# LEGENDRIAN AND TRANSVERSE KNOTS AND THEIR INVARIANTS

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Bulent Tosun

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# LEGENDRIAN AND TRANSVERSE KNOTS AND THEIR INVARIANTS

Approved by:

Professor John B. Etnyre, Advisor
School of Mathematics
*Georgia Institute of Technology*

Professor Dan Margalit
School of Mathematics
*Georgia Institute of Technology*

Professor Igor Belegradek
School of Mathematics
*Georgia Institute of Technology*

Professor Mohammad Ghomi
School of Mathematics
*Georgia Institute of Technology*

Professor William H. Kazez
Department of Mathematics
*University of Georgia, Athens*

Date Approved: June 7, 2012

*Zikê zaroka tijeye lê zimanê wan nagere...*

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# Chapter I

# INTRODUCTION

In this thesis we study Legendrian and transverse *s*implicity and *n*on-simplicty problem under the cabling operation. The terminology and more precise statements of the following theorems will be given in order in Chapter 2, 3, 4, 5 and 6.

## 1.1 Sufficiently positive and negative cables

An $(r, s)$-curve on the boundary of a solid torus refers to the curve $r[\lambda] + s[\mu]$, where $\lambda, \mu$ is the longitude-meridian basis for the homology of the torus, and we denote this by the fraction $\frac{s}{r}$. The $(r, s)$-cable of a knot type $\mathcal{K}$, denoted $\mathcal{K}_{(r,s)}$, is the knot type obtained by taking the $(r, s)$-curve on the boundary of a tubular neighborhood of a representative of $\mathcal{K}$.

**Theorem 1.1.1** ([32])**.** *If $\mathcal{K}$ is Legendrian simple and $\omega(\mathcal{K}) \in \mathbb{Z}$, then $\mathcal{K}_{(r,s)}$ is Legendrian and transverse simple, provided $\frac{r}{s} > \omega(\mathcal{K})$.*

**Theorem 1.1.2** ([32])**.** *If $\mathcal{K}$ is Legendrian simple and $\ell\omega(\mathcal{K}) \in \mathbb{Z} \cup \infty$, then $\mathcal{K}_{(r,s)}$ is Legendrian and transverse simple, provided $\frac{r}{s} < \ell\omega(\mathcal{K})$.*

Moreover, in both cases the classification of Legendrian and transverse knots in the knot type $\mathcal{K}_{(r,s)}$ is determined by the classification of such knots in the knot type $\mathcal{K}$.

Note that for the unknot $\mathcal{U}$ we have $w(\mathcal{U}) = 0$ and $\ell\omega(\mathcal{U}) = \infty$. So, as an immediate corollary of Theorem 1.1.1 and 1.1.2 we obtain

**Corollary 1.1.3.** *Torus knots are Legendrian and transverse simple.*

This result was originally proved by Etnyre and Honda in [14]. Also observe that in case $w(\mathcal{K}) = \ell\omega(\mathcal{K}) \in \mathbb{Z}$ we get that $\mathcal{K}$ is uniformly thick and recover a result of Etnyre and Honda in [13] that says $\mathcal{K}_{(r,s)}$ is Legendrian simple if $\mathcal{K}$ is Legendrian simple and $\mathcal{K}$ is uniformly thick. The last observation prompts the following question.

**Question 1.1.4.** *Is there a knot type $\mathcal{K}$ which is not uniformly thick? If so what can one say about the classification of Legendrian and transverse knots in $\mathcal{K}_{(r,s)}$ when $w(\mathcal{K}) < \frac{r}{s} < \ell\omega(\mathcal{K})$?*

## *1.2 Non-simple cables*

We address Question 1.1.4 for the $(2,3)$- torus knot and then in joint work with Etnyre and LaFountain we extend the result to the other positive torus knots. We begin with the notation

$$\mathcal{L}_{(r,t)}(\mathcal{K}) = \{L \in \mathcal{L}(\mathcal{K}) : \mathrm{tb}(L) = t \text{ and } \mathrm{r}(L) = r\}.$$

We similarly denote the set of transverse knots isotopic to $\mathcal{K}$ by $\mathcal{T}(\mathcal{K})$ and the ones having self-linking number $s$ by $\mathcal{T}_s(\mathcal{K})$.

We first consider cables of the right handed trefoil, that is, the $(2,3)$-torus knot. We have a complete classification indicated in Figure 1, but since the statement is technical we state a corollary of the classification here.

**Theorem 1.2.1** (Etnyre, LaFountain and Tosun [16])**.** *Let $\mathcal{K}$ be the positive trefoil knot in $S^3$. The knot $\mathcal{K}_{(r,s)}$ formed by $(r,s)$-cabling $\mathcal{K}$ is Legendrian simple if and only if $\frac{s}{r} \notin (1,\infty)$. Furthermore, given positive integers $k$, $m$, and $n$, where $n > 1$ and $\gcd(k,m) = 1$, there exists a slope $\frac{s}{r} \in (1,\infty)$ such that $\mathcal{L}_{(u,t)}(\mathcal{K}_{(r,s)})$ contains $n$ Legendrian knots for some pair of integers $(u,t)$ with $t = \overline{\mathrm{tb}}(\mathcal{K}_{(r,s)}) - m$; moreover, one of these does not destabilize, and they remain distinct when stabilized fewer than $k$ times (and there are $k$ stabilizations that will make them isotopic).*

Note that this theorem gives the first example of a knot type with non-destabilizable Legendrian knots with Thurston-Bennequin invariant arbitrarily far from the maximal Thurston-Bennequin invariant. It gives yet another family of knots which have arbitrarily many Legendrian knots with fixed classical invariants. We also observe that this theorem gives the first set of prime Legendrian knots with the same invariants that require arbitrarily many stabilizations before becoming Legendrian isotopic. See Figure 1.

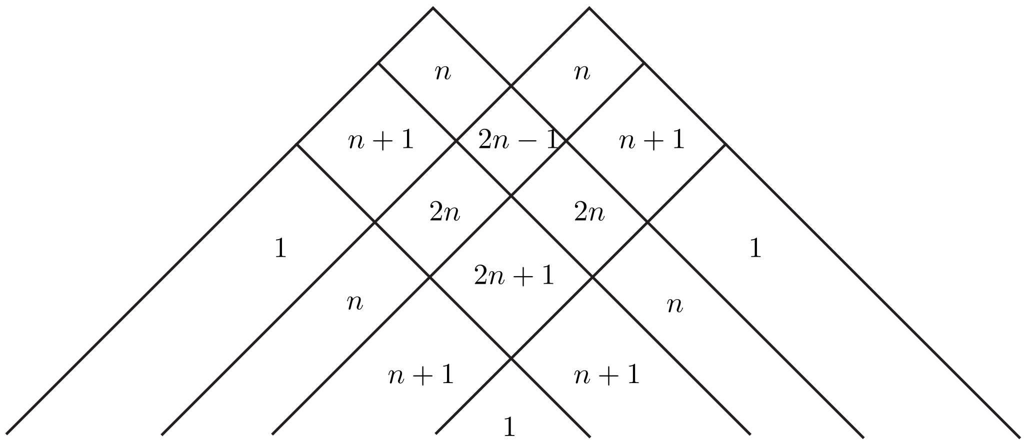


**Figure 1:** The image of $\mathcal{L}(\mathcal{K}_{(r,s)}) \to \mathbb{Z}^2 : L \mapsto (\mathrm{r}(L), \mathrm{tb}(L))$ for non-simple cablings of the positive trefoil with $\frac{s}{r} \in (n, n+1)$. The number of Legendrian knots realizing each point in $\mathbb{Z}^2$ whose coordinates sum to an odd number is indicated in the figure. The exact width of each region depends on the pair $(r, s)$.

**Theorem 1.2.2** (Etnyre, LaFountain and Tosun [16]). *Let $\mathcal{K}$ be the positive trefoil knot in $S^3$. The knot $\mathcal{K}_{(r,s)}$ formed by $(r,s)$-cabling $\mathcal{K}$ is transversely simple if and only if $\frac{s}{r} \notin (1, \infty)$. Furthermore, given positive integers $k$, $m$, and $n$, where $n > 2$ and $\gcd(k, m) = 1$, let $p = k(n-1) + m(n-2)$. Then there is some $\frac{s}{r} \in (1, \infty)$ such that $\mathcal{T}(\mathcal{K}_{(r,s)})$ contains $(n-1)$ distinct transverse knots with $sl = \overline{sl}(\mathcal{K}_{(r,s)}) - 2p$, of which $(n-2)$ are non-destabilizable, and such that there is another non-destabilizable knot with $sl = \overline{sl}(\mathcal{K}_{(r,s)}) - 2(p+m)$. Moreover, these non-destabilizable knots must be stabilized until their self-linking number is $\overline{sl}(\mathcal{K}_{(r,s)}) - 2(p+m+k)$ before they become transversely isotopic.*

Note that Theorem 1.2.2 gives not just an infinite family of transversely non-simple prime knot types, but also demonstrates three new phenomena concerning transverse knots that were not previously known. More precisely, it gives the first example of knot types that have transverse knots with the same self-linking number that require arbitrarily many stabilizations before they become transversely isotopic, and it also gives the first examples where there are non-destabilizable transverse knots whose self-linking number is arbitrarily far from maximal. Finally, the theorem gives the first knot type where there are non-destabilizable knots with distinct self-linking numbers.

We recall that a Legendrian knot in an overtwisted contact manifold is caled *non-loose* if it has tight complement. The complete classification of Legendrian cables of the positive trefoil above also supply the following interesting result.

**Theorem 1.2.3.** *Given an integer $k$, there is an overtwisted contact manifold $(M, \xi)$ and distinct prime non-loose Legendrian knots $L_1, ..., L_k$ in the same knot type with the same Thurston-Bennequin number and rotation number.*

We note that previously known (arbitrarily many) examples were all non-prime examples. See [12].

With all the interesting and complicated behavior exhibited by cables of the right handed trefoil knot, one would expect to see behavior at least as complicated for cables of other positive torus knots. Surprisingly, cables of such knots turn out to be relatively simple.

**Theorem 1.2.4** (Etnyre, LaFountain and Tosun [16])**.** *Let $\mathcal{K}$ be a positive $(p,q)$-torus knot with $(p,q) \neq (2,3)$. Then for any rational number $\frac{s}{r}$ and any $(u,t)$ with $t+u$ odd, there are at most 3 Legendrian knots in $\mathcal{L}_{(u,t)}(\mathcal{K}_{(r,s)})$ and at most 2 for all but one pair $(u,t)$.*

**Theorem 1.2.5** (Etnyre, LaFountain and Tosun [16])**.** *Let $\mathcal{K}$ be a positive $(p,q)$-torus knot with $(p,q) \neq (2,3)$. Then for any rational number $\frac{s}{r}$ there are at most two transverse knots isotopic to the $(r,s)$-cable of $\mathcal{K}$ with the same self-linking number. However, for any positive integers $n$ and $m$ with $\gcd(m,n) = 1$, there is a rational* *number $\frac{s}{r} > 0$ for which there is a non-destabilizable transverse knot with self-linking number at most $\overline{sl}(\mathcal{K}_{(r,s)}) - 2n$ and it must be stabilized exactly $m$ times to become isotopic to the destabilizable transverse knot with the same self-linking number.*

## 1.3 Non-thickenable and partially thickenable neighborhoods

A key feature in the knot classification results above is a complete understanding of not only non-thickenable tori but also *partially thickenable* tori (See Chapter 3), that is tori with convex boundary that thicken, but not to a maximally thick torus in the given knot type. The existence of such tori has not been observed before, but it is clear that such tori will be key to future Legendrian classification results. In addition, it is likely they will be important in understanding contact surgeries on Legendrian and transverse knots (See Chapter 7).

**Theorem 1.3.1.** *Let $S$ be a solid torus in the knot type of a positive $(p, q)$-torus knot. In the standard tight contact structure $\xi_{std}$ on $S^3$ suppose that $\partial S$ is convex with two dividing curves of slope $\frac{s}{r}$. Then $S$ thickens unless $\frac{s}{r}$ is an exceptional slope*

$$e_k = \frac{k}{pq - p - q},$$

*for some positive integer $k$, in which case it might or might not thicken.*

*Moreover for each positive integer $k > 1$ there are, up to contact isotopy, exactly two solid tori $N_k^{\pm}$ with convex boundary having $2n_k$ dividing curves of slope $e_k$ that do not thicken, where $n_k = \gcd(pq - p - q, k)$. For $k = 1$ there is exactly one solid torus $N_1$ with convex boundary having two dividing curves of slope $e_1$. This solid torus is a standard neighborhood of a Legendrian $(p, q)$-torus knots with maximal Thurston-Bennequin invariant and it does not thicken.*

The following theorem shows that partially thickenable tori exist for the $(2, 3)$-torus knot.

**Theorem 1.3.2.** *Let $\mathcal{K}$ be a positive $(2,3)$-torus knot and let $e_k = k$ be the exceptional slopes. Let $I_k = [k, \infty)$ (clearly $I_k \subset I_{k+1}$). All solid tori below will represent the knot type $\mathcal{K}$.*

1. *Any solid torus $S$ with convex boundary thickens to $N_k^{\pm}$ or to $N_1$ (that is a neighborhood of the maximal Thurston-Bennequin invariant $(2,3)$-torus knot).*

2. *Any solid torus inside $N_k^{\pm}$ with convex boundary having dividing slope in $I_k$ does not thicken past the slope $e_k$.*

3. *Any solid torus inside $N_k^{\pm}$ with convex boundary having negative (or infinite) dividing slope will thicken to a neighborhood of the maximal Thurston-Bennequin invariant $(2,3)$-torus knot.*

From this theorem we can classify solid tori in the knot type of $(2,3)$-torus knot.

**Corollary 1.3.3.** *Let $\mathcal{K}$ be the $(2,3)$-torus knot,*

1. *Given a slope $s > 1$ there is some integer $n$ such that $n \leq s < n+1$ and there are exactly $2n$ solid tori representing the knot type $\mathcal{K}$ with convex boundary having dividing slope $s$ and two dividing curves, only two of which thicken to a standard neighborhood of a Legendrian knot.*

2. *Given any negative slope $s$ there is some negative integer $n < 0$ such that $\frac{1}{n+1} < s < \frac{1}{n}$. A solid torus with convex boundary having dividing slope $s$ and two dividing curves will thicken to a solid torus that is a standard neighborhood of a $\mathrm{tb} = n+1$ Legendrian knot.*

The classification of solid tori in the knot types of positive torus knots, other than trefoil, is also obtained in [16]. We include the statement of this result and refer [16] for its proof.

**Theorem 1.3.4.** *Let $\mathcal{K}$ be a positive $(p,q) \neq (2,3)$-torus knot and let $e_k = \frac{k}{pq-p-q}$ be the exceptional slopes. Let $I_k = [e_k, e_k^a)$ and $\mathcal{I} = \{n \in \mathbb{Z} : n > 1$ and $\gcd(n, pq-p-q) = 1\}$. All solid tori below will represent the knot type $\mathcal{K}$.*

1. *For any $k \notin \mathcal{I}$, any solid torus $S$ inside $N_k^{\pm}$ with either boundary slope different from $e_k$, or less than $2n_k$ dividing curves, thickens to $N_1$.*

2. *All the $I_k$ with $k \in \mathcal{I}$ are disjoint.*

3. *Any solid torus $S$ with convex boundary having dividing slope in $I_k$ thickens to $N_k^{\pm}$ or to $N_1$ (that is a neighborhood of the maximal Thurston-Bennequin invariant $(p,q)$-torus knot).*

4. *Any solid torus inside $N_k^{\pm}$ for some $k \in \mathcal{I}$, and with convex boundary having dividing slope in $I_k$, does not thicken past the slope $e_k$.*

5. *Any solid torus inside $N_k^{\pm}$ with convex boundary having dividing slope outside of $I_k$ (that is greater than or equal to $e_k^a$ or negative) will thicken to a neighborhood of the maximal Thurston-Bennequin invariant $(p,q)$-torus knot.*

From this theorem one can classify solid tori in the knot types of positive torus knots.

**Corollary 1.3.5.** *Let $\mathcal{K}$ be a positive $(p,q)$-torus knot and let $e_k = \frac{k}{pq-p-q}$ be the exceptional slopes. Let $I_k = [e_k, e_k^a)$ and $\mathcal{I} = \{n \in \mathbb{Z} : n > 1$ and $\gcd(n, pq-p-q) = 1\}$. Given any slope $s \geq \frac{1}{pq-p-q}$ we have the following.*

1. *If there is some integer $n > 0$ such that $\frac{1}{n} < s < \frac{1}{n-1}$ and $s \notin I_k$ for any $k \in \mathcal{I}$, then there are exactly $2(pq-p-q-n+1)$ solid tori representing the knot type $\mathcal{K}$ with convex boundary having dividing slope $s$ and two dividing curves each of which thickens to a standard neighborhood of a Legendrian knot with $\mathrm{tb} = n$.*

2. *If there is some integer $n > 0$ such that $\frac{1}{n} < s < \frac{1}{n-1}$ and $s \in I_k$ for any $k \in \mathcal{I}$, then there are exactly $2(pq - p - q - n + 1) + 2$ solid tori representing the knot type $\mathcal{K}$ with convex boundary having dividing slope $s$ and two dividing curves, all but two of which thicken to a standard neighborhood of a Legendrian knot with* $\text{tb} = n$.

3. *If there is some $n > 0$ such that $s = \frac{1}{n}$, then there are exactly $pq-p-q-n+1$ solid tori representing the knot type $\mathcal{K}$ with convex boundary having dividing slope $s$ and two dividing curves and they each represent a standard neighborhood of a Legendrian knot with* $\text{tb} = n$.

This thesis organized as follows. In Chapter 2 we collect needed preliminaries, including facts about continued fractions and convex surfaces, and we outline a strategy for classifying Legendrian knots. In Chapter 3 we give the more precise statements of Theorem 1.1.1, 1.1.2 and their proofs. In Chapter 4 we classify embeddings of solid tori representing the $(2, 3)$-torus knot, that is we prove Theorem 1.3.1, 1.3.2 and Corollary 1.3.3. In Chapter 5 we provide more precise statements of Theorem 1.2.1, 1.2.2 and then establish classifications for all non-simple cables of the positive trefoil and provide the proof of Theorem 1.2.3. In Chapter 6 we provide more precise statements of Theorem 1.2.4, 1.2.5 and establish classifications for all non-simple cables of positive torus knots and finally, in Chapter 7, we give future directions in the light of what have been developed in this thesis.

# Chapter II

# BACKGROUND

A *contact structure* on a 3–manifold is a 2–plane field $\xi$ in the tangent bundle that is maximally non-integrable. This means that the 2–planes are not tangent, even locally, to a foliation. Studying contact structures on 3–manifolds is important in its own right but also has a crucial role in our understanding of topology and geometry in 3–dimensions. For example, Eliashberg and Thurston have found relations between contact geometry and foliation theory [9]. Giroux has found a close relation between contact structures and fibered links [19]. Eliashberg has used them to understand diffeomorphisms of $S^3$ [8]. Contact geometry was also an important ingredient of the following beautiful results. Kronheimer and Mrowka's proof that all non-trivial knots satisfy property $P$, Ozsváth and Szabó's proof that that the unknot, trefoil and figure eight knots are all determined by surgery, and Ozsváth and Szabó's proof that Heegaard-Floer invariants detect the Thurston norm of a manifold and the minimal Seifert genus of a knot. Contact structures on 3–manifolds fall into two disjoint classes: *overtwisted* and *tight*. Eliashberg has shown [7] that studying overtwisted contact structures amounts to studying homotopy class of 2–plane fields and hence algebraic in nature, while tight contact structures are more subtle and more intimately related to the topology of 3–manifolds.

## *2.1 Legendrian and transverse knots in tight contact structures*

The study of knots that respect a contact structure in a certain way has been very important, because, they capture the geometry and topology of underlying contact structures very well. For example, the classical invariants $\mathrm{tb}(L)$ and $\mathrm{r}(L)$ (see below) associated to a *Legendrian knot* $L$ (those tangent to contact planes) have been used by Eliashberg [8] (following the work of Bennequin [2]) to prove that a contact structure is *tight* if and only if for any knot type $\mathcal{K}$ and all Legendrian knot $L \in \mathcal{K}$ the following inequality holds

$$\mathrm{tb}(L) + |\mathrm{r}(L)| \leq 2g(\mathcal{K}) - 1, \tag{1}$$

where $g(\mathcal{K})$ is the genus of the knot $\mathcal{K}$. Rudolph [31] further extended this result to find obstructions to slicing a knot. *Transverse knots* (those transverse to contact planes), on the other hand, have shown to be very powerful in the work of Giroux [19] on fibered links and work of Bennequin [2] and Birman-Menesco [3] on braid theory. Despite their importance, not very much is known concerning the classification of Legendrian and transverse knots in general. There are two simple invariants of a Legendrian knot $L$, the *Thurston-Bennequin* number, denoted $\mathrm{tb}(L)$ which measures the framing $\xi$ gives to $L$ and *rotation number*, denoted by $\mathrm{r}(L)$ which is more or less a relative Euler class, and there is only one invariant of a transverse knot $T$, the *self-linking number*, denoted $sl(T)$. We say a knot type is *Legendrian simple* (respectively *transversely simple*) if Legendrian knots (respectively transverse knots) in the knot type are determined by their simple invariant(s).

One major problem in 3–dimensional contact geometry is classification of Legendrian/transverse knots up to Legendrian/transverse isotopy, that is an isotopy through Legendrian/transverse knots. Traditionally this problem has been either worked for some nice class of knots [14, 17] or under certain toplogical operations [15, 13]. We want to study this problem under cabling operation. Studying Legendrian and transverse knots in cabled knot types has been very fruitful. For example, in [1] cabling was used to better understand open book decompositions of contact structures; in particular, leading to non-positive monodromy maps supporting Stein fillable contact structures, monoids in the mapping class group associated to contact geometry and procedures to construct open books on manifolds after allowable transverse surgery (from an open book for the original contact manifold). Moreover, the first classification of a non-transversely simple knot type was done in [13] for the $(2,3)$-cable of the $(2,3)$-torus knot. In that paper it was also shown that studying solid tori with convex boundary that represent a given knot type (that is, their core curves are in a given knot type) is key to understanding cables; such an analysis for solid tori representing negative torus knots yielded simple Legendrian and transverse classifications for cables of negative torus knots. Tori representing iterated cables of torus knots were further studied in [29, 30] as well as [32].

In the following we will give a detailed explanations of background material.

## 2.2 Continued fractions and interval of influence

In this section we collect various facts about continued fractions that will be needed throughout our work.

Given a rational number $u > 0$ we may represent it as a continued fraction

$$u = a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 \ldots - \cfrac{1}{a_n}}}$$

with $a_0 \geq 1$ and the other $a_i > 1$. We will denote this as $u = [a_0; a_1, \ldots, a_n]$. If we know that $u = [a_0; a_1, \ldots, a_n]$ then we define

$$u^a = [a_0; a_1, \ldots, a_{n-1}],$$

with the convention that if $n = 0$ then $u^a = \infty$; we also define

$$u^c = [a_0; a_1, \ldots, a_n - 1].$$

**Lemma 2.2.1.** *The number $u^a$ is the largest rational number bigger than $u$ with an edge to $u$ in the Farey tessellation and $u^c$ is the smallest rational number less than $u$*

*with an edge to u in the Farey tessellation. Moreover there is an edge in the Farey tessellation between* $u^a$ *and* $u^c$ *and* $u$ *is the mediant of* $u^a$ *and* $u^c$*, that is if* $u^a = \frac{p^a}{q^a}$ *and* $u^c = \frac{p^c}{q^c}$ *then*

$$u = \frac{p^a + p^c}{q^a + q^c}.$$

*Proof.* Define $\frac{p_k}{q_k} = [a_0; a_1, \ldots, a_k]$ , $p_{-1} = 1, q_{-1} = 0.$ and $p_{-2} = 0, q_{-2} = -1$. One may easily verify using induction that

$$p_{k+1} = a_{k+1}p_k - p_{k-1}, \text{ and } q_{k+1} = a_{k+1}q_k - q_{k-1}.$$

From this one can inductively deduce that

$$p_{k+1}q_k - p_k q_{k+1} = -1.$$

Thus there is an edge in the Farey tessellation between $u = \frac{p}{q} = \frac{p_n}{q_n}$ and $u^a = \frac{p^a}{q^a} = \frac{p_{n-1}}{q_{n-1}}$. Similarly, let $\frac{c_k}{d_k} = [a_k; a_{k+1}, \ldots, a_n]$ and $\frac{c'_k}{d'_k} = [a_k; a_{k+1}, \ldots, a_n - 1]$ and notice that $c_n d'_n - d_n c'_n = a_n - (a_n - 1) = 1$. Now we see that

$$\frac{c_k}{d_k} = a_k - \frac{1}{c_{k+1}/d_{k+1}} = \frac{a_k c_{k+1} - d_{k+1}}{c_{k+1}}$$

and a similar expression for $\frac{c'_k}{d'_k}$ and induction yield $c_k d'_k - d_k c'_k = 1$. In particular, there is an edge in the Farey tessellation between $u = \frac{c_0}{d_0}$ and $u^c = \frac{c'_0}{d'_0}$.

Finally by setting $\frac{c''_k}{d''_k} = [a_k; a_{k+1}, \ldots, a_{n-1}]$ and noticing that $c''_{n-1}d'_{n-1} - d''_{n-1}c'_{n-1} = 1$, we can use the above formulas, and analogous ones, to inductively prove that $c''_{k-1}d'_{k-1} - d''_{k-1}c'_{k-1} = 1$. This establishes an edge in the Farey tessellation between $u^c = \frac{c'_0}{d'_0}$ and $u^a = \frac{c''_0}{d''_0}$. Since there is an edge in the Farey tessellation between each pair of numbers in the set $\{u, u^a, u^c\}$ the lemma is established by noticing that the numerators (and denominators) of $u^a$ and $u^c$ are both smaller than the numerator (and denominator) of $u$. □

Given a rational number $u = \frac{s}{r} > 0$ let $u^a$ be the largest rational number with an edge in the Farey tessellation to $u$. See Figure 2. (The $a$ superscript stands for "anti-clockwise", as $u^a$ is anti-clockwise of $u$ in the Farey tessellation.) Similarly the smallest rational number with an edge in the Farey tessellation to $u$ will be denoted by $u^c$. A formula for computing these numbers will be given in Subsection 2.2. We will refer to the interval $(u^c, u^a)$ as the *interval of influence* for $u$.

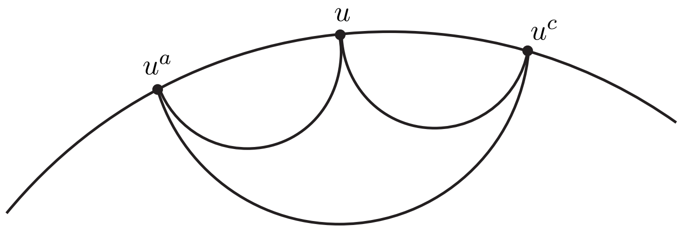


**Figure 2:** Given a rational number $u$, the numbers $u^a$ and $u^c$ are determined by the above figure in the Farey tessellation.

Given a positive $(p, q)$-torus knot and $k$ a positive integer, define

$$e_k = \frac{k}{pq - p - q}$$

We will see in Subsection 4.1 that such $e_k$ represent boundary slopes of *non-thickenable* solid tori, and that the half-intervals of influence $(e_k, e_k^a)$ will represent boundary slopes of *partially thickenable* solid tori when $\gcd(k, pq - p - q) = 1$. We will refer to the $e_k$ as *exceptional slopes.* If we think of the fractions $e_k^*$ as representing curves on a torus, we denote the homological intersection of $(r, s)$ curves with the $e_k^*$ curves by

$$\frac{s}{r} \cdot e_k^*.$$

**Lemma 2.2.2.** *Fix some positive integer $n$ and set $e_k = \frac{k}{n}$ for $k \in \{1, 2, \ldots\}$ and $\mathcal{I} = \{k \in \mathbb{Z} : k > 1$ and $\gcd(n, k) = 1\}$. If $n \neq 1$ then the intervals $J_k = (e_k^c, e_k^a)$ for $k \in \mathcal{I}$ are all disjoint. If $n = 1$ then the intervals are nested $J_{k+1} \subset J_k$.*

*If $r$ is a positive rational number less than $e_k^c$ or greater than $e_k^a$ then for any $s \in \overline{J_k}$ we have*

$$|r \cdot s| \geq \min\{|r \cdot e_k^a|, |r \cdot e_k^c|\}$$

*with equality only if $s = e_k^a$ or $e_k^c$.*

*If* $r \in (e_k^c, e_k)$ *and* $s \in (e_k, e_k^a)$, *then*

$$|r \cdot s| > |r \cdot e_k^a|.$$

## 2.3 Convex surfaces, bypasses and the Farey tessellation

Recall a surface $\Sigma$ in a contact manifold $(M, \xi)$ is *convex* if it has a neighborhood $\Sigma \times I$, where $I = (-\epsilon, \epsilon)$ is some interval, and $\xi$ is $I$-invariant in this neighborhood. Any closed surface can be $C^\infty$-perturbed to be convex. Moreover if $L$ is a Legendrian knot on $\Sigma$ for which the contact framing is non-positive with respect to the framing given by $\Sigma$, then $\Sigma$ may be perturbed in a $C^0$ fashion near $L$, but fixing $L$, and then again in a $C^\infty$ fashion away from $L$ so that $\Sigma$ is convex.

Given a convex surface $\Sigma$ with $I$-invariant neighborhood let $\Gamma_\Sigma \subset \Sigma$ be the multicurve where $\xi$ is tangent to the $I$ factor. This is called the *dividing set* of $\Sigma$. If $\Sigma$ is oriented it is easy to see that $\Sigma \setminus \Gamma = \Sigma_+ \cup \Sigma_-$ where $\xi$ is positively transverse to the $I$ factor along $\Sigma_+$ and negatively transverse along $\Sigma_-$. If $L$ is a Legendrian curve on a $\Sigma$ then the framing of $L$ given by the contact planes, relative to the framing coming from $\Sigma$, is given by $-\frac{1}{2}(L \cdot \Gamma)$. Moreover if $L = \partial\Sigma$ then the rotation number of $L$ is given by $\mathrm{r}(L) = \chi(\Sigma_+) - \chi(\Sigma_-)$.

### 2.3.1 Convex tori

A convex torus $T$ is said to be in *standard form* if $T$ can be identified with $\mathbb{R}^2/\mathbb{Z}^2$ so that $\Gamma_T$ consists of $2n$ horizontal curves (note $\Gamma_T$ will always have an even number of curves and we can choose a parameterization to make them horizontal) and the characteristic foliations consists of $2n$ vertical lines of singularities ($n$ lines of sources and $n$ lines of sinks) and the rest of the foliation is by non-singular lines of slope $s$. SeeFigure 3

The lines of singularities are called *Legendrian divides* and the other curves are called *ruling curves*. We notice that the Giroux Flexibility Theorem allows us to isotope any convex torus into standard form.

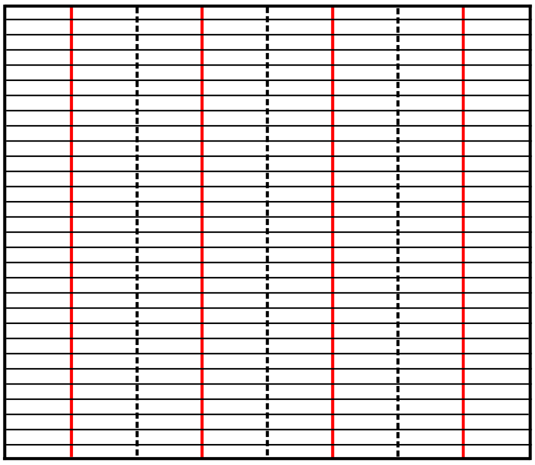

**Figure 3:** Standard convex tori. The thicker dashed curves are dividing curves. The horizontal thin lines are rulling curves of slope 0.

### 2.3.2 Bypasses and tori

Let $\Sigma$ be a convex surface and $\alpha$ a Legendrian arc in $\Sigma$ that intersects the dividing curves $\Gamma_\Sigma$ in 3 points $p_1, p_2, p_3$ (where $p_1, p_3$ are the end points of the arc). Then a *bypass for $\Sigma$ (along $\alpha$)*, is a convex disk $D$ with Legendrian boundary such that

1. $D \cap \Sigma = \alpha$,
2. $tb(\partial D) = -1$,
3. $\partial D = \alpha \cup \beta$,
4. $\alpha \cap \beta = \{p_1, p_3\}$ are corners of $D$ and elliptic singularities of $D_\xi$.

The bypass attachment operation is the basic unit of isotopy of surfaces and will be crucial in our proofs. It is given in the following theorem.

**Theorem 2.3.1** (Honda 2000, [22]). *Let $\Sigma$ be a convex surface, $D$ a bypass for $\Sigma$ along vertical $\alpha$ in $\Sigma$ (Figure 3), then there exists a neighborhood of $\Sigma \cup D \subset M$ diffeomorphic to $\Sigma \times [0,1]$, such that $\Sigma = \Sigma_0$, $\Sigma_1$ are convex, $\Sigma \times [0, \epsilon]$ is $I$–invariant and $\Gamma_\Sigma$ is related to $\Gamma_{\Sigma_1}$ as in Figure 4.*

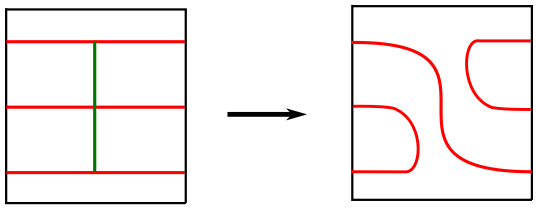

**Figure 4:** Original surface $\Sigma$ with bypass arc $\alpha$, on the left. The surface $\Sigma_1$ after isotoping $\Sigma$ across $D$, on the right.

A surface $\Sigma$ locally separates the ambient manifold. If a bypass is contained in the (local) piece of $M \setminus \Sigma$ that has $\Sigma$ as its oriented boundary then we say the bypass will be attached to the front of $\Sigma$ otherwise we say it is attached to the back of $\Sigma$.

When a bypass is attached to a torus $T$ then either the dividing curves do not change, their number increases by two, or decreases by two, or the slope of the dividing curves changes. The slope of the dividing curves can change only when there are two dividing curves. If the bypass is attached to $T$ along a ruling curve then either the number of dividing curves decreases by two or the slope of the dividing curves changes. To understand the change in slope we need the following. Let $\mathbb{D}$ be the unit disk in $\mathbb{R}^2$. Recall the *Farey tessellation* of $\mathbb{D}$ is constructed as follows. Label the point $(1,0)$ on $\partial\mathbb{D}$ by $0 = \frac{0}{1}$ and the point $(-1,0)$ with $\infty = \frac{1}{0}$. Now join them by a geodesic. If two points $\frac{p}{q}$, $\frac{p'}{q'}$ on $\partial\mathbb{D}$ with non-negative $y$-coordinate have been labeled then label the point on $\partial\mathbb{D}$ half way between them (with non-negative $y$-coordinate) by $\frac{p+p'}{q+q'}$. Then connect this point to $\frac{p}{q}$ by a geodesic and to $\frac{p'}{q'}$ by a hyperbolic geodesic. Continue this until all positive fractions have been assigned to points on $\partial\mathbb{D}$ with non-negative $y$-coordinates. Now repeat this process for the points on $\partial\mathbb{D}$ with non-positive $y$-coordinate except start with $\infty = \frac{-1}{0}$. See Figure 6.

The key result we need to know about the Farey tessellation is given in the following theorem. See Figure 6.

**Theorem 2.3.2** (Honda 2000, [22])**.** *Let $T$ be a convex torus in standard form with $|\Gamma_T| = 2$, dividing slope $s$ and ruling slope $r \neq s$. Let $D$ be a bypass for $T$ attached to*

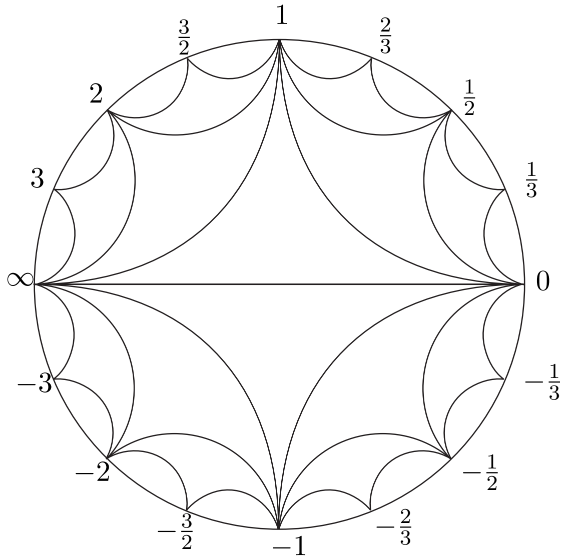


**Figure 5:** The Farey tessellation.

*the front of $T$ along a ruling curve. Let $T'$ be the torus obtained from $T$ by attaching the bypass $D$. Then $|\Gamma_{T'}| = 2$ and the dividing slope $s'$ of $\Gamma_{T'}$ is determined as follows: let $[r, s]$ be the arc on $\partial\mathbb{D}$ running from $r$ counterclockwise to $s$, then $s'$ is the point in $[r, s]$ closest to $r$ with an edge to $s$.*

*If the bypass is attached to the back of $T$ then the same algorithm works except one uses the interval $[s, r]$ on $\partial\mathbb{D}$.* □

### 2.3.3 The Imbalance Principle

As we see that bypasses are useful in changing dividing curves on a surface we mention a standard way to try to find them called the Imbalance Principle. Suppose that $\Sigma$ and $\Sigma'$ are two disjoint convex surfaces and $A$ is a convex annulus whose interior is disjoint from $\Sigma$ and $\Sigma'$ but its boundary is Legendrian with one component on each surface. If $|\Gamma_\Sigma \cdot \partial A| > |\Gamma_{\Sigma'} \cdot A|$ then there will be a dividing curve on $A$ that cuts a disk off of $A$ that has part of its boundary on $\Sigma$. It is now easy to use the Giroux Flexibility Theorem to show that there is a bypass for $\Sigma$ on $A$.

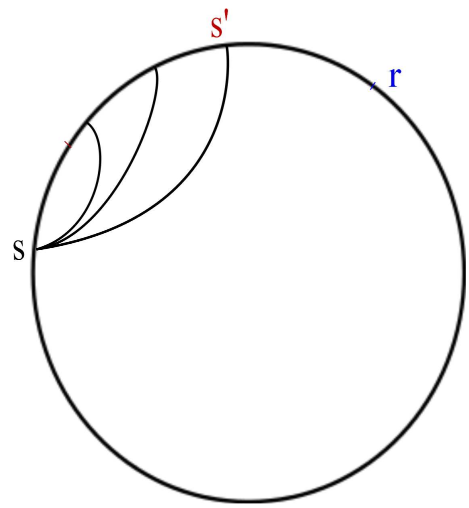


**Figure 6:** The Farey tessellation on the left. Schematic of the change in the dividing slope from $s$ to $s'$ after bypass attachment along a Legendrian rulling curve of slope $r$ on the Farey tessellation on the right.

### 2.3.4 Discretization of Isotopy

We will frequently need to analyze what happens to the contact geometry when we have a topological isotopy between two convex surfaces $\Sigma$ and $\Sigma'$. This can be done by the technique of *Isotopy Discretization* [4] (see also [14] for its use in studying Legendrian knots). Given an isotopy between $\Sigma$ and $\Sigma'$ one can find a sequence of convex surfaces $\Sigma_1 = \Sigma, \Sigma_2, \ldots, \Sigma_n = \Sigma'$ such that

1. all the $\Sigma_i$ are convex and
2. $\Sigma_i$ and $\Sigma_{i+1}$ are disjoint and $\Sigma_{i+1}$ is obtained from $\Sigma_i$ by a bypass attachment.

Thus if one is trying to understand how the contact geometry of $M \setminus \Sigma$ and $M \setminus \Sigma'$ relate, one just needs to analyze how the contact geometry of the pieces of $M \setminus \Sigma_i$ changes under bypass attachment. In particular, many arguments can be reduced from understanding a general isotopy to understanding an isotopy between two surfaces that cobound a product region.

There is also a relative version of Isotopy Discretization where $\Sigma$ and $\Sigma'$ are convex surfaces with Legendrian boundary consisting of ruling curves on a convex torus. If

$\partial\Sigma = \partial\Sigma'$ and there is a topological isotopy of $\Sigma$ to $\Sigma'$ relative to the boundary then we can find a discrete isotopy as described above. (Note that during the discrete isotopy the boundary of the surface is not fixed but is allowed to move among the ruling curves on the convex torus. One could slightly rephrase item (2) in the above definition of a discretized isotopy to keep the boundary fixed, but we find it more natural to allow the boundary to move even though the original isotopy is relative to the boundary.)

## *2.4 Classifying knots in a knot type*

### 2.4.1 Standard neighborhoods of knots

Given a Legendrian knot $L$, a *standard neighborhood* of $L$ is a solid torus $N$ that has convex boundary with two dividing curves of slope $1/\operatorname{tb}(L)$ (and of course we will usually take $\partial N$ to be a convex torus in standard form). Conversely given any such solid torus it is a standard neighborhood of a unique Legendrian knot (*cf*. [25]). Up to contactomorphism one can model a standard neighborhood.

One may understand stabilizations and destabilizations of a Legendrian knot $L$ in terms of the standard neighborhood. Specifically, inside the standard neighborhood $N$ of $L$, $L$ can be positively stabilized to $S_+(L)$, or negatively stabilized to $S_-(L)$. Let $N_\pm$ be a neighborhood of the stabilization of $L$ inside $N$. As above we can assume that $N_\pm$ has convex boundary in standard form. It will have dividing slope $\frac{1}{\operatorname{tb}(L)-1}$. Thus the region $N \setminus N_\pm$ is diffeomorphic to $T^2 \times [0,1]$ and the contact structure on it is easily seen to be a *basic slice*, see [22]. There are exactly two basic slices with given dividing curves on their boundary and as there are two types of stabilization of $L$ we see that the basic slice $N \setminus N_\pm$ is determined by the type of stabilization done, and vice versa. Moreover if $N$ is a standard neighborhood of $L$ then $L$ destabilizes if the solid torus $N$ can be thickened to a solid torus $N_d$ with convex boundary in standard form with dividing slope $\frac{1}{\operatorname{tb}(L)+1}$. Moreover the sign of the destabilization will be determined by the basic slice $N_d \setminus N$. Finally, we notice that using Theorem 2.3.2 we can destabilize $L$ by finding a bypass for $N$ attached along a ruling curve whose slope is clockwise of $1/(\mathrm{tb}(L)+1)$ (and anti-clockwise of 0).

Furthermore, by using this neighborhood one can talk about the *positive/negative transverse push-off*, $T_{\pm}(L)$ of a Legendrian knot $L$. The only classical invariant of these transverse knots, the self linking number, can be computed for transverse push-offs as (*cf*. [18])

$$sl(T_{\pm}(L)) = \mathrm{tb}(L) \mp \mathrm{r}(L).$$

As in [14] two Legendrian knots $L$ and $L'$ are called *stably isotopic* if there is some $n$ and $n'$ such that $S_{-}^{n}(L)$ and $S_{-}^{n'}(L')$ are Legendrian isotopic. Note that $\mathrm{tb}(L) - \mathrm{r}(L) = \mathrm{tb}(S_{-}(L)) - \mathrm{r}(S_{-}(L))$. A knot type $\mathcal{K}$ is called *stably simple* if Legendrian knots in this knot type are stably isotopic. The key result that we need concerning the transverse classification of a knot type is the following theorem of Epstein, Fuchs and Meyer from [18] (also [14] for general manifolds) which reduces the classification of transverse knots up to transverse isotopy to the classification Legendrian knots up to Legendrian isotopy and their negetaive stabilizations.

**Theorem 2.4.1** (Epstein-Fuchs-Meyer [18], Etnyre-Honda [14]). *A knot type $\mathcal{K}$ is stably simple if and only if it is transversely simple.*

We want to note that in the proofs we will use the following classical strategy, first proposed by Etnyre in [10] and efficiently used for almost all known results concerning the clasification of Legendrian knots.

1. Find a formula that computes $\overline{\mathrm{tb}}(\mathcal{K}_{(p,q)})$ and $\mathrm{r}(K)$ where $K \in \mathcal{K}_{(p,q)}$ with $\mathrm{tb}(K) = \overline{\mathrm{tb}}(\mathcal{K}_{(p,q)})$.

2. Classify Legendrian knots with maximal Thurston-Bennequin invariant.

3. Show that all Legendrian representatives of $\mathcal{K}_{(p,q)}$ of non-maximal Thurston-Bennequin invariant admit destabilization or determine those that cannot be destabilized.

4. Understand the relationship between the stabilizations of two non-destabilizable representatives of $\mathcal{K}_{(p,q)}$.

Recall that the Bennequin inequality implies that, there are finitely many distinct $L_1,\ L_2,\ ...\ , L_n \in \mathcal{L}(\mathcal{K})$, called peaks, with $tb(L_i) = \overline{tb}(\mathcal{K})$, $i = 1, 2, ..., n$. These are distuinguished by their rotation numbers $\mathrm{r}(L_i)$. Without loss of generality we can assume $\mathrm{r}(L_1) < \mathrm{r}(L_2) < ... < \mathrm{r}(L_n)$. Moreover, the contactomorphism $(x, y, z) \longmapsto (-x, y, -z)$ of $(\mathbb{R}^3, ker(dz - ydx))$ shows that $\mathrm{r}(L_i) = \mathrm{r}(L_{n-i})$. Also recall that, the positive (resp. negative) stabilization operation, as discussed above, decreases tb by 1 and increases (resp. decreases) r by 1. Hence once we enter all these values of $(\mathrm{r}, \mathrm{tb})$, the image of $\mathcal{L}(\mathcal{K})$ we see look like a mountain range. See Figure 7

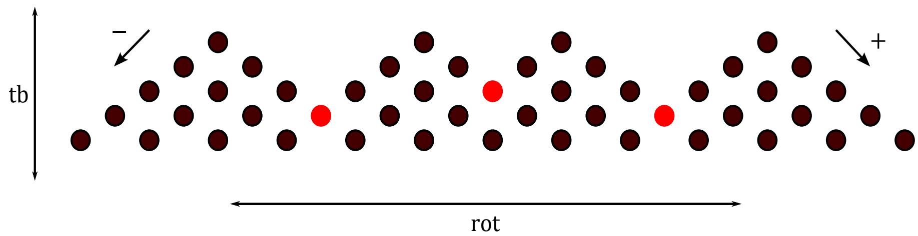


**Figure 7:** The image of $\mathcal{L}(\mathcal{K})$ under $(\mathrm{r}, \mathrm{tb})$. The diagonal arrows stands for $\pm$ stabilizations.

As stabilization of a Legendrian knot is well defined and positive and negative stabilizations commute, it is clear that these steps will yield a classification of Legendrian knots in the knot type $\mathcal{K}$.

Second part of the strategy is facilitated by the observation above that bypasses attached to appropriate ruling curves of a standard neighborhood of a Legendrian knot yield destabilizations. Similarly, if $L$ is a Legendrian knot contained in a convex surface $\Sigma$ (and the framing given to $L$ by $\Sigma$ is less than or equal to the framing given by a Seifert surface) and there is a bypass for $L$ on $\Sigma$ then this leads to a destabilization of $L$. Moreover one can find such a bypass in some cases by the Imbalance Principle discussed above.

Last part of the strategy require to show that each of the non-maximal representatives of $\mathcal{L}(\mathcal{K})$ shown red, called a valley, in the Figure 7 destabilizes to the two adjecent peaks.

### 2.4.2 Contact isotopy and contactomorphism

We begin by recalling a result of Eliashberg concerning the contactomorphism group of the standard contact structure $\xi_{std}$ on $S^3$. Fix a point $p$ in $S^3$ and let $\mathit{Diff}_0(S^3)$ be the group of orientation-preserving diffeomorphisms of $S^3$ that fix the plane $\xi_{std}(p)$, and let $\mathit{Diff}_{\xi_{std}}$ be the group of diffeomorphisms of $S^3$ that preserve $\xi_{std}$.

**Theorem 2.4.2** (Eliashberg 1992, [8]). *The natural inclusion of*

$$\mathit{Diff}_{\xi_{std}} \hookrightarrow \mathit{Diff}_0(S^3)$$

*is a weak homotopy equivalence.* □

Using this fact it is clear that if one has a contactomorphism $\phi$ of $(S^3, \xi_{std})$ that takes a set $S \subset S^3$ to $S' \subset S^3$, then there is a contact isotopy of $(S^3, \xi_{std})$ that takes $S$ to $S'$. In particular, if one is trying to show that two embeddings of a contact structure on a torus are contact isotopic then one merely needs to construct a contactomorphism that takes one torus to the other. Similarly to show two Legendrian knots are Legendrian isotopic one only needs to construct a contactomorphism that takes one knot to the other (or takes a standard neighborhood of one of the knots to the other, that is understand the contactomorphism type of the complement of the standard neighborhood).

## 2.5 *Contact width, uniform thickness property and lower width*

The contact width of a knot $\mathcal{K}$ is

$$\omega(\mathcal{K}) = \sup \frac{1}{slope(\Gamma_{\partial(S^1\times D^2)})},$$

where the supremum is taken over all $S^1 \times D^2 \hookrightarrow S^3$ representing $\mathcal{K}$ with $\partial(S^1 \times D^2)$ convex.

In order to make sense of slopes of homotopically non trivial curves on $\partial(S^1 \times D^2)$ we identify $\partial(S^1 \times D^2) = \mathbb{R}^2/\mathbb{Z}^2$ where the meridian has slope 0 and the well-defined longitude (as $\mathcal{K}$ is in $S^3$) has slope $\infty$. More details will be given in the next Subsection.

**Definition 2.5.1.** A topological knot type $\mathcal{K}$ is said to satisfy the *uniform thickness property* (UTP) if the following hold:

1. $\overline{\mathrm{tb}}(\mathcal{K}) = \omega(\mathcal{K})$
2. Every embedded solid tori $S^1 \times D^2 \hookrightarrow S^3$ representing $\mathcal{K}$ can be thickened to a standard neighborhood of a maximal tb Legendrian knot.

We remind also a theorem of Etnyre and Honda that was main motivation of the work in this thesis.

**Theorem 2.5.2** (Etnyre-Honda,[13])**.** *If $\mathcal{K}$ is Legendrian simple knot type and satisfies the UTP, then all of its cables are Legendrian simple.*

We say that a solid torus $S^1 \times D^2$ with convex boundary representing $\mathcal{K}$ is *non-thickenable*, if there is no $N'$ containing $S^1 \times D^2$ (whenever we discuss solid torus contained in another we assume they have the same core) with $slope(\Gamma_{N'}) \neq slope(\Gamma_N)$. Since there are knots with this property (see Chapter 4), we define another invariant of a Legendrian knot type, the lower contact width, to be

$$\ell w(\mathcal{K}) = \inf \frac{1}{\mathrm{slope}(\Gamma_{\partial(S^1\times D^2)})},$$

where $S^1 \times D^2$ ranges over all non-thickenable solid tori representing $\mathcal{K}$ with convex boundary.

## *2.6 Framings for cables*

One can talk about two coordinate systems for $\mathcal{K}_{(r,s)}$ on its neighborhood, $\partial N(\mathcal{K}_{(r,s)})$. The first coordinate system, which is denoted by $\mathcal{C}$, has the meridian slope 0 and the well-defined longitude, coming from the intersection of a Seifert surface for $\mathcal{K}_{(r,s)}$ with $\partial N(\mathcal{K}_{(r,s)})$, has slope $\infty$. In the second coordinate system, denoted $\mathcal{C}'$, the meridian has slope 0 and slope $\infty$ comes from the surface $\partial N(\mathcal{K})$ on which $\mathcal{K}_{(r,s)}$ sits. That is we take an annulus $A$ on $\partial N(\mathcal{K})$ that intersects $\partial N(\mathcal{K}_{(r,s)})$ along its boundary with $\partial N(\mathcal{K}_{(r,s)}) \setminus A$ has two disjoint annuli components $B_1$ and $B_2$ such that $A \cup B_i$ is isotopic to $\partial N(\mathcal{K})$. Now $A \cap \partial N(\mathcal{K})$ has slope $\infty$. As explained in [13] one can relate these two framings for $\partial N(\mathcal{K}_{(r,s)})$ and deduce the following relation between the twisting numbers of $L_{(r,s)} \in \mathcal{K}_{(r,s)}$:

$$t(L_{(r,s)}, \mathcal{C}') + rs = t(L_{(r,s)}, \mathcal{C}) = \mathrm{tb}(L_{(r,s)}). \tag{2}$$

Given two embedded closed curves $\gamma$ and $\gamma'$ on a torus $T$ we denote their minimal intersection by $\gamma \bullet \gamma'$. If the slope of $\gamma$, respectively $\gamma'$, is $s = \frac{r}{t}$, respectively $s' = \frac{r'}{t'}$, then

$$s \bullet s' = |rt' - tr'|.$$

## *2.7 Computations of* $\mathrm{tb}, \mathrm{r}$ *and* $\overline{\mathrm{tb}}$

In this subsection we collect various facts that are useful in computing the classical invariants of Legendrian knots on tori.

### 2.7.1 Rotation numbers for curves on convex tori

Let $T$ be a convex torus in a contact manifold $(M, \xi)$, where $\xi$ has Euler class 0. Now we define an invariant of homology classes of curves on $T$. Let $v$ be any globally non-zero section of $\xi$ and $w$ a section of $\xi|_T$ that is transverse to and twists (with $\xi$) along the Legendrian ruling curves and is tangent to the Legendrian divides. If $\gamma$ is a closed oriented curve on $T$ then set $f_T(\gamma)$ equal to the rotation of $v$ relative $w$ along $\gamma$. One may check the following properties (cf. [10, 14]).

1. The function $f_T$ is well-defined on homology classes.

2. The function $f_T$ is linear.

3. The function $f_T$ is unchanged if we isotope $T$ through convex tori in standard form.

4. If $\gamma$ is a $(r, s)$-ruling curve or Legendrian divide then $f_T(\gamma) = r(\gamma)$.

### 2.7.2 Legendrian knots on tori

We recall two simple lemmas from [13]. The first concerns the computation of the Thurston-Bennequin invariant for cables and follows immediately from (2).

**Lemma 2.7.1.** *Let $\mathcal{K}$ be a knot type and $N$ a solid torus representing $\mathcal{K}$ whose boundary is a standard convex torus. Suppose that $L \in \mathcal{L}(\mathcal{K}_{(r,s)})$ is contained in $\partial N$.*

1. *Suppose $L_{(r,s)}$ is a Legendrian divide and slope$(\Gamma_{\partial N(\mathcal{K})}) = \frac{s}{r}$. Then*

$$\operatorname{tb}(L_{(r,s)}) = rs.$$

2. *Suppose $L_{(r,s)}$ is a Legendrian ruling curve and slope$(\Gamma_{\partial N(\mathcal{K})}) = \frac{s'}{r'}$. Then*

$$\operatorname{tb}(L_{(r,s)}) = rs - |rs' - sr'|.$$

□

A simple consequence of the discussion in Subsection 2.7.1 yields the following computation of the rotation number for cables.

**Lemma 2.7.2.** *Let $\mathcal{K}$ be a knot type and $N$ a solid torus representing $\mathcal{K}$ whose boundary is a standard convex torus. Suppose that $L \in \mathcal{L}(\mathcal{K}_{(r,s)})$ is contained in $\partial N$. Then*

$$r(L_{(r,s)}) = r \cdot r(\partial D) + s \cdot r(\partial \Sigma),$$

*where $D$ is a convex meridional disk of $N$ with Legendrian boundary on a contact-isotopic copy of the convex surface $\partial N$, and $\Sigma$ is a convex Seifert surface with Legendrian boundary in $\mathcal{L}(\mathcal{K})$ which is contained in a contact-isotopic copy of $\partial N(\mathcal{K})$.* $\square$

# Chapter III

# SUFFICIENTLY POSITIVE AND NEGATIVE CABLES ARE SIMPLE

In this section we give the proofs of Theorem 1.1.1 and 1.1.2.

## *3.1 Sufficiently positive cables.*

We first start to give a more precise statement of Theorem 1.1.1, then work our way up to the proof of Theorem 3.1.1 through a series of lemmas.

**Theorem 3.1.1.** *If $\mathcal{K}$ is Legendrian simple and $\omega(\mathcal{K}) \in \mathbb{Z}$. Then its $(r,s)$-cable, $\mathcal{K}_{(r,s)}$, is Legendrian simple and admits a classification in terms of the classification of $\mathcal{K}$, provided $\frac{r}{s} > \omega(\mathcal{K})$. Moreover the maximal Thurston-Bennequin invariant is*

$$\overline{\mathrm{tb}}(\mathcal{K}_{(r,s)}) = rs - |\overline{\mathrm{tb}}(\mathcal{K}) \bullet \frac{r}{s}|,$$

*and the set of rotation numbers associated to $L \in \mathcal{K}_{(r,s)}$ with $\mathrm{tb}(L) = \overline{\mathrm{tb}}(\mathcal{K}_{(r,s)})$ is*

$$\mathrm{r}(L) = \{s \cdot \mathrm{r}(K) |\ K \in \mathcal{L}(\mathcal{K})\ ,\ \mathrm{tb}(K) = \overline{\mathrm{tb}}(\mathcal{K})\}$$

*If $K \in \mathcal{L}(\mathcal{K})$ is a non-destabilizable with $\mathrm{tb}(K) = n < \overline{\mathrm{tb}}(K)$, then there is non-destabilizable $L \in \mathcal{L}(\mathcal{K}_{(r,s)})$ with $\mathrm{tb}(L) = rs - |\frac{1}{n} \bullet \frac{r}{s}|$ and the set of rotation numbers associated to non-destabilizable $L \in \mathcal{K}_{(r,s)}$ with $\mathrm{tb}(L) = rs - |\frac{1}{n} \bullet \frac{r}{s}|$ is*

$$\mathrm{r}(L) = \{s \cdot \mathrm{r}(K) |\ K \in \mathcal{L}(\mathcal{K})\ ,\ \mathrm{tb}(K) = n\}.$$

**Lemma 3.1.2.** *Under the hyphothesis of Theorem 3.1.1 the maximal Thurston-Bennequin invariant is $\overline{\mathrm{tb}}(K_{(r,s)}) = rs - |\overline{\mathrm{tb}}(\mathcal{K}) \bullet \frac{r}{s}|$. The set of rotation numbers realized by $L \in \mathcal{K}_{(r,s)}$ with $\mathrm{tb}(L) = \overline{\mathrm{tb}}$ is*

$$\mathrm{r}(L) = \{s \cdot \mathrm{r}(K) |\ K \in \mathcal{L}(\mathcal{K}), \mathrm{tb}(K) = \overline{\mathrm{tb}}(K)\}.$$

*Proof.* During the proof we will use the $\mathcal{C}'$ coordinate system. Note that $tw(L, \mathcal{C}') < 0$ for all $L \in \mathcal{L}(K_{(r,s)})$. If not, we can assume there is $L' \in \mathcal{L}(K_{(r,s)})$ with $tw(L') = 0$. Then there exists a solid torus $S$ with $\partial S$ convex such that $L'$ is a Legendrian divide on $\partial S$ which implies that slope of dividing set is $s/r$ when measured with respect to $\mathcal{C}$ but this contradicts the assumption that $\frac{r}{s} > \omega(\mathcal{K})$

Thus, there exists a solid torus $S$ representing $\mathcal{K}$ with $\partial S$ convex, $L \subset \partial S$ and the slope of $\Gamma_{\partial S}$ equal to $t$.

Recall in our Theorem 3.1.1 it is assumed that $\omega(\mathcal{K}) \in \mathbb{Z}$. Since $\overline{\mathrm{tb}}(\mathcal{K}) \leq \omega(\mathcal{K}) \leq \overline{\mathrm{tb}}(\mathcal{K}) + 1$. We have either $\omega(\mathcal{K}) = \overline{\mathrm{tb}}(\mathcal{K})$ or $\omega(\mathcal{K}) = \overline{\mathrm{tb}}(\mathcal{K}) + 1$. Hence there are two cases to check.

*Case 1.* $\omega(\mathcal{K}) = \overline{\mathrm{tb}}(\mathcal{K})$: We claim the following inequality holds under the assumptions of Theorem 3.1.1

$$|\frac{1}{t} \bullet \frac{r}{s}| \geq |\omega(\mathcal{K}) \bullet \frac{r}{s}| \tag{3}$$

and equality holds iff $\frac{1}{t} = \omega(\mathcal{K})$.

To see this note that, since $\omega(\mathcal{K}) \in \mathbb{Z}$ we know that on the Farey tesellesion there is an edge from 0 to $\frac{1}{\omega(\mathcal{K})}$. Moreover, by definition of the contact width we have, $\frac{1}{t} < \omega(\mathcal{K})$. Now by using the oriented diffeomorphism of $\partial S$, we can normalize the slopes by sending 0 to 0 and $\frac{1}{\omega(\mathcal{K})}$ to $\infty$. Such a diffeomorphism will preserve order and hence force $s'/r' > 0$ and $\frac{1}{t'} \in [-\infty, 0)$ where $s'/r'$ and $\frac{1}{t'}$ denotes the images of $s/r$ and $\frac{1}{t}$ under this diffeomorphism, respectively.

Observe that $\frac{1}{t'} \in (-\infty, 0)$ means

$$\frac{1}{t'} = m \begin{pmatrix} 0 \\ -1 \end{pmatrix} + n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} n \\ -m \end{pmatrix}$$

where $n,\ m > 0$. Hence as slope $\frac{1}{t'} = -\frac{m}{n}$. Now we easily get Inequality (3)

$$|\frac{1}{t} \bullet \frac{r}{s}| = |\frac{1}{t'} \bullet \frac{r'}{s'}| = |\frac{-m}{n} \bullet \frac{r'}{s'}| = |r'n + s'm| > s' = |\frac{-1}{0} \bullet \frac{r'}{s'}| = |\frac{1}{t} \bullet \frac{r}{s}|.$$

Therefore $t(L, \mathcal{C}') \leq -|\omega(\mathcal{K}) \bullet \frac{r}{s}|$. Now any Legendrian ruling on $\partial S$, where $S$ is solid torus representing $\mathcal{K}$ of maximal thickness (i.e. slope$\Gamma_{\partial S} = \frac{1}{\overline{\mathrm{tb}}(\mathcal{K})}$), realizes the equality. By Equation (2) we see that

$$\overline{\mathrm{tb}}(\mathcal{K}_{(r/s)}) = rs - |\omega(\mathcal{K}) \bullet r/s| = rs - \left|\overline{\mathrm{tb}}(\mathcal{K}) \bullet r/s\right|.$$

*Case 2.* $\omega(\mathcal{K}) = \overline{\mathrm{tb}}(\mathcal{K}) + 1$: The same proof as in *Case 1* is true when $s < \frac{1}{\overline{\mathrm{tb}}(\mathcal{K})+1}$ except in Inequality (3) equality holds iff $\frac{1}{t} = \overline{\mathrm{tb}}(\mathcal{K})$. When $t \in [\frac{1}{\overline{\mathrm{tb}}(\mathcal{K})}, \frac{1}{\overline{\mathrm{tb}}(\mathcal{K})+1}]$, then first observe that for any such $t \in [\frac{1}{\overline{\mathrm{tb}}(\mathcal{K})}, \frac{1}{\overline{\mathrm{tb}}(\mathcal{K})+1})$ we have

$$|\frac{1}{t} \bullet \frac{r}{s}| \geq |\overline{\mathrm{tb}}(\mathcal{K}) \bullet \frac{r}{s}|. \tag{4}$$

Moreover, we cannot have $s = \frac{1}{\overline{\mathrm{tb}}(\mathcal{K})+1}$ as otherwise we would have $L \in \mathcal{L}(\mathcal{K})$ with $\mathrm{tb}(L) = \overline{\mathrm{tb}}(\mathcal{K}) + 1$.

Therefore $t(L, \mathcal{C}') \leq -|\overline{\mathrm{tb}}(\mathcal{K}) \bullet \frac{r}{s}|$ and any Legendrian ruling curve of slope $s/r$ on $\partial N$, where $N$ is solid torus representing $\mathcal{K}$ convex boundary and $s(\Gamma_{\partial N}) = \frac{1}{\overline{\mathrm{tb}}(\mathcal{K})}$ will realize the equality in Inequality (4)

Next we compute the rotation numbers associated to this representatives. Take $L \in \mathcal{L}(\mathcal{K}_{(r,s)})$ with $\mathrm{tb}(L) = \overline{\mathrm{tb}}(\mathcal{K}_{(r,s)})$. Then there exist a solid torus $S$ with convex boundary, where slope$(\Gamma_{\partial S}) = \frac{1}{\overline{\mathrm{tb}}(\mathcal{K})}$ and $L$ is Legendrian ruling curve on $\partial S$.

Such a solid torus is a standard neighborhood of Legendrian knot $K \in \mathcal{L}(\mathcal{K})$. Thus by Formula (2.7.2) we have

$$\mathrm{r}(L) = r \cdot \mathrm{r}(\partial D) + s \cdot \mathrm{r}(K) = s \cdot \mathrm{r}(K)$$

as $\mathrm{r}(\partial D) = 0$.

□

**Lemma 3.1.3.** *The* $L \in \mathcal{L}(\mathcal{K}_{(r,s)})$ *with* $\mathrm{tb}(L) = \overline{\mathrm{tb}}$ *are classified by their rotation numbers.*

*Proof.* If $L, L' \in \mathcal{L}(\mathcal{K}_{(r,s)})$ with $\mathrm{tb}(L) = \mathrm{tb}(L') = \overline{\mathrm{tb}}$, then there exist solid tori $S$ and $S'$ which represent $K, K' \in \mathcal{L}(\mathcal{K})$, respectively. Since $tw(L, \partial S) < 0$ (similarly

$tw(L', \partial S') < 0$) we can make $\partial S$ (and $\partial S'$) convex and $L$, $L'$ are Legendrian ruling curve on $S$ and $S'$, respectively. Moreover since $L$ and $L'$ are maximal tb representatives there are only two dividing curves of slope $\frac{1}{\overline{\mathrm{tb}}(\mathcal{K})}$ on $\partial S$ and $\partial S'$.

If $\mathrm{r}(L) = \mathrm{r}(L')$, then by Lemma 3.1.2, $\mathrm{r}(K) = \mathrm{r}(K')$ and hence $K$ and $K'$ are Legendrian isotopic by Legendrian simplicity of the underlying knot type $\mathcal{K}$. Thus we may assume $K$ and $K'$ are the same. Let $S$ and $S'$ be the standard neighborhoods of the $K = K'$ on which $L$ and $L'$, respectively, sit. Since $K = K' \subset S \cup S'$, there exist a solid torus $S''$ sitting inside both $S$ and $S'$ and with $\partial S''$ convex and $\mathrm{slope}(\Gamma_{\partial S''}) = \frac{1}{\overline{\mathrm{tb}}(\mathcal{K})}$. Since $\overline{S - S''}$ and $\overline{S' - S''}$ are $I$-invariant neighborhoods, we can assume $L$, $L'$ are (slope $s/r$) Legendrian rulings on $\partial S''$. Finally, $L$ and $L'$ are Legendrian isotopic through the other Legendrian rulings. $\square$

*Remark* 3.1.4. If the knot type $\mathcal{K}$ satisfies UTP property, then a classification as in Figure 8 is impossible, i.e. either there is single representative at maximal tb (hence has $\mathrm{r} = 0$) or several representatives at maximal tb which are distuinguished by their rotation numbers. Since in our case we are dealing with the knot types that do not necessarily satisfy UTP, it is possible to have a picture as in Figure 2, though we do not know any example of it. In other words, there might be a knot type $\mathcal{K}$ that is Legendrian simple and has a Legendrian classification such that some $K' \in \mathcal{L}(\mathcal{K})$ has $\mathrm{tb}(K') = n < \overline{\mathrm{tb}}$ but cannot be destabilized to $L$ with $\mathrm{tb}(L) = \overline{\mathrm{tb}}$. We note that Chongchitmate and Ng have conjectural examples in [5] of this phenomena.

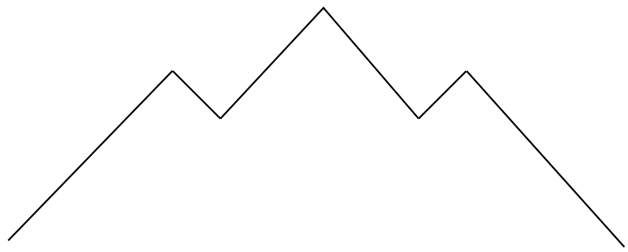

**Figure 8:** Possible non-standart $(\mathrm{tb}, \mathrm{r})$– Mountain range for a knot type $\mathcal{K}$

**Lemma 3.1.5.** *For each non-destabilizable $K \in \mathcal{L}(\mathcal{K})$ with Thurston-Bennequin invariant* $\mathrm{tb}(K) = n < \overline{\mathrm{tb}}$*, there exists a unique, up to Legendrian isotopy, non-destabilizable $L$, a $(r,s)$-ruling curve on the standard neighborhood $N$ of $K$ with* $\mathrm{tb}(L) = rs - |\frac{1}{n} \bullet \frac{r}{s}|$ *and the set of rotation numbers associated to such $L$ is*

$$\mathrm{r}(L) = \{s \cdot \mathrm{r}(K) |\ K \in \mathcal{L}(\mathcal{K})\ ,\ \mathrm{tb}(K) = n\}.$$

*Proof.* Let $K \in \mathcal{L}(\mathcal{K})$ be such representative. Since $\mathrm{tb}(K) = n < \overline{\mathrm{tb}}$ we can have an $L \in \mathcal{L}(K_{(r,s)})$ which is a Legendrian ruling on $\partial N'$ where $N$ is the standard neighborhood of $K \in \mathcal{L}(K)$ with $s(\Gamma_{\partial N}) = \frac{1}{n}$ and $n < \overline{\mathrm{tb}}(\mathcal{K})$. Now we want to show that $L$ does not admit a destabilization. Suppose that $L$ admits a destabilization. This implies the existence of a convex torus $\Sigma$ which is (topologically) isotopic to $\partial N$ and contains $L$ and a bypass for $L$. Now isotope the annulus $A = \partial N - L$ to $A' = \Sigma - L$ relative to the boundary $L$. By the Isotopy Discretization technique in [24, Lemma 3.10], we know such isotopy corresponds to a sequence of bypass attachments. Now we show that all potential bypass attachment are trivial, that is dividing set of $A$ will not change and hence we cannot reach $A'$. To end this, observe that a nontrivial bypass attachment from the outside will corresponds to a thickening of $\partial N$ and it cannot be thickened to some solid torus $N'$ with $s(\Gamma_{\partial N'}) = \frac{1}{n+1}$ since this will corresponds to a destabilization of $K \in \mathcal{L}(K)$ which is impossible. Hence a nontrivial bypass attachments will give a thickening of $\partial N$ to some solid tori $N'$ with $s(\Gamma_{\partial N'}) = t$ where $\frac{1}{n+1} < t < \frac{1}{n}$. An important observation is that since bypass attachment happens in the complement of $L$, any bypass attachments to $A$ cannot increase the intersection number of the dividing set with $L$. On the other hand, as in Case 1 in Lemma 3.1.2, one can easily show

$$\left|\frac{s}{r} \bullet t\right| > \left|\frac{s}{r} \bullet \frac{1}{n}\right|. \tag{5}$$

Thus, bypass attachment to $A$ from the outside must increase intersection number of the dividing set with $L$. Similarly bypass attachment to $A$ from the inside would increase the intersection of the dividing set with $L$. Hence, we cannot reach $A'$ and so $L$ does not destabilize $\square$

**Lemma 3.1.6.** *If $L \in \mathcal{L}(\mathcal{K}_{(r,s)})$ with $\mathrm{tb}(L) < \overline{\mathrm{tb}}(K_{(r,s)})$, then either $L$ admits a destabilization or L is one of the non-destabilizable representative from Lemma 3.1.5.*

*Proof.* Given such an $L$ there is a solid torus $S$ representing $K$ with convex boundary, containing $L$ and dividing slope $s$. If $L$ does not intersect the dividing set $\Gamma_{\partial S}$ efficiently, then we can destabilize $L$ with a bypass on $\partial S$. So we now assume $L$ intersects $\Gamma_{\partial S}$ efficiently. We know $s \neq \frac{1}{\omega(\mathcal{K})}$, since $\mathrm{tb}(L) < \overline{\mathrm{tb}}(\mathcal{K}_{(r,s)})$. If $S$ has boundary slope $\frac{1}{n}$, then either $K \in \mathcal{L}(\mathcal{K})$ is non-destabilizable and we are in situation of Lemma 3.1.5 or, as the underlying knot type $\mathcal{K}$ is Legendrian simple, $K \in \mathcal{L}(\mathcal{K})$ admits a destabilization and hence get a thickening of $S$. Now we can take a convex annulus $A = L \times [0,1]$ in $\partial S \times [0,1]$ and using the Imbalance Principle, we get a destabilization for $L$. Finally, suppose $s(\Gamma_{\partial S}) = t$ and $S$ is non thickenable. Shrink $S$ to a solid torus $N'$ with $\partial N'$ convex and $s(\Gamma_{\partial N}) = \frac{1}{n'}$. By using Equation (5) we get that $|s'/r' \bullet t| = |s'/r' \bullet (-n/m)| = |r'n + s'm| > |r'n - s'nn'| > |r' - s'n'| = |s'/r' \bullet \frac{1}{n'}|$. Thus, we again get a destabilization for $L$. $\square$

Finally we want to show for pairs $(\mathrm{tb}, \mathrm{r})$ obtained from stabilizations of multiple different non-destabilizable Legendrian knots (i.e. maximal tb representatives or Legendrian knots from Lemma 3.1.5), there is unique Legendrian with that tb and r. More precisely we prove

**Lemma 3.1.7.** *If $L, L' \in \mathcal{L}(\mathcal{K}_{(r,s)})$ with $\mathrm{tb}(L) = \mathrm{tb}(L') = \overline{\mathrm{tb}}(\mathcal{K}_{(r,s)})$ and $\mathrm{r}(L) = \mathrm{r}(L') + 2sn$, then $S_-^{sn}(L)$ and $S_+^{sn}(L')$ are Legendrian isotopic. Also If $\mathrm{tb}(L) = \overline{\mathrm{tb}}(\mathcal{K}_{(r,s)})$ and $L'$ is from Lemma 3.1.5 with $\mathrm{r}(L) = \mathrm{r}(L') + s(n-m)$, then and $S_-^{sk}(L)$ and $S_+^{sl}(L')$, $k + l = n - m$, are Legendrian isotopic.*

*Proof.* We need to show that $S_-^{sn}(L) = S_+^{sn}(L')$. Observe that $L$ and $L'$ sit on standard neighborhood of $K$ and $K'$, respectively, where $K$ and $K'$ of $\mathcal{L}(\mathcal{K})$ have maximal tb and $\mathrm{r}(K) = \mathrm{r}(K') + 2n$, by the assumption and Lemma 3.1.2. As $\mathcal{K}$ is Legendrian simple, we have $S_-^n(K) = S_+^n(K')$. On the other hand since $L$ is in $\mathcal{L}(\mathcal{K}_{(r,s)})$ is Legendrian ruling curve of slope $\frac{s}{r}$ on the standard neighborhood, say $N(K)$, of $K$ in which we have the standard neighborhood, $N(S_{(K)})$, of $S_{(K)}$. Let $L_0$ be a Legendrian ruling curve of slope $s/r$ on $\partial N(S_{(K)})$ and let $A$ be a convex annulus between $N(K)$ and $N(S_{(K)})$ with $L$ and $L'$ being its boundary. A quick computation of tb shows that the dividing set on $A$ has to have $s$-boundary parallel arcs on $L_0$ side and no boundary parallel arcs on $L$ side (as otherwise we would be able to isotop $L$ along this bypass disks and end up with a representative with less twisting and contradict with the maximality of $L$). Now the boundary parallel arcs on $L_0$ side are all either positive or all negative, giving two kinds of destabilization of $L_0$. Therefore, we can easily conclude that $S_-^s(L)$ sits on a standard neighborhood of $S_-(K)$. In a similar way $S_+^s(L')$ sits on the standard neighborhood of $S_+(K')$. One can induct this argumennt to see that $S_-^{sn}(L)$ and $S_+^{sn}(L')$ sit on the standard neighborhoods of $S_-^n(K) = S_+^n(K')$. Using the arguments as in the proof of Lemma 3.1.3, we conclude that $L$ and $L'$ are Legendrian isotopic.

By using similar argument we see can see that $L, L' \in \mathcal{L}(\mathcal{K}_{(r,s)})$ with $tb(L) = \overline{\mathrm{tb}}(\mathcal{K}_{(r,s)})$ and $L'$ is from Lemma 3.1.5 and $\mathrm{r}(L) = \mathrm{r}(L') + s(n-m)$ stabilizes to same Legendrian knot.

□

*Proof of Theorem 3.1.1.* Lemma 3.1.3, Lemma 3.1.5 and Lemma 3.1.6 give a complete list of non-destabilizable Legendrian knots in $\mathcal{K}_{(r,s)}$ and they are all determined by tb and r, by Lemma 3.1.7 □

## *3.2 Sufficiently negative cables.*

Now we give the proof of the Therem 1.1.2. Once again we start with a more precise statement, then Now establish the proof through the sequence of lemmas.

**Theorem 3.2.1.** *If $\mathcal{K}$ is Legendrian simple and $\ell\omega(\mathcal{K}) \in \mathbb{Z}$. Then $\mathcal{K}_{(r,s)}$ is also Legendrian simple, provided $\frac{r}{s} < \ell\omega(\mathcal{K})$. Moreover*

$$\overline{\mathrm{tb}}(\mathcal{K}_{(r,s)}) = rs = \omega(\mathcal{K}_{(r,s)}),$$

*and the set of rotation numbers realized by*

$$\{L_{(r,s)} \in \mathcal{L}(\mathcal{K}_{(r,s)}) : \mathrm{tb}(L) = \overline{\mathrm{tb}}(\mathcal{K}_{(r,s)})\}$$

*is*

$$\{\pm(r + s(n + r(L)) : L \in \mathcal{L}(\mathcal{K}), \mathrm{tb}(L) = -n\}$$

*where $n$ is the integer that satisfies*

$$-n - 1 < \frac{r}{s} < -n.$$

**Lemma 3.2.2.** *If $\frac{r}{s} < \ell\omega(\mathcal{K})$ and $\ell\omega(\mathcal{K}) \in \mathbb{Z}$, then*

$$\overline{\mathrm{tb}}(\mathcal{K}_{(r,s)}) = rs = \omega(\mathcal{K}_{(r,s)}).$$

*Moreover the set of rotation numbers realized by*

$$\{L_{(r,s)} \in \mathcal{L}(\mathcal{K}_{(r,s)}) : \mathrm{tb}(L) = \overline{\mathrm{tb}}(\mathcal{K}_{(r,s)})\}$$

*is*

$$\{\pm(p + q(n + r(L)) : L \in \mathcal{L}(\mathcal{K}), \mathrm{tb}(L) = -n\}$$

*where $n$ is the integer that satisfies*

$$-n - 1 < \frac{r}{s} < -n.$$

*Proof.* We will use the $\mathcal{C}'$ coordinate system. Observe that since $\frac{r}{s} < \ell\omega(\mathcal{K})$, there is a convex torus of slope $s/r$, parallel to $\partial N$, inside solid torus $N$ representing $\mathcal{K}$, with convex boundary. Now a Legendrian divide on this convex torus is a representative $L_{(r,s)} \in \mathcal{L}(\mathcal{K}_{(r,s)})$ with twisting number zero. Thus $\bar{t}(L_{(r,s)}, \mathcal{C}') \geq 0$.

For the equality it is enough to show that $\omega(\mathcal{K}_{(r,s)}, \mathcal{C}') = 0$ since $\bar{t}(L_{(r,s)}, \mathcal{C}') \leq \omega(\mathcal{K}_{(r,s)}, C')$. The proof below is essentially the same as Claim 4.2 in [13]. The key point is showing that the knot type $\mathcal{K}_{(r,s)}$ satisfies the first condition of the UTP.

Let $N_{(r,s)}$ be a solid torus representing $\mathcal{K}_{(r,s)}$ and has convex boundary with $s(\Gamma_{\partial(N_{(r,s)})}) = t$. We want to show $t = 0$. Suppose $t > 0$. After thinning the solid tori $N_{(r,s)}$ we may take $t$ to be a large positive integer and $\#\Gamma_{\partial(N_{(r,s)})} = 2$. We use Giroux's Flexibility Theorem, [20], to arrange charecteristic foliation on $\partial N_{(r,s)}$ to be in standart form with Legendrian ruling of slope $\infty$ and consider convex annulus $A$ with Legendrian boundary of slope $\infty$ on $\partial N_{(r,s)}$ such that a thickening $R = N_{(r,s)} \cup (A \times [-\epsilon, \epsilon]) \cong T^2 \times [1, 2]$ has $\partial R = T_1 \cup T_2$ parallel to $N(\mathcal{K})$, where $N(\mathcal{K})$ is a solid torus representing $\mathcal{K}$ with convex boundary of slope $s/r$, $T_2$ is isotopic to $\partial N$ and $T_1 \subset N(\mathcal{K})$. Note that $\Gamma_A$ must consists of parallel non-seperating arcs, otherwise we can attach the bypass corresponding to boundary parallel arcs onto $\partial(N_{(r,s)})$ to increase $t$ to $\infty$ by Theorem 2.3.2. This will result excessive twisting inside $N(\mathcal{K}_{(r,s)})$ and hence would result contact structure to be overtwisted. Moreover, we can take an identification of $\partial N(\mathcal{K})$ so that $slope(\Gamma_{T_1}) = -t$ and $slope(\Gamma_{T_2}) = 1$. To see this, we note that $T_1$ and $T_2$ are each obtained by gluing one half of $\partial N(\mathcal{K}_{(r,s)})$ to the annulus $A$ and now since $t$ is a positive integer, it is clear that $\Gamma_{T_1}$ is obtained from $\Gamma_{T_2}$ by performing $t+1$ right-handed Dehn twists.

Let $N'$ be a solid torus of maximal thickness containing $R$. By [22, Proposition 4.1], such a neighborhood has exactly two universally tight contact structures. On the other hand, any tight contact structure on $R$ can be layered into two basic slices at the torus $T_{1.5}$ parallel to $T_i$, $i = 1, 2$, with $slope(\Gamma_{T_{1.5}}) = \infty$ which is $s/r$ when measured with respect to $\mathcal{C}$ coordinate system. Moreover, a quick computation of the Poincare duals of the relative Euler classes for each of this basic slices shows that there are four possible tight contact structures on $R$ (two for each basic slices) which are given by $\pm(1,0) \pm (1, 1-t)$ and the universally tight ones are the ones that has no mixing of sign (i.e. either $+(1,0) + (1, 1-t)$ or $-(1,0) - (1, 1-t)$ ). We want to determine if the tight contact structure $\xi$ we start with, has a mixing of sign or not. To end this, we compute the Euler class. Let $\gamma$ be a Legendrian ruling curve of slope $\infty$ on $A$ and let $A' = \gamma \times [-\epsilon, \epsilon]$. We easily see that the dividing set on $A'$ is made of $2t$ parallel curves (as $A'$ is $(-\epsilon, \epsilon)$–invariant), we use this to get that $< e(\xi), A' >= \chi(A'_+) - \chi(A'_-) = 0$, this gives then $PDe(\xi) = \pm(0, 1-t)$. So, there is a mixing of sign. But this cannot happen inside $N'$. Thus, $t = 0$ and we get $\omega(\mathcal{K}_{(r,s)}, \mathcal{C}') = 0$, passing $\mathcal{C}$ coordinate system we have $\overline{\mathrm{tb}}(\mathcal{K}_{(r,s)}) = rs$.

Now we want to compute rotation numbers of $L_{(r,s)}$ in $\mathcal{L}(\mathcal{K}_{(r,s)})$ realizing maximal Thurston-Bennequin number. Let $T^2_{1.5} = \partial N$ which contains $L_{(r,s)}$ with $\mathrm{tb}(L_{(r,s)}) = rs$. Since $\frac{r}{s} < \ell\omega(\mathcal{K})$, we can take a thickening of tori $T^2_{1.5}$, $T^2 \times [1,2]$ such that boundary tori have slope $\mathrm{slope}(\Gamma_{T^2_1}) = -\frac{1}{n-1}$ and $\mathrm{slope}(\Gamma_{T^2_1}) = -\frac{1}{n}$ where $n$ is the integer that satisfies $-n-1 < \frac{r}{s} < -n$ (note that $n$ may equal to $\ell\omega(\mathcal{K})$). But now the solid tori of boundary slopes $-\frac{1}{n-1}$ and $-\frac{1}{n}$ are the standard neighborhoods of $L$ and $S_\pm(L)$, respectively. We can now make the relative Euler class computation as above and then use Lemma 2.7.2 to get desired formula for the rotation number computation. $\square$

**Lemma 3.2.3.** *Legendrian knots with maximal* tb *in $\mathcal{L}(\mathcal{K}_{(r,s)})$ are determined by their rotation numbers.*

*Proof.* Let $L$ and $L'$ be two Legendrian knots in $\mathcal{L}(\mathcal{K})$ with maximal tb and $\mathrm{r}(L) = \mathrm{r}(L')$, then we have associated solid tori $N$ and $N'$ with convex boundary on which $L$ and $L'$ sit as Legendrian divides. The classification of tight contact structures on the solid torus in [21, 22] says that the contactomorphism type of a tight contact structure on a solid torus with convex boundary is determined by the number of the positive bypasses on the meridional disk. Hence, determined by the rotation number of $L$ and $L'$, respectively, which are the same by the assumption. Thus, we get a contactomorphism $f : N \to N'$. We may extend $f$ to a contactomorphism of $S^3$ that takes $\partial N$ to $\partial N'$. Furthermore, by using Eliashberg's result in [8], there is a contact isotopy of $S^3$ that takes $\partial N$ to $\partial N'$. So we will now think $L$ and $L'$ are Legendrian divides on same solid torus, say $N$, with convex boundary. We now want to form a Legendrian isotopy between $L$ and $L'$. To end this, we recall from Lemma 3.2.2 that $\partial N$ is siting inside a thickened torus $T^2 \times [1, 2]$ such that boundary tori have $\text{slope}(\Gamma_{T_1^2}) = -\frac{1}{n-1}$ and $\text{slope}(\Gamma_{T_2^2}) = -\frac{1}{n}$. Now as the consequence of the classification of tight contact structure on thickened tori (see [22, Corollary 4.8]), we know there is also a pre-Lagrangian torus, (still) denote by $\partial N$, which has linear characteristic foliation and the same boundary slope as convex torus does. Thus, we can take $L$ and $L'$ to be two leaves on this pre-Lagrangian torus. Now, $L$ and $L'$ are Legendrian isotopic through this linear characteristic foliation. $\square$

**Lemma 3.2.4.** *If $L' \in \mathcal{K}_{(r,s)}$ with $\text{tb}(L') < \overline{\text{tb}}$, then $L'$ admits a destabilization.*

*Proof.* We can put $L'$ on a solid torus $S$ with $\partial S$ convex and *slope*$(\Gamma_{\partial S}) = t$. By the above lemma and the assumption that $\frac{r}{s} < \ell\omega(\mathcal{K})$ we can deduce that $L'$ is a Legendrian ruling on $S$ (clearly we can assume $L'$ intersects $\Gamma_{\partial S}$ efficiently otherwise destabilization is immediate) and $\frac{1}{t} \neq \ell\omega(\mathcal{K})$. If $t < \frac{1}{\ell\omega(\mathcal{K})}$, then, as in Equation (3.1.2), we easily see that $|s/r \bullet t| > |s/r \bullet 1/\ell\omega|$. Hence,by using the Imbalance Principle, we get a destabilaztion of $L'$. If $t > \frac{1}{\ell\omega(\mathcal{K})}$, then we can thicken $S$ to a solid tori $S'$ with $\partial S'$ convex and *slope*$(\Gamma_{\partial S'}) = \frac{1}{\ell\omega(\mathcal{K})}$. Hence taking a convex annulus $A$ with one boundary component on $L'$ in $\partial S \times [0, 1] = \overline{S' - S}$ and applying the Imbalance Principle again we find a bypass for $L'$ which gives a destabilization for $L'$. $\square$

**Lemma 3.2.5.** *If $L^+_{(r,s)}, L^-_{(r,s)} \in \mathcal{L}(\mathcal{K}_{(r,s)})$ with $\text{tb}(L^+_{(r,s)}) = \text{tb}(L^-_{(r,s)})$ and $\text{r}(L^+_{(r,s)}) =$*

$\mathrm{r}(L^-_{(r,s)}) + 2r + 2sn$ *(or* $\mathrm{r}(L^+_{(r,s)}) = \mathrm{r}(L^-_{(r,s)}) + 2ks - 2r - 2sn$*), then* $S_+^{r+sn}(L^-_{(r,s)}) = S_-^{r+sn}(L^+_{(r,s)})$ *(or* $S_+^{ks-r-sn}(L^-_{(r,s)}) = S_-^{ks-r-sn}(L^+_{(r,s)})$*).*

*Proof.* There are two cases to concern based on rotation number computation in Lemma 3.2.2

*Case 1:* $L \in \mathcal{L}(\mathcal{K})$ in Lemma 3.2.2 has $\mathrm{r}(L) = 0$. In this case $L^\pm_{(r,s)}$ are the only maximal tb representatives of $\mathcal{L}(\mathcal{K}_{(r,s)})$ with $r(L^+_{(r,s)}) = -r - sn$ and $r(L^-_{(r,s)}) = r+sn$. Clearly by doing $-r-sn$ positive ( respectively negative) stabilization on $L^-_{(r,s)}$ (respectively on $L^+_{(r,s)}$) we end up at Legendrian knots with the same $(\mathrm{tb}, \mathrm{r})$ pair. We also have $L'_{(r,s)} \in \mathcal{L}(\mathcal{K}_{(r,s)})$ with $\mathrm{tb}(L'_{(r,s)}) = \overline{\mathrm{tb}}(\mathcal{K}_{(r,s)})\}+r+sn$ number and $\mathrm{r}(L'_{(r,s)}) = s\,\mathrm{r}(L) = 0$. We know by Lemma 3.2.4, such a $L'_{(r,s)}$ admits a destabilization. We want to show, these are Legendrain isotopic, i.e. $S_+^{-r-sn}(L^+_{(r,s)}) = L'_{(r,s)} = S_-^{-r-sn}(L^-_{(r,s)})$. Recall that $L^\pm_{(r,s)}$ are the Legendrian divide on a convex torus $T_{1.5}$ with boundary slope $\frac{s}{r}$ inside $T^2 \times [1,2] = N(L) - N(S_\pm(L))$ (See the remark at the end of the statement of [13, Lemma 3.8]). Hence $L'_{(r,s)}$ is a Legendrian ruling curve of slope $\frac{s}{r}$ on the standard neighborhood $N(L)$ of $L \in \mathcal{L}(\mathcal{K})$ with $\mathrm{tb}(L) = -n$. Note that, $S_+^{-r-sn}(L^+_{(r,s)})$ and $S_-^{-r-sn}(L^-_{(r,s)})$ are also Legendrian ruling curve on $N(L)$. Hence, $L'_{(r,s)}$ is Legendrian isotopic to $S_+^{-r-sn}(L^+_{(r,s)})$ and $S_-^{-r-sn}(L^-_{(r,s)})$ through the other ruling curves. Indeed, by taking a convex annulus $A = L_{(r,s)} \times [1.5, 2]$ between $T_{1.5}$ and $N(L)$ with $\partial A$ is Legendrian curves of slope $\frac{s}{r}$ on $T_{1.5}$ and $N(L)$, we easily see $L'_{(r,s)}$ destabilizes in two ways.

*Case 2:* $L \in \mathcal{L}(\mathcal{K})$ in Lemma 3.2.2 has $\mathrm{r}(L) \neq 0$. In this case, $L^\pm_{(r,s)} \in \mathcal{L}(\mathcal{K}_{(r,s)})$ coresponds to $L^\pm \in \mathcal{L}(\mathcal{K})$ where $\mathrm{tb}(L^+) = \mathrm{tb}(L^-) = -n$ and $r(L^+) \neq r(L^-)$. Without loss genarility we can assume that $\mathrm{r}(L^-) < \mathrm{r}(L^+)$ and there is no $L^0$ with $\mathrm{r}(L^-) < \mathrm{r}(L^0) < \mathrm{r}(L^+)$ , then $\mathrm{r}(L^+) - \mathrm{r}(L^-) = 2k$, $k \in \mathbb{Z}_{>0}$. Thus $\mathrm{r}(L^-_{(r,s)}) = s\,\mathrm{r}(L^-) + r + sn$ and $\mathrm{r}(L^+_{(r,s)}) = s\,\mathrm{r}(L^+) - r - sn = q\,\mathrm{r}(L^-) - (2ks + r + sn)$. This extra depth $ks$ comes from the underlying knot type puts us precisely in the situation of Lemma 3.1.7. Namely, the $L'_{(r,s)}$ with $\mathrm{tb}(L'_{(r,s)}) = rs - (ks + r + sn)$ and $\mathrm{r}(L'_{(r,s)}) = s\,\mathrm{r}(L^+) + ks = s\,\mathrm{r}(L^-) - ks$

is the Legendrian ruling curve of slope $\frac{s}{r}$ on the standard neighborhood $S_+^k(L^+) = S_-^k(L^-)$ (as $\mathcal{K}$ is Legendrian simple). Therefore, a Legendrian isotopy through the other ruling curves gives that $L'_{(r,s)} = S_+^{ks-r-sn}(L^-_{(r,s)}) = S_-^{ks-r-sn}(L^+_{(r,s)})$. $\square$

*Proof of Theorem 3.2.1.* Lemma 3.2.2 and Lemma 3.2.3 give a complete list of non-destabilizable Legendrian knots in $\mathcal{K}_{(r,s)}$ and show they are all determined by their tb and *rot*. By Lemma 3.2.4, every $L'_{(r,s)}$ in $\mathcal{L}(\mathcal{K}_{(r,s)})$ with non-maximal tb invariant can be written as $S_-^k S_+^l(L_{(r,s)})$ for some $L^{\pm}_{(r,s)} \in \mathcal{L}(\mathcal{K}_{(r,s)})$ with maximal tb. Finally, Lemma 3.2.5 shows any two $L^{\pm}_{(r,s)}$ with maximal tb and $\mathrm{r}(L^-_{(r,s)}) < \mathrm{r}(L^+_{(r,s)})$ (and no $L^0_{(r,s)}$ with $\mathrm{r}(L^-_{(r,s)}) < \mathrm{r}(L^0_{(r,s)}) < \mathrm{r}(L^+_{(r,s)})$ ), stabilize to same $L'_{(r,s)}$ in $\mathcal{L}(\mathcal{K}_{(r,s)})$. $\square$

# Chapter IV

# CONTACT NEIGHBORHOODS OF THE POSITIVE TORUS KNOTS

In Section 4.1 we classify non-thickenable tori in the knot types of the positive torus knots, and in Subsection 4.2 we classify the partially thickenable tori in the knot type of the $(2,3)$-torus knot.

Let $S$ be a solid torus in a manifold $M$. We say $S$ *is in the knot type* $\mathcal{K}$, or *represents* $\mathcal{K}$, if the core curve of $S$ is in the knot type $\mathcal{K}$.

We say a solid torus $S$ with convex boundary in a contact manifold $(M,\xi)$ *thickens* if there is a solid torus $S'$ that contains $S$, has the same core curve as $S$ (in particular $\overline{S'-S}$ is a thickened torus) and such that $S'$ has convex boundary with dividing slope different from $S$. The existence of non-thickenable tori was first observed in [13]; the following theorem shows that non-thickenable tori exist for all positive $(p,q)$-torus knots.

## *4.1 Non-thickenable tori*

This account to prove the followings

- Find necessary condition on a solid torus representing $\mathcal{K}$ that does not thicken, i.e. obtain possible list of slopes for its dividing set.
- Prove that all these canditate neighborhoods do exist in $(S^3, \xi_{std})$.
- Prove the potential canditates you obtained in the first part are non-thickenable, i.e. any one of them does not thicken to any other. Moreover, prove this is a complete list.

**Lemma 4.1.1.** *Suppose that the solid torus $N$ represents the knot type of a positive $(p, q)$-torus knot $\mathcal{K}$. If $N$ has convex boundary then $N$ will thicken unless it has dividing slope*

$$e_k = \frac{k}{pq - p - q}$$

*for some $k \in \{1, 2, \ldots\}$, and $2n_k$ dividing curves where $n_k = \gcd(pq - p - q, k)$.*

*Proof.* We begin by ignoring the contact structure and building a topological model for the complement of $N$. See Figure 9. The knot $\mathcal{K}$ can be thought to sit on a torus $T$ that separates $S^3$ into two solid tori $V_1$ and $V_2$, each of which can be thought of as a neighborhood of an unknot $F_1$ and $F_2$. As $N$ is a neighborhood of $\mathcal{K}$, we can isotope $T$ so that it intersects $N$ in an annulus and thus $A' = T \setminus (T \cap N)$ is an annulus in the complement of $N$ with boundary on $\partial N$. Moreover, there is a small neighborhood of $A'$, which we denote $N(A')$ such that $S^3 \setminus (N \cup N(A'))$ consists of two solid tori, which we may think of as $V_1$ and $V_2$. Turning this construction around $V_1 \cup V_2 \cup N(A')$ is the complement of $N$. We can identify $N(A')$ as a neighborhood of an annulus $A$ that has one boundary component a $(p, q)$ curve on $\partial V_1$ and the other boundary component a $(q, p)$ curve on $\partial V_2$. Thus, topologically, the complement of $N$ can be built as the neighborhood of two unknots (that form a Hopf link) union the neighborhood of an annulus $A$.

Bringing the contact structure back into the picture we can assume that $L_i$, $i = 1, 2$, is a Legendrian representative of $F_i$ in the complement of $N$. Let $tb(L_i) = -m_i$, where $m_i > 0$. If $N(L_i)$ is a regular neighborhood of $L_i$, then $\text{slope}(\Gamma_{\partial N(L_i)}) = -1/m_i$ with respect to $\mathcal{C}_{F_i}$.

Notice that $S^3 \setminus (N(L_1) \cup N(L_2))$ is diffeomorphic to $S = T^2 \times [0, 1]$ and contains $N$. We wish to change coordinates on $T^2$ so that $N$ is a vertical solid torus in $S$. Specifically, $T^2$ inherits coordinates as the boundary of $N(L_1)$, that is using the coordinate system coming from the framing $\mathcal{C}_{F_1}$. We change coordinates so that the $(p, q)$ curve on $T^2$ becomes the $(0, 1)$ curve (which can be thought of as the longitude

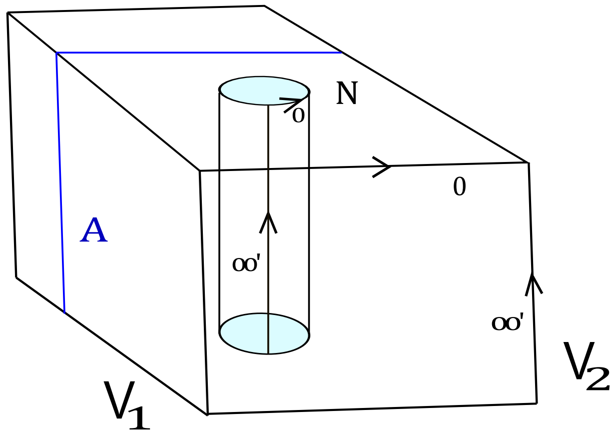


**Figure 9:** The cube in the picture represent $T^2 \times [0,1]$ (the top and bottom are identified and the front and back are also identified), thought of as the complement of the Hopf link $F_1 \cup F_2$. We see the square $\partial V_1$ on the left face that bounds the solid tori $V_1$ and the square $\partial V_2$ on the right face that bounds $V_2$ (minus their cores) and the annulus $A$ from $V_1$ to $V_2$. We have chosen coordinates on the torus (as specified in the figure) so that the $(p,q)$ curve is vertical, i.e. $\infty'$ with respect to $\mathcal{C}'$ coordinate system.

in the $\mathcal{C}'$ framing). This can be done by sending the oriented basis $((p,q),(p',q'))$ for $T^2$, where $pq' - qp' = 1$, to the basis $((0,1),(-1,0))$. This corresponds to the map $\phi_1 = \begin{pmatrix} q & -p \\ q' & -p' \end{pmatrix}$. Then $\phi_1$ maps $(-m_1, 1) \mapsto (-qm_1 - p, -q'm_1 - p')$. Since we are only interested in slopes, we write this as $(qm_1 + p, q'm_1 + p')$.

Similarly, we change from $\mathcal{C}_{F_2}$ to $\mathcal{C}'$. The only thing we need to know here is that $(-m_2, 1)$ maps to $(pm_2 + q, p'm_2 + q')$. Thus $S$ is a thickened torus $T^2 \times [0,1]$ with dividing slope $\frac{q'm_1+p'}{qm_1+p}$ on $T \times \{0\}$ and $\frac{p'm_2+q'}{pm_2+q}$ on $T \times \{1\}$.

Now suppose $qm_1+p \neq pm_2+q$. This would mean that the twisting of Legendrian ruling representatives of $K$ on $\partial N(L_1)$ and $\partial N(L_2)$ would be unequal. Then we could apply the Imbalance Principle to a convex annulus $A$ in $S^3 \backslash N$ between $\partial N(L_1)$ and $\partial N(L_2)$ to find a bypass along one of the $\partial N(L_i)$. This bypass in turn gives rise to a thickening of $N(L_i)$, allowing, by the twist number lemma [22], the increase of $tb(L_i)$ by one. Hence, eventually we arrive at $qm_1 + p = pm_2 + q$ and a standard convex annulus $A$; that is, the dividing curves on $A$ run from one boundary component of $A$

to the other.

Since $m_i > 0$, the smallest solution to $qm_1 + p = pm_2 + q$ is $m_1 = m_2 = 1$. All the other positive integer solutions are therefore obtained by taking $m_1 = pj + 1$ and $m_2 = qj + 1$ with $j$ a non-negative integer. We can then compute the boundary slope of the dividing curves on $\partial(\widetilde{N})$ where $\widetilde{N} = N(L_1) \cup N(L_2) \cup N(A)$. This will be the boundary slope for the solid torus $\widetilde{N}$ containing $N$. We have

$$-\frac{q'(pj+1)+p'}{pqj+p+q} + \frac{p'(qj+1)+q'}{pqj+p+q} - \frac{1}{pqj+p+q} = -\frac{j+1}{pqj+p+q} \tag{6}$$

After changing from $\mathcal{C}'_{\mathcal{K}}$ to $\mathcal{C}_{\mathcal{K}}$ coordinates, and setting $k = j+1$, these slopes become $k/(pq-p-q)$ as desired. We also notice that $\partial\widetilde{N}$ has $2\gcd(pq-p-q,k)$ dividing curves. Thus any solid torus $N$ will thicken unless it satisfies the conditions stated in the lemma. $\square$

We have not yet proved that tori as described in the above lemma actually exist. It is clear that for $k = 1$, we have slope $\frac{1}{pq-p-q}$, which is the slope of the standard neighborhood of maximal tb representative of $\mathcal{K}$. So we have at least one non-thickenable neighborhood exists. To contruct the others we first make some observations. Note that by the classification of tight contact structures on solid tori due to Honda [22], there are precisely two universally tight contact structures on $N$ (except $k = 1$ in which case there is unique (universally) tight contact structure [25]) and moreover the convex meridional disks all have bybasses of the same sign. Let $N_k^{\pm}$ denote $N$ with one of its universally tight contact structure on $N$ with slope$\Gamma_{\partial N_k^{\pm}} = \frac{k}{pq-p-q}$ and $\#\Gamma_{\partial N_k^{\pm}} = 2n_k$. This notation makes sense as the observation above shows that the two contact structures on $N_k^{\pm}$ differ by $-Id$. We will still denote $N_1^{\pm}$ for two (universally) tight contact structures on $N_1$ even though they are same. In the following we will drop $\pm$ from the notation and assume $N_k$ has one of its two (universally) tight contact sturctures.

**Lemma 4.1.2.** *The neighborhoods $N_k$ can be embedded in $(S^3, \xi_{std})$ for all positive integers $k$.*

*Proof.* Let $R = N_k \cup N(A')$ where $A'$ is an annulus that has boundary on $N_k$, $N(A') = A' \times [-\epsilon, \epsilon]$ is its product neighborhood with a $[-\epsilon, \epsilon]$-invariant contact structure on $N(A')$. Clearly $R$ is diffeomorphic to $T^2 \times [0,1]$ and the closed curves which run parallel to the core curve of $A'$ gives a $S^1$ fibration of $R$. Note that $\partial R$ has two parallel components $T_1$ and $T_2$ each of which is an unknotted torus. As in the proof above a product neighborhood $N(A)$ of an annulus $A$ that has one boundary component a $(p,q)$-curve on $\partial V_1$ and the other boundary component a $(q,p)$-curve on $\partial V_2$ can be thought of as a neighborhood of an annulus $A'$. Moreover the union of $N_k$ and $N(A)$ is a thickened torus $T^2 \times [0,1]$ whose complement is two standard neighborhood of unknots $V_1 \cup V_2$. and the dividing set on $\partial R$ and these standard neighborhoods match. For either choice of contact structure on $N_k$, the contact structure on $R$ which is obtained by extending the chosen one on $N_k$ can be isotoped to be transverse to the fibers of $R$, while preserving the dividing set on $\partial R$. It is well known, see for example [23], that such a horizontal contact structure is universally tight. Moreover, we see the boundary conditions on $R$ are $\#\Gamma_{T_1} = \#\Gamma_{T_2} = 2$ and (with appropriately chosen dividing curves on $A'$) slope$(\Gamma_{T_1}) = -\frac{1}{m_1}$, slope$(\Gamma_{T_2}) = -m_2$ when using the coordinates on $T^2$ coming from the framing $\mathcal{C}_{F_1}$.

We know that there are exactly two universally tight contact structures on $T^2 \times [0,1]$ with these dividing curves, differing by $-Id$, and their horizontal annuli contain bypasses all of the same sign; one can easily see they correspond to the two choices of universally tight contact structures on $N_k$. We know that each of these universally tight contact structures on $R$ embeds in the standard tight contact structure as the region between a Legendrian realization of the Hopf link $F_1 \cup F_2$. Thus the standard tight contact structure on $S^3$ minus $R$ give standard neighborhoods of a Legendrian realization $L_1$ of $F_1$, and $L_2$ of $F_2$. Moreover, we know that if $F_1$ and $F_2$ are oriented so that their linking is $+1$ then for one choice of universally tight contact structure on $R$ we have that $L_1$ and $L_2$ are both obtained from maximal Thurston-Bennequin unknots by only positive stabilizations and for the other choice of universally tight contact structure on $R$ we have only negative stabilizations. $\square$

We first notice that these $N_k^{\pm}$ just constructed in $S^3$ are non-thickenable solid tori. In the course of the proof we will include an easier argument in the case of the positive trefoil essentially from [13].

**Lemma 4.1.3.** *The tori $N_k^{\pm}$ from Construction 4.1.2 are non-thickenable.*

*Proof.* By Lemma 4.1.1, it suffices to show that $N_k$ does not thicken to any $N_{k'}$ for $k' < k$. (We drop the $\pm$ from the notation for $N_k$ for the remainder of this proof and just assume one choice of sign is fixed throughout.) To this end, observe that the $(p,q)$-torus knot is a fibered knot over $S^1$ with fiber a Seifert surface $\Sigma$ of genus $g = (p-1)(q-1)/2$ (see [27]). Moreover, the monodromy map $\phi$ of the fibration is periodic with period $pq$. Thus, $M_k$ has a $pq$-fold cover $\widetilde{M_k} \cong S^1 \times \Sigma$. If one thinks of $M_k$ as $\Sigma \times [0,1]$ modulo the relation $(x,0) \sim (\phi(x),1)$, then one can view $\widetilde{M_k}$ as $pq$ copies of $\Sigma \times [0,1]$ cyclically identified via the same monodromy. Now note that in $M_k$, the $\infty'$-longitude intersects any given Seifert surface $pq$ times efficiently. It is therefore evident that we can view $M_k$ as a Seifert fibered space with two singular fibers (the components of the Hopf link). The regular fibers are topological copies of the $\infty'$-longitude, which itself is a Legendrian ruling curve on $\partial M_k = \partial N_k$ with twisting $-(pq(k-1)+p+q)$.

*Case 1: The $(2,3)$-torus knot*

In the case of the trefoil, $\Sigma$ is a punctured torus. Note that we have a contact structure $\xi$ on $\widetilde{M_k}$ coming from pulling back the standard contact structure $\xi_{std}$ on $M_k$. It is not difficult to see in case of $\Sigma$ is punctured torus, $\xi$ is the restriction of contact structure $\xi_{-6k+1}$ on $T^3$ (which is characterized by the fact that the maximal twisting of any Legendrian $S^1$ which is isotopic to a fiber in $T^3$ is $-6k+1$ with respect to product framing). If $N_k$ does thicken to any $N_{k'}$ for $k' < k$ (This is the only possibility all otherwise it would correspond to thinning, see last paragraph before Lemma 4.1.4). Then there exists a rulling curve $L$ of slope $\infty'$ on $N_{k'}$ with $tw(L) = -6k'+1 > -6k+1$. As we assumed that $N'_k \supset N_k$ we have $L$ is a regular fiber in $M_k$. We claim that this cannot happen. To see this, we pull back $L$ to $\widetilde{M_k}$ and still have it is isotopic to a regular fiber with twisting $> -6k+1$. Now we close up $\widetilde{M_k}$ by gluing a solid torus to get $T^3$ and extend the contact structure so that all the circle fibers are Legendrian with twisting $-6k+1$. By classification of tight contact structures on $T^3$ due to Giroux and Kanda ([20, 25]), we conclude that the maximal twisting of a fiber is $-6k+1$. This contradiction shows that $N_k$ cannot be thickened to any $N_{k'}$ for $k' < k$.

For the positive torus knots other than trefoil we need further work. The essential reason is that the classification of tight contact structures on $S^1$ bundle over a closed surface $\Sigma_g$ with $g \geq 2$ is more complicated than $g = 1$ case. This is why closing up $S^1 \times (\Sigma_g \setminus D^2)$, the $pq$-fold cover of $M_k$ explained above, unfortunately does not work. But we still have a way to turn around mainly due to an idea first appeared in [25].

*Case 2: The $(p, q)$-torus knot*

We claim the pullback of the tight contact structure to $\widetilde{M_k}$ admits an isotopy where the $S^1$ fibers are all Legendrian and have twisting number $-(pq(k-1)+p+q)$ with respect to the product framing. To see this we consider the contact structure on $V_i$, the neighborhood of the Legendrian unknot $L_i$ (we will use notation from Construction 4.1.2). In the $pq$-cover of $M_k$ the torus $V_1$ will lift to $p$ copies of the $q$-fold cover $\widetilde{V}_1$ of $V_1$ and similarly $V_2$ will lift to $q$ copies of the $p$-fold cover $\widetilde{V}_2$ of $V_2$. We can assume that $\partial V_1$ has ruling slope $\frac{q}{p}$ (that is the ruling curves are Legendrian isotopic to a Legendrian $\infty'$-curve on $\partial M_k$) and similarly for $\partial V_2$. The ruling curves lift to curves of slope $\frac{1}{p}$ in $\widetilde{V}_1$. In particular they are longitudes and have twisting

$-(pq(k-1)+p+q)$. Moreover the dividing curves on $\widetilde{V}_1$ are also longitudinal (a different longitude of course). Thus we see that the contact structure on $\widetilde{V}_1$ is just a standard neighborhood of one of the ruling curves (pushed into the interior of the solid torus) as well as one of the dividing curves (pushed into the interior of the solid torus). (One may easily see this by considering the following model for a standard neighborhood: $D^2 \times S^1$ with the contact structure $\ker(\sin 2n\phi\, dx + \cos 2n\phi\, dy)$ where $(x, y)$ are coordinates on $D^2$ and $\phi$ is the coordinate on $S^1$.) Similarly for $\widetilde{V}_2$. Thus each of these tori is foliated by Legendrian curves isotopic to the ruling curves. As $\widetilde{M_k}$ is made from copies of the $\widetilde{V}_i$ and copies of covers of the convex neighborhoods of the annuli $A$ we see the claimed isotopy of $\widetilde{M_k}$ so that the $S^1$ fibers are all Legendrian.

If $N_k$ can be thickened to $N_{k'}$, then there exists a Legendrian curve topologically isotopic to the regular fiber of the Seifert fibered space $M_k$ with twisting number greater than $-(pq(k-1)+p+q)$, measured with respect to the Seifert fibration. Pulling back to the $pq$-fold cover $\widetilde{M_k}$, we have a Legendrian knot which is topologically isotopic to a fiber but has twisting greater than $-(pq(k-1)+p+q)$. Call this Legendrian knot with greater twisting $\gamma$. We will obtain a contradiction, thus proving that $N_k$ cannot be thickened to $N_{k'}$.

Since $\Sigma$ is a punctured surface of genus $g$, we can cut $\Sigma$ along $2g$ disjoint arcs $\alpha_i$, all with endpoints on $\partial\Sigma$, to obtain a polygon $P$. Thus we have a solid torus $S^1 \times P$ embedded in $\widetilde{M_k}$. We first calculate $\text{slope}(\Gamma_{\partial(S^1\times P)})$ as measured in the product framing. To do so, note that a longitude for this torus intersects $\Gamma$, $2(pq(k-1)+p+q)$ times, and a meridian for this torus is composed of 2 copies each of the $2g$ arcs $\alpha_i$, as well as $4g$ arcs $\beta_i$ from $\partial\Sigma$. Now since $\partial\Sigma$ is a preferred longitude downstairs in $M_k$, we know that $\Gamma$ intersects these $\beta_i$, $2(pq-p-q) = 2(2g-1)$ times positively. Thus the dividing curves on $\partial(S^1 \times \Sigma)$ have slope $(2g-1)/(pq(k-1)+p+q)$. Cutting along the $2g$ curves $\alpha_i$ and rounding will result in dividing curves on $\partial(S^1 \times P)$ with $\text{slope}(\Gamma_{\partial(S^1\times P)}) = -1/(pq(k-1)+p+q)$.

Now as in Lemma 3.2 in [23], we take $\widetilde{M}_k = S^1 \times \Sigma$ and pass to a (new) finite cover of the base by tiling enough copies of $P$ together so that $\gamma$ is contained in a solid torus $S^1 \times (\bigcup P)$. We notice that $S^1 \times (\bigcup P)$ is foliated by Legendrian knots with twisting $-(pq(k-1)+p+q)$ that are isotopic to the $S^1$ fibers in the product structure and that the dividing curves on the boundary of the solid torus are longitudinal. Thus $S^1 \times (\bigcup P)$ is a standard neighborhood of a Legendrian curve with twisting $-(pq(k-1)+p+q)$ with respect to the product structure. We know that inside any such solid torus any Legendrian isotopic to the core of the torus has twisting less than or equal to $-(pq(k-1)+p+q)$ (or else one could violate the Bennequin bound). Thus $\gamma$ cannot exist. $\square$

We now observe that if $N_k$ admits thickening to some $N$ with convex boundary representing the positive $(p,q)$-torus knot, then $N$ either admits further thickening to $N_{k'}$ or is non-thickenable, we claim that in latter case $N$ has to be in the list of $N_{k'}$. To end this, as in Lemma 4.1.1, we can find Legendrian unknots $L_i$ in $S^3 \setminus N = V_1 \cup V_2$ which are isotopic to core curves $F_i$ of unknotted tori $V_i$ (which we can think of it as standard neighborhoods of $L_i$). Let $A$ denote a convex annulus from $V_1$ to $V_2$ and $N(A) = A \times [-\epsilon, \epsilon]$ is its $[-\epsilon, \epsilon]$-invariant neighborhood. If $L_i$ are maximizing twisting number in $\mathcal{L}(F_i)$ and $N$ is non-thickenable, then the region between $N$ and $V_1 \cup V_2 \cup N(A)$ is an $I$-invariant neighborhood of $\partial N$. So, the dividing slope of $\partial N$ is same as the dividing slope of $\partial(V_1 \cup V_2 \cup N(A))$ which is $\frac{k'}{pq-p-q}$ for some positive $k'$. Hence, If $N$ does not thicken then $N$ must be isotopic to one of the $N_k^{\pm}$ from Construction 4.1.2.

In addition, we compute what the rotation numbers of Legendrian curves on $\partial N_k^{\pm}$ are.

**Lemma 4.1.4.** *If $\partial N_k^{\pm}$ is isotoped so that the ruling curves are meridional then the meridional curves will have rotation number $\pm(k-1)$, and if $\partial N_k^{\pm}$ is isotoped so that the ruling curves are $\infty$-longitudes then the $\infty$-longitudes have rotation number 0.*

*Proof.* From the proof of Lemma 4.1.1 we see that $\text{tb}(L_1) = -(p(k-1)+1)$ and $\text{tb}(L_2) = -(q(k-1)+1)$ for some positive integer $k$. We can assume that $\partial A$ are ruling curves on the tori $\partial V_1$ and $\partial V_2$. Ruling curves on $A$ provide a Legendrian isotopy form $K_1$ to $K_2$. Thus $K_1$ and $K_2$ have the same rotation numbers. From this and the discussion at the end of Lemma 4.1.1 we see that the signs of the stabilizations must be the same, thus $r(L_1) = \pm p(k-1)$ and $r(L_2) = \pm q(k-1)$. Moreover we see that $N^{\pm} \cup N(A')$ must be a universally tight contact structure on $T^2 \times [0,1]$ (or else we could find a bypass for one of the $L_i$ and hence thicken $N_k^{\pm}$).

The statement about meridional ruling curves is obvious. To verify the statement for the $\infty$-longitudes we need to use the function $f_T$ that measures the rotation numbers of curves on convex tori $T$ that was discussed in Subsection 2.7.1. Recall $L_1$ is a Legendrian unknot obtained from the maximal Thurston-Bennequin unknot by $p(k-1)$ positive (resp negative) stabilizations. Thus if $V_1$ is a standard neighborhood of $L_1$ and $K$ is a $(p,q)$-ruling curve on $\partial V_1$ then we see

$$f_{\partial V_1}(K) = p f_{\partial V_1}(\mu') + q f_{\partial V_1}(\lambda'') = \pm qp(k-1),$$

where $\mu'$ is a meridional curve on $\partial V_1$ and $\lambda''$ is a longitude.

If we isotope $\partial N_k^{\pm}$ (by this we mean for either choice of one of two univerally tight contact structure) so that the ruling curves are $\infty'$-curves then there is a convex annulus $A''$ in $S^3$ from the curve $K$ on $\partial V_1$ to an $\infty'$-longitude $\lambda'$ on $\partial N_k^{\pm}$ that has dividing curves that run from one boundary component to the other. Thus we can rule $A''$ by curves parallel to $K$ and $\lambda'$ and see that $K$ and $\lambda'$ are Legendrian isotopic. In particular $f_{\partial N_k^{\pm}}(\lambda') = r(\lambda') = \pm qp(k-1)$. Let $\lambda$ denote a $\infty$-longitude on $\partial V_k^+$. Since we know that $\lambda = \lambda' - pq\mu$ where $\mu$ is a meridian on $\partial V_k^+$ with $f_{\partial N_k^{\pm}}(\mu) = \pm(k-1)$ we see that

$$f_{\partial N_k^{\pm}}(\lambda) = f_{\partial N_k^{\pm}}(\lambda') - pq f_{\partial N_k^{\pm}}(\mu) = 0.$$

□

*Proof of Theorem 1.3.1.* The theorem collects the statements of Lemmas 4.1.1, 4.1.3, and 4.1.4, together with Construction 4.1.2. □

## *4.2 Partially thickenable tori*

**Proposition 4.2.1.** *Let $N_k^s$ be a solid torus in $N_k^\pm$ with standard convex boundary having dividing slope $s \in [e_k = k, \infty)$. Then $N_k^s$ will thicken to a solid torus $N'$ of slope $e_k = k$ but not beyond. Moreover, $N'$ is isotopic to $N_k^\pm$.*

*Proof.* Note that we already proved in Lemma 4.1.3 that $N'$ is isotopic to $N_k^\pm$ for some $k$. For the first statement, recall that by Corollary 4.8 in [22] we already know that for each $s > k$ we can find a solid torus of boundary slope $s$ inside $N_k^\pm$. As explained above $M_k^\pm = S^3 \setminus N_k^\pm$ is a Seifert Fibered space and has a degree 6 cover, $\widetilde{M}_k^\pm$, diffeomorphic to $S^1$ times a punctured torus (Seifert surface of $\mathcal{K}$) so that the $S^1$ fibers are the lift of the longitudal rulling curve downstairs and (via an isotopy of the pullback of the tight contact structure to $\widetilde{M}_k^\pm$) they can be made Legendrian with twisting number $-6k+1$ with respect to the product framing. Let $A$ be a $T^2 \times I$ layer between $\partial N_k^\pm$ and $\partial N$. Clearly $A$ is the union of the basic slices of the same sign as the tight contact structures on $N_k^\pm$ are universally tight. Observe that we can always thicken $N$ to $N_k^\pm$ and hence $S^1$ fibers of a 6-fold cover $M_k^\pm = S^3 \setminus N$ still has maximal twisting $-6k+1$. Suppose now, we can thicken $N$ further to $N_{k'}^\pm$ with $k' < k$. This would imply now in the cover we have a Legendrian curve which is isotopic to a $S^1$ fiber with twisting $tw > -6k+1$. On the other hand, by gluing a solid torus $D^2 \times S^1$ to $\widetilde{M}_k^\pm$ we get $T^3$ and extend the contact structure so that all the $S^1$ fibers are Legendrian with twisting exactly $-6k+1$ and the classification of tight contact structures on $T^3$ implies that $-6k+1$ is maximal twisting number. Hence we get a contradiction which finishes the proof of the first statement. □

**Proposition 4.2.2.** *Let $N$ be a solid torus in $N_k^\pm$ with standard convex boundary having dividing slope $s \notin [k, \infty)$. Then $N$ will thicken to the solid torus $N_1$ (which* *is a standard neighborhood of the maximal Thurston-Bennequin invariant Legendrian* $(2,3)$*-torus knot).*

*Proof.* Given such a torus $N$ we know from the construction and discussion in Subsection 4.1 that we can thicken $N$ to a solid torus $N'$ whose boundary is convex with two dividing curves of slope $\infty$. Note that $N'$ is the standard neighborhood of a Legendrian $(2,3)$-torus knot, say $L$, which obtained by stabilizing the maximal Thurston-Bennequin invariant Legendrian $(2,3)$-torus knot once and by being a standard neighborhood it is has a unique tight contact structure on it. Moreover, since the $(2,3)$-torus knot is Legendrian simple, all non-maximal *tb* invariant Legendrian $(2,3)$-torus knot, in particular $L$, destabilize to the maximal *tb* invariant Legendrian $(2,3)$-torus knot. In other words $N'$ thickens to maximally thickened neighborhood $N_1$. $\square$

We are now ready to establish the main results stated in the introduction concerning partially thickenable tori.

*Proof of Theorem 1.3.2.* The statements in the theorem just collect the facts from Proposition 4.2.1 and Lemma 2.2.2. $\square$

*Proof of Corollary 1.3.3.* For statement (1) notice that if $n \leq s < n+1$ then a convex torus with two dividing curves of slope $s$ will lie inside one of the $N_m^{\pm}$ for $m = 2, \dots n$ or $N_1$. From the classification of the $N_m^{\pm}$ we know there is a convex torus with two dividing curves and infinite dividing slope inside each of the $N_m^{\pm}$ and it will cobound with $\partial N_m^{\pm}$ a unique basic slice, [22]. Moreover there are two distinct such tori in $N_1$ and each of these two will cobound with $\partial N_1$ a unique basic slice. Inside a basic slice there is a unique, up to contactomorphism, convex torus of slope $s$. Thus given any convex torus $T$ with two dividing curves of slope $s$ we can use this data to construct a contactomorphism of $S^3$ taking $T$ to one of the tori described above. Then the discussion in Subsection 2.4.2 gives a contact isotopy from $T$ to one of these tori. As there are $2n$ such tori this establishes statement (1) of the theorem.

The other statements in the corollary have analogous proofs. $\square$

# Chapter V

# LEGENDRIAN AND TRANSVERSE CABLES OF THE POSITIVE TREFOIL

In the next two subsections we state and prove the precise classification theorems that lead to the qualitative results in Theorem 1.2.1 and 1.2.2.

**Theorem 5.0.3.** *Let $\mathcal{K}$ be the $(2,3)$-torus knot. Then the $(r,s)$-cable of $\mathcal{K}$, $\mathcal{K}_{(r,s)}$, is Legendrian simple if and only if $\frac{s}{r} \notin (1,\infty)$, and the classification of Legendrian knots in the knot type $\mathcal{K}_{(r,s)}$ is given as follows.*

1. *If $\frac{s}{r} \in (0,1]$ then there is a unique Legendrian knot $L \in \mathcal{L}(\mathcal{K}_{(r,s)})$ with Thurston-Bennequin invariant $\mathrm{tb}(L) = rs+s-r$ and rotation number $r(L) = 0$. All others are stabilizations of $L$.*

2. *If $\frac{s}{r} < 0$, then the maximal Thurston-Bennequin invariant for a Legendrian knot in $\mathcal{L}(\mathcal{K}_{(r,s)})$ is $rs$ and the rotation numbers realized by Legendrian knots with this Thurston-Bennequin invariant are*

$$\{\pm(r+s(n+k)) \mid k = (1+n), (1+n)-2, \ldots, -(1+n)\},$$

   *where $n$ is the integer that satisfies*

$$-n-1 < \frac{r}{s} < -n.$$

   *All other Legendrian knots $L \in \mathcal{L}(\mathcal{K}_{(r,s)})$ are stabilizations of these. Two Legendrian knots with the same $\mathrm{tb}$ and $\mathrm{r}$ are Legendrian isotopic.*

3. *If* $1 < \frac{s}{r} < \infty$, $s > r > 1$, *then* $\mathcal{K}_{(r,s)}$ *is not Legendrian simple and has the following complete classification. See Figure.*

   (a) *There are exactly* $2n$, $n = \lfloor \frac{s}{r} \rfloor$, *pairwise Legendrian non-isotopic maximal Thurston-Bennequin representatives* $L^i_\pm \in \mathcal{L}(\mathcal{K}_{(r,s)})$, $i = 1, 2, ..., n$, *with*

   $$\mathrm{tb}(L^i_\pm) = rs \ \ and \mathrm{r}(L^i_\pm) = \pm(s - r)$$

   (b) *If* $r \neq 1$ *then there are exactly two non-destabilizable non-maximal Thurston-Bennequin representatives* $K_\pm \in \mathcal{L}(\mathcal{K}_{(r,s)})$ *with*

   $$tb(K_\pm) = rs - |r(n+1) - s| \ \ and \mathrm{r}(K_\pm) = \pm(s - r + |r(n+1) - s|) = \pm rn$$

   (c) *Every other* $L \in \mathcal{L}(\mathcal{K}_{(r,s)})$ *is a stabilization of one of* $K_+$, $K_-$, $L^i_+$ *or* $L^i_-$.

   (d) $S^{s-r}_+(L^1_-) = S^{s-r}_-(L^1_+)$ *but* $S^{s-r}_+(L^i_-)$ *is not Legendrian isotopic to* $S^{s-r}_-(L^i_+)$ *for* $i = 2, ..., n$.

   (e) $S^{m-n}_-(K_-) = S^m_-(L^i_-)$ *and* $S^{m-n}_+(K_+) = S^m_+(L^i_+)$.

   (f) $S^j_-(K_-)$ *is not Legendrian isotopic to* $S^{j+n}_-(L^i_-)$, *similarly* $S^j_+(K_+)$ *is not Legendrian isotopic to* $S^{j+n}_+(L^i_+)$ *for* $j = 1, ..., m$ *and* $i = 1, 2, ..., n$.

   (g) $S^e_+(K_-)$ *is not Legendrian isotopic to* $S^e_+ S^n_-(L^i_-)$ *and* $S^e(K_+)$ *is not isotopic to* $S^e S^n_+(L^i_+)$ *for all* $e \in \mathbb{Z}_{>0}$. *Also* $S^{n+1}_+(K_-)$ *is not isotopic to* $S^{n+1}_-(K_+)$.

   (h) $S^e_+(S^j_- K_-)$ *is not Legendrian isotopic to* $S^e_+ S^{j+n}_-(L^i_-)$ *and* $S^e(S^j_+ K_+)$ *is not isotopic to* $S^e S^{j+n}_+(L^i_+)$ *for all* $e \in \mathbb{Z}_{>0}$ *and* $j = 1, 2, ..., m - n$.

Note that by Theorem 2.4.1, Items 3d- 3h of Theorem 5.0.3 yield the following

**Theorem 5.0.4.** *Let* $\mathcal{K}$ *be the* $(2,3)$*-torus knot. If* $\frac{s}{r} \notin (1, \infty)$ *then* $\mathcal{K}_{(r,s)}$ *is transversely simple and all transverse knots are stabilizations of the one with maximal self-linking number* $rs + s - r$.

*If* $\frac{s}{r} > 1$ *and* $\frac{s}{r} \in [n, n+1)$ *for a positive integer* $n$ *then* $\mathcal{K}_{(r,s)}$ *is not transversely simple and has the following classification.*

1. *The maximal self-linking number is $rs+s-r$, and there is a unique transverse knot in $\mathcal{T}(\mathcal{K}_{(r,s)})$ with this self-linking number.*

2. *There are $n-1$ distinct transverse knots in $\mathcal{T}(\mathcal{K}_{(r,s)})$ that do not destabilize and have self-linking number $rs+r-s$.*

3. *If $\frac{s}{r} \neq n$ then there is a unique transverse knot in $\mathcal{T}(\mathcal{K}_{(r,s)})$ that does not destabilize and has self-linking number $rs+r-s-2|(n+1)r-s|$.*

4. *All other transverse knots in $\mathcal{T}(\mathcal{K}_{(r,s)})$ destabilize to one of the ones listed above.*

5. *None of the transverse knots listed above become transversely isotopic until they have been stabilized to have self-linking number $rs-s-r$. There is a unique transverse knot in $\mathcal{T}(\mathcal{K}_{(r,s)})$ with self-linking number less than or equal to $rs-s-r$.*

We note that by Theorem 1.1.1 we get that if $r, s$ are relatively prime integers with

$$\frac{r}{s} = \frac{1}{s/r} > w(\mathcal{K}) = 1,$$

then $\mathcal{K}_{(r,s)}$ is Legendrian simple. Moreover, by Theorem 1.1.2 we get that if $r, s$ are relatively prime integers with $s > 1$ and

$$\frac{r}{s} = \frac{1}{s/r} < lw(\mathcal{K}) = 0,$$

then $\mathcal{K}_{(r,s)}$ is also Legendrian simple.

Hence, we left to classify all non-simple cables. We follow the standard approach to classifying Legendrian knots used above.

- *Identify the maximal Thurston-Bennequin invariant of the knot type and classify Legendrian knots realizing this:*

**Lemma 5.0.5.** $\overline{\mathrm{tb}}(\mathcal{K}_{(r,s)}, \mathcal{C}) = rs$ *and* $rot(L) = \pm(s-r)$ *where* $L \in \mathcal{L}(\mathcal{K}_{(r,s)})$ *with* $\mathrm{tb}(L) = \overline{\mathrm{tb}}(\mathcal{K}_{(r,s)})$.

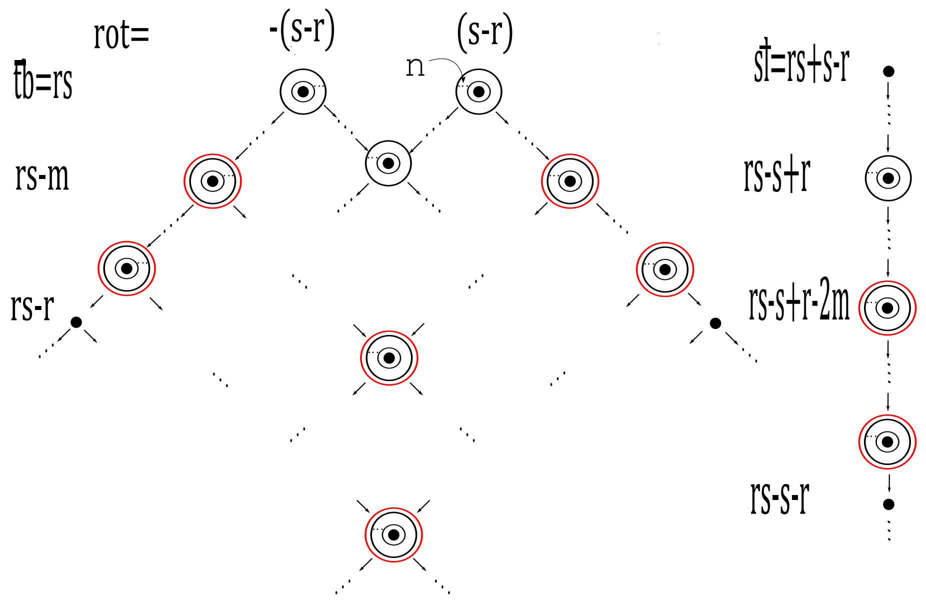


**Figure 10:** The image of $\mathcal{L}(\mathcal{K}_{(r,s)}) \to \mathbb{Z}^2 : L \mapsto (\mathrm{r}(L), \mathrm{tb}(L))$ for non-simple cablings of the positive trefoil with $\frac{s}{r} \in (n, n+1)$ on the left and $\mathbb{T}(\mathcal{K}_{(r,s)}) \to \mathbb{Z} : T \mapsto sl(T)$ on the right. The number of Legendrian knots realizing each point in $\mathbb{Z}^2$ whose coordinates sum to an odd number is indicated in the figure. The concentric circles stand for Legendrian knots in $\mathcal{L}(\mathcal{K}_{(r,s)})$ that have the same (r, tb) but pairwise Legendrian non-isotopic. The red circles stands for the non-destabilizable, non-maximal representatives where $n = \lfloor \frac{s}{r} \rfloor$, $m = |\lceil \frac{s}{r} \rceil \bullet \frac{s}{r}|$.

*Proof.* One way of the $\overline{\mathrm{tb}}$ computation is the adaptation of Lemma 3.2.2. But we include another relatively easier proof for the cables with $s > r > 1$. Observe that by assumption, the cabling coefficient is sufficiently negative; i.e. $\frac{r}{s} < \omega(\mathcal{K})$. Hence one can find Legendrian representatives in $\mathcal{L}(\mathcal{K}_{(r,s)})$ which appear as Legendrian divides on a convex torus parallel to $\partial N$, where $N$ is any solid torus of maximal thickness representing $\mathcal{K}$ and its boundary is convex. Hence $\overline{\mathrm{tb}}(\mathcal{K}_{(r,s)}) \geq rs$ by (2). For the converse, assume $\overline{\mathrm{tb}}(\mathcal{K}_{(r,s)}) > rs$. That means there is an $L$ in $\mathcal{L}(\mathcal{K}_{(r,s)})$ with $tb(L) = rs + 1$. Now by attaching a 2-handle to $D^4 = \partial S^3$ along $L$ with framing $rs$, we obtain a Stein manifold $W$. It is well known that the boundary 3-manifold is $\partial W = S^3_L(rs) = S^3_K(r/s)\#L(s,r)$ where $K$ is in $\mathcal{L}(\mathcal{K})$. Obviously there is a 2–sphere $S$ in $S^3_L(rs) = S^3_K(r/s)\#L(s,r)$. On the other hand, a theorem of Eliashberg in [6] claims that if Stein 4–manifold $W$ has an embedded 2-sphere in $\partial W$, then there must be an embedded 3–ball $D$ in $W$ such that $\partial D = S$. We prove now that this is simply not possible. Hence get a desired contradiction. Assume there is such a ball, then there are essentially two posibilities for $W$. Either $W$ has a 1-handle, i.e. $W = W' \cup$1-handle or $W$ is the boundary sum of two 4– manifolds, say $W = W_1 \natural W_2$. The former possibility is imposible as this would imply that our simply connected $W$ has $H_1(W) \neq 0$. The latter possibility is also impossible as otherwise a simple Mayer-Vietoris argument $(H_2(W_1) \oplus H_2(W_2) \cong H_2(W))$ would imply that one of the summands, say $W_1$, has $H_2(W_1) = 0$. Recall $\partial W = S^3_K(r/s) \# L(s,r)$. Let $\partial W_1 = L(s,r)$ but long exact sequence of the pair $(W_1, \partial W_1)$

$$H_2(\partial W_1) \to H_2(W_1) \to H_2(W_1, \partial W_1) \to H_1(\partial W_1) \to H_1(W_1) \to 0$$

gives

$$H_2(W_1, \partial W_1) \cong H_1(\partial W_1) \cong \mathbb{Z}_s$$

and recall by the assumption that $s > 1$. On the other hand first by the Poincare duality, then the Universal coefficient theorem we get that

$$H_2(W_1, \partial W_1) \cong H^2(W_1) \cong H_2(W_1) = 0$$

Therefore, with this contradiction in hand we conclude that $\overline{\mathrm{tb}}(\mathcal{K}_{(r,s)}) = rs.$. Note that the possibility $\partial W_1 = S^3_K(r/s)$ can be handled similarly as $r > 1$.

Now we compute the rotation number associated to maximal *tb* representatives. See Figure 5 for following computation.

Take the thickened tori $T^2 \times [1,2]$ such that

1. $T_{1.5} = \partial N(K)$ and $s(\Gamma_{T_{1.5}}) = \frac{s}{r}$ and $L_{(r,s)} \subset \partial N(K)$ is a Legendrian divide
2. $s(\Gamma_{T_1}) = \infty$ and $s(\Gamma_{T_2}) = k$ and $T^2 \times [1, 1.5] \subset \partial N(K)$

Clearly $T^2 \times [1,2]$ is a basic slice and there are two possible tight contact structures

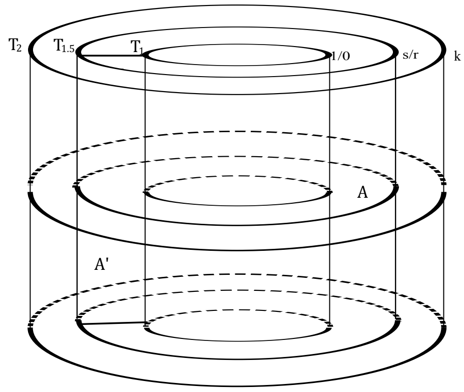


**Figure 11:** $T^2 \times I$

([22]) that are distinguished by their relative Euler class. We also have

$$PD(e(\xi)) = \mp \begin{pmatrix} 1 \\ k \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mp \begin{pmatrix} 1 \\ k-1 \end{pmatrix}. \tag{7}$$

Since both tight contact structures are universally tight, it follows from the classification ([22], [21]) that such tight contact structure can be obtained by evaluating PD(e($\xi$)) on $T^2 \times [1, 1.5]$ and $T^2 \times [1.5, 2]$ respectively and by doing this we get

$$\mp \left[ \begin{pmatrix} r \\ r \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = \mp \begin{pmatrix} r \\ s-1 \end{pmatrix} \tag{8}$$

$$\mp \left[ \begin{pmatrix} 1 \\ k \end{pmatrix} - \begin{pmatrix} r \\ s \end{pmatrix} \right] = \mp \begin{pmatrix} 1-r \\ k-s \end{pmatrix} \tag{9}$$

We need to know what $rot\,(\mu_2)$ and $rot\,(\lambda_2)$ are to compute $rot\,(L)$

- $rot\,(\mu_2) = rot\,(\mu_1) + \langle e\,(\xi)\,, A\rangle = 0 \mp (s-1)$ where $A$ is annulus between $\partial$–slope $\infty$ solid tori and $\partial$–slope $\frac{s}{r}$ solid tori.

  Moreover by using innermost solid tori in Figure 3 we get

- $rot\,(\lambda_2) = rot\,(\lambda_1) + \langle e\,(\xi)\,, A\rangle = rot\,(\lambda_1) \mp r$ where $A'$ is annulus between $\partial$–slope $\infty$ solid tori and $\partial$–slope $\frac{s}{r}$ solid tori.

By using Equation (2.7.2) we get

$$rot\,(L) = srot\,(\lambda_1) \pm sr + r(\pm(s-1)) = srot\,(\lambda_1) \pm r = \begin{cases} \pm(s+r) \\ \pm(s-r) \end{cases}$$

Observe that the second equality is because the solid tori with $\partial$–slope $\infty$ is the standart neighborhood of $K \in \mathcal{L}(\mathcal{K})$ with $tb = 0$ and $rot = \pm 1$, and hence we have $rot\,(\lambda_1) = \pm 1$.

By a careful analysis of the signs in (4.2) and (4.3) above, one can determine that $\mathrm{r}(L) = \pm(s-r)$. We may alternatively rule out the cases $\pm(s+r)$ by the following easy calculation. Recall that the Euler characteristic for the Seifert surface of cabling is

$$\chi(L) = s\chi(K) + r - rs = -rs - s + r.$$

Now by the Bennequin inequality [2], [10] we have

$$\mathrm{tb} + |\,\mathrm{r}\,| \leq rs + s - r$$

putting $\mathrm{tb} = rs$ in this inequality we get that $\mathrm{r} = \pm(s-r)$ as claimed. □

We know give classification of maximal Thurston-Bennequin invariant knots in $\mathcal{L}(\mathcal{K}_{(r,s)})$. If $K \in \mathcal{L}(K_{(r,s)})$ with $\mathrm{tb}(K) = rs$ then $K$ sits on a convex torus with dividing slope $\frac{s}{r}$.

**Lemma 5.0.6.** *There are exactly $2n$ pairwise Legendrian non-isotopic maximal tb representatives in $\mathcal{L}(\mathcal{K}_{(r,s)})$, call $L^i_{\pm}$, which have $r(L^i_{-}) = -(s-r)$ and $r(L^i_{+}) = (s-r)$ where $n < \frac{s}{r} < n+1$; $i = 0, 2, ..., n-1$.*

*Proof.* We separate the proof into the three cases.

*Case 1:* $1 < \frac{s}{r} < 2$

If $L \in \mathcal{L}(\mathcal{K}_{(r,s)})$ with maximal tb, then it can be realized as a Legendrian divide on the boundary of a solid torus $N_{\frac{s}{r}}$ representing $\mathcal{K}$ with $slope(\Gamma_{N_{\frac{s}{r}}}) = \frac{s}{r}$ and $\#\Gamma_{N_{\frac{s}{r}}} = 2$.

Any such solid torus can be thickened to $N_1$ with boundary slope 1, which is the standard neighborhood of $K \in \mathcal{L}(\mathcal{K})$ with maximal Thurston-Bennequin number and hence carries a unique tight contact structure. On the other hand, by classification of tight contact structures on solid tori, $N_{\frac{s}{r}}$ has two tight contact structures (both of them are universally tight). Once the extension to $N_1$ is determined the complement $S^3 \backslash N_1$ is unique up to contact isotopy as $\mathcal{L}(\mathcal{K})$ is Legendrian simple. Hence we have at most two Legendrian representatives at maximal tb for $\mathcal{L}(\mathcal{K}_{(r,s)})$. On the other hand the rotation number computation in Lemma 5.0.5 shows, indeed, there are exactly two Legendrian representatives at maximal *tb* which are distuinguished by their rotation numbers. This finishes the proof of Case 1.

*Case 2:* $2 \leq n \leq \frac{s}{r} < n+1$.

We have $n$ (including the maximally thickened solid torus, $N_1$) non-thickenable solid torus outside of slope $\frac{s}{r}$. Let $L^i_{\pm}$ be a maximal tb representative of $\mathcal{L}(\mathcal{K}_{(r,s)})$ which can be realized as a Legendrian divide on partially thickenable solid tori $\partial N^i_{\frac{s}{r}}$, $i = 1, 2, ..., n$ that thickens to $N_i$ but not further. By Theorem 4.2.1, we have $n$ different partial thickenings. There are two universally tight contact structures on each $N^i_{\frac{s}{r}}$ and $N_i$. By partial thickenning of each $N^i_{\frac{s}{r}}$ to $N_i$, we get at most 4 Legendrian representatives of $\mathcal{L}(\mathcal{K}_{(r,s)})$ at maximal *tb*. But as we explained in the last part of Lemma 5.0.5 two of them are rule out and we get 2 Legendrian representatives, call them $L^i_{\pm}$. If we apply Claim 5.0.7 below to each $L^i_{\pm}$, we get $2n$ maximal tb representatives in $\mathcal{L}(\mathcal{K}_{(r,s)})$ which finishes the proof of Lemma 5.0.6. $\square$

**Claim 5.0.7.** *$S^e_+(L^j_-)$ and $S^e_+(L^i_-)$ are not Legendrian isotopic. Similarly $S^e_-(L^j_+)$ and $S^e_-(L^i_+)$ are not Legendrian isotopic for any positive integer $e$. Consequently, $L^i_-$ is not Legendrian isotopic to $L^j_-$, and similarly $L^i_+$ is not Legendrian isotopic to $L^j_+$ whenever $j \neq i = 1, 2, ..., n$.*

*Proof.* Observe that $S^e_+(L^i_-)$ (for $e >> 0$) can be realized on $\partial N_i$ as a non-efficient rulling curve. On the other hand $S^e_+(L^j_-)$ can be realized on $\partial N_j$. Now let $A' =$

$\partial N_i \backslash S^e_+(L^i_-)$ and $A = \partial N_j \backslash S^e_+(L^j_-)$ and suppose $S^e_+(L^j_-)$ is Legendrian isotopic to $S^e_+(L^i_-)$, then such isotopy can be considered as an isotopy from $A'$ to $A$. Now by Isotopy Discretisation technique [24], Lemma 3.10, there is a sequence of bypass attachments that starts at $A'$ and end at $A$. Observe that there are no non-trivial bypass attachments from the outside as $\partial N_i$ is non thickenable. From the inside, however, we can have non-trivial bypass attachments but any such bypass will result a slope $t \in (i, \infty]$. It is not difficult to see that $t \notin (\frac{s}{r}, \infty]$ because of $(tb, rot)$-count (as in the proof of Lemma 5.0.13). If $t \in [i, \frac{s}{r})$ and $\tilde{L}$ is a Legendrian curve of slope $\frac{s}{r}$ on $\tilde{T}$ with $rot(\overline{L}) = rot(S^e_+(L^i_-))$, then, by Theorem 4.2.1, any bypass attachments to $\tilde{A}$ will give a torus $T'$ with $slope(\Gamma_{T'}) \in [i, \frac{s}{r})$. Hence we cannot reach $A'$. This contradiction shows that $S^e_+(L^j_-)$ and $S^e_+(L^i_-)$ are not Legendrian isotopic whenever and $j \neq i = 1, 2, ..., n$. □

• *Identify and classify the non-destabilizable, non-maximal Thurston-Bennequin Legendrian knots in $\mathcal{L}(\mathcal{K}_{(r,s)})$ and then show the rest destabilize to one of these or a maximal Thurston-Bennequin Legendrian knot:*

Let $N_k$ be the non-thickenable solid tori representing $\mathcal{K}$ that were constructed in Chapter 4.

**Lemma 5.0.8.** *Let $K$ be a Legendrian rulling curve of slope $\frac{s}{r}$ on $N_k^{\pm}$. The knot $K$ does not admit any destabilization. Since $K$ has twisting number $t(K, \mathcal{C}) = |(n+1) \bullet \frac{s}{r}|$, it is a non-maximal tb representatives of $\mathcal{L}(\mathcal{K}_{(r,s)})$. Also there are exactly two of them with the following classical invariants*

$$tb(K_\pm) = rs - \left|(n+1) \bullet \frac{s}{r}\right| = rs - (n+1)r - s,$$

$$r(K_\pm) = \pm(s - r + \left|(n+1) \bullet \frac{s}{r}\right|) = \pm rn.$$

*Proof.* Suppose $K$ admits a destabilization. Then there is a $K' \subset \Sigma'$ such that $S_\pm(K')$ is also in $\Sigma'$ and isotopic to $K$. Recall in $S^3$ Legendrian isotopy is equivalent to ambient isotopy [11]. Hence an isotopy from $S_{\pm}(K') \subset \Sigma'$ to $K$ extend to a global isotopy $\Psi_t$ such that $\Psi_1(S_{\pm}(K')) = K$. Now set $\Sigma = \Psi_1(\Sigma')$. By assumption $\Sigma$ both contains $S_{\pm}(K') = K$ and a bypass for it. Moreover, $\Sigma$ is topologically isotopic to $\partial N_{(n+1)}$. We may isotop $\Sigma$ to $\partial N_{n+1}$ relative to $K$ which is equivalent to isotoping the annulus $A = \partial N_{n+1} - K$ to $A' = \Sigma - K$. By Isotopy Discretisation technique [24], Lemma 3.10, such an isotopy can be discritized, i.e. we can get from $\Sigma$ to $\partial N_{n+1}$ by a sequence of bypass attachments. There may be two kind of bypass attachments to $A$, either from the outside or from the inside. We show in either case the only bypass attachments are triavial ones. First of all, since $N_{n+1}$ is non thickenable there cannot be any nontrivial bypass attachment onto $A$ from the outside. On the other hand any bypass attachments from the inside will increase the slope of $A$ and since the slope $\frac{s}{r}$ is shilded by an edge from a slope greater than $n+1$, the new dividing set will have more intersection with boundary rulling curve $K$ of slope $\frac{s}{r}$ but this is not possible. Hence we cannot change the boundary slope of $A$ and hence there cannot be an isotopy from $A$ to $A'$ which finishes the first part of the proof.

Secondly, we want to compute the classical invariants; *tb* computation is clear by Equation (2). For rotation number, on the other hand, first by Equation (2.7.2) we have

$$rot(K) = r \cdot rot(\partial D) + s \cdot rot(\partial \Sigma)$$

where $D$ is convex meridional disk on $N_k$, which has $rot = \pm n$ as the universally tightness of contact structures on $N_k$ implies all the bypasses are the same sign, on the other hand $\Sigma$ is convex Seifert surface of longitudal curve (which is isotopic to the core curve of $N_k$ ) on $\partial N_k$ and $rot(\partial \Sigma) = 0$ as $N_k$ is non thickenable, there is no boundary parallel dividing arcs. Hence we get $rot(K) = \pm(rn)$ which finishes the proof. $\square$

**Lemma 5.0.9.** *The only non-destabilizable representatives of $\mathcal{L}(\mathcal{K}_{(p,q)})$ are $L^i_{\pm}$ and $K_{\pm}$ from Lemma 5.0.6 and Lemma 5.0.8, respectively.*

*Proof.* Let $K$ be a non-destabilizable representative of $\mathcal{L}(\mathcal{K}_{(r,s)})$. We can place $K$ on a convex torus $\Sigma = \partial N(\mathcal{K})$. Let $slope(\Gamma_\Sigma, \mathcal{C}) = s$. Now $s \in [1,\infty] \cup [-\infty, 0)$ We claim that the only boundary slope that the convex torus $\Sigma$ can have is either $\frac{s}{r}$ or $(n+1)$. All the other cases we show that there is a convex torus $\Sigma'$ isotopic to $\Sigma = \partial N(\mathcal{K})$ and disjoint from it and $|\frac{s}{r} \bullet \Gamma_{\Sigma'}| < |\frac{s}{r} \bullet \Gamma_\Sigma|$. Then by applying the imbalance principle ([22]) to the annulus $A$ in between $\Sigma$ and $\Sigma'$ with $\partial A$ is Legendrian rulling curve of slope $\frac{s}{r}$ we show that $K$ admits a destabilization which is excluded by assumption. First of all observe that if $s = \frac{s}{r}$, then $K$ is Legendrian divide and hence we are in the situation of Lemma 5.0.6. Thus we can take $K$ as Legendrian rulling curve on $\Sigma$ and $K$ intersect $\Gamma_\Sigma$ efficiently, otherwise we get an immediate destabilization. If $s = k+1$, then we are in situation of Lemma 5.0.8 and now we show these are the only boundary slopes for $\Sigma = \partial N(\mathcal{K})$. To this end, if $s \in [1, \frac{s}{r})$, then there is a convex torus $\Sigma' \subset N(\mathcal{K})$ with $slope(\Gamma_{\Sigma'}, \mathcal{C}) = s' = \frac{s}{r}$. If $s \in (\frac{s}{r}, n+1)$ , then the same convex torus $\Sigma'$ (which in this case is outside of $N(\mathcal{K})$, but it is possible as $N(\mathcal{K})$ can be thickened to solid torus with boundary slope $\frac{s}{r}$) can be used to get a destabilization for $K$. Next, if $s \in (n+1, \infty)$, then there is a convex torus $\Sigma' \subset N$ of slope $\infty$.

$n+1 < s < \infty$ means $s = l\begin{pmatrix} 1 \\ n+1 \end{pmatrix} + m\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} l \\ ln+l+m \end{pmatrix}$ where $m, l > 0$. Then,

$$\left|\frac{s}{r} \bullet s\right| > \left|sl - rln - rl - rm\right| > \left|rm\right| > \left|r\right| = \left|\frac{s}{r} \bullet \frac{1}{0}\right|.$$

The second strict inequality is because $\frac{s}{r} < k+1$ and hence $sl - rln - rl < 0$. Thus we get a destabilization for $K$. Next, if $s \in (-\infty, 0)$, then there is a convex torus $\Sigma'$ of slope $\infty$ outside of $N$. Finally, if $s = \infty$, then in this case the solid torus $N(\mathcal{K})$ with boundary slope $\infty$ is the standard neighborhood of once stabilized maximal *tb* representative of underlying knot type $\mathcal{K}$ which is Legendrian simple by assumption. Now take the convex torus $\Sigma'$ with $s(\Gamma_{\Sigma'}) = 1$ (which is isotopic and disjoint from $\partial N(\mathcal{K})$) and we are done. $\square$

- *Determine which stabilizations of the $K_\pm$ and $L^i_\pm$ are Legendrian isotopic:*

The stabilizations of the $L^1_\pm$ are shown to be Legendrian isotopic when they have the same classical invariants

**Lemma 5.0.10.** $S_+^{s-r}(L^1_-) = S_-^{s-r}(L^1_+)$. *On the other hand* $S_+^{s-r}(L^i_-)$ *is not Legendrian isotopic to* $S_-^{s-r}(L^i_+)$ *where* $i = 2, 3, ..., n$ *and* $n < \frac{q}{p} < n+1$.

*Proof.* For the first part observe that $L^1_\pm$ are Legendrian divides on $\partial N_{\frac{s}{r}}$ and $N_{\frac{s}{r}}$ can be thickened all the way to $N_1$ (which is the maximally thickened solid torus representing $K \in \mathcal{L}(\mathcal{K})$ with $tb(K) = \overline{\mathrm{tb}}$). Simple tb count shows $S_+^{s-r}(L^1_-)$ and $S_-^{s-r}(L^1_+)$ are Legendrian rulling curves on $\partial N_1$ and now by using Legendrian simplicity of underlying knot types and the 1-parameter family of rulling curves we get that $S_+^{s-r}(L^1_-)$ and $S_-^{s-r}(L^1_+)$ are Legendrian isotopic. On the other hand, observe that for each $i$, $L^i_-$ and $L^i_+$ are Legendrian divides on the same partially thickenable solid torus $N_i^{\frac{s}{r}}$ and distuinguished by their rotation numbers. Moreover as a result of Lemma 5.0.5 we see, $S_+^{s-r}(L^i_-)$ and $S_-^{s-r}(L^i_+)$ are also can only be put (as a non-efficient curve) on $N_i^{\frac{s}{r}}$. They have the same tb and both have $rot = 0$. But now since stabilization operation is well defined, i.e. $S_+S_- = S_-S_+$, Claim 5.0.7 results that $S_+^{s-r}(L^i_-)$ and $S_-^{s-r}(L^i_+)$ are not Legendrian isotopic for $i = 2, 3, ..., n$. □

Next we understand the relationship between the stabilizations of $K_\pm$ and $L^i_\pm$;

**Lemma 5.0.11.** $S_-^{s-rn}(K_-) = S_-^r(L^i_-)$ *and* $S_+^{s-rn}(K_+) = S_+^r(L^i_+)$ *where* $i = 1, 2, ..., n$ *and* $n < \frac{s}{r} < n+1$ .

*Proof.* Observe that since $K_-$ is a Legendrian rulling curve on $N_{n+1}$, its $s - rn$-times negative stabilizition, $S_-^{s-rn}(K_-)$, is a rulling curve on slope $\infty$ solid torus $N_\infty \subset N_{n+1}$. On the other hand, as in Lemma 5.0.6, for each $i$, $L^i_-$ is a Legendrian divide on $N^i_{\frac{s}{r}}$ and $S_-^r(L^i_-)$, as a Legendrian rulling, also sits on a solid torus of slope $\infty$, $N_\infty \subset N^i_{\frac{s}{r}}$. But $N_\infty$ is the standard neighborhoods of Legendrian knot $K \in \mathcal{L}(\mathcal{K})$

with $tb(K) = 0$ and such a neighborhood is unique up to isotopy. Moreover by using Lemma 5.0.6 and Formula (2.7.2), one can see the associated rotation number is $rot(K) = -1$. Now since $\mathcal{K}$ is Legendrian simple we get that $S_-^{s-rn}(K_-)$ can be isotoped to $S_-^r(L_-)$ through the other Legendrian rullings on the . Similarly we obtain $S_+^{s-rn}(K_+) = S_+^r(L_+^i)$. $\square$

**Lemma 5.0.12.** *$S_-^j(K_-)$ is not Legendrian isotopic to $S_-^{rn+r-s+j}(L_-^i)$, similarly $S_+^j(K_+)$ is not Legendrian isotopic to $S_+^{rn+r-s+j}(L_+^i)$ for any $i = 1, ..., n$ and $j = 1, ..., s - rn$.*

*Proof.* Observe that for each $i$ $S_-^{rn+r-s}(L_-^i)$ is a Legendrian rulling on $N_{\frac{s}{r}}^k$ which admits a (partial) thickening and all other stabilizations $S_-^{rn+r-s+j}(L_-^i)$, for $j = 1, ..., s - rn$, are Legendrian rulling curve on some solid torus $S_j$ (they exist as $N_{\frac{s}{r}}^k$ admits a thickening) with $\partial S_j$ are convex and $slope(\Gamma_{\partial S_j}) = s_j$ such that $s_j < n + 1$. On the other hand $K_-$ is Legendrian rulling curve on $N_k'$ which cannot be thickened. We claim that $S_-^j(K_-)$, for $j = 1, ..., s - rn$, cannot sit on a convex torus $S_j'$ with boundary slope $s_j < n + 1$. Assume there are such torus then again by Isotopy Discretization technique [24], Lemma 3.10, such an isotopy can be discritized,i.e, there should be an isotopy which is given by sequence of bypass attachments from the annulus $A = \partial N_k' \backslash K_-$ to $A' = \partial S_j' \backslash S_-^j(K_-)$. This is simply impossible as any possible bypass attachment from the inside would have to result a boundary slope greater than $n + 1$ and an edge on the Farey teselation to $K_-$, the only such slope is $n + 1$. On the other hand any bypass attachments from the outside would result a thickening of $N_k'$ but it is non-thickenable. Hence we cannot reach $A'$. $\square$

Finally we want to give the proof of the last two statements in Theorem 5.0.3 part(a) and hence proof of Theorem 5.0.4.

**Lemma 5.0.13.** *$S_+^e(K_-)$ is not Legendrian isotopic to $S_+^e(S_-^{rn+r-s}(L_-))$ and $S^e(K_+)$ is not isotopic to $S^e S_+^{rn+r-s}(L_+)$ for all $e \in \mathbb{Z}_{>0}$. Also $S_+^r(K_-)$ is not isotopic to $S_-^r(K_+)$.*

*Proof.* Recall from Lemma 5.0.9 that $K_-$ is a Legendrian ruling curve on the convex torus $\Sigma_k = \partial N'_k(\mathcal{K})$, where $\#\Gamma_{\Sigma_k} = 2$ and $s(\Gamma_{\Sigma_k}) = k+1$. Now by a finger move we can create nonefficient intersection of $K_-$ with $\Gamma_{\Sigma_k}$ and hence realize all its (say positive) stabilizations, call $K = S^e_+(K_-)$, on $\Sigma_k$. Thus if we show that any other convex torus $\Sigma$ containing $K$ and isotopic to $\Sigma_k$ has slope $k+1$, then by the Lemma 5.0.9 we would get that $S^e_+(K_-)$ is not Legendrian isotopic to $S^e_+(S^{rk+r-s}_-(L^i_-))$ for all $e \in \mathbb{Z}_{>0}$. Observe that the creation of nonefficient intersections of $K_-$ with $\Gamma_{\Sigma_k}$ that we mentioned above geometrically corresponds to having an annulus $A = \Sigma \backslash K$ so that its dividing set contains exactly $k$ negative (and respectively positive) boundary parallel arcs on the left (and repectively on the right) hand side edge. Now assume $\Sigma$ is another convex torus containing $K$. Then we can use the other incompressible torus in Seifert fibered space to topologically isotop $\Sigma$ to $\Sigma_k$ relative to $K$. Now we want to show that under any isotopy relative to $K$ the slope of the dividing set remains the same. By Isotopy Discretization technique [24], Lemma 3.10, such an isotopy can be discritized, that is, the basic unit for this isotopy is a bypass attachment. Now we prove that all potential bypasses are trivial by induction. To this end observe that, we already know that $K$ can be placed on $\Sigma_k$. We assume inductively that $\Sigma$ satisfies the following assumptions:

- $\Sigma$ is a convex torus which contains $K$ and satisfies $2 \leq \#\Gamma_\Sigma \leq 2n+2$ and $s(\Gamma_\Sigma) = n+1$.
- $\Sigma$ is contained in a $I$-invariant $T^2 \times I$ with $s(\Gamma_{T_0}) = s(\Gamma_{T_1}) = n+1$ and $\#\Gamma_{T_0} = \#\Gamma_{T_1} = 2$
- There is a diffeomorphism of $S^3$ that takes the above $I$-invariant neighborhood $T^2 \times I$ to standart $I$-invariant neighborhood of $\Sigma$ and matches up their complement.

Hence it is enough to show that under any bypass attachment the above conditions are preserved.

We will use the method in Lemma 5.0.8 and Lemma 5.0.10 to show that the first condition is preserved. To this end, suppose $\Sigma$ satisfy the above hypothesis and consider the annulus $A = \Sigma - K$. First of all there cannot be any nontrivial bypass attachments (i.e. which changes the slope of the attached convex torus or increases the number of dividing curves) from the outside as such a bypass will result a thickening for our non-thickenable solid torus $N'_k(\mathcal{K})$. On the other hand any nontrivial bypass attachment from the inside will result a convex torus $\Sigma'$ with boundary slope $s' \in (n+1, \infty]$. Now recall from Lemma 5.0.9 that there is a convex torus $\Sigma''$ with boundary slope, $s'' = \infty$ and $\#\Gamma_{\Sigma''} = 2$, such that $|K'' \cap \Gamma_{\Sigma''}| \leq |K' \cap \Gamma_{\Sigma'}|$ where $K'$ and $K''$ are Legendrian ruling curves, parallel and disjoint from $K$, on $\Sigma'$ and $\Sigma''$, respectively. By applying the Imbalance Principle to annulus $A$ in between $\Sigma'$ and $\Sigma''$ with $\partial A = K' \cup K''$ we get bypasses disjoint from $K$. Hence we reduce to the situation that $K$ sits on a convex torus $\Sigma''$ with boundary slope $\infty$. Now we want to show that this is not possible. Recall that in Lemma 5.0.10 we showed that a Legendrian rulling curve of slope $\frac{s}{r}$ on $\Sigma''$ must be $S^j_-(K_-)$, for some positive integer $j$, where $K_-$ is the non-maximal, non-destabilizable representative (since the convex torus $\Sigma''$ with boundary slope $\infty$ bounds a standard neighborhood of a Legendrian knot in $\mathcal{L}(\mathcal{K})$ with $tb = 0$ and $rot = -1$). Hence $K \subset \Sigma''$ must be a stabilization of $S^j_-(K_-)$. On the other hand, $tb(K) = tb(K_-) - e$ and $rot(K) = rot(K_-) + e$ although the last conclusion $K = S_\pm(S^j_-(K_-))$ gives $tb(K) = S_\pm(S^j_-(K_-)) = tb(K_-) - j - 1$ and $rot(K) = rot(K_-) - j \pm 1$. This is a contradiction as $tb$ and $rot$ numbers do not match. Therefore the first condition of the induction hypothesis is preserved. We want to show that the second and the third conditions of the induction hypothesis are preserved under bypass attachment. The argument comes from Lemma 6.8 in [13] and we recall it for the sake of completeness. Suppose $\Sigma'$ is obtained from $\Sigma$ by a non-trivial bypass attachment. We have already showed that $\Sigma$ and $\Sigma'$ must have the same slope. Hence this bypass may only change the number of dividing curves. It either increases or decreases the number of dividing curves by 2 (cf. [22]). Now there are two situations to handle. First suppose that $\Sigma' \subset N$ where $\partial N = \Sigma$ and also suppose $\Sigma = T^2_{1/2}$ inside $T^2 \times [0,1]$ which satisfies the induction hypotheses. Now we will modify this thickened torus. First form the new $T^2 \times [1/2, 1]$ by adjoining the old $T^2 \times [1/2, 1]$ and the thickened torus between $\Sigma'$ and $\Sigma$. We know that $\Sigma'$ bounds a solid torus $N'$ and by using the classification of tight contact structures on solid torus, we can factor a nonrotative outer layer which gives the new $T^2 \times [0, 1/2]$ for $\Sigma'$. Now suppose that $\Sigma' \subset (S^3 \backslash N)$. Observe that $S^3 \backslash N$ is the union of neighborhoods $N(F_1)$ and $N(F_2)$ of $F_1$ and $F_2$, respectively (which are the core curves of genus 1 Heegaard splitting $V_1 \cup_T V_2$ of $S^3$), and the vertical annulus between $N(F_1)$ and $N(F_2)$. Now by thickening $N(F_1)$ and $N(F_2)$ to their maximal thickness inside $S^3 \backslash N'$ and rounding the edges we get a convex torus in $S^3 - N'$ parallel to $\Sigma'$ with $\#\Gamma_{\Sigma'} = 2$ as $F_1$ and $F_2$ are outside of $S^3 \backslash N$ and their maximally thickened neighborhoods are the standart neighborhoods $N(L_1)$ and $N(L_2)$, where $L_i$, $i = 1, 2$ are the Legendrian representatives of $F_i$, $i = 1, 2$ which maximize $tb(L_i)$ in $S^3 - N$, and $L_1 \cup L_2$ is isotopic to $F_1 \cup F_2$ . Hence we get a nonrotative outer layer $T^2 \times [1/2, 1]$. Therefore this says induction hypothesis preserved under any bypass attachment and this completes the proof of the lemma. □

**Lemma 5.0.14.** *$S^e_+(S^j_-(K_-))$ is not Legendrian isotopic to $S^e_+(S_-^{(rn+r-s)+j}(L_-))$ and $S^e(S^j_+(K_+)$ is not isotopic to $S^e(S_+^{(rn+r-s)+r}(L_+))$ for all $e \in \mathbb{Z}_{>0}$, $j = 1, 2, ...s - rn$.*

*Proof.* The proof is a corollary of Lemma 5.0.12 and Lemma 5.0.13 above. Recall by Lemma 5.0.11 we have proved that $S^j_-(K_-)$ is not Legendrian isotopic to $S_-^{(rk+r-s)+j}(L^i_-)$ for each $j = 0, 1, 2, ...s - rk - 1$. The reason was $S^j_-(K_-)$ can only be obtained by putting $K_-$ on $N'_k$ as a Legendrian which intersect with $\Gamma_{\partial N'_k}$ non-efficiently exactly $j$-times. On the other hand $S_-^{(rn+r-s)+j}(L^i_-)$ sits on a convex torus

$\Sigma$ with $s(\Gamma_\Sigma) = s > n+1$. Now by the same argument we used in Lemma 5.0.13 we know any convex torus $\Sigma'$ containing $S_+^e(S_-^j(K_-))$ and isotopic to $\partial N_k'$ must have slope $k+1$. Moreover by using this we immediately get that $S_+^e(S_-^j(K_-))$ is not Legendrian isotopic to $S_+^e(S_-^{(rn+r-s)+j}(L_-^i))$ for all $e \in \mathbb{Z}_{>0}$ and $j = 1, 2, ..., s-rn$. The proof of the second statement is similar. $\square$

*Proof of Theorem 1.2.1 and Theorem 5.0.3.* Theorem 5.0.3 simply collects the results from Lemma 5.0.5- 5.0.14. For Theorem 1.2.1 we can choose $\frac{s}{r} = \frac{kn+m(n-1)}{k+m}$. One may easily check using Theorem 5.0.3 that $\mathcal{L}(\mathcal{K}_{(r,s)})$ contains $n-1$ Legendrian knots $L_1, \ldots, L_{n-1}$ with maximal Thurston-Bennequin invariant (which will be $rs$ in this case) and rotation number $s-r$. It also contains one non-destabilizable knot $L'$ with tb $= rs - |\frac{s}{r} \cdot n| = rs - m$ and rotation number $s - r + m$. Moreover, one must stabilize $L'$ positively $k$ times before it becomes isotopic to a stabilization of one of the $L_i$. $\square$

*Proof of Theorem 1.2.2 and Theorem 5.0.4.* Theorem 2.4.1 tells us that the classification of transverse knots is equivalent to the classification of Legendrian knots up to negative stabilization. Thus the Theorem 5.0.4 is a corollary of Theorem 5.0.3. Turning to Theorem 1.2.2 we see that choices similar to those in the previous proof yield the desired result. $\square$

*Proof of Theorem1.2.3.* Let $K_i \in \mathcal{L}(\mathcal{K}_{(r,s)})$ denote Legendrian knots obtained in Item 3 of Theorem5.0.3 with $tb(K_i) = rs - m$ and $rot(K_i) = (s - r + m)$. Let's $(S^3_{+1}(K_i), \xi_i)$ denote contact +1-surgery along $K_i$ in $(S^3, \xi_{std})$. Since all but one of $K_i$, say $K_1$ (which is the non-destabilizable representative of $\mathcal{L}(\mathcal{K}_{(r,s)})$), comes from the stabilization, $\xi_i$) are overtwisted for all $i \neq 1$ for sure. But on the other hand, according to Theorem 1 in [12], $(S^3_{+1}(K_i), \xi_i)$ are all contactomorphic manifolds. Hence $\xi_1$ is overtwisted, too. Moreover since all the $K_i$ have the same *classical* invariants, all the

$\xi_i$) are homotopic as 2-plane fields. Let $(M, \xi)$ denote this common overtwisted manifold. The Legendrian knots $K_i'$'s are core curve of the surgery solid torus associated to surgery on $K_i$ and they have the same *classical* invariants (including the kot type) as well. Moreover, $K_i'$ are Legendrian *non-loose* as $M \backslash K_i'$ is contactomorphic to $S^3 \backslash K_i$. Finally, since $K_i$'s in $S^3$ are distinct, by Theorem 2.13 in [11], the contact structures on $S^3 \backslash K_i$ and hence $M \backslash K_i'$ are distinct for all $i$. Therefore $K_i'$ are all distinct.

$\square$

# Chapter VI

# LEGENDRIAN AND TRANSVERSE CABLES OF THE POSITIVE TORUS KNOTS OTHER THAN TREFOIL

We can now state the precise classification theorems for cables of general positive $(p, q)$-torus knots.

**Theorem 6.0.15.** *Let $\mathcal{K}$ be a $(p, q)$-torus knot with $(p, q) \neq (2, 3)$. Let*

$$\mathcal{I} = \{n \in \mathbb{Z} : n > 1 \text{ and } \gcd(n, pq - p - q) = 1\}$$

*and*

$$J = \cup_{n \in \mathcal{I}} J_n$$

*where $J_n = (e_n^c, e_n^a)$ is the interval of influence for the exceptional slope $e_n$, Figure 12. The $J_n$ are all disjoint.*

*The classification of Legendrian knots in the knot type $\mathcal{K}_{(r,s)}$ is then given as follows.*

1. *If $\frac{s}{r} \notin J$ then $\mathcal{K}_{(r,s)}$ is Legendrian simple. Moreover, in this case we have the following classification.*

   (a) *If $\frac{s}{r} \in (0, \frac{1}{pq-p-q}]$ then there is a unique Legendrian knot $L \in \mathcal{L}(\mathcal{K}_{(r,s)})$ with Thurston-Bennequin invariant $\mathrm{tb}(L) = rs + s(pq - p - q) - r$ and rotation number $r(L) = 0$. All others are stabilizations of $L$.*

   (b) *If $\frac{s}{r} > \frac{1}{pq-p-q}$ or $\frac{s}{r} < 0$, then the maximal Thurston-Bennequin invariant for a Legendrian knot in $\mathcal{L}(\mathcal{K}_{(r,s)})$ is $rs$ and the rotation numbers realized by Legendrian knots with this Thurston-Bennequin invariant are*

   $$\{\pm(r+s(n+k)) \mid k = (pq-p-q-n), (pq-p-q-n)-2, \ldots, -(pq-p-q-n)\},$$

*where $n$ is the least integer bigger than $\frac{r}{s}$. All other Legendrian knots $L \in \mathcal{L}(\mathcal{K}_{(r,s)})$ are stabilizations of these. Two Legendrian knots with the same* $\mathrm{tb}$ *and* $\mathrm{r}$ *are Legendrian isotopic.*

2. *If $\frac{s}{r} \in [\frac{n}{pq-p-q}, e_n^a)$, then there is some $m \geq 0$ such that $\frac{1}{m-1} > \frac{s}{r} > \frac{1}{m}$ and $\mathcal{K}_{(r,s)}$ is not Legendrian simple. The classification of Legendrian knots in $\mathcal{K}_{(r,s)}$ is as follows.*

   (a) *There are exactly $2(pq-p-q-m)+2$ pairwise Legendrian non-isotopic maximal Thurston-Bennequin representatives of $\mathcal{L}(\mathcal{K}_{(r,s)})$, call them $L^j_\pm$ and $K_\pm$. Then they satisfy $\mathrm{tb}(L^j_\pm) = \mathrm{tb}(K_\pm) = rs$ and the set of rotation numbers realized by $\left\{L^j_\pm \in \mathcal{K}_{(r,s)} \middle| \, \mathrm{tb}(L^j) = \overline{\mathrm{tb}}(\mathcal{K}_{(r,s)})\right\}$ is*

   $$\{\pm(r + s(-m + \mathrm{r}(K)))|K \in \mathcal{L}(\mathcal{K}), \mathrm{tb}(K) = m\}$$

   *and*

   $$\mathrm{r}(K_\pm) = \pm(r - s(pq-p-q)),$$

   *where $m$ is the integer satisfies $m-1 < r/s < m$ and $j = 0, 1, ..., pq-p-q-m-1$.*

   (b) *Every other $L \in \mathcal{L}(K_{(r,s)})$ is either a stabilization of one of $L^j_\pm$ or $K_\pm$ where $j = 0, 1, ..., pq-p-q-m-1$.*

   (c) *If $L^{j_0}_\pm \in \mathcal{L}(\mathcal{K}_{(r,s)})$ with $tb(L^{j_0}_\pm) = rs$ and $rot(L^{j_0}_\pm) = \pm(r - s(pq-p-q))$, then $S^e_+(L^{j_0}_-)$ is not Legendrian isotopic to $S^e_+(K_-)$ and similarly $S^e_-(L^{j_0}_+)$ is not Legendrian isotopic to $S^e_-(K_+)$ for all $e \geq 0$.*

3. *If $\frac{s}{r} \in (e_n^c, \frac{n}{pq-p-q})$, then there is some $m \geq 0$ such that $\frac{1}{m-1} > \frac{s}{r} > \frac{1}{m}$ and $\mathcal{K}_{(r,s)}$ is not Legendrian simple. The classification of Legendrian knots in $\mathcal{K}_{(r,s)}$ is as follows.*

*(a) Let $L, L' \in \mathcal{L}(\mathcal{K}_{(r,s)})$ with $\mathrm{tb}(L) = \mathrm{tb}(L') = \overline{\mathrm{tb}}(\mathcal{K}_{(r,s)}) = rs$. Then $L$ and $L'$ are Legendrian isotopic if and only if $\mathrm{r}(L) = \mathrm{r}(L')$. We have precisely $2(pq-p-q-m)$ Legendrian representatives of $\mathcal{L}(\mathcal{K}_{(r,s)})$ at $\overline{\mathrm{tb}}$ distuinguished by their rotation numbers. Moreover the set of rotation numbers realized by $\left\{L^j_{\pm} \in \mathcal{K}_{(r,s)} \,|\, \mathrm{tb}(L^j) = \overline{\mathrm{tb}}(\mathcal{K}_{(r,s)})\right\}$ is*

$$\{\pm(r + s(-m + \mathrm{r}(K)))|K \in \mathcal{L}(\mathcal{K}), \mathrm{tb}(K) = m\},$$

*where $m$ is the integer satisfies $m-1 < r/s < m$ and $j = 1, 2, ..., pq - p - q - m$.*

*(b) There are exactly two non-destabilizable non-maximal Thurston-Bennequin representatives $K_{\pm} \in \mathcal{L}(\mathcal{K}_{(r,s)})$ with*

$$\mathrm{tb}(K_{\pm}) = rs - n \qquad \textit{and} \qquad \mathrm{r}(K_{\pm}) = \pm r(k-1)$$

*where $n = \left|\frac{k}{pq-p-q} \bullet \frac{s}{r}\right|$ and $k \geq 0$.*

*(c) Every other $L \in \mathcal{L}(K_{(r,s)})$ is either a stabilization of one of $K_+$, $K_-$ from Item (b) or one of $L^j_+$, $L^j_-$ from Item (a).*

*(d) $S_-^{s(m-1)-r}(L^j_-) = S_-^{s(m-1)-r-n}(K_-)$ and $S_+^{s(m-1)-r}(L^j_+) = S_+^{s(m-1)-r-n}(K_+)$*

*(e) $S^e_+(K_-)$ is not Legendrian isotopic to $S^e_+ S^n_-(L^j_-)$ and $S^e_-(K_+)$ is not isotopic to $S^e_- S^n_+(L^j_+)$ for all $e \in \mathbb{Z}_{>0}$.*

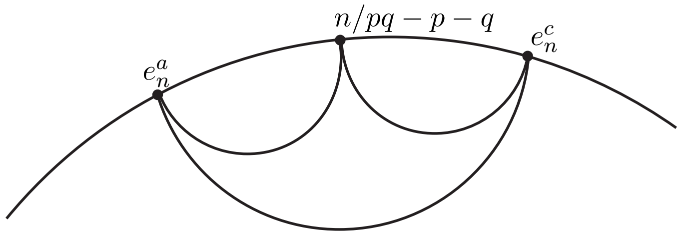


**Figure 12:** Given a rational number $\frac{n}{pq-p-q} = [a_0; a_1, ..., a_k]$ with $a_0 \geq 0$ and the other $a_i > 1$, the numbers $e^a_n$ and $e^c_n$ are determined in the Farey tessellation as $e^a_n = [a_0; a_1, ..., a_{n-1}]$ and $e^c_n = [a_0; a_1, ..., a_n - 1]$ with the convention that $e^a_n = \infty$ if $k = 0$

From this theorem we can easily derive the transverse classification.

**Theorem 6.0.16.** *Let $\mathcal{K}$ be a $(p,q)$-torus knot with $(p,q) \neq (2,3)$. Using notation from Theorem 6.0.15 we have the following classification of transverse knots in $\mathcal{T}(\mathcal{K}_{(r,s)})$.*

1. *If $\frac{s}{r} \notin J_n$ for any $n \in \mathcal{I}$ then $\mathcal{K}_{(r,s)}$ is transversely simple and all transverse knots in this knot type are stabilizations of the one with self-linking number $rs - r + s(pq - p - q)$.*

2. *If $\frac{s}{r} \in J_n$ for some $n \in \mathcal{I}$ then $\mathcal{K}_{(r,s)}$ is not transversely simple. There is a unique transverse knot $T$ in this knot type with maximal self-linking number, which is $rs - r + s(pq - p - q)$. There is also a unique non-destabilizable knot $T'$ in this knot type and it has self-linking number $rs + r - s(pq - p - q)$. All other transverse knots in $\mathcal{T}(\mathcal{K}_{(r,s)})$ destabilize to either $T$ or $T'$ and the stabilizations of $T$ and $T'$ stay non-isotopic until they are stabilized to the point that their self-linking numbers are*

$$rs + r - s(pq - p - q) - 2\left(\frac{s}{r} \cdot e_n^a\right)$$

*in the case of $\frac{s}{r} \in [e_n, e_n^a)$, and*

$$rs + r - s(pq - p - q) - 2\left(\frac{s}{r} \cdot e_n^a - \frac{s}{r} \cdot e_n\right)$$

*in the case of $\frac{s}{r} \in (e_n^c, e_n)$.*

## 6.1 Simple cables of the positive torus knots (other than the trefoil)

We note that again by Theorem 1.1.1 we get that if $r, s$ are relatively prime integers with

$$\frac{r}{s} = \frac{1}{s/r} > w(\mathcal{K}) = pq - p - q,$$

then $\mathcal{K}_{(r,s)}$ is Legendrian simple. By Theorem 1.1.2 we get that if $r, s$ are relatively prime integers with $s > 1$ and

$$\frac{s}{r} = \frac{1}{s/r} < lw(\mathcal{K}) = 0,$$

then $\mathcal{K}_{(r,s)}$ is also Legendrian simple. Moreover we have, contrary to the trefoil case, infinitely many subdomains in $(\frac{1}{pq-p-q}, \infty)$ such that for $s/r$ in these domains the $\mathcal{K}_{(r,s)}$ is Legendrian simple. More precisely we have

**Theorem 6.1.1.** *Suppose $\mathcal{K}$ is a positive $(p, q)$-torus knot with $(p, q) \neq (2, 3)$. If $r, s$ are relatively prime positive integers with $0 < \frac{r}{s} < w(\mathcal{K}) = pq - p - q$ but $\frac{s}{r} \notin J$, where $J$ is as in Theorem 6.0.15, then $\mathcal{K}_{(r,s)}$ is also Legendrian simple. Moreover, $\overline{tb}(\mathcal{K}_{(r,s)}) = rs$ and the set of rotation numbers realized by $\{L \in \mathcal{L}(\mathcal{K}_{(r,s)}) | tb(L) = \overline{tb}(\mathcal{K}_{(r,s)})\}$ is*

$$\{\pm(r + s(-n + k)) \mid k = (pq - p - q - n), (pq - p - q - n) - 2, \ldots, -(pq - p - q - n)\},$$

*where $n$ is the integer that satisfies*

$$n - 1 < \frac{r}{s} < n.$$

*All other Legendrian knots destabilize to one of these maximal Thurston-Bennequin knots.*

*Proof.* Establishing the classification of maximal Thurston-Bennequin Legendrian knots in this knot type can be done exactly as in Chapter 4, except when $\frac{s}{r} \in [e_n, e_n^a)$ for some $n$ not relatively prime to $pq - p - q$. If $L$ is a Legendrian knot in the knot type $\mathcal{K}_{(r,s)}$ for such an $\frac{s}{r} \neq e_n$ and $L$ has maximal Thurston-Bennequin invariant, then, as discussed above, $L$ will sit as a Legendrian divide on a convex torus $T$ in the knot type $\mathcal{K}$. Such a torus bounds a solid torus $S$ that can be thickened to a solid torus with convex boundary having two dividing curves of slope $e_n$. As mentioned in Corollary 1.3.3, we see that this torus further thickens to $N_1$. Thus the reasoning in Theorem 3.6 in [13] applies. If $L$ is a Legendrian knot in the knot type $\mathcal{K}_{(r,s)}$ with $\frac{s}{r} = e_n$, then it again sits as a Legendrian divide on a convex torus $T$. If $T$ is not $\partial N_n^\pm$ then according to Corollary 1.3.3 it will bound a solid torus that thickens to $N_1$. If $T = \partial N_n^\pm$ then since $e_n \notin J$, by assumption, we know $\gcd(n, pq - p - q) \neq 1$ and hence $T$ has more than two dividing curves. Below we show that we can find a torus $T'$, inside the solid torus $T$ bounds, with two less dividing curves on which $L$ also sits. Of course this new torus will thicken to $N_1$ and hence we are done as above. To find $T'$ notice that according to the classification of contact structures on thickened tori we can find a convex torus $T_0$ inside of $S$, the solid torus $T$ bounds, with two dividing curves of slope $e_n$. Let $B = T_0 \times [0,1]$ be the thickened torus that $T$ and $T_0$ cobound. Take a simple closed curve $\gamma$ on $T_0$ that intersects a curve of slope $e_n$ one time. Let $A = \gamma \times [0,1]$ be an annulus in $B$ running from $\gamma$ on $T_0$ to $T$. We can arrange that $\partial A$ consists of ruling curves on $T_0$ and $T$. Now if $\gcd(n, pq-p-q) > 2$ then there will be at least 2 non-adjacent bypasses on $A$ for $T$. Thus one of them will be disjoint from $L$. Pushing $T$ across this bypass will result in the torus $T'$ with fewer dividing curves than $T$ and on which $L$ sits. Since we are considering $(p,q)$-torus knots notice that $pq-p-q$ is odd and thus $\gcd(n, pq-p-q)$ cannot be even, thus the condition that $\gcd(n, pq-p-q) > 2$ is satisfied.

We are left to show that any Legendrian knot with non-maximal Thurston-Bennequin invariant destabilizes. Let $K$ be a Legendrian knot in the knot type $\mathcal{K}_{(r,s)}$ with $\mathrm{tb}(K) < rs$. We know that $K$ can be put on a convex torus $T$ that bounds a solid torus $S$ representing the knot type $\mathcal{K}$. Let $a$ be the dividing slope of $T$. If $a > \frac{s}{r}$ then there is a torus $T'$ parallel to $T$ inside $S$ with dividing slope $\frac{s}{r}$. We can use an annulus that cobounds $K$ and a Legendrian divide on $T'$ to show that $K$ destabilizes. Now suppose that $a < \frac{s}{r}$. If $a \in I_n = [e_n, e_n^a)$ for some $n$ then from Lemma 2.2.2 we see that $|a \cdot \frac{s}{r}| \geq |e_n^a \cdot \frac{s}{r}|$ with equality if and only if $a = e_n^a$. Since $a \neq e_n^a$ we can let $T'$ be a torus inside $S$ that is parallel to $T$ and has dividing slope $e_n^a$ and use an annulus between $K$ and a ruling curve on $T'$ to show $K$ destabilizes. If $a$ is not in $I_n = [e_n, e_n^a)$ for any $n$ then from Theorem 1.3.2 we know there is a torus $T'$ outside $S$ that is parallel to $T$ and has dividing slope $\frac{1}{pq-p-q}$. Thus between $T$ and $T'$ we have a convex torus $T''$ with dividing slope $\frac{s}{r}$. As above we can use this torus to show $K$

destabilizes. $\square$

## 6.2 Non-simple cables of the positive torus knots (other than the trefoil)

To complete the proof of Theorem 6.0.15 we need to classify Legendrian knots in the $(r, s)$-cable of the $(p, q)$-torus knot type $\mathcal{K}$ when $\frac{s}{r} \in J_n$ for some $n \in \mathcal{I}$. We do this first for the case when $\frac{s}{r} \in [e_n, e_n^a)$, and then for the case when $\frac{s}{r} \in (e_n^c, e_n)$.

We follow the standard approach to classifying Legendrian knots in a given knot type outlined in Chapter 2. For an example of classification picture see Figure 13. An arbitrary cable of an arbitrary torus knot concerned in this section is going to have a classification picture same as in Figure 13 except there might be extra peaks distinguished by their rotation numbers.

*Case 1.* $\frac{s}{r} \in [\frac{n}{pq-p-q}, e_n^a)$

*Identify the maximal Thurston-Bennequin invariant of the knot type and classify Legendrian knots realizing this:*

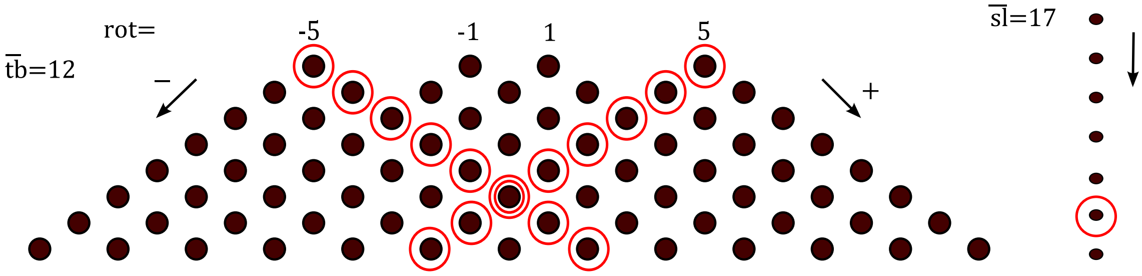


**Figure 13:** The image of the $(4, 3)$-cable of the $(2, 5)$-torus knot under $(\mathrm{r}, \mathrm{tb})$ on the left and under $sl$ on the right. The diagonal arrows stands for $\pm$ stabilizations. The red circle and the black dot at $(\mathrm{r} = -5, \mathrm{tb} = 12)$ are Legendrian non-isotopic and stay Legendrian non-isotopic under any number of positive stabilizations. Similarly the red circle and the black dot at $(\mathrm{r} = 5, \mathrm{tb} = 12)$ are Legendrian non-isotopic and stay Legendrian non-isotopic under any number of negative stabilizations. Hence give rise to transversely non-isotopic representatives in the same knot type at $sl = 7$

The computation of the maximal Thurston-Bennequin invariant is done in Lemma 3.1.2 as well as in Lemma 5.0.5

- *Construction of maximal Thurston-Bennequin invariant knots in $\mathcal{L}(\mathcal{K}_{(r,s)})$:*

Let $N_m^{\pm}$ be the non-thickenable solid tori representing $\mathcal{K}$ that were constructed in Subsection 4.1. Recall $N_1$ is a standard neighborhood of the maximal Thurston-Bennequin invariant Legendrian $(p,q)$-torus knot $L$ (and that there is only one $N_1$ so the $\pm$ is ignored here). Inside $N_1$ there are solid tori corresponding to stabilizing $L$, $(pq-q-p)-k$ times. The range of the rotation numbers for the Legendrian $(p,q)$-torus knots represented by these tori is $S = \{(pq-p-q-k), (pq-p-q-k)-2, \ldots, -(pq-p-q-k)\}$. Denote these tori $S_l$ for $l \in S$. Inside each $S_l$ there are two tori $S_l^{\pm}$ that come from positively or negatively stabilizing the Legendrian knot corresponding to $S_l$. In the thickened torus $S_l - S_l^{\pm}$ there is a unique convex torus $T_l^{\pm}$ with dividing slope $\frac{s}{r}$. Let $i = sl \pm m$ where $m = r - sk > 0$ is the remainder. Denote by $L_i$ a Legendrian divide on $T_l^{\pm}$. We clearly have that $\operatorname{tb}(L_i) = rs$ and the computation in the proof of Lemma 3.8 in [13] (or similar to the one given below for $K_{\pm}$) gives that $\operatorname{r}(L_i) = i$.

Now consider the two tori $N_n^{\pm}$. Inside each one there is a convex torus $T^{\pm}$ with dividing slope $\frac{s}{r}$. Let $K_{\pm}$ be a Legendrian divide on $T^{\pm}$. Again it is clear that $\operatorname{tb}(K_{\pm}) = rs$. Recall that from Lemma 2.7.2 we know that

$$\operatorname{r}(K_{\pm}) = r\,\operatorname{r}(\partial D) + s\,\operatorname{r}(\partial \Sigma)$$

where $D$ is a meridional disk for $T^{\pm}$ with Legendrian boundary and $\Sigma$ is a surface, outside the solid torus $T^{\pm}$ that bounds, with Legendrian boundary on $T^{\pm}$. If $D'$ and $\Sigma'$ are the corresponding surfaces for $\partial N_n^{\pm}$ then we know from Lemma 4.1.4 that $\operatorname{r}(\partial D') = \pm(n-1)$ and $\operatorname{r}(\partial \Sigma') = 0$. Thus the rotation number of an $(r,s)$-ruling curve on $\partial N_n^{\pm}$ is $\pm r(n-1)$. To compute the rotation number for the Legendrian divide on $T^{\pm}$ we use the classification of tight contact structures on thickened tori, as given in [22], and the fact that $N_n^{\pm}$ is universally tight. In particular, we can compute the relative Euler class $e$ of the thickened torus cobounded by $N_n^{\pm}$ and $T^{\pm}$:

$$P.D.(e) = \pm((r,s) - (pq - p - q, n)) \in H_1(T^2 \times I; \mathbb{Z}),$$

where $P.D.$ stands for the Poincaré Dual and we are using the basis for $H_1$ given by the meridian and longitude. We can use this to compute the difference between the rotation number of the $(r,s)$ curve on $\partial N_n^{\pm}$ and on $T^{\pm}$ which is $\pm(r(s-n) - s(r-(pq-p-q))$. Thus we have that $\mathrm{r}(K_{\pm}) = \pm(s(pq-p-q) - r)$.

*Classification of maximal Thurston-Bennequin invariant knots in $\mathcal{L}(\mathcal{K}_{(r,s)})$:*

If $K \in \mathcal{L}(K_{(r,s)})$ with $\mathrm{tb}(K) = rs$ then $K$ sits on a convex torus with dividing slope $\frac{s}{r}$. Theorem 1.3.2 and Corollary 1.3.3 say that such a torus is one of the ones considered when constructing $K_{\pm}$ and $L_i$. Thus, a by now standard argument, see [14] and Subsection 2.4.2 above, says the torus must be isotopic to one of the ones used in those constructions from which we can also conclude that $K$ is isotopic to one of $K_{\pm}$ or $L_i$.

- *Prove all non-maximal Thurston-Bennequin invariant knots in $\mathcal{L}(\mathcal{K}_{(r,s)})$ destabilize:*

Let $K$ be any Legendrian knot in $\mathcal{L}(\mathcal{K}_{(r,s)})$ with Thurston-Bennequin invariant less than $rs$. Let $T$ be a torus bounding a solid torus $S$ in the knot type $\mathcal{K}$ on which $K$ sits. Since $\mathrm{tb} < rs$ we know that we can perturb $T$ relative to $K$ so that it is convex. If the dividing slope $t$ of $T$ is equal to $\frac{s}{r}$ then $K$ intersects the dividing curves inefficiently and we can find a bypass for $K$ on $T$. Thus we can destabilize $K$. If $t \neq \frac{s}{r}$ then we have three cases to consider. Case one is when $t \notin [e_m, e_m^a)$ for any $m$. In this case Theorem 1.3.2 tells us that $S$ can be thickened to a standard neighborhood of a maximal Thurston-Bennequin knot in $\mathcal{L}(\mathcal{K})$. Thus there is a convex torus $T'$ parallel to $T$ (either inside $S$ or outside $S$ depending on $t$) with dividing slope $\frac{s}{r}$. We can use an annulus between $T$ and $T'$ with boundary on $K$ and a Legendrian divide on $T'$ to find a bypass for $K$ and hence $K$ destabilizes. Case two is when $t \in [e_m, e_m^a)$ for $m \neq n$. Lemma 2.2.2 says that $|t \cdot \frac{s}{r}|$ is strictly greater than $|\frac{s}{r} \cdot e_m^a|$ and $|\frac{s}{r} \cdot e_m^c|$ (since

$t$ is on the interior of $[e^c_m, e^a_m]$). Thus there is a torus $T'$ in $S$ with dividing slope $e^a_m$. Using an annulus between $K$ on $T$ and a $\frac{s}{r}$ ruling curve on $T'$ we find a bypass for $K$ and hence a destabilization. Finally in case three we consider $t \in [e_n, e^a_n)$. In this case we can find a torus $T'$ as in case one to destabilize $K$.

- *Determine which stabilizations of the $K_\pm$ and $L^j$ are Legendrian isotopic:*

Recall that $L^{j_0}_-$ and $K_-$ are Legendrian divides on $N_{s/r}$ and $N^k_{s/r}$, respectively. Since $N^k_{s/r}$ is partially thickenable, i.e. we can thicken it to $k$th non-thickenable solid tori $N^k$ but not further, we can realize $S^e_+(K_-)$ on non-thickenable $N^k$ as a Legendrian rulling curve for some $e > 0$. Similarly $S^e_-(L^{j_0}_-)$ can be realized on thickenable $N^{k'}$ (it exisits because $N_{s/r}$ can be thickened). Assume $S^e_+(K_-) = S^e_-(L^{j_0}_-)$, but then this would imply that $\partial N^k$ is isotopic to $\partial N^{k'}$ relative to $S^e_+(K_-) = S^e_-(L^{j_0}_-)$. In other words the annulus $A = \partial N^k - S^e_+(K_-)$ can be isotoped to $A' = \partial N^{k'} - S^e_-(L^{j_0}_-)$ relative to the boundary. By state transition technique introduced in [23], one can discritize this isotopy such that each step is a bypass attachment either from the outside or the inside. There are no nontrivial bypass attachment from the outside as $N^k$ is non-thickenable. On the other hand we could have bypasses from the inside. Assume such a bypass exists, then we get a convex torus $T_1$. By $(tb, rot)$-count (as in Claim 5.0.7) shows that $slope(\Gamma_{T_1}) \in \left[\frac{k}{pq-p-q}, \frac{s}{r}\right)$. On the other hand by partial thickenability of $N^k_{s/r}$, we conclude that any bypass atachment to $T_1$ will result sequence of convex tories $T_i$ such that $slope(\Gamma_{T_i}) \in \left[\frac{k}{pq-p-q}, \frac{s}{r}\right)$. Hence we cannot reach $A'$. This completes the proof.

We now classify cables arise in the right portion of the interval influence around $e_n$. As it will be explained below, the reason of non-simplicity for the cables that can be realized in the right portion of the non-simple domain is very different than the ones in the left portion of non-simple domains. For an example of classification picture see Figure 14. An arbitrary cable of an arbitrary torus knot concerned in the following case is going to have a classification picture same as in Figure 14 except the non-stabilizable, non-maximal representatives might have tb arbitrary less than $\overline{\text{tb}}$.

*Case 2.* $\frac{s}{r} \in (e_n^c, e_n)$

- *Identify the maximal Thurston-Bennequin invariant of the knot type and classify Legendrian knots realizing this:*

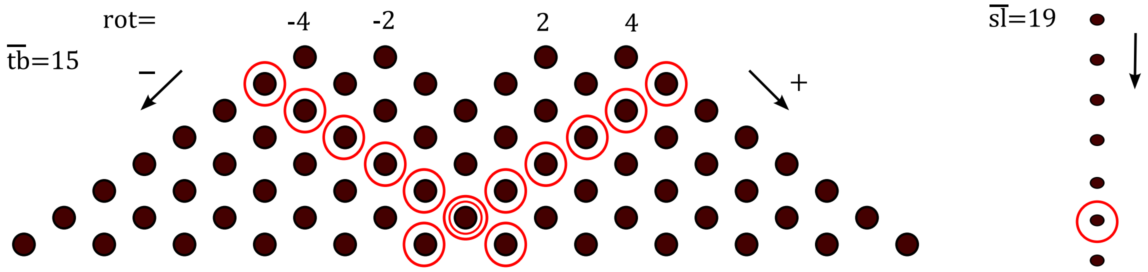


**Figure 14:** The image of the $(5,3)$-cable of the $(2,5)$-torus knot under $(\mathrm{r}, \mathrm{tb})$ on the left and under $sl$ on the right. The diagonal arrows stands for $\pm$ stabilizations of Legendrian representatives. The red circles stands for the non-destabilizable non-maximal Thurstaon-Bennequin representatives. Moreover the red circle and the black dot at $(\mathrm{r} = -5, \mathrm{tb} = 14)$ are Legendrian non-isotopic and stay Legendrian non-isotopic under any number of positive stabilizations. Similarly the red circle and the black dot at $(\mathrm{r} = 5, \mathrm{tb} = 14)$ are Legendrian non-isotopic and stay Legendrian non-isotopic under any number of negative stabilizations. Hence give rise to transversely non-isotopic representatives in the same knot type at $sl = 9$

The computation of the maximal Thurston-Bennequin invariant is done in Lemma 3.1.2.

*Construction of maximal Thurston-Bennequin invariant knots in* $\mathcal{L}(\mathcal{K}_{(r,s)})$*:*

This is identical to part of the construction in the previous case. Let $N_1$ be a standard neighborhood of the maximal Thurston-Bennequin invariant Legendrian $(p,q)$-torus knot. Inside $N_1$ there are solid tori corresponding to stabilizing $L$, $(pq-q-p)-k$ times. The range of the rotation numbers for the Legendrian $(p,q)$-torus knots represented by these tori is $S = \{(pq-p-q-k), (pq-p-q-k)-2, \ldots, -(pq-p-q-k)\}$. Denote these tori $S_l$ for $l \in S$. Inside each $S_l$ there are two tori $S_l^{\pm}$ that come from positively or negatively stabilizing the Legendrian knot corresponding to $S_l$. In the thickened torus $S_l - S_l^{\pm}$ there is a unique convex torus $T_l^{\pm}$ with dividing slope $\frac{s}{r}$. Let

$i = sl \pm m$ where $m = r - sk > 0$ is the remainder. Denote by $L_i$ a Legendrian divide on $T_l^{\pm}$. We clearly have that $\mathrm{tb}(L_i) = rs$ and the computation in the proof of Lemma 3.8 in [13] gives that $\mathrm{r}(L_i) = i$.

*Classification of maximal Thurston-Bennequin invariant knots in $\mathcal{L}(\mathcal{K}_{(r,s)})$:*

If $K \in \mathcal{L}(K_{(r,s)})$ with $\mathrm{tb}(K) = rs$ then $K$ sits on a convex torus with dividing slope $\frac{s}{r}$. Theorem 1.3.2 and Corollary 1.3.3 say that such a torus is one of the ones considered when constructing the $L_i$. Thus, a by now standard argument, see [14], says the torus must be isotopic to one of the ones used in those constructions from which we can also conclude that $K$ is isotopic to one of $L_i$.

• *Identify and classify the non-destabilizable, non-maximal Thurston-Bennequin Legendrian knots in $\mathcal{L}(\mathcal{K}_{(r,s)})$ and then show the rest destabilize to one of these or a maximal Thurston-Bennequin Legendrian knot:*

Let $N_m^{\pm}$ be the non-thickenable solid tori representing $\mathcal{K}$ that were constructed in Subsection 4.1.

*Constructing the non-destabilizable Legendrian knots:*

Consider the two tori $N_n^{\pm}$. Let $K_{\pm}$ be a ruling curve of slope $(r, s)$ on $\partial N_n^{\pm}$. It is clear that the twisting of the contact planes along $K_{\pm}$ with respect to the framing of $K_{\pm}$ coming from $\partial N_n^{\pm}$ is

$$-\frac{1}{2}\left|K_{\pm} \cdot \Gamma_{\partial N_n^{\pm}}\right| = -\left|\frac{s}{r} \cdot e_n\right|.$$

Thus the Thurston-Bennequin invariant (that is the twisting with respect to the Seifert surface for $K_{\pm}$) is

$$\mathrm{tb}(K_{\pm}) = rs - \left|\frac{s}{r} \cdot e_n\right|.$$

Just as in the previous case we compute

$$\mathrm{r}(K_{\pm}) = \pm r(n-1).$$

*Proving all non-maximal Thurston-Bennequin invariant knots either destabilize or have* $\mathrm{tb} = rs - |\frac{s}{r} \cdot e_n|$ *and sit as a ruling curve on $\partial N_n^{\pm}$:*

Let $L$ be a Legendrian knot in $\mathcal{L}(\mathcal{K}_{(r,s)})$ with $\text{tb}(L) < rs$. Let $S$ be a solid torus representing the knot type $\mathcal{K}$ that contains $L$ in its boundary. We know that the twisting of the contact planes with respect to $\partial S$ is negative so we can make $\partial S$ convex without moving $L$. If $L$ does not intersect the dividing curves $\Gamma_{\partial S}$ minimally (for curves in their homology classes) then we will see a bypass for $L$ on $\partial S$ and hence $L$ destabilizes. So we can assume that $L$ intersects $\Gamma_{\partial S}$ minimally.

Now if the dividing slope $t$ of $\partial S$ is not $e_n$ then there are three cases to consider. Case one is when $t \notin [e_m, e_m^a)$ for any $m$. In this case Theorem 1.3.2 tells us that $S$ can be thickened to a standard neighborhood of a maximal Thurston-Bennequin knot in $\mathcal{L}(\mathcal{K})$. Thus there is a convex torus $T$ parallel to $\partial S$ (either inside $S$ or outside $S$ depending on $t$) with dividing slope $\frac{s}{r}$. We can use an annulus between $T$ and $\partial S$ with boundary on $L$ and a Legendrian divide on $T$ to find a bypass for $L$ and hence $L$ destabilizes. Case two is when $t \in [e_m, e_m^a)$ for $m \neq n$. Lemma 2.2.2 says that $|t \cdot \frac{s}{r}|$ is strictly greater than $|\frac{s}{r} \cdot e_m^a|$ and $|\frac{s}{r} \cdot e_m^c|$ (since $t$ is on the interior of $[e_m^c, e_m^a]$). Thus there is a torus $T$ in $S$ with dividing slope $e_m^a$. Using an annulus between $K$ on $T$ and a $\frac{s}{r}$ ruling curve on $T$ we find a bypass for $L$ and hence a destabilization. Finally in case three we consider $t \in (e_n, e_n^a)$. In this case we have that $|\frac{s}{r} \cdot t| > |\frac{s}{r} \cdot e_n|$. We can thus use an annulus between $L$ on $\partial S$ and a $\frac{s}{r}$ ruling on $\partial N_n^{\pm}$ to find a bypass for $L$.

If $t = e_n$ then $L$ is a ruling curve on $\partial S$. If $S$ is not $N_n^{\pm}$ then $S$ will thicken to $N_1$ and thus we can again destabilize $L$ as in case one of the previous paragraph. So we see that $L$ will destabilize unless it is a ruling curve on $N_n^{\pm}$. Of course in this case $\text{tb}(L) = rs - |\frac{s}{r} \cdot e_n|$.

- *Proving the knots $K_{\pm}$ do not destabilize:*

If $K_{\pm}$ destabilized then by the above work they would be stabilizations of one of the $L_i$. Thus $K_{\pm}$ could be put on some convex torus other than $\partial N_n^{\pm}$, but this contradicts

*Proving any Legendrian knots with* $\mathrm{tb} = rs - |\frac{s}{r} \cdot e_n|$ *either destabilize or are isotopic to* $K_\pm$*:* This is immediate from the work above and Corollary 1.3.3.

- *Determine which stabilizations of the* $K_\pm$ *and* $L_i$ *are Legendrian isotopic:*

Note that $L_-^{j_0}$ is the Legendrian divide on $N$ and $s(m-1)-r$ is the necessary number of negatif stabilization to realize $L_-$ as Legendrian rulling curve on $\partial N' \subset N$ where $N'$ is the standard neighborhood of a Legendrian knot $K \in \mathcal{L}(\mathcal{K})$ with $tb(K) = m-1$. Similarly since $K_-$ is a Legendrian rulling curve on the $k$th non-thickenable $N^k$, i.e. $slope(\Gamma_{\partial T^k}) = \frac{k}{pq-p-q}$, $S_-^{s(m-1)-r-n}(K_-)$ is a Legendrian rulling curve on $N_1'$, where $N_1'$ is also a standard neighborhood of Legendrian knot $K' \in \mathcal{L}(\mathcal{K})$ with $tb(K') = m-1$. Moreover $rot(K) = rot(K') = pq-p-q-m+1$. Since underlying knot type $\mathcal{K}$ is Legendrian simple we conclude that there is a global contact isotopy of $(S^3)$ that takes $N'$ to $N_1'$. Now we may Legendrian isotope $S^{s(m-1)-r}(L_-)$ to $S^{s(m-1)-r}(K_-)$ through rulling curves. Similarly one can conclude that $S^{s(m-1)-r}(L_+)$ to $S^{s(m-1)-r}(K_+)$.

We now prove $S_+^e(K_-)$ is not Legendrian isotopic to $S_+^e S_-^n(L_-^j)$ and $S_-^e(K_+)$ is not isotopic to $S_-^e S_+^n(L_+^j)$ for all $e \in \mathbb{Z}_{>0}$, where $n = |\left\lceil \frac{k}{pq-p-q} \right\rceil \bullet \frac{s}{r}|$.

By using the very similar argument we used in Theorem 5.0.13 (Or Claim 6.5 in the proof of Theorem 1.7 from [13]) we would like to show that any convex torus which contains $S_+^e(K_-)$ and is isotopic to $T^k = \partial N^k$, $k$th-non-thickenable tori, has slope $\frac{k}{pq-p-q}$ and $\#\Gamma = 2$. In other words every positive stabilization of $K_-$ has to be obtained by creating non-efficient intersection of $K_-$ with $\Gamma_T$. Assuming for while we have this. On the other hand, since this will not be a case for further stabilizations of $S_-^n(L_-^j)$,we will conclude that $S_+^e(K_-)$ is not Legendrian isotopic to $S_+^e S_-^n(L_-^j)$ for all $e \in \mathbb{Z}_{>0}$ by previous step. Note that we may always put $S_+^e(K_-)$ on $T^k$ so that the dividing set on annulus $A = T^k \backslash S_+^e(K_-)$ has $e$ boundary parallel arc of positive/negative sign on the left boundary and $e$ boundary parallel arc of negative/positive sign on the right boundary. Now let $T$ be another convex torus that contains $S^e_+(K_-)$ and having base case already satisfied by $T^k$ we assume $T$ satisfies the following induction hypothesis; (1) $T$ is a convex torus which contains $S^e_+(K_-)$ and satisfies $2 \leq \#\Gamma_T \leq 2e+2$ and $s(\Gamma_T) = \frac{k}{pq-p-q}$ (2) $T$ is contained in a $I$-invariant $T^2 \times I$ with $s(\Gamma_{T_0}) = s(\Gamma_{T_1}) = \frac{k}{pq-p-q}$ and $\#\Gamma_{T_0} = \#\Gamma_{T_1} = 2$ (3) There is a diffeomorphism of $S^3$ that takes the above $I$-invariant neighborhood $T^2 \times I$ to standart $I$-invariant neighborhood of $\Sigma$ and matches up their complement.

Hence it is enough to show that under any bypass attachment the above conditions are preserved. We will prove for (1) and refer the identical proof of Lemma 5.0.13 for (2) and (3). Let $T$ satisfy the inductive hypothesis. Since $N^k$ is non-thickenable no bypass attachment to $T \backslash S^e_+(K_-)$ from the outside will change the slope of dividing set. On the other hand we could have non-trivial bypass attachment from the insidea and any such bypass will result a convex torus $T'$ with $slope(\Gamma_{T'}) \in \left[\frac{k}{pq-p-q}, \infty\right]$. We want to show that after further bypass attachment to $T'$ we can obtain convex torus $T''$ such that it has slope $\frac{1}{m-1}$ for some $m = 1, 2, ..., pq-p-q$. For a while assume we have this, i.e. $S^e_+(K_-)$ sits on $T''$ with $slope(\Gamma_{T''}) = \frac{1}{m-1}$ for some $m$ and $\#\Gamma_{T''} = 2$. Note that $T''$ bounds the standard neighborhood of a Legendrian knot in $\mathcal{L}(\mathcal{K})$ with $\mathrm{tb} = m-1$ and $\mathrm{r} = pq-p-q-m+1$. We know $\mathrm{tb}(S^e_+(K_-)) = \mathrm{tb}(K_-) - e$ and $\mathrm{r}(S^e_+(K_-)) = \mathrm{r}(K_-) + e$. On the other hand by Step (4) we find that slope $s/r$ rulling curve on $T''$ must be $S_-^{s(m-1)-r-n}(K_-)$ which implies that $S^e_+(K_-) = S_\pm S_-^{s(m-1)-r-n}(K_-)$ which gives $\mathrm{tb}(S^e_+(K_-)) = \mathrm{tb}(K_-) - (s(m-1)-r-n) - 1$ and $\mathrm{r}(S^e_+(K_-)) = \mathrm{r}(K_-) - (s(m-1)-r-n) \pm 1$ that contradicts our $(\mathrm{tb}, \mathrm{r})$ computation. Now we prove the assumption that caused this contradiction.

Assume $slope(\Gamma_{T'}) \in \left[\frac{k}{pq-p-q}, \frac{1}{m-1}\right)$ for some $m = 1, 2, ..., pq-p-q$. Then take a efficient Legendrian curve $L'$ parallel to and disjoint from $S^e_+(K_-)$. Also take $L''$ on $T''$. Once again by using the Farey tesselation as in Lemma 5.0.9 and Lemma 5.0.13 we get that $|\Gamma_{T'} \cap L'| > |\Gamma_{T''} \cap L''|$. Thus we get bypasses for $T'$ by using the above inequality and the Imbalance Principle. We can further continue on this to get succesive bypasses that finally gives a convex torus with slope $\frac{1}{m-1}$, for $m = 1, 2, .., pq - p - q$. Therefore this proves that $slope(\Gamma_T)$ is preserved under any bypass attachment.

*Proof of Theorem 1.2.4 and Theorem 6.0.15 .* Theorem 6.0.15 is an immediate consequence of *Case 1* and *Case 2* together with Theorems 1.1.1, 1.1.2 and 6.1.1. Theorem 1.2.4 is clear from the statement of Theorem 6.0.15. $\square$

*Proof of Theorem 1.2.5 and Theorem 6.0.16.* Theorem 2.4.1 tells us that the classification of transverse knots is equivalent to the classification of Legendrian knots up to negative stabilization. Thus the Theorem 6.0.16 is a corollary of Theorem 6.0.15. Theorem 1.2.5 follows from Theorem 6.0.16 once one observes that that if we choose $\frac{s}{r} = me_k + ne_k^a$ (where the addition is done as on the Farey tessellation), then $\frac{s}{r} \cdot \frac{1}{pq-p-q} > n$. As a result, the non-destabilizable transverse knot will have self-linking number at least $2n$ less than maximal; furthermore, it will take $\frac{s}{r} \cdot e_k^a = m$ stabilizations before it becomes isotopic to a stabilization of the maximal self-linking number transverse knot. $\square$

# Chapter VII

# FUTURE PLANS

In this chapter we list some of the problems we want to study in the future.

Let $\mathcal{K}$ be a knot type in $(S^3, \xi_{std})$ that realize Bennequin bound, i.e. $\overline{\mathrm{tb}}(\mathcal{K}) + |\,\mathrm{r}(\mathcal{K})| = 2g(\mathcal{K}) - 1$ and $\mathrm{r}(\mathcal{K}) \neq 0$. Let $(Y_n(L), \xi_n^-)$ denote contact 3–manifold obtained by doing contact $n$ - surgery along $L \in \mathcal{L}(\mathcal{K})$ in $(S^3, \xi_{std})$.

**Problem 7.0.1.** Determine necessary and sufficient condition on integer $n$ that results $\xi_n^-$ to be overtwisted.

One way is to attack this problem is showing that the contact class $c(\xi_n^-) \in HF^o(-Y_n(L), s_{\xi_n^-})$ does not vanish for all $n$. Hence concluding that $\xi_n^-$ is tight. We can give a necessary and sufficient condition for vanishing of $c(\xi_n^-)$ in terms of $n$. So, tightness is not the automatic for all $n$. Moreover, this vanishing result, of course, gives a clue for overtwistedness but nothing more. As, unfortunately, vanishing of the contact class does not necessarily implies the overtwistedness. The iterated torus knots treated in Chapter 5 and 6 are, at least to the author, only known examples of knot types that realize Bennequin bound and have $\mathrm{r} \neq 0$.

**Problem 7.0.2.** Study on the classification problem of tight contact structures on small Seifert Fibered spaces, $M(e_0; r_1, r_2, r_3)$ where $e_0 = -1$ and $r_i \in \mathbb{Q} \cap (0, 1)$ with $r_1 \geq r_2 \geq r_3$.

The case $r_1 \geq r_2 \geq \frac{1}{2}$ was completed by Lisca-Ghiggini-Stipsicz in [26]. The usage of non-thickenable (and possibly partially thickenable) neighborhhods is already suggesting some classification results for certain Briskorn spheres.

**Problem 7.0.3.** Given a knot type $\mathcal{K}$ in $(S^3, \xi_{std})$, is there an integer $n$ such that any $L \in \mathcal{L}(\mathcal{K})$ with $\mathrm{tb}(L) < n$ admits a destabilization?

In our classification of Legendrian cables of the positive trefoil, we locate a non-destabilizable representative arbirtary far below maximum tb invariant, by arranging the cabling coefficients. But this effort also result a very high maximum tb invariant. As a result, it is not difficult to see the integer asked in the problem above is indeed 0 for Legendrian cables of the positive torus knots. It is very likely that answer will be "yes" to Problem 7.0.3 and one way to attack this addressing the following.

**Problem 7.0.4.** Let $\mathcal{K}$ be a knot type with $\omega(\mathcal{K}) \neq \overline{\mathrm{tb}}(\mathcal{K})$, is $\mathcal{K} =$ *the unknot*? Moreover, if $\mathcal{K}$ is not the unknot, is $\ell\omega(\mathcal{K}) < \infty$?

**Problem 7.0.5.** Study Legendrian simplicity under some other statalite constructions, in particular under Whitehead doubling.

The last problem in particular will be very helpful in terms of better understanding the uniform thickness property.

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