Faculty Working Papers

SEQUENTIAL MULTI-PARAMETER ESTIMATION: A DYNAMIC PROGRAMMING APPROACH

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#281

College of Commerce and Business Administration
University of Illinois at Urbana-Champaign

FACULTY WORKING PAPERS

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October 14, 1975

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ABSTRACT

An algorithm for the scheduling of information acquisition is explored in the new product evaluation context. The decision process proposed indicates the need for an orderly scheduling process in conjunction with a specified objective function. Simulations employing a variety of parameter inputs are presented for illustrative purposes.

INTRODUCTION

We consider the decision-making framework to involve an option on the decision-maker's part to accept an alternative, reject an alternative or defer the decision until additional information is collected that presumably has a bearing on alternative acceptance or rejection. One area dealing with information collection and utilization is Bayesian decision-making as proposed in the work of Raiffa [6] and Schlaifer [7]. (In the marketing research context, see [1]). Utilizing this framework, work in the sequential sampling area has extended the "collect information" option to the situation where the decision maker collects information, notes the outcome and then elects to accept, reject, or collect additional information [3]. The presumption is that information collection is a costly operation and that information should not be collected beyond the point where the cost exceeds the value.

Previous work in this area has generally involved only the single parameter decision problem. Consideration of a decision involving many parameters poses an additional dimension to the strategic framework. Now the decision maker must decide between accept, reject, and information collection in which he has a number of alternative parameters to estimate. If we use the word "experiment" to refer to information collection for the sake of parameter estimation, then it becomes critical to specify the first experiment to be performed as well as a subsequent experimental order which is dependent on the prior experimental outcomes.

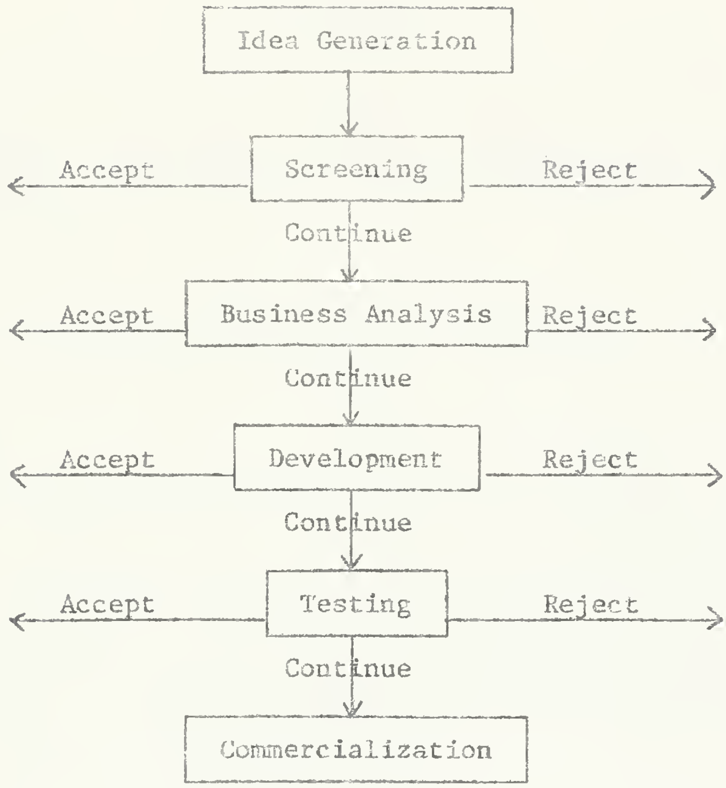
Since only the experimental outcomes, and not the order of completed experiments is critical to subsequent experimental selection,

a dynamic programming recursion algorithm will be proposed. Basic inputs are the same as in any single parameter Bayesian decision framework: the cost of each experiment, the prior probabilities of the unknown parameter, conditional probabilities that relate experimental outcomes to the true value of the parameter, and an objective function specifying relationships between variables and the objective.

The problem of multi-parameter estimation are neither trivial nor infrequent in management. An aircraft manufacturer has a number of tests that may need to be performed on a new or modified product; the experiments are costly and, thus, savings with this proposed solution may be realized. New product evaluations are also most difficult, and the number of experiments range from inexpensive breakeven analysis to expensive market-testing. The high rate of failure among new products indicates that some products should have undergone more extensive experimentation prior to experimentation. The large number of products that are abandoned after market testing causes one to question whether these products could have been screened out prior to expensive experimentation. Finally, one must wonder about the possibly larger number of products that would have been successful but were abandoned when firms faced the high cost of further testing; this statistic will never be known.

Solutions to the problem of new product evaluation has been proposed [2,5]. Unfortunately, these provide only a "rule of thumb" for the scheduling of the steps (see Figure 1), or methodology for performing each experiment. In general, the steps scheduled earliest are those which are least expensive; one might also expect the earlier steps to be very efficient in eliminating poor ideas. Nevertheless,

Figure 1


Idea Generation
Accept
Screening
Reject
Continue
Accept
Business Analysis
Reject
Continue
Accept
Development
Reject
Continue
Accept
Testing
Reject
Continue
Commercialization

the assumption of this prior work is that all steps will be followed until rejection and that the order specified will be rigidly adhered to.

An Example

To continue the new product evaluation example, assume that a new product development team formulated the following objective function for its new product idea:

$$(1) \quad Z = \{\sum_{t=0}^{T} \frac{1}{(1+i)^t} PS \left[D_t (50 - C_t) - FC_t\right]\} - K$$

where,

PS = probability of technological success (i.e., probability of developing a manufacturable prototype)

D_t = demand for new product (if developed) in year t

C_t = unit variable cost associated with the production of the new product in year t

FC_t = fixed costs associated with new product in year t

i = cost of capital

K = a minimum return or cost of implementing the new product

Thus we see that the new product team faces uncertainty about the new product idea which, if developed, would sell for $50 per unit. It is also apparent that the timing of cash flows is critical (this is utilized in present value calculations) and that some expenditures may vary over the years. For example, fixed cost expenditures are typically very heavy in year zero (e.g., purchase of capital equipment) while demand and manufacturing costs are incurred over the life of the product.

The objective function featured in the example is indicative of why analytical solutions to problems of this nature may not be feasible. Expansion of equation (1) yields a term which is the interaction of three parameters, $PS \cdot D_t \cdot C_t$. The value of reducing uncertainty for C_t, for example, is dependent upon the parameter estimates and uncertainty about PS and D_t. Although variables may clearly interact in the objective function, the assumption is made that experimental outcomes for different parameters are independent. This assumption is made to simplify the simulations presented in a later section; it is by no means limited to the general method of solution and the program could be easily modified to handle the condition of dependence.

The Programming Approach

To expand on the proposed approach to the multiparameter sequential information collection problem, consider a decision situation with four unknown parameters, X_1, X_2, X_3, and X_4. Let us introduce the following notation and decision inputs: If the alternative is accepted, our profit will be

$$P\ (X_1,\ X_2\ .\ .\ .\ X_n)$$

where X_1, X_2, X_3, X_4 . . . are the actual (unknown) values of the parameters

We assume that the following data can be determined:

(a) c_i, the cost of performing an experiment on the i^{th} parameter

(b) $f_i(x_i)$, the prior density function of the i^{th} parameter

(c) $h_{i|x_i}(\hat{x}_i)$, the conditional density function of the outcome of the experiment on parameter i, $\hat{x}_i$, given the true value of the parameter is x_i.

Then (b) and (c) above can be used with Bayes theorem to compute, $g_{i|\hat{x}_i}(x_i)$, the conditional density function on parameter i given the experimental outcome $\hat{x}_i$.

Next we develop the dynamic programming recursion to determine the optimal order [4]. First consider the situation when our objective function has four parameters and all experiments have been completed with experimental outcomes $\hat{x}_1$, $\hat{x}_2$, $\hat{x}_3$, $\hat{x}_4$. Since we must now reach a terminal decision, we would calculate expected profit with acceptance as:

(2) $$E\,(P|\hat{x}_1,\hat{x}_2,\hat{x}_3,\hat{x}_4) = \int_{x_1}\int_{x_2}\int_{x_3}\int_{x_4} P(x_1x_2x_3x_4)\, g_{1|\hat{x}_1}(x_1) \ldots g_{4|\hat{x}_4}(x_4)\, dx_1 \ldots dx_4$$

The value associated with rejection is 0. Thus the decision with the greater expected value would be selected.

Working back one step, consider the case where only three experiments, 1, 3, and 4, have been completed with experimental outcomes $\hat{x}_1$, $\hat{x}_3$, $\hat{x}_4$; three options are available: we can accept, reject (with expected value of 0) or perform experiment 2. The expected value of immediately accepting is,

(3) $$A = E\,(P|\hat{x}_1, \hat{x}_3, \hat{x}_4) = \int_{x_1}\int_{x_2}\int_{x_3}\int_{x_4} P(x_1x_2x_3x_4)\, g_{1|\hat{x}_1}(x_1)\, f_2(x_2)\, g_{3|\hat{x}_3}(x_3)\, g_{4|\hat{x}_4}(x_4)\, dx_1 \ldots dx_4$$

or utilizing equation (2) we can rewrite equation (3) as,

(4) $$A = \int_{\hat{x}_2} E\,(P|\hat{x}_1,\hat{x}_2,\hat{x}_3,\hat{x}_4)\, \left[\int_{x_2} h_{2|x_2}(\hat{x}_2)\, f_2(x_2)\, dx_2\right] d\hat{x}_2$$

The above is merely performing the integration over a different variable and the quantity in brackets represents the pre-posterior distribution of $\hat{x}_2$. In addition to the terminal act options, we may elect to perform experiment 2, thereby delaying our terminal act, but also incurring the cost of the second experiment, c_2:

$$(5)\quad C = -c_2 + \int_{\hat{x}_2} \max\left[E(P|\hat{x}_1,\hat{x}_2,\hat{x}_3,\hat{x}_4),\ 0\right] \left[\int_{x_2} h_{2|x_2}(\hat{x}_2) f_2(x_2)\,dx_2\right] d\hat{x}_2$$

It is apparent that the cost of c_2 must be weighed against the advantage of choosing the better terminal decision after observing $\hat{x}_2$.

To simplify future expansion, let us introduce the following notation:

Let K represent the set of indices of all possible experiments

$$K = \{1, \ldots, n\}$$

Let $J_{i,k,m}$ represent the set of experiments which have been performed

$J_{i,k,m} = K - \{i,k,m\}$ (i.e., all but i, k, m have been performed)

i.e., for $n = 4$

$$K = \{1,2,3,4\}$$

$J_{2,4} = K - \{2,4\} = \{1,3\}$ (i.e., experiments 2 and 4 have not been performed)

Let $Y_K = \{\hat{x}_1 \ldots \hat{x}_n\}$

And $Y_{J_{2,4}} = \{\hat{x}_i | i\varepsilon J_{2,4}\} = \{\hat{x}_1,\hat{x}_3\}$ (i.e., the experimental outcomes for the experiments performed)

Let A_Y = the expected value associated with termination by acceptance given experimental outcomes Y

Then

$$A_{Y_{J_{2,4}}} = \int_{x_1}\int_{x_2}\int_{x_3}\int_{x_4} P(x_1x_2x_3x_4)\ g_{1|\hat{x}_1}(x_1)\ f_2\ (x_2)\ g_{3|\hat{x}_3}(x_3)\ f_4(x_4)\ dx_1dx_2dx_3dx_4$$

For $i \epsilon J$ let

C_{i,Y_J} = the expected value associated with continuing with experiment i.

Thus,

$$C_{4,Y_{J_{2,4}}} = -c_4 + \int_{\hat{x}_4} \max \left(A_{Y_{J_2}},\ R,\ C_{2,Y_{J_2}}\right)$$

$$\left[\int_{x_4} h_{4|x_4}(\hat{x}_4)\ f_4(x_4)\ dx_4\right] d\hat{x}_4$$

where

$R = 0$

Intuitively the above integral considers the expected value of the best decision associated with each experimental outcome $\hat{x}_4$, and weights the expected value by the probability of the $\hat{x}_4$ outcome. These are then summed over all possible $\hat{x}_4$ outcomes to yield the expected value of continuing with the fourth experiment.

The general form of the dynamic programming recursion can now be written for the n parameter case:

1. start with $J = K = \{1, \ldots, n\}$ and

   compute A_{Y_J} for all possible Y_J

   Compute $EV_{Y_J} = \max \{A_{Y_J}, R\}$

2. Let $J_i = \{K\} - \{i\}$. Then for all i

   compute $A_{Y_{J_i}}$, $C_{i,Y_{J_i}}$, and let

   $EV_{Y_{J_i}} = \max \{A_{Y_{J_i}}, R, C_{i,Y_{J_i}}\}$

   This represents the case where only one experiment remains to be done.

3. Let $J_{i,k} = J_i - \{k\} = J_k - \{i\}$. For all $k \neq i$

   we then compute $A_{Y_{J_{i,k}}}$, $C_{k,Y_{J_{i,k}}}$

   Then $EV_{Y_{J_{i,k}}} = \max \{A_{Y_{J_{i,k}}}, R, C_{i,Y_{J_{i,k}}}\}$

4. Continue to recursion to and including the case $J_{1, \ldots n}$. This represents the case when none of the experiments has been performed and the decision maker has selected the best option: either accept, reject, or continue (and the best experiment to perform first).

It should be noted that, in the foregoing, we have assumed all density functions are for continuous random variables. Clearly, if this is not the case, then we merely replace the appropriate integral with the corresponding summation. From a computational point of view, however, we must assume that we have discrete density functions, at least for all but the most trivial situations. Thus to perform the actual computations outlined above, we would approximate all density functions by discrete density functions.

Sample Runs of Experimental Scheduling

To explore the impact and provide a qualitative feel for the model's scheduling algorithm, eight runs of the model are presented in Table 1. These runs embody a number of input parameters from which generalizations will be discussed. To conform to the previous example as well as the programming notation, we considered a four experiment case in which the decision maker must determine whether to obtain information regarding the probability of technological success, demand, unit cost, or fixed cost for a potential new product. Thus we have the following objective function:

$$Z = \sum_{t=0}^{10} \frac{1}{(1+i)^t} PS \left[D_t (50-C_t) - FC_t\right] - 5000$$

where all variables are defined as before and

$i = .10$

PS = a discretely distributed random variable

D_t, C_t = normally distributed random variables, equal to 0 in period 0 and constant over time periods 1 to 10.

FC_t = uniformly distributed random variable in period 0 and equal to 0 in periods 1 to 10.

The computations were performed by dividing the range of the density function into intervals and then massing the total probability in each interval at the midpoint of the interval. The conditional distributions $h_{i|x_i}(\hat{x}_i)$ were approximated by massing all the probability in an interval at the midpoint. In particular, the range for D_t was divided into deciles and the ranges for FC_t and C_t were divided into quintiles. Obviously, as we use finer intervals, we will get a better approximation, but our cost of computation will also rise. In practice, we would have to check that the discrete distributions reasonably reflect our initial data. No transformations were necessary in the case of PS as this variable was initially a discrete variable.

Table 1 provides a summary of the input and output of the computer runs. For experiment 1, the cost of the experiment and prior probability of success is indicated (the probability of failure is simply 1 minus this value) as well as the probability of ultimate success given a successful outcome of the experiment and the probability of ultimate failure given an unsuccessful outcome of the experiment. The latter two values capture

Table 1

RESULTS OF EXPERIMENTAL ORDERING SIMULATIONS

	INPUT PARAMETERS																EXPERIMENTAL SCHEDULE							
	Experiment 1 Prob. of Success				Experiment 2 Demand				Experiment 3 Unit Cost				Experiment 4 Fixed Cost				First Exp.	Second Exp.		Third Exp. Prob.				
Run	Cost	P(S)	$P(S\|\hat{S})$	$P(F\|\hat{F})$	Cost	Pr.Mean	Pr.St.Dev.	Post St.Dev.	Cost	Pr.Mean	Pr.St.Dev.	Post St.Dev.	Cost	Pr.Mean	Pr.St.Dev.	Post St.Dev.		Exp.	Prob.	1	2	3	4	T
1 (base)	700	.5	.95	.9	6000	10000	1667	400	5000	30	3.33	1.67	1000	1200000	400000	100000	1	1						
																		2	.525			.603	.351	.046
																		3						
																		4	.475					1.0
																		T[b]						
2	700	.5	.95	.9	8000[a]	10000	1667	400	5000	30	3.33	1.67	1000	1200000	400000	100000	1	1						
																		2						
																		3	.525		.423		.526	.05
																		4	.475					1.0
																		T						
3	700	.5	.95	.9	6000	10000	1250[a]	400	5000	30	3.33	1.67	1000	1200000	400000	100000	1	1						
																		2						
																		3	.525		.423		.476	.10
																		4	.475					1.0
																		T						
4	700	.5	.95	.9	6000	10000	625[a]	400	5000	30	3.33	1.67	1000	1200000	400000	100000	1	1						
																		2						
																		3	.525				.661	.3[illegible]
																		4	.475					1.0
																		T						
5	700	.5	.95	.9	6000	10000	1667	667[a]	5000	30	3.33	1.67	1000	1200000	400000	100000	1	1						
																		2						
																		3	.525		.661		.288	.05
																		4	.475					1.0
																		T						
6	3000[a]	.5	.95	.9	6000	10000	1667	400	5000	30	3.33	1.67	1000	1200000	400000	100000	4	1	.400			.525		.475
																		2	.600			.609		.391
																		3						
																		4						
																		T						
7	700	.75[a]	.95	.9	6000	10000	1667	400	5000	30	3.33	1.67	1000	1200000	400000	100000	2	1						
																		2						
																		3	.603				.274	.726
																		4	.321	.200		.269		.531
																		T	.076					
8	700	.5	.7[a]	.7[a]	6000	10000	1667	400	5000	30	3.33	1.67	1000	1200000	400000	100000	2	1						
																		2						
																		3	.444				.542	.45[illegible]
																		4	.399	.080		.480		.44[illegible]
																		T	.157					

[a]This parameter value has been changed from the base run.

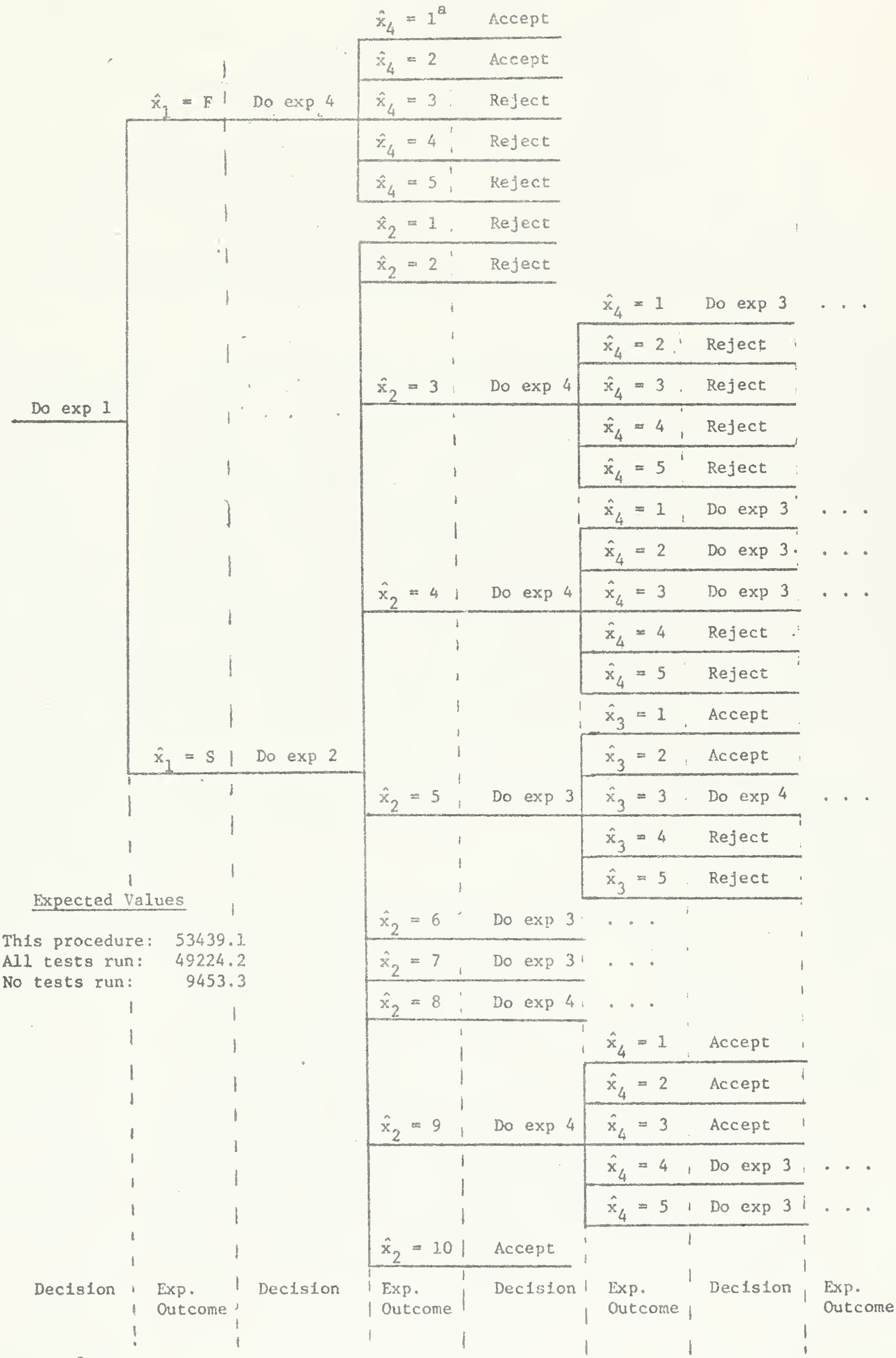
[b]T indicates a terminal decision of acceptance or rejection.

the reliability of experiment 1 in predicting ultimate technological success. For the remaining three experiments, input parameters are the cost of the experiment, the prior mean, the prior standard deviation, and the posterior standard deviation. For the sake of simplicity, it was assumed that the posterior distribution was from the same family as the prior distribution and that the experiment was unbiased (e.g., $E(X|\hat{x}) = \hat{x}$, where X is the basic parameter and $\hat{x}$ is the experimental outcome).

Run 1 represents the base run from which generalizations regarding the other simulations will emerge. As indicated in the summary table, experiment 1 is the first experiment scheduled; depending on the experimental outcome, experiments 2 and 4 have probabilities of being scheduled second of .525 and .475 respectively. Figure 2 is an abbreviated version of the entire decision-experimentation tree for the base run. This decision tree is indicative of the infeasibility of analytical solutions or anything but broad generalizations to this method of problem solving. For example, experiment 1 is scheduled first because of the reliability of the experiment in predicting success or failure as well as the comparatively low cost. If a failure is predicted, then we continue with experiment 4 which leads to terminal acts of either acceptance or rejection. If the outcome of experiment 1 points to success, then we continue with experiment 2. If demand is estimated to be very low (first or second decile) or very high (tenth decile), we terminate. The decision tree also provides a visual presentation of the non-monotonic relationship between outcomes of experiment 2 and the optimal third experiment. If the experiment 2 outcome is in the 3rd, 4th, 8th, or 9th deciles, we perform experiment 4; outcomes in the 5th, 6th, or

Figure 2

Partial Decision Tree of Base Run (Run 1)



[a]Numbers represent quintile or decile outcomes resulting from experimentation

7th deciles warrant experiment 3. Thus analytical solutions or guesswork are likely to yield non-optimal decision procedures. A comparison of expected values of the alternative decision rules indicates that, for this set of parameters, an approximately 10 per cent gain can be expected from using the algorithm as compared with running all tests and then choosing the best course of action. The algorithm results in an even more substantial gain when compared with the simple decision rule to do no experimentation.

A comparison of runs 1 and 2 indicates the effect of experimental cost. As the cost of experiment 2 is increased, the scheduling moves the experiment from a likely second step to a possible third step.

In runs 3 and 4, the prior standard deviation of the parameter 2 value has been decreased from 1667 to 1250 and 625 respectively. In this fashion, as the prior standard deviation approaches the posterior standard deviation, the experiment has either a low probability of taking place or, in the case of run 4, no chance at all of being completed. Similarly when experimental reliability is decreased (i.e., posterior standard deviation is increased) as in run 5, the probability of the experiment being done diminishes.

The previous runs have shown how experimental costs and value of information have moved experiment 2 from a likely second step to an improbable third step. In a similar manner, experiment 1 (which was scheduled first in the base run) can be shifted out of the first scheduling position. In run 6, the cost of experiment 1 has been increased so that experiment 1 is a possible second step. Run 7 illustrates the effect of the prior parameter of experiment 1; in this case, the likelihood of a success has been increased from .5 to .75 and experiment 1 has only a

a very small probability of occurring before the fourth step. Run 8 indicates the effect of decreased information reliability. In this case, experiment 1 has no chance of being within the first three experimental steps.

Conclusion

The dynamic programming Bayesian approach is a flexible technique that may be applied to a number of decision-information acquisition problems that involve multiple testing of several parameters. One obvious use is in the area of new product evaluations which must consider a number of relevant parameters to the overall profit function. Extensions to the areas of research and development or medical testing may yield additional promise. The nature of the objective function is quite flexible and the independence of tests/parameter estimation is by no means limiting.

The value of the organized problem solution is threefold:

1. Management/researchers must translate qualitative judgments about the quality of information, prior information, and cost of information into quantitative parameter inputs. Where high uncertainty exists, sensitivity analysis is available to detect parameters that greatly affect the scheduling outcome.
2. The nature of possible complex objective functions suggest that analytical or rules of thumb solutions to this problem will result in less than optimal decision rules. (See [8] for a description of subjects' inabilities to make judgments of this nature for a simple problem.)

3. The process only schedules information that will have a bearing on the final decision and the ambiguity or potential source of conflict in test evaluation is thereby eliminated.

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