. Requiring differentiability significantly reduces the class of economies for which their analysis may be applied. Further, their proof of sufficiency contains an oversight (proof of Lemma 1, page 247), which we explain in the conclusion. These observations motivate an approach to the problem using convex analysis, in the standard fashion established by Arrow (1951) for economies with private goods only.

Such an analysis was offered by Foley (1970) in the course of formalizing the general notion of Lindahl equilibrium. However, he requires in his definition that allocations be in the relative interior of the private goods subspace of the consumption set of each agent. Thus, corner allocations are not dealt with by Foley, either. Khan and Vohra (1987) generalize Foley (1970) to allow general preferences and nonconvexities, but their aim is to present the second welfare theorem and they do not examine the refinements needed to deal with boundary allocations.

In this paper we provide efficiency conditions for economies with a finite number of private and public goods, without assuming differentiability. We so not require that commodities be goods. This allows us to consider important real world cases such as the one in which a public project (a garbage incinerator, for example) benefits some agents while imposing costs on others. We require only that agents are not locally satiated. In addition, we do not assume that the preference relations are complete or transitive. Our analysis deals with corner and interior allocations in a unified way. Unlike Campbell and Truchon (1988), we do not need to appeal to the Karush-Kuhn-Tucker theorem, and our proofs are simple and geometric in nature. We develop the most general from of the Samuelson conditions in a simple and operational form, and we further show the existence of fully (Lindahl) supporting prices at any Pareto efficient allocation, for all agents who are allowed a cheaper point by the Samuelson prices corresponding to the allocation. As corollaries to these efficiency conditions we show first and second welfare theorems.

2. The Model

We consider an economy with L private goods and M public goods, I individual consumers, and F firms. We use the convention $\mathcal{I} \equiv \{1, \ldots, I\}$, and similarly for $\mathcal{L}, \mathcal{M}$ and $\mathcal{F}$. Superscripts are used to represent firms and consumers and subscripts to represent goods.

Each agent $i \in \mathcal{I}$ is characterized by an endowment $\omega^i \in \mathbb{R}^L_+$, and a preference relation $\succ^i$ over the consumption set $C^i \equiv \mathbb{R}^{L+M}_+$. A typical consumption bundle will be written (x, y) where x is a bundle of private goods, and y is a bundle of public goods. We remark that assuming the consumption set to be the nonnegative orthant is not less general than Campbell and Truchon's introduction of a nonnegative lower bound for the consumption of the private good by each agent, since we can always translate the preferences in order to make this lower bound zero. It is also possible to generalize the results in this paper to bounded below, convex consumption sets at the cost of complicating the proofs.

We make the following assumptions on $\succ^i$ for all $i \in \mathcal{I}$.

A1) $\succ^i$ is irreflexive.

A2) $\succ^i$ is continuous (the strict upper and lower preferred sets are open).

A3) If $(x, y) \succ^i (\tilde{x}, \tilde{y})$, then for all $\lambda \in (0, 1)$, $\lambda(x, y) + (1 - \lambda)(\tilde{x}, \tilde{y}) \succ^i (\tilde{x}, \tilde{y})$. (Weak convexity)

A4) For all $(x, y) \in C^i$ and all $\epsilon > 0$ there exists $(\tilde{x}, \tilde{y}) \in C^i$ such that $\| (x, y) - (\tilde{x}, \tilde{y}) \| < \epsilon$ and $(\tilde{x}, \tilde{y}) \succ^i (x, y)$. [1] (Local nonsatiation)

We normalize supporting prices to sum to one, but do not assume that prices are positive:

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[1] The three kinds of vector inequalities are represented by $\geq, >$, and $\gg$.

$$\Pi \equiv \left\{ (p, q^1, \ldots, q^I) \in \mathbb{R}^{L+IM} \mid \sum_\ell p_\ell + \sum_m \sum_i q^i_m = 1 \right\}.$$

Define the *marginal rate of substitution correspondence for consumer i*, $\text{MRS}^i : C^i \to\to \Pi$, by:

$$\text{MRS}^i(x, y) \equiv$$

$$\left\{ (p, q) \in \Pi \mid (p, q^i) \cdot (x, y) < (p, q^i) \cdot (\tilde{x}, \tilde{y}) \; \forall (\tilde{x}, \tilde{y}) \in C^i \text{ s.t. } (\tilde{x}, \tilde{y}) \succ^i (x, y) \right\}.$$

Define also the *weak marginal rate of substitution correspondence for consumer i*, $\text{WMRS}^i : C^i \to\to \Pi$, by:

$$\text{WMRS}^i(x, y) \equiv$$

$$\left\{ (p, q) \in \Pi \mid (p, q^i) \cdot (x, y) \leq (p, q^i) \cdot (\tilde{x}, \tilde{y}) \; \forall (\tilde{x}, \tilde{y}) \in C^i \text{ s.t. } (\tilde{x}, \tilde{y}) \succ^i (x, y) \right\}.$$

Note that the marginal rate of substitution set $\text{MRS}^i(x, y)$ is always a subset of the weak marginal rate of substitution set $\text{WMRS}^i(x, y)$, and $\text{MRS}^i(x, y)$ can be empty. The WMRS correspondence is never empty-valued, as we indicate in our proof. Also note that if $(p, q) \in \text{MRS}^i(x, y)$ and the agent has income $(p, q^i) \cdot (x, y)$, then (x, y) is a preference maximizing choice over the budget set. On the other hand if $(p, q) \in \text{WMRS}^i(x, y)$ then we are only guaranteed that (x, y) minimizes expenditure over the set of consumption bundles that are not inferior to (x, y).

[Figure 1 here]

In the example depicted in Figure 1, the agent's indifference curves intersect the public good axis with an vertical slope, and terminate at their intersection with this axis. Otherwise, the preferences are standard, satisfying all of the assumptions A and furthermore all other assumptions commonly made on preferences. At every point on the public good axis, the weak marginal rate of substitution correspondence has a singleton value of $(1,0)$.[2] In other words, the vertical axis supports the preferred set. Since the WMRS correspondence contains the MRS correspondence, and the unique line of support intersects the preferred set, the marginal rate of substitution correspondence is empty-valued.

We represent each firm $f \in \mathcal{F}$ by a production set $P^f \subset \mathbb{R}^L \times \mathbb{R}^M_+$. A typical production plan will be written (z,y), where z is a net output vector of private goods and y is a the output vector of public goods.

Define the *marginal rate of transformation correspondence for* P^f, MRT^f : $P^f \to\to \Pi$, as follows:

$$\mathrm{MRT}^f(z,y) \equiv$$
$$\left\{(p,q) \in \Pi \mid (p, \sum_i q^i) \cdot (z^f, y^f) \geq (p, \sum_i q^i) \cdot (\tilde{z}^f, \tilde{y}^f) \; \forall \, (\tilde{z}^f, \tilde{y}^f) \in P^f \right\}.$$

The *comprehensive hull* of a set in $\mathbb{R}^L \times \mathbb{R}^M_+$ is defined as follows:

$$\mathrm{comp}(Z) \equiv \left\{(z,y) \in \mathbb{R}^L \times \mathbb{R}^M_+ \mid \exists (\tilde{z}, \tilde{y}) \in Z \text{ s.t. } (z,y) \leq (\tilde{z}, \tilde{y})\right\}.$$

For all $f \in \mathcal{F}$ we assume:

B1) P^f is a nonempty, closed set.

B2) P^f is a convex set.

B3) $P^f = \mathrm{comp}(P^f)$ (Free disposal).

We define the *global production set* in the usual way:

$$P \equiv \left\{ (z,y) \in \mathbb{R}^L \times \mathbb{R}^M_+ \;\middle|\; (z,y) = \sum_f (z^f, y^f) \text{ and } (z^f, y^f) \in P^f \; \forall \, f \in \mathcal{F} \right\},$$

and we define the *aggregate marginal rate of transformation correspondence* MRT : $P \to\to \Pi$ by

$$\mathrm{MRT}(z,y) \equiv \left\{(p,q) \in \Pi \mid (p, \sum_i q^i) \cdot (z,y) \geq (p, \sum_i q^i) \cdot (\tilde{z}, \tilde{y}) \; \forall \, (\tilde{z}, \tilde{y}) \in P \right\}.$$

[2] Properly speaking, we should have indicated all the elements of the supporting vector here, in accordance with the definition of MRS and WMRS. However, in all discussions of examples we only indicate the components relating to the goods consumed by the agent in question, to enhance clarity.

We make the additional assumption:

B4) P is closed.

Notice that P inherits convexity and comprehensiveness from the individual P^f sets.

An allocation is list $a = ((x^1, y^1), \ldots, (x^I, y^I), (z^1, y^1) \ldots (z^F, y^F)) \in C^1 \times \cdots \times C^I \times P^1 \times \cdots \times P^F$. Let A denote the set of feasible allocations:

$$A \equiv \left\{ a \in C^1 \times \cdots \times C^I \times P^1 \times \cdots \times P^F \right|$$

$$\left. \sum_f z^f = \sum_i (\omega^i - x^i) \text{ and } \sum_f y^f = y^i \, \forall \, i \in \mathcal{I} \right\}.$$

The set of *Pareto efficient allocations* is defined as

$$PE \equiv$$

$$\left\{ a \in A \mid \nexists \, \hat{a} \in A \text{ s.t. } \forall i \in \mathcal{I}, (x^i, y^i) \not\succ^i (\hat{x}^i, \hat{y}^i) \text{ and } \exists \, j \in \mathcal{I} \text{ s.t. } (\hat{x}^j, \hat{y}^j) \succ^j (x^j, y^j) \right\}.$$

Let Δ^{I-1} denote the $I - 1$ dimensional simplex:

$$\Delta^{I-1} \equiv \left\{ \theta \in \mathbb{R}^I \mid \sum_i \theta^i = 1, \text{ and } \theta^i \geq 0 \, \forall \, i \in \mathcal{I} \right\}.$$

We denote a profit share system for a private ownership economy by $\theta = (\theta^1, \ldots \theta^f, \ldots \theta^F) \in \Delta^{I-1} \times \ldots \times \Delta^{I-1} \equiv \Theta$ where $\theta^{i,f}$ is interpreted as consumer i's share of the profits of firm f.

An allocation and price vector $(a, p, q) \in A \times \Pi$ is said to be a *Lindahl equilibrium relative to the endowment* $\omega \in \mathbb{R}^{I \times L}$ *and profit shares* $\theta \in \Theta$ if and only if:

a. for all $f \in \mathcal{F}$, $(p, \sum_i q^i) \in \mathrm{MRT}^f(z^f, y^f)$.

b. for all $i \in \mathcal{I}$, $(p, q) \in \mathrm{MRS}^i(x^i, y^1)$ and $(p, q^i) \cdot (x^i, y) = p \cdot \omega^i + \sum_f \theta^{i,f}(p, \sum_i q) \cdot (z^f, y^f)$.

Note that given the definitions of MRS^i and MRT^f, and the fact that local nonsatiation implies that each agent will exhaust his income, these are equivalent to profit and preference maximization, respectively. Feasibility is already required by the definition of an allocation. Define the Lindahl equilibrium allocation correspondence $LE : \mathbb{R}^{I \times L} \times \Theta \rightarrow\rightarrow A$ as follows:

$$LE(\omega, \theta) \equiv$$

$$\{a \in A \mid \text{ for some } (p, q) \in \Pi, (a, p, q) \text{ is a Lindahl equilibrium for } \omega \text{ and } \theta\}.$$

3. Results

We start with a simple statement of our main results. The following two conditions are necessary for an allocation $a \in A$ to be Pareto efficient:

There exists a price vector $(p, q^1, \ldots, q^n) \in \Pi$ *such that*

a. *for all* $f \in \mathcal{F}$ $(p, \sum_{i=1}^{I} q^i) \in \mathrm{MRT}^f(z^f, y^f)$,

b. *for all* $i \in \mathcal{I}$, $(p, q) \in \mathrm{WMRS}^i(x^i, y)$.

Alternatively, for $a \in A$, if for all $i \in \mathcal{I}$ there is a cheaper point than (x^i, y) in C^i (this would be true for example if every agent's consumption bundle was in the interior of his consumption set), the following conditions are necessary for a to be a Pareto efficient allocation:

There exists a price vector $(p, q^1, \ldots, q^n) \in \Pi$ *such that*

a. *for all* $f \in \mathcal{F}$ $(p, \sum_{i=1}^{I} q^i) \in \mathrm{MRT}^f(z^f, y^f)$,

b. *for all* $i \in \mathcal{I}$, $(p, q) \in \mathrm{MRS}^i(x^i, y)$.

Finally, in both cases the following two conditions are sufficient for an allocation $a \in A$ to be Pareto efficient :

There exists a price vector $(p, q^1, \ldots, q^n) \in \Pi$ *such that*

a. *for all* $f \in \mathcal{F}$ $(p, \sum_{i=1}^{I} q^i) \in \mathrm{MRT}^f(z^f, y^f)$,

b. *for all* $i \in \mathcal{I}$, $(p, q) \in \mathrm{MRS}^i(x^i, y)$.

Notice that if we assume differentiability, then MRT^f and MRS^i are singletons and we have the familiar Samuelson conditions.

We begin our demonstration of these claims by showing that private goods prices must be nonnegative.

Lemma 1. *For all* $(z, y) \in P$ *and all* p *such that there exists* q *with* $(p, q) \in \mathrm{MRT}(z, y)$, $p \geq 0$.

Proof/

Suppose not; then for some $(z, y) \in P$ and p, q such that $(p, q) \in \mathrm{MRT}(z, y)$, there is a private good $\ell \in \mathcal{L}$ such that $p_\ell < 0$. By free disposal, for all $\delta > 0$ $(z_1, \ldots, z_\ell - \delta, \ldots, z_L, y) \in P$. But $(p, \sum_i q^i)(z_1, \ldots, z_\ell - \delta, \ldots, z_L, y) > (p, \sum_i q^i)(z, y)$, contradicting the definition of $\mathrm{MRT}(z, y)$.

•

The following lemma states given an allocation $a \in A$ and prices $(p, q) \in \Pi$, (z, y) maximizes profits over the global production set P at prices $(p, \sum^i q^i)$ if and only if (z^i, y^i) maximizes the profits of each firm $f \in \mathcal{F}$ at these prices. This allows us to state the subsequent theorems in terms of maximizing profits over the global production set instead of going to the extra step of considering each firm.

Lemma 2. *Given* $(z, y) \in P$ *and* $(p, q) \in \Pi$, $(p, q) \cdot (z, y) \geq (p, q) \cdot (\bar{z}, \bar{y})$ *for all* $(\bar{z}, \bar{y}) \in P$ *if and only if for all* $f \in \mathcal{F}$ *there exists* (z^f, y^f) *such that* $(p, q) \cdot (z^f, y^f) \geq (p, q) \cdot (\bar{z}^f, \bar{y}^f)$ *for all* $(\bar{z}^f, \bar{y}^f) \in P^f$ *and* $\sum_f (z^f, y^f) = (z, y)$.

Proof/

1. Necessity: Suppose not, then for all $f \in \mathcal{F}$ there exists (z^f, y^f) such that $(p,q) \cdot (z^f, y^f) \geq (p,q) \cdot (\bar{z}^f, \bar{y}^f)$ for all $(\bar{z}^f, \bar{y}^f) \in P^f$ and $\sum_f (z^f, y^f) = (z, y)$, but for some $(\bar{z}, \bar{y}) \in P$, $(p,q) \cdot (z,y) < (p,q) \cdot (\bar{z}, \bar{y})$. By definition, there exists a collection of production vectors $(\bar{z}^f, \bar{y}^f) \in P^f$ such that $\sum_f (\bar{z}^f, \bar{y}^f) = (\bar{z}, \bar{y})$. However, by hypothesis for all $f \in \mathcal{F}$, $(p,q) \cdot (z^f, y^f) \geq (p,q) \cdot (\bar{z}^f, \bar{y}^f)$. But this implies $(p,q) \cdot (z,y) = \sum_f (p,q) \cdot (z^f, y^f) \geq (p,q) \cdot (\bar{z}, \bar{y}) = \sum_f (p,q) \cdot (\bar{z}^f, \bar{y}^f)$, a contradiction.
2. Sufficiency: Suppose not, then there exists $(z,y) \in P$ such that $(p,q) \cdot (z,y) \geq (p,q) \cdot (\bar{z}, \bar{y})$ for all $(\bar{z}, \bar{y}) \in P$, and plan for each firm $(z^f, y^f) \in P^f$ such that $\sum_f (z^f, y^f) = (z,y)$, but for some $f' \in \mathcal{F}$, there exists $(\bar{z}^{f'}, \bar{y}^{f'}) \in P^{f'}$ such that $(p,q) \cdot (z^{f'}, y^{f'}) < (p,q) \cdot (\bar{z}^{f'}, \bar{y}^{f'})$. But then $\sum_{f \neq f'} (z^f, y^f) + (\bar{z}^{f'}, \bar{y}^{f'}) \in P$, and $(p,q) \cdot \sum_{f \neq f'} (z^f, y^f) + (p,q) \cdot (\bar{z}^{f'}, \bar{y}^{f'}) > (p,q) \cdot \sum_f (z^f, y^f)$, a contradiction.

•

We now give the first necessity theorem.

Theorem 1. *If $a \in A$ is a Pareto efficient allocation, then there exists a price vector $(p, q^1, \ldots, q^n) \in \Pi$ such that (a) $(p, \sum_i q^i) \in \mathrm{MRT}(\sum_i (x^i - \omega^i), y)$ and, (b) for all $i \in \mathcal{I}$, $(p,q) \in \mathrm{WMRS}^i(x^i, y)$.*

Proof/

Following Foley, we define an artificial production set in which public goods are treated as strictly jointly produced private goods:

$$AP \equiv \left\{ (\tilde{z}, \tilde{y}^1, \ldots, \tilde{y}^I) \mid \tilde{y}^1 = \ldots \tilde{y}^I = \tilde{y} \text{ and } (\tilde{z}, \tilde{y}) \in P \right\}.$$

AP is closed, convex, and nonempty as a consequence of P possessing these properties. Next we define the socially preferred set of the allocation a:

$$SP(a) \equiv \left\{ (\tilde{z}, \tilde{y}^1, \ldots, \tilde{y}^I) \in \mathbb{R}^L \times \mathbb{R}^{IM} \;\middle|\; \tilde{z} = \sum_{i=1}^{I} (\tilde{x}^i - \omega^i), \right.$$

$$\forall\, i \in \mathcal{I},\ \forall\, (x^i, y^i) \in C^i, (x_i, y) \not\succ^i (\tilde{x}^i, \tilde{y}^i) \text{ and } \exists j \in \mathcal{I} \text{ s.t. } (\tilde{x}^j, \tilde{y}^j) \succ^j (x_j, y) \Big\}.$$

The socially preferred set inherits convexity, and by continuity and nonsatiation it has nonempty interior.

a. Since a is Pareto efficient by assumption, $SP(a) \cap AP = \emptyset$. Then by the Minkowski Separation Theorem (Takayama (1985, p. 44)), there exists a price vector $(p, q^1, \ldots, q^I) \neq 0$ with $\| p \| < \infty$, and a scalar r, such that:

(i) For all $(\tilde{z}, \tilde{y}^1, \ldots, \tilde{y}^I) \in AP$, $p \cdot \tilde{z} + \sum_i q^i \cdot \tilde{y} \leq r$ (where $\tilde{y}^1 = \cdots = \tilde{y}^I = \tilde{y}$.)

(ii) For all $(\tilde{z}, \tilde{y}^1, \ldots, \tilde{y}^I) \in \text{closure}(SP(a))$, $p \cdot \tilde{z} + \sum_i q^i \cdot \tilde{y} \geq r$.

By continuity and nonsatiation, $(z, y^1, \ldots, y^I) \in \text{closure}(SP(a))$. By hypothesis, $(z, y^1, \ldots, y^I) \in AP$. It follows from (i) and (ii) that $p \cdot z + \sum_i q^i \cdot y = r$. Therefore, for all $(\tilde{z}, \tilde{y}) \in P$:

$$p \cdot z + \sum_i q^i \cdot y = r \geq p \cdot \tilde{z} + \sum_i q^i \cdot \tilde{y}.$$

Since it is possible to renormalize these prices to be elements of Π, this establishes part (a) of the theorem.

b. Now suppose that part (b) is false. Then for all $j \in \mathcal{I}$ there exists $(\bar{x}^j, \bar{y}^j) \in C^j$ such that $(\bar{x}^j, \bar{y}^j) \succ^j (x^j, y)$ and $(p, q^j) \cdot (\bar{x}^j, \bar{y}^j) < (p, q^j) \cdot (x^j, y)$. Hence,

$$\left(\sum_{i \neq j} (x^i - \omega^i) + (\bar{x}^j - \omega^j), y^1, \ldots, \bar{y}^j, \ldots, y^I \right) \in SP(a)$$

and

$$p \cdot \sum_{i \neq j} (x^i - \omega^i) + p \cdot (\bar{x}^j - \omega^j) + \sum_{i \neq j} q^i \cdot y^i + q^j \cdot \bar{y}^j$$

$$< (p, q^1, \ldots, q^I) \cdot \left(\sum_i (x^i - \omega^i), y^1, \ldots, y^I \right),$$

a contradiction to (ii) above.

•

As a corollary to this we state a version of the second welfare theorem. In particular, we show that we can decentralize any Pareto efficient allocation through prices for some set of endowments and profit shares such that the production of each firm is profit maximizing under the prices, and each agent's consumption bundle minimizes expenditure over the weakly preferred set. This is not quite the same thing as decentralizing the allocation as a Lindahl equilibrium since agents are not necessarily maximizing preferences over the budget set. To get this stronger result, slightly stronger conditions as needed. We show this below. See Debreu (1959) for details.

Corollary 1.1 *(weak second welfare theorem) If $a \in A$ is a Pareto efficient allocation, then there exists a price vector $(p, q^1, \ldots, q^n) \in \Pi$ an endowment vector $\hat{\omega}$, and a profit share system θ such that (a) $(p, \sum_i q^i) \in \mathrm{MRT}(\sum_i (x^i - \omega^i), y)$, (b) for all $i \in \mathcal{I}$, and all $(\tilde{x}^i, \tilde{y}) \not\succ (x^i, y)$, $(p, q^i) \cdot (\tilde{x}^i, \tilde{y}) \leq p \cdot \omega^i + \sum_f \theta^{i,f}(p, \sum_i q) \cdot (z^f, y^f)$ and, (c) $\sum_i \omega_i = \sum_i \hat{\omega}_i$*

<u>Proof/</u>

We know by Theorem 1, there exist prices $(p, q) \in \Pi$ such that (a)$(p, \sum_i q^i) \in \mathrm{MRT}(\sum_i (x^i - \omega^i), y)$ and (b) for all $i \in \mathcal{I}$, $(p, q^i) \in \mathrm{WMRS}^i(x^i, y)$. Notice that:

$$\sum_i (p, q^i) \cdot (x^i, y^i) =$$

$$p \cdot \sum_i x^i + y \cdot \sum_i q^i =$$

$$p \cdot (\sum_f z^f + \sum_i \omega_i) + \sum_f y^f \cdot \sum_i q^i =$$

$$(p, \sum_i q^i) \cdot (\sum_f z^f, \sum_f y^f) + p \cdot \sum_i \omega_i .$$

In words, the total cost of consumption equals the value of the endowment plus the profit shares. It only remains to show that we can redistribute endowment and profit shares in a way that satisfies (c) such that the cost of the Pareto efficient consumption bundle (x^i, y) for each agent equals the implied income under these prices. But clearly this is possible since total income to society does not change when we vary the distribution, we can continuously vary the income distribution over the full range of possibilities, and we know from the above that there is exactly enough endowment so that when it is fully distributed, society's budget balances.

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Next we give a second necessity theorem. We strengthen the hypothesis to require that all agents have a cheaper point in the consumption set. This allows us to conclude that there will exist supporting prices in the MRS correspondence of each agent, instead of just the WMRS. This means that the prices are fully decentralizing.

Theorem 2. *If $a \in \mathcal{A}$ is a Pareto efficient allocation, then for every $i \in \mathcal{I}$ such that (a) $p \cdot x^i > 0$, or (b) $\exists m$ s.t. $q^i_m < 0$, or (c) $\exists$ m s.t. $q^i_m > 0$ and $y_m > 0$, where p, q^i are the prices established by Theorem 1, we have that $(p, q) \in \mathrm{MRS}^i(x^i, y)$.*

Proof/

(a) Suppose that for some $i \in \mathcal{I}$, $p \cdot x^i > 0$ and $(p, q) \notin \mathrm{MRS}^i(x^i, y)$. The latter implies that there exists $(\bar{x}^i, \bar{y}^i) \in C^i$ such that $(\bar{x}^i, \bar{y}^i) \succ^i (x^i, y)$ and $(p, q^i) \cdot (x^i, y) \geq (p, q^i) \cdot (\bar{x}^i, \bar{y}^i)$. Since $p \geq 0$ by Lemma 1, and $x^i \geq 0$ because $(x^i, y) \in C^i$, $p \cdot x^i > 0$ implies that there exists $\ell \in \mathcal{L}$ such that $p_\ell > 0$ and $x^i_\ell > 0$.

Denote the open line segment between two points as follows:

$$L((x^i, y), (\bar{x}^i, \bar{y}^i)) \equiv$$

$$\left\{(\tilde{x}^i, \tilde{y}^i) \mid \exists \lambda \in (0,1) \text{ and } (\tilde{x}^i, \tilde{y}^i) = \lambda(x^i, y) + (1 - \lambda)(\bar{x}^i, \bar{y}^i)\right\}.$$

By the convexity of preferences and the linearity of the budget constraint, for all $(\tilde{x}^i, \tilde{y}^i) \in L((x^i, y), (\bar{x}^i, \bar{y}^i))$, we have $(\tilde{x}^i, \tilde{y}^i) \succ^i (x^i, y)$ and $(p, q^i) \cdot (x^i, y) \geq (p, q^i) \cdot (\tilde{x}^i, \tilde{y}^i)$. For $(\tilde{x}^i, \tilde{y}^i)$ close enough to (x^i, y) (λ close enough to 1), $\tilde{x}^i_\ell > 0$. By the continuity of preferences, there exists $\epsilon > 0$ such that $(\tilde{x}^i_1, \ldots, \tilde{x}^i_\ell - \epsilon, \ldots, \tilde{x}^i_L, \tilde{y}^i) \succ$ (x^i, y). Since $p_\ell > 0$, there follows $(p, q^i) \cdot (\tilde{x}^i_1, \ldots, \tilde{x}^i_\ell - \epsilon, \ldots, \tilde{x}^i_L, \tilde{y}^i) < (p, q^i) \cdot$ $(\tilde{x}^i, \tilde{y}^i) \leq (p, q^i) \cdot (x^i, y)$, leading to a contradiction to (ii) in the proof of Theorem 1 in the same manner as in that proof.

(b) Suppose now that for some $i \in \mathcal{I}$, $\exists m$ s.t. $q^i_m < 0$ and $(p, q^i) \notin \text{MRS}^i(x^i, y)$. The latter implies that there exists $(\bar{x}^i, \bar{y}^i) \in C^i$ such that $(\bar{x}^i, \bar{y}^i) \succ^i (x^i, y)$ and $(p, q^i) \cdot (x^i, y) \geq (p, q^i) \cdot (\bar{x}^i, \bar{y}^i)$. By the continuity of preferences, there exists $\epsilon > 0$ such that $(\bar{x}^i, \bar{y}^i_1, \ldots, \bar{y}^i_m + \epsilon, \ldots, \bar{y}^i_M) \succ^i (x^i, y)$. Since $q^i_m < 0$, this leads to the same contradiction as before.

(c) Finally, suppose that for some $i \in \mathcal{I}$, $\exists m$ s.t. $q^i_m > 0$ and $y_m > 0$ and $(p, q^i) \notin \text{MRS}^i(x^i, y)$. The latter implies that there exists $(\bar{x}^i, \bar{y}^i) \in C^i$ such that $(\bar{x}^i, \bar{y}^i) \succ^i (x^i, y)$ and $(p, q^i) \cdot (x^i, y) \geq (p, q^i) \cdot (\bar{x}^i, \bar{y}^i)$. We can now mimic the proof of (a) above, with y^i_m in the place of x^i_ℓ and q^i_m in the place of p_ℓ.

•

The reason that the extra assumption is required to obtain the full support is illustrated in the following example. Consider an economy with two agents, one private and one public good, one firm with one-to-one linear technology, and endowment of one unit of the private good for each agent. Agent 1 has preferences exactly as in Figure 1, and agent 2 has translated Cobb-Douglas preferences such that the slope of agent 2's indifference curve at $(x^1, y^1) = (1/2, 3/2)$ is -1. Then the allocation $(x^1, y^1, x^2, y^2, z^1, y^1) = (0, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ is Pareto efficient, but $\text{WMRS}^1(0, \frac{1}{2})$ contains only the vector $(1, 0)$, which intersects the strictly preferred set of agent 1. Therefore, the Samuelson prices arising from Theorem 1 are not separating prices, and this failure occurs for agent 1 who violates all three of the conditions of Theorem 2. This allows us to state a stronger second welfare theorem.

Corollary 2.1 *(strong second welfare theorem) If $a \in A$ is a Pareto efficient allocation such that for all agents $i \in \mathcal{I}$, (x^i, y) is in the interior of C^i, then there exists* *a price vector* $(p, q^1, \ldots, q^n) \in \Pi$ *an endowment vector* $\hat{\omega}$, *and a profit share system* θ *such that* $a \in LE(\hat{\omega}, \theta)$ *and* $\sum_i \omega_i = \sum_i \hat{\omega}_i$.

Proof/

Since for all agents $i \in \mathcal{I}$, (x^i, y) is in the interior of C^i, the hypothesis of Theorem 2 is satisfied. Therefore, there exist prices $(p, q) \in \Pi$ such that (a)$(p, \sum_{i=1}^{I} q^i) \in \text{MRT}(\sum_{i=1}^{I}(x^i - \omega^i), y)$ and (b) for all $i \in \mathcal{I}$, $(p, q) \in \text{MRS}^i(x^i, y)$. But (a) means all firms profit maximize under the prices, and (b) means each consumer i chooses (x^i, y) when he maximizes his preferences while having income $(p, q^i) \cdot (x^i, y)$. But we know from the argument given in the proof of Corollary 1.1 that it is possible to divide endowments and profits so that each agent has exactly his income, and the social endowment is exactly exhausted.

•

Now we give our sufficiency theorem.

Theorem 3. *If* $a \in A$ *is an allocation and there exists a price vector* $(p, q^1, \ldots, q^I)$ *such that (a)* $(p, \sum_{i=1}^{I} q^i) \in \text{MRT}(\sum_{i=1}^{I}(x^i - \omega^i), y)$ *and (b) for all* $i \in \mathcal{I}$, $(p, q) \in \text{MRS}^i(x^i, y)$, *then* a *is Pareto efficient.*

Proof/

Suppose that the hypotheses of the Theorem are met but a is not Pareto efficient. Then there exists a (Pareto dominant) feasible allocation $\tilde{a} \in A$ such that for no $i \in \mathcal{I}$ is it the case that $(x^i, y^i) \succ^i (\tilde{x}^i, \tilde{y}^i)$ and for some $j \in \mathcal{I}$, we have $(\tilde{x}^j, \tilde{y}^j) \succ^i (x^i, y^i)$. Then by (b) and summing up over all agents,

$$\sum_i p \cdot \tilde{x}^i + \sum_i q^i \cdot \tilde{y} > \sum_i p \cdot x^i + \sum_i q^i \cdot y. \qquad (i)$$

But by (a),

$$(p, \sum_i q^i) \cdot (\sum_i (\tilde{x}^i - \omega^i), \tilde{y}) \leq (p, \sum_i q^i) \cdot (\sum_i (x^i - \omega^i), y). \qquad (ii)$$

Since (ii) contradicts (i), the proof is finished.

•

Finally, we get the first welfare theorem as an immediate consequence of this.

Corollary 3.1 *If $a \in LE(\omega, \theta)$, a is Pareto efficient.*

Proof/

By the definition of Lindahl equilibrium, there exists a price vector $(p, q^1, \ldots, q^I)$ such that (a) $(p, \sum_{i=1}^{I} q^i) \in \text{MRT}(\sum_{i=1}^{I}(x^i - \omega^i), y)$ and (b) for all $i \in \mathcal{I}$, $(p, q) \in \text{MRS}^i(x^i, y)$. But then by Theorem 3, a is Pareto efficient.

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4. Some comments on the literature

The oversight in Campbell and Truchon (1988) occurs on page 247, in the proof of Lemma 1. The first inequality of the last sequence of inequalities in that proof does not hold, because some the dx_i may be negative.

Condition *GOC* in Campbell and Truchon (1988) attempts to deal with agents pushed against the public good axis by weighting their (unique, under their differentiability assumption) MRS by a nonnegative coefficient that can be less than unity. In such a case, if we allow the coefficient to run from 0 to 1, we trace our WMRS, as shown in Figure 2.

[Figure 2 here]

Khan and Vohra (1987) have general assumptions on preferences and they allow for nonconvexities in preferences and production, but they require all public goods to be desired by all agents, so that the example with the incinerator mentioned in our introduction falls outside their coverage. They prove a version of the second welfare theorem employing a notion of supporting vector set equivalent, under convexity, to our WMRS(x^i, y) (Khan and Vohra 1987, page 236).

Finally, two notes on papers that are tangentially relevant. Saijo (1990) addresses a quite different point arising from Campbell and Truchon than we do; namely, he shows that the robustness of boundary Pareto efficient allocations observed by Cambpell and Truchon is not a phenomenon specific to public good economies, since it also happens in exchange economies. Manning (1993, chapter 3) contains an extension of Foley's (1970) results to economies with local public goods, using assumptions based on Foley's, such as constant returns to scale and ruling out the private goods boundary.

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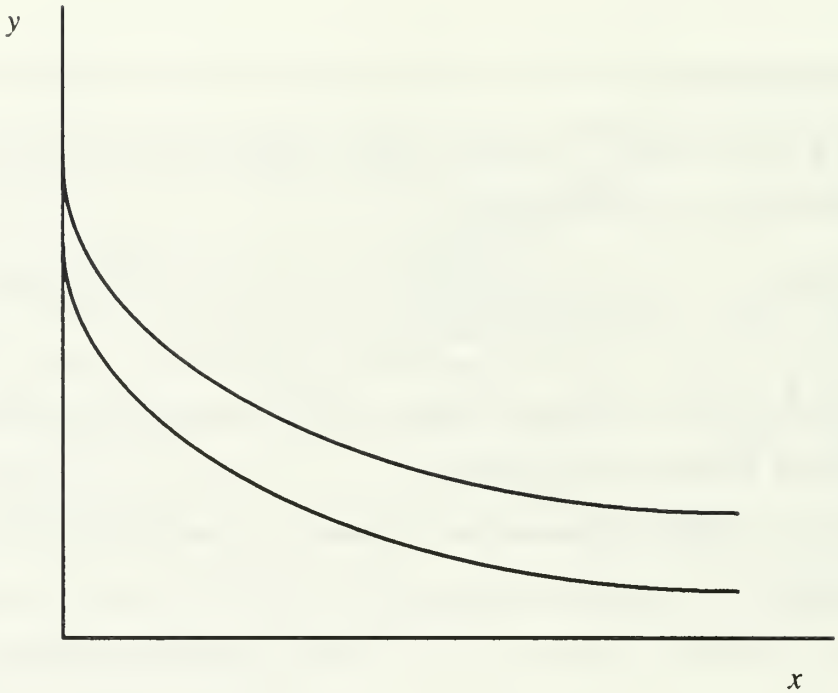
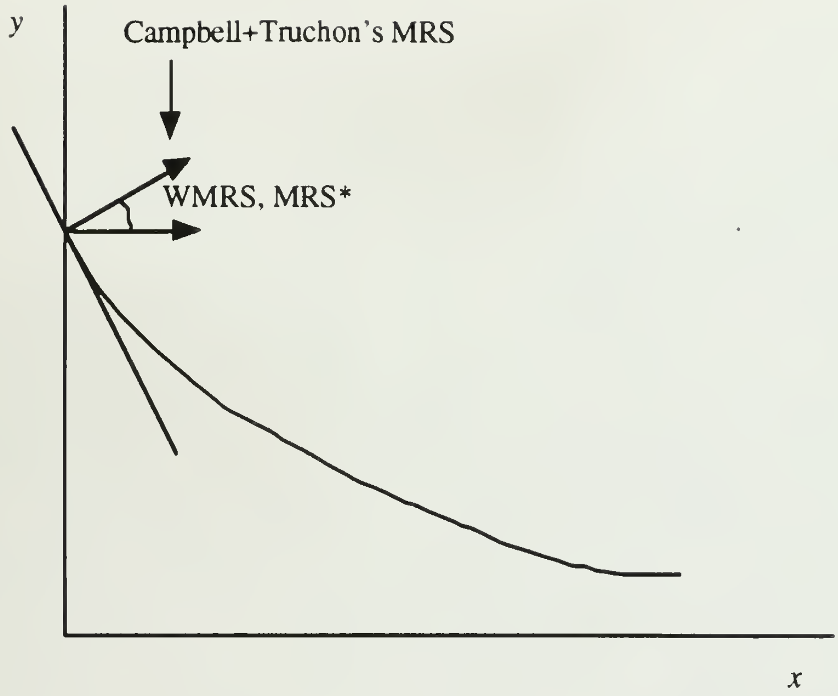


Figure 1



*: The horizontal edge of WMRS is not in MRS

Figure 2