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OBTAINING CONSENSUS PROBABILITY
DISTRIBUTIONS AND THE PARI-MUTUAL METHOD

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In 1959 Eisenberg and Gale (hereafter E & G) presented the mathematical formulation of a mechanical (as opposed to behavioral) method for aggregating individual subjective probability distributions to achieve a "consensus" distribution -- the Pari-Mutual method. However, beyond casual reference to its being an "important device" (Winkler, 1968) and "clever" (Hogarth, 1975), the E & G model has not been operationalized, experimented with or empirically tested. We believe it is the lack of operationalization of the model which has resulted in the lack of testing or experimentation. That is, it can be demonstrated that neither the problem addressed nor this particular method lacks significance. Hence, it is the primary objective of this paper to provide the requisite implementation vehicle for the Pari-Mutual method and therefore facilitate research on this potentially useful aggregation technique.

Section I discusses the important position the subjective probability aggregation problem occupies within the overall framework of decision making. Section II is devoted to a brief review of the various theoretical and experimental methods which have been identified to date in dealing with the "consensus element of decision making problems. Section III examines the E & G model, its characteristics and its advantages and disadvantages. A relatively simple computer simulation model is also presented which can (and does) operationalize the E & G model. Further, this model circumvents the reasonably sophisticated -- and correspondingly cumbersome -- mathematical techniques which that model involves. In the final section of the paper, Section IV, various categories of research which appear desirable in light of the potential usefulness of the Pari-Mutual aggregation scheme, are identified.

I.   CONSENSUS IN DECISION MAKING

The process of arriving at the likelihood of an event occurring, of
a variable taking on one or more values, or of a particular state of nature
obtaining, are indigeneous to most examples of decision making problems.
The consensus problem exists in any decision problem where uncertainty is
involved and where it is also deemed desirable to consult more than one
"expert" or opinion.   In practice, groups of experts are routinely consulted
or convened to arrive at a decision involving uncertainty.   If one accepts
that "the most detailed and most interesting representation of an expert's
judgement pertaining to an uncertain quantity is the probability function
he assigns to it," (Morris, 1974, p. 1234) then group consensus more often
than not means an aggregate subjective probability distribution.   Thus, since
the general conclusion arising from research is that composite distributions
show greater predictive ability than most single expert opinions, one finds
both a common and important problem (Hogarth,   1975, p. 282; Gustafson, et. al.,
1973,   p. 281; Winkler, 1971).

Obviously there are many different approaches to decision making under
uncertainty.   However, greater emphasis is being placed on formal mechanistic
(versus hueristic) modes of analysis or methods of processing information.
This movement appears justified since the general conclusions from research --
with respect to the information processing (combining) element of decision
making -- is that the mechanical mode of combination is superior to the clinical
mode of combination (Einhorn, 1972, p. 87).   Thus, recent times have found
a proliferation of Baysian, Markovian and other forms of analysis being offered
as applicable to numerous specific business and other types of decisions and
problems.   Yet the implementation of these stochastic methodologies requires

prior probability distributions, transitional probability matrices and the like. In the absence of actuarial data, these probabilities must either be postulated (subjectively) or elicited from individuals in the form of subjective probability distributions. In situations where the decision maker or analyst has little knowledge of the parameters of interest, he is almost forced to consult a number of experts to construct the requisite probability distribution. Since only a single distribution (or likelihood ratio) can be input to the mechanistic models, the aggregation issue takes on singular significance.

Hence, whether a mechanistic analysis of a decision or other problem is anticipated, or one is simply interested in arriving directly at a decision where a group is involved, a consensus estimate is also likely to be involved. This consensus takes the form of a subjective probability distribution. Several approaches have been used or proposed for reaching consensus, and each type has its advantages and disadvantages. These are considered in the following section.

II. METHODS OF ASCERTAINING CONSENSUS DISTRIBUTIONS

As Rowse (1974, pp. 274-5) suggests, a final group estimate may be obtained either by "behavioral consensus" -- where group members interact with each other either verbally or by way of correspondence or feedback -- or by "mathematical consensus" (aggregation of individual member's estimates). Within the "behavioral consensus" group of approaches, one finds the Delphi technique, variations on that theme, and other group methods which involve either interaction or some form of inter-member communication -- e.g., the simple committee meeting. Within the mathematical group fall various averaging techniques, mathematical models and the Pari-Mutual method.

The Delphi technique as a generic form typically involves repeated interrogation or questionnaire inquiry of the experts, making it both a costly and time consuming process. In its favor -- since it avoids confrontation --

it does not involve the many restrictive and dysfunctional effects which have

been associated with the group dynamics of other behavioral approaches

involving member interaction (Dalkey and Helmer, 1963, p. 459; Gustafson,

et.al., 1973, p. 282; Rowse, 1974, p. 275). Yet with those behavioral

consensus approaches involving feedback, there is always the problem of the

extent and form of the feedback. Specifically, such feedback may have to

take the form of aggregate distributions. Further, there is always the

possibility that no consensus (convergence) will occur.

In general, the mathematical consensus approaches are the most appealing --

particularly in terms of cost, simplicity, implementation, time consumed and

the number of experts which can be handled effectively. Since no actual group

dynamics are involved, the aforementioned potential problems are completely

avoided. Also, most mathematical aggregation techniques permit a differential

weighting of the individual opinions or distributions being combined. This

feature is important since the level of expertise is likely to vary among

the members of any group. Such weightings might be derived from self-ratings,

inter-judge ratings, assigned subjectively by the decision maker ultimately

responsible, or may be derived from applying "scoring rules" to previously

assessed distributions and actual outcomes (Winkler, 1968, 1969). Alternatively,

the Delphi technique does not involve an identification (in the feedback process)

with respect to which expert holds what opinion. Finally, in the other behavioral

approaches, there is no guarantee that "expertise" is being considered in a

systematic fashion.

The most commonly used aggregation techniques are average and weighted-

average mathematical models. These models are also the simplest and least

costly to implement. Moreover, some experimental evidence exists which suggests

they work better than behavioral aggregations -- perhaps for the very reason

that they avoid group dynamics (Rowse, 1974). Other more complex approaches

to aggregation have also been proposed. For example, DeGroot (1974) proposes a mathematical model whose theoretical basis resembles the Delphi technique but which involves both the weighting of individual opinions and the application of Markov techniques. Winkler (1968), on the other hand, has proposed a "natural conjugate" method. This technique involves successive application of Bayes' theorem to arrive at a group consensus -- in which expert opinions are basically treated as additional sample evidence. Lastly, it is in this "more complex" category of aggregation schemes that Eisenberg and Gale's model can be classified. However, it is not clear that E & G were anticipating this type of classification when their work was published.

III. THE PARI-MUTUAL MODEL

The Eisenberg and Gale Pari-Mutual model of consensus is probably most analogous to DeGroot's. In both cases a mathematical model of a real-world referent is involved. In the case of DeGroot, the empirical process envisioned resembles the Delphi process. In the case of the E & G model, the process envisioned is the dynamics of the pari-mutual horse race betting market. Analytically, this market can be viewed as one in which large groups of bettor's subjective probability distributions (across horses) are voted by way of betting decisions and aggregated in the form of odds (prices). As the betting process in horse racing is an interactive process, so also is the E & G model. That is, their model is a mathematical process of step-wise convergence to the equilibrium odds which should obtain given that all bettors are expected monetary return maximizers.

Though a more comprehensive treatment of the E & G model (with examples) may be found in Appendix A, certain general characteristics warrant attention here -- particularly with regard to its appeal as a scheme for arriving at

consensus subjective probability distributions. Incorporated in the model
is the view of the market process as one where: (1) each bettor has a
subjective probability distribution across horses, (2) is then exposed to
continually revised pay-off distributions (tote board odds), and (3) has
wealth constraints limiting the size of his bet. Comparatively speaking,
these elements correspond to: (1) the initial subjective probability dis-
tribution that a member of a decision group possesses, (2) feedback with
regard to other group members beliefs, and (3) the wealth constraint which
may be variously translated as the power or weight the individual might
wield in the total group; or the weight he, his fellow members, or an analyst-
aggregator might accord his opinion. In the model, or the actual market, two
factors determine equilibrium odds (though equilibrium in a normative sense
may not actually be reached in the market since the betting period is arbitrar
cut off), if a homogeneity of decision models is assumed. These includ.
(1) the subjective probability distributions held by the bettors, and (2) the
size of their bets. Again speaking comparatively, these are the primary
factors a mathematical consensus should reflect.

Hence, as a method of estimating consensus distributions, the Pari-mutual
method has all the advantages of other mathematical consensus models -- and
then some. It can accommodate any number of expert opinions in probability
form and, as E & G prove, will always work to a unique solution. Weights
derived in any of the methods described earlier, may easily be incorporated
by way of wealth constraints. Naturally, all problems of group dynamics are
avoided but a feedback element is none the less incorporated in the model.
Finally, there is reason to believe that the Pari-Mutual method can generate
reliable and reasonably accurate consensus distributions <u>for the very reason
that its referent is a market</u>. That is, we already have considerable evidence

that organized markets -- in particular securities markets which are not
all that dissimilar from the pari-mutual markets -- tend to be efficient
in the sense of generating unbiased estimates in the form of prices. In-
deed, there is even some existing empirical evidence that pari-mutual markets
generate fairly unbiased prices (odds) in terms of their mapping onto actual
outcomes (Griffith, 1949). The authors are currently involved in an extensive
empirical study on this and related phenomena regarding the pari-mutual
market and preliminary results tend to support this conclusion. At the least,
it is important to note that evidence of the accuracy and reliability of this
aggregation scheme will not entirely have to come from the artificial environ--
ments of laboratory experiments which is the typical case.

The major drawback to the Pari-Mutual method is that the mathematics
involved in the Eisenberg and Gale paper are fairly sophisticated. For example,
even the three bettor (e.g., expert), three horse (e.g., outcomes, events,
states) case is difficult to deal with computationally. Appendix A describes
more fully (with an example) the computations which are necessary. Simply
phrased, what is involved is the mathematical searching of the faces and edges
of an N-dimensional space for global optimums. As N becomes large this becomes
a quite tedious process. However, since there is a real-world referent, with
specifiable mechanics and trading rules, a simulation approach is an attractive
alternative.

Again, the flowchart and technical details of the computer simulation
which has been developed have been set forth in an appendix (Appendix B).
(The actual program can be supplied upon request.) However, like the dynamics
of the pari-mutual market itself, the simulation is an iterative process.
Further, like the E & G mathematical model, it assumes a homogeneous market
of expected monetary return maximizers. Of greater interest, perhaps, is

that we have computationally employed the E & G model for several cases -- up to a four by four situation (bettors and horses) -- and under both equal and unequal wealth assignment. In all cases the output from the simulation was successfully compared with the E & G model solutions. Thus it would appear that the simulation constitutes an acceptable vehicle for implementation of the Pari-Mutual method.

IV. RESEARCH APPLICATIONS OF THE PARI-MUTUAL METHOD

Research addressing the relative accuracy and reliability of the Pari-Mutual method in generating consensus distributions appears necessary -- both in the form of experimentation and direct empirical testing. As noted earlier, the very fact that validation may be attempted with reference to both "real" and "created" settings enhances the method's initial attractiveness. The market referent which underlies the Pari-Mutual approach provides a readily available data base for large sample testing of the accuracy and reliability of these "large group" subjective probability distributions. The distributions themselves take the form of final odds which can easily be converted to probabilities and compared with actual outcomes. This comparison can take the form of comparison across all races, with the accuracy concept related to repetitive-type decision settings. Alternatively, one may look at the association of individual race odds distributions and individual race outcomes, where the accuracy concept relates to the single or unique one-time decision setting.

At the same time, experimentation with "small group" consensus involving comparison of the Pari-Mutual method with other mathematical consensus approaches and behavioral consensus approaches is also desirable. For example, prior studies could be extended or replicated to include the Pari-Mutual method.

Winkler (1971) studied football betting by comparing individual estimates of point spreads and various forms of consensus estimates. If that data remains available, it would be an easy task to generate a Pari-Mutual consensus for additional comparison. The Rowse, et.al., (1974) study of the accuracy and reliability of various aggregation techniques using firemen could also be broadened to encompasses the Pari-Mutual method.

Of course, new experiments also are desirable; not only because the potentially useful Pari-Mutual method remains untested, but also because a great need exists overall for experimental work concerning probability assessments by groups (Hogarth, 1975). For example, as an accounting researcher, one can perceive the assessment of input or output market values by experts (appraisors, real estate agents) as a particularly attractive setting--since the discipline may well be entering the "replacement cost" or "current value" era. Both accountants and auditors may well be concerned with aggregating subjective probability judgements of experts on value. However, any setting involving uncertainty and an actuarial data base for accuracy comparisons is a satisfactory setting for the needed experimental work. As such, the simulation model identified in this study would appear to provide a vehicle for extensive future research.

APPENDIX A

THE EISENBERG AND GALE MODEL

Eisenberg and Gale, in formulating their model, assume m individuals (bettors) are betting on a race with n horses. After careful study, the bettor determines his personal estimation of each horse's probability of winning. These are expressed as an m⨯n probability matrix (p_{ij}).

After determining these probabilities, the bettor places his bets in a way which maximizes his subjective expectation. The bettor, of course, does not usually bet all of his fixed wealth, b_i, on the horse for which his subjective probability is largest. Instead, he waits until the track probabilities π_j, are announced and then places his bets on the horses for which the ratio p_{ij}/π_j is a maximum. Therefore, the π_j depend on the bets and the bets depend on the π_j.

To solve for the track probabilities and, hence, the individual bets, Eisenberg and Gale define a function ϕ and show that the variables which maximize it yield a unique solution. This function has mn arguments ξ_{ij} and is defined by the rule:

$$\phi\ (\xi_{11},\ldots,\xi_{mn}) = \sum_{i=1}^{m} b_i \log \sum_{j=1}^{n} p_{ij}\cdot\xi_{ij}$$

the variables ξ_{ij} subject to the constraints:

$$\xi_{ij}\geq 0$$
$$\sum_{i=1}^{m} \xi_{ij} = 1$$

In order to simplify this ϕ function, consider the function $\psi = e^{\phi}$. Thus

$$\psi\ (\xi_{11},\ldots,\xi_{mn}) = \prod_{i=1}^{m} (\ \sum_{j=1}^{n} p_{ij}\cdot\xi_{ij})$$

clearly, maximizing ψ is equivalent to maximizing ϕ.

In particular, consider ψ for a case of three bettors with equal wealth and three horses. The probability matrix for the bettors' subjective probability

is assumed to be:

$$p_{ij} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 3/4 & 1/8 & 1/8 \\ 3/5 & 1/5 & 1/5 \end{bmatrix}$$

The problem then reduces to:

$$\max \; \psi = (1/2 \; \xi_{11} + 1/2 \; \xi_{12}) \cdot (3/4 \; \xi_{21} + 1/8 \; \xi_{22} + 1/8 \; \xi_{23}) \cdot$$

$$(3/5 \; \xi_{31} + 1/5 \; \xi_{32} + 1/5 \; \xi_{33})$$

subject to:

$$\sum_{i=1}^{3} \xi_{ij} = 1 \qquad j = 1,2,3$$

$$\xi_{ij} \geq 0 \quad \forall_{i,j}$$

The method of LaGrange Multipliers gives contradictory equations and, thus, the maximum must be on the boundary of the constrained region.

Solving the constraints for $i = 3$ (note that ξ_{13} can be assumed to be zero since $p_{13} = 0$) and substituting into the objective function ψ we have:

$$\max \; \psi = \frac{1}{80} \; (\xi_{11} + \xi_{12}) \cdot (6\xi_{21} + \xi_{22} + \xi_{23}) \cdot (5 - 3\xi_{11} - 3\xi_{21} - \xi_{12} - \xi_{22} - \xi_{23})$$
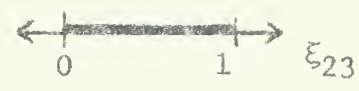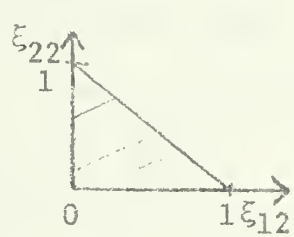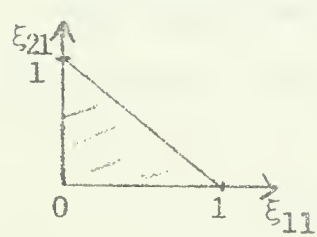
subject to:

$$0 \leq \xi_{11} + \xi_{21} \leq 1$$

$$0 \leq \xi_{12} + \xi_{22} \leq 1$$

$$0 \leq \xi_{23} \leq 1$$

For simplicity, the constant $1/80$ may be dropped during the maximization search. Graphically, the constraints are:

The maximum of ψ can be found by considering the maximum of ψ for each possible combination of the edges of these figures. The final maximum will be the maximum of these maxima.

For example, consider the first of the nine possible cases; $\xi_{11} = 0$, $\xi_{22} = 0$. The problem now is:

$$\max \psi = \xi_{12}(6\xi_{21} + \xi_{23})\cdot(5-3\xi_{21} -\xi_{12} -\xi_{23})$$

subject to $\quad 0\leq \xi_{21} \leq 1$

$$0\leq \xi_{12} \leq 1$$

$$0\leq \xi_{23} \leq 1$$

The domain is, thus,
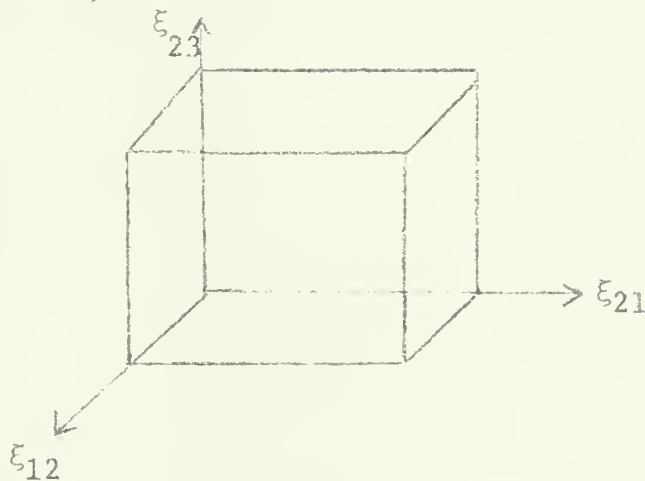
and the maximum is on an edge of this figure. Again reduce the problem by individual consideration of each of the twelve edges.

The maximum of these twelve subcases is $\xi_{12}=1$, $\xi_{21}=2/3$, $\xi_{23}=0$ Hence, the solution for case 1 is:

$$\psi_{max} = 8$$

$$\bar{\xi}_{11}=0, \quad \bar{\xi}_{12}=1, \quad \bar{\xi}_{21}=2/3, \bar{\xi}_{22}=0, \quad \bar{\xi}_{23}=0$$

Upon comparison with the other eight cases, this is shown to be the final maximum. Solving for the other $\bar{\xi}_{ij}$'s we have:

$$\xi_{ij} = \begin{pmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 0 \\ 1/3 & 0 & 1 \end{pmatrix}$$

Substituting into the equation for π_1:

$$\pi_i = \max_i \frac{b_i p_{ij}}{\sum\limits_S p_{is} \ \bar{\varepsilon}_{is}} \qquad b_i = 1/3 \ \forall \ i$$

we find:

$$\pi_1 = \max (1/3, \ 1/2, 1/2) = 1/2$$

$$\pi_2 = \max (1/3, \ 1/12, \ 1/6) = 1/3$$

$$\pi_3 = \max (0, \ 1/12, \ 1/6) = 1/6$$

APPENDIX B

THE SIMULATION MODEL

DECISION MODEL

The simulation model employed within this study replicates a pari-mutual market. All bettors utilize a decision model which dictates they bet all horses where the expected return[ER] of that bet is greater than one. Symbolically, this condition can be stated as:

$$ER > 1 \qquad (1)$$

The expected return can be decomposed thusly:

$$ER_{i,j} = SP_{i,j} \cdot ODDS_j \qquad (2)$$

Where: i = Bettor
  j = Horse
  SP = Subjective probability
  $ODDS$ = Equivalent odds based on all previous wagers

INPUTS

The simulation model--illustrated in Figure One--utilizes a series of inputs which relates to the specific events under study: Symbolically,

JJ = Number of "Bettors" (i.e., judges)
II = Number of "Horses" (i.e., events)
W_j = Wealth of each bettor (i.e., amount of relative "influence" of a given judge)
SP_j = Subjective probability vector of a given bettor with respect to the success of each horse.

The only input requiring discussion here is W_j. This variable represents the relative influence of each judge vis-a-vis one another. For example, if judge one is assigned twice as much influence as judge two, then the following values might be input to the simulation model: W_1 = 20,000 and W_2 = 10,000.

INITIALIZATION

The next major component of the model requires the initialization of the model parameters. While several are simply intrinsic to the specific

Start

READ:
(1) # of "judges"→JJ
(2) # of "events"→II
(3) "Influence" of each judge→W_j
(4) Subjective probability of each judge to each event→$SP_{i,j}$

Initialize parameters (including "overnight" consensus probabilities) $ODDS_{o,j}$

C → Begin "Market" Iteration→M where M=1, 1000

B → Begin "Judge" Iteration→J where J = 1,JJ

Begin "Event" Iteration→I where I = 1, II

Calculate "Expected Return" (ER) for each event and judge
$ER_{i,j} = SP_{i,j} \times ODDS_{m-1,j}$
where: ODDS = current value of market determined consensus probability (stated as odds)

is I >II ? → No / yes

Calculate "total bet" (TB) for Mth "market" iteration for judge "j"
$TB_j = W_j \cdot X$
where: $X = M / \sum_{i=1}^{1000} M_i$

Determine number of events (K_j) where $ER_{i,j} > 1.0$

Calculate amt. of individual "bet" (B_j)→$B_j = TB_j / K_j$

A

A

"Bet" on all events where $ER_{i,j} > 1$

Calculate total "bet" to date (across all judges) on event "i"
$POOL_{m,k} = POOL_{m-1,i} + B_j$

Any more events where $ER_{i,j} > 1.0$? → Yes / No

Calculate remaining "influence" for judge "j"
$W_{m,j} = W_{m-1,j} - TB_j$

is J > JJ ? → No → B / Yes

Calculate total bets (across all judges) through iteration "m"
$TP_m = \sum_{j=1}^{JJ} POOL_{m,j}$
where TB = total bets

Calculate consensus probability (ODDS) for each event "j"
$ODDS_{m,j} = \left(\frac{TP_m - POOL_{m,j}}{POOL_{m,j}} \right) + 1.0$

is J > JJ ? → No / Yes

is M > 1000 ? → No → C / Yes

Calculate & print final (π_i) consensus probabilities for all events.

END

computer program written, one which was not relates to "overnight odds".
That is, in a pari-mutual market a series of odds are determined by handi-
cappers which are placed on the tote board prior to actual betting. Regard-
less of whether o¯ not overnight odds imp¯ct upon the subjective probabili-
ties of "real world" bettors, the simulation result indicated they had no
impact upon the final consensus probabilities. (Random sets of overnight
odds were employed for several runs of the program with the final results
all being identical.) However, given the bettors' decision model, an
initial set of odds is required to begin the process.

ITERATIONS

Upon completion of parameter initialization, a series of iterations
begin. The first--termed "market" iteration -- encompasses a complete cycle
of the entire process. The program utilized one thousand market iterations.
While this number was arbitrarily selected, it was chosen with the rationale
that a large number was required in order for this market to reach an
equilibrium point.

The second iteration -- termed "judge" iteration -- simulates the
entire decision process (including wagering) for a given bettor. Several
phases were included within this iteratio... First, each bettor calculates
an expected return for horse$_j$ based on the formulation in equation 2. Second,
the better then calculates his total bet within the given market iteration.
This calculation is based upon the following formulation:

$$TB_{m,j} = W_{m,j} X \qquad (3)$$

Where: m ≠ Market iteration number

TB = Total bet

W = Remaining wealth

X = Proportion of remaing wealth bet on this round

The variable "X" is expressed as follows:

$$X = m \left/ \sum_{i=1}^{1000} m_i \right. \qquad (4)$$

This betting scheme basically states that in each market iteration the bettor will wager an infinitesimally larger portion of his remaining wealth. In total--over all market iterations--he wagers his entire initial wealth. (It should be emphasized that other wagering schemes were attempted. However, all produced the same final unique set of consensus probabilities.)

The third phase of the "judge" iteration required the bettor to bet an equal share of his total wager (i.e., TB from equation 3) on all horses where his expected return was greater than one (i.e., the condition expressed in equation 1). The fourth phase entailed updating various registers to maintain cumulative totals of all wagers (across all bettors) on each horse. Finally, the bettor's remaining wealth was adjusted to reflect his total wagers in that particular "market" iteration.

Once all bettors have made their wagers for a particular market iteration, a new set of odds are calculated. This process is completely analogous to the method found at a race track (excluding "cuts" to taxing bodies and track commissions) and can be stated as follows:

$$ODDS_{m,j} = \left[\frac{TP_m - POOL_{m,j}}{POOL_{m,j}} \right] + 1 \qquad (5)$$

Where: TP = Total bets across all horses

POOL = Total bets on an individual horse

These calculated odds are then used in the next "market" iteration. Two points should then be made with respect to equation 5. First, the number stated parenthetically is increased by one to represent the return of capital-- as in the "real world" pari-mutual market. Second, odds--rather than probabilities--

are calculated for purposes of the expected return decision model. These

odds are later converted to probabilities by simply taking their reciprocal.

Finally, upon completion of the 1000 "market" iterations, the final set of

consensus probabilities--π_i, per E & G's notation--are output.

SENSITIVITY ANALYSIS

While a true sensitivity analysis of the model is not included within

this presentation, the underlying process represented in this program is

relatively stable. That is, different aspects of the model were changed with

no variation in the final consensus probabilities. These aspects included:

(1) different methods of initializing the overnite odds, (2) different

wagering strategies--in terms of calculating the amount bet, and (3) different

numbers of market iterations. As such, it would appear the key elements

within the model--the expected return decision model and the calculation of

market odds--are the factors which drive the model to converge upon the

unique consensus probabilities. Moreover, it is these properties which

Eisenberg and Gale employed as essential components to their mathematical

proof of the unique set of final probabilities. As such this simulation

appears to capture the essence of their analysis.

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