AC Transmission System Planning Choosing Lines from a Discrete Set

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***Abstract*—Transmission system planning (TSP) is a difficult nonlinear optimization problem involving non-convex quadratic terms, as well as discrete variables. We extend prior results on linear relaxations, drawing on a preliminary notional model of the power grid for the State of Florida. Realistic line choices necessitate a binary formulation, which is at the same time substantially more expensive than the mixed-integer counterpart and more accurate. In many cases, our relaxation directly generates a feasible solution; where it does not, we apply a practical load-deflation heuristic to recover strong solutions.**

***Index Terms*—Transmission system planning, linear relaxation, AC power flow, binary formulation.**

I. Introduction

Transmission system planning (TSP) is a classic problem in power distribution; from the first installations more than a century ago it has been desirable to minimize build costs and line losses, subject to discrete choices in the components, and to AC physics [6]. The problem is typified by non-convex quadratic terms in the complex power equations, and in previous work we developed linear relaxations using lift-and-project procedures [12][1], and stronger second-order cone and semidefinite relaxations [11] . An advantage of linear relaxations is that one can keep the discrete variables intact, making use of powerful cutting plane techniques; in our mixed-integer formulation, relaxed solutions are realistic for $O(100)$ busses. At the other extreme, heuristic methods have been widely applied to the problem [10],[8]. It is well-known that the DC approximation is weak [2].

The present paper extends our earlier linear formulation to include more complex user choices. In particular, we are motivated by the process of planning expansions of the AC power grid in the State of Florida. New lines cost several million dollars per mile, with line lengths in Florida reaching several hundred miles. Further, in this scenario the lines are chosen from a discrete and exclusive set of options: four different line types are available, each with a different voltage and current rating, resistance per mile, and so on. Binary variables are therefore a necessity in this problem.

Nomenclature is provided in Table 1. Section II develops the relaxation from the power equations in rectangular form, and Section III applies it to representative problems of differing size. The new relaxation is strong.

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This work was supported by the Office of Naval Research, Grant N00014-02-1-0623, monitored by Dr. T. Ericsen.
[1] This work provides a number of additional references.

II. AC Power Flow Optimization

Let $\mathcal{E}$ denote the set of edges being considered for expansion and $\mathcal{E}^o$ the set of existing lines. Neglecting shunt elements, we have

$$\underset{v,s,z,e,\overline{v}}{\text{minimize}} \sum_{ij\,\in\,\mathcal{E}} c_{ij} \tag{1}$$

$$\text{subject to} \quad c_{ij} = \sum_{k=1}^{K+1} z_{kij}\mathbb{C}_{kij} \tag{2}$$

$$(s_{ij}-s_{ji}) = (v_i v_i^* - v_i v_j^*)\left(\frac{1}{R_{ij}-jX_{ij}} + \frac{1}{R^o_{ij}-jX^o_{ij}}\right), \quad ij \in \mathcal{E}^o \cap \mathcal{E} \tag{3}$$

$$(s_{ij}-s_{ji}) = (v_i v_i^* - v_i v_j^*)\left(\frac{1}{R_{ij}-jX_{ij}}\right), \quad ij \in \mathcal{E}\setminus\mathcal{E}^o$$

$$(s_{ij}-s_{ji}) = (v_i v_i^* - v_i v_j^*)\left(\frac{1}{R^o_{ij}-jX^o_{ij}}\right), \quad ij \in \mathcal{E}^o\setminus\mathcal{E}$$

$$|s_{ij}-s_{ji}| \;\le\; \frac{\sqrt{2}}{2} V_{ij} I_{ij} \tag{4}$$

$$\underline{p}_i \;\le \sum_{j,\, ij\,\in\,\mathcal{E}\cup\mathcal{E}^o} \mathrm{Re}\,(s_{ij}-s_{ji}) \;\le\; \overline{p}_i \tag{5}$$

$$\underline{q}_i \;\le \sum_{j,\, ij\,\in\,\mathcal{E}\cup\mathcal{E}^o} \mathrm{Im}\,(s_{ij}-s_{ji}) \;\le\; \overline{q}_i$$

$$\overline{v}_i = \sum_{l=1}^{L} \mathcal{V}_l e_{li} \tag{6}$$

$$f\overline{v}_i \;\le\; |v_i| \;\le\; \overline{v}_i \tag{7}$$

$$\sum_{l=1}^{L} e_{li} = 1 \qquad \sum_{k=1}^{K+1} z_{kij} = 1 \tag{8}$$

$$V_{ij} = \sum_{k=1}^{K+1} \mathbb{V}_k z_{kij} \qquad I_{ij} = \sum_{k=1}^{K+1} \mathbb{I}_k z_{kij} \tag{9}$$

$$R_{ij} = \sum_{k=1}^{K+1} \mathbb{R}_k z_{kij} \qquad X_{ij} = \sum_{k=1}^{K+1} \mathbb{X}_k z_{kij} \tag{10}$$

$$e_{li}, z_{kij} \in \{0,1\}\,. \tag{11}$$

The formulation includes products of binary variables in (2), which can be easily reduced to a linear function, e.g., via $ze = \frac{1}{2}(-|z-e|+z+e)$. The admittance equations (3), for edges in $\mathcal{E}\cap\mathcal{E}^o$, expand into the real and imaginary parts

$$R^o_{ij}[(p_{ij}-p_{ji})R_{ij}+(q_{ij}-q_{ji})X_{ij}]- \\ X^o_{ij}[(q_{ij}-q_{ji})R_{ij}-(p_{ij}-p_{ji})X_{ij})] = \\ (w_i^2+x_i^2-w_iw_j-x_ix_j)(R^o_{ij}+R_{ij})+ \\ (w_ix_j-w_jx_i)(X_{ij}+X^o_{ij}) \tag{12}$$

$$X^o_{ij}[(p_{ij}-p_{ji})R_{ij}+(q_{ij}-q_{ji})X_{ij}]+ \\ R^o_{ij}[(q_{ij}-q_{ji})R_{ij}-(p_{ij}-p_{ji})X_{ij}] = \\ (w_i^2+x_i^2-w_iw_j-x_ix_j)(-X_{ij}-X^o_{ij})+ \\ (w_ix_j-w_jx_i)(R^o_{ij}+R_{ij}). \tag{13}$$

For edges in $\mathcal{E}\setminus\mathcal{E}^o$, the admittance equations are

$$(p_{ij}-p_{ji})R_{ij}+(q_{ij}-q_{ji})X_{ij} = w_i^2+x_i^2-w_iw_j-x_ix_j \tag{14}$$

$$-(p_{ij}-p_{ji})X_{ij}+(q_{ij}-q_{ji})R_{ij} = w_ix_j-w_jx_i. \tag{15}$$

Edges $\mathcal{E}^o\setminus\mathcal{E}$ are a trivial variant on this, replacing R_{ij} and X_{ij} with R^o_{ij} and X^o_{ij}. In all of these cases, because R_{ij} and X_{ij} depend on z_{kij}, we see that these expressions involve products of binary and linear terms on the left side, and products of binary and quadratic terms on the right. The former are easy to handle in linear programming, using the following well-known trick:

$$y = zx \quad\longleftrightarrow\quad \begin{array}{l} y\le \overline{x}z,\ \ y\le x-\underline{x}(1-z), \\ y\ge \underline{x}z,\ \ y\ge x-\overline{x}(1-z), \end{array}$$

where z and x represent binary and continuous variables, and $\{\underline{x},\overline{x}\}$ the upper and lower bounds of x. We give a more specific expansion of these terms below.

The main difficulty in the admittance equations is the quadratic voltage terms. Along the same lines, the line power and nodal voltage magnitude constraints ((4) and (7)) in the program are quadratic:

$$(p_{ij}-p_{ji})^2+(q_{ij}-q_{ji})^2 \;\le\; \frac{1}{2}V_{ij}^2I_{ij}^2 \tag{16}$$

$$f^2\overline{v}_i^2 \;\le\; w_i^2+x_i^2 \;\le\; \overline{v}_i^2. \tag{17}$$

We will not relax the power variables, since they appear only in the power magnitude limit and are therefore suitable for a piecewise linear approximation; a simple example is a circumscribed octagon approximation

$$|p_{ij}-p_{ji}|+\frac{\sqrt{2}}{2-\sqrt{2}}|q_{ij}-q_{ji}| \;\le\; \frac{2}{(2-\sqrt{2})\sqrt{1+\frac{\sqrt{2}}{2}}}V_{ij}I_{ij} \tag{18}$$

$$|p_{ij}-p_{ji}|+\left(\sqrt{2}-1\right)|q_{ij}-q_{ji}| \;\le\; \frac{\sqrt{2}}{\sqrt{1+\frac{\sqrt{2}}{2}}}V_{ij}I_{ij}.$$

We formulate a lift-and-project linear relaxation for the quadratic voltage terms by introducing new variables and constraints. Let

$$\phi_{ij} \longleftrightarrow w_i^2+x_i^2-w_iw_j-x_ix_j \tag{19}$$

$$\mu_{ij} \longleftrightarrow w_ix_j-w_jx_i \tag{20}$$

$$\alpha_i \longleftrightarrow w_i^2+x_i^2. \tag{21}$$

The admittance constraints for edges in $\mathcal{E}\cap\mathcal{E}^o$ become

$$R^o_{ij}((p_{ij}-p_{ji})R_{ij}+(q_{ij}-q_{ji})X_{ij})- \\ X^o_{ij}((q_{ij}-q_{ji})R_{ij}-(p_{ij}-p_{ji})X_{ij}) = \\ \phi_{ij}(R^o_{ij}+R_{ij})+\mu_{ij}(X_{ij}+X^o_{ij}) \tag{22}$$

$$X^o_{ij}((p_{ij}-p_{ji})R_{ij}+(q_{ij}-q_{ji})X_{ij})+ \\ R^o_{ij}((q_{ij}-q_{ji})R_{ij}-(p_{ij}-p_{ji})X_{ij}) = \\ \phi_{ij}(-X_{ij}-X^o_{ij})+\mu_{ij}(R^o_{ij}+R_{ij}). \tag{23}$$

The admittance constraints for edges in $\mathcal{E}\setminus\mathcal{E}^o$ are

$$(p_{ij}-p_{ji})R_{ij}+(q_{ij}-q_{ji})X_{ij} = \phi_{ij} \\ -(p_{ij}-p_{ji})X_{ij}+(q_{ij}-q_{ji})R_{ij} = \mu_{ij}, \tag{24}$$

and as before, the case of edges in $\mathcal{E}^o\setminus\mathcal{E}$ is an extension without the binary variables

$$(p_{ij}-p_{ji})R^o_{ij}+(q_{ij}-q_{ji})X^o_{ij} = \phi_{ij} \\ -(p_{ij}-p_{ji})X^o_{ij}+(q_{ij}-q_{ji})R^o_{ij} = \mu_{ij}. \tag{25}$$

Through the same mechanism the voltage magnitude constraints become

$$f^2\overline{v}_i^2 \;\le\; \alpha_i \;\le\; \overline{v}_i^2 \;=\sum_{l=1}^{L}\mathcal{V}_l^2e_{li}. \tag{26}$$

The relaxation variables possess an intrinsic structure that allows us to add the important constraint set

$$\phi_{ij}-\phi_{ji} = \alpha_i-\alpha_j \\ \mu_{ij}+\mu_{ji} = 0. \tag{27}$$

These arise from the fact that indices can be switched in products of variables.

Next, the product $p_{ij}R_{ij}$ can be expanded using the definition of $R_{ij}=\sum_{k=1}^{K+1}\mathbb{R}_kp_{ij}z_{kij}$. We define new intermediate variables in order to rewrite all products of binary and continuous variables as sets of linear constraints:

$$\epsilon_{kij}=p_{ij}z_{kij} \quad \theta_{kij}=q_{ij}z_{kij}, \\ \lambda_{kij}=\phi_{ij}z_{kij} \quad \gamma_{kij}=\mu_{ij}z_{kij}. \tag{28}$$

If we assume $\underline{p}\le p_{ij}\le\overline{p}$, then ϵ_{kij} can be constrained as:

$$\epsilon_{kij} \le \overline{p}z_{kij} \\ \epsilon_{kij} \le p_{ij}-\underline{p}(1-z_{kij}) \\ \epsilon_{kij} \ge \underline{p}z_{kij} \\ \epsilon_{kij} \ge p_{ij}-\overline{p}(1-z_{kij}). \tag{29}$$

The cases of $q:\theta$, $\phi:\lambda$, $\mu:\gamma$ are completely analogous. Making the substitutions, the admittance constraints for edges in $\mathcal{E}\cap\mathcal{E}^o$ become

$$\sum_{k=1}^{K+1}\left(R^0_{ij}\left(\mathbb{R}_k(\epsilon_{kij}-\epsilon_{kji})+\mathbb{X}_k(\theta_{kij}-\theta_{kji})\right)\right) \tag{30}$$
$$\sum_{k=1}^{K+1} X^0_{ij}\left(\mathbb{R}_k(\theta_{kij}-\theta_{kji})-\mathbb{X}_k(\epsilon_{kij}-\epsilon_{kji})\right)=$$
$$R^0_{ij}\phi_{ij}+X^0_{ij}\mu_{ij}+\sum_{k=1}^{K+1}\left(\mathbb{R}_k\lambda_{kij}+\mathbb{X}_k\gamma_{kij}\right)$$
$$\sum_{k=1}^{K+1}\left(-X^0_{ij}\left(\mathbb{R}_k(\epsilon_{kij}-\epsilon_{kji})+\mathbb{X}_k(\theta_{kij}-\theta_{kji})\right)\right)$$
$$\sum_{k=1}^{K+1} R^0_{ij}\left(\mathbb{R}_k(\theta_{kij}-\theta_{kji})-\mathbb{X}_k(\epsilon_{kij}-\epsilon_{kji})\right)=$$
$$-X^0_{ij}\phi_{ij}+R^0_{ij}\mu_{ij}+\sum_{k=1}^{K+1}\left(-\mathbb{X}_k\lambda_{kij}+\mathbb{R}_k\gamma_{kij}\right).$$

Admittance constraints for edges in $\mathcal{E}\setminus\mathcal{E}^o$ are

$$\sum_{k=1}^{K+1}\left(\mathbb{R}_k(\epsilon_{kij}-\epsilon_{kji})+\mathbb{X}_k(\theta_{kij}-\theta_{kji})\right)=\phi_{ij} \tag{31}$$
$$\sum_{k=1}^{K+1}\left(-\mathbb{X}_k(\epsilon_{kij}-\epsilon_{kji})+\mathbb{R}_k(\theta_{kij}-\theta_{kji})\right)=\mu_{ij}.$$

and the case on $\mathcal{E}^o\setminus\mathcal{E}$ is already given in Equation 25. The minimization problem can now be formulated as a mixed binary linear program

$$\begin{aligned}
\underset{p,q,\phi,\mu,\alpha,z,e}{\text{minimize}} \quad & \sum_{ij\,\in\,\mathcal{E}} c_{ij}\\
\text{subject to} \quad & (2)\text{ with expanded binary products}\\
& (30),(31),(25),(18),(5)\\
& (6),(26),(27),(8)-(11)\\
& (29)\text{ and its analogs for } q,\phi,\mu.
\end{aligned}$$

Remark The above model does not include costs of transformers that would be needed to manage different voltage levels. This is not difficult to include, however, and does not change the number of binary variables. Let $\mathbb{T}_{kl}$ be the transformer cost for line type k connected to node of voltage level l. Then the cost (2) is modified to

$$c_{ij}=\sum_{k=1}^{K+1} z_{kij}\left[\mathbb{C}_{kij}+\sum_{l=1}^{L}\mathbb{T}_{kl}e_{li}+\mathbb{T}_{kl}e_{lj}\right],$$

and one has to accordingly distinguish between node voltages and voltages on the lines.

Validation on a Benchmark System. We checked the new binary algorithm on the six-bus Garver benchmark (see [9]), with voltage limits and no pre-existing lines. We obtained a directly feasible solution with cost 190, employing [1,2,1,2,2] lines on edges {1,5}, {2,3}, {2,6}, {3,5}, {4,6}, respectively. This outcome is a significant improvement on the objective of 260 reported by [9], who used a constructive heuristic algorithm. Several other papers have achieved good objectives of 190 [7] [8] and 200 [5] on the same problem, but these solutions involve capacitors or reactive power elements added on some buses; the best objective reported without these additions is 200 [7], which employs [1,1,2,2,2] lines on edges {1,5}, {2,3}, {2,6}, {3,5}, {4,6}, respectively. We conclude that our new formulation is the strongest available to date for this particular problem.

III. Computational Tests with Florida Data

In this section we use the lift-and-project binary model to generate new lines for sample systems drawn from a data set for a notional model[2] representative of the Florida grid [1]. We find and confirm optimal solutions for four-bus systems, and then describe feasible solutions for sizes up to fifteen buses, employing simple heuristics for cases in which the relaxed solution is infeasible (unlike the Garver result above). Along with characterizing the behavior of the relaxation and the heuristics, a second major question we address in this section is scalability.

Line types are common to all of our cases, and characterized in Table II. This gives each line's ratings, resistance per mile, reactance per mile, cost per mile, cost per VA, and relative cost. The cost per VA is the cost per mile divided by the line power rating in MVA, and the relative cost is the cost per VA normalized by the cost per VA of Line Type 1. This number shows, for instance, that Line Type 4 is much cheaper per VA than any other line type. This is an aspect of the problem that would be difficult to capture with integer variables. Cases involving four to ten busses used the same physical locations, with distances indicated in Table III. The fifteen-bus case used node locations drawn randomly from the full 154 available in the Florida data set. In all cases, there are no initial lines given.

We solved the mixed binary linear programming model using the commercial solvers AMPL [4] and CPLEX [3]; we checked feasibility of the resulting decisions using MATPOWER [13]. MATPOWER assumes a π-transmission line model, consistent with Section II. Computation times reported are for a representative 2011 laptop.

A. Procedure for Each Trial

We ran five different trials at each system size from four to ten nodes, and one trial for a fifteen-node system. For each trial, generation and load levels were first chosen at random from the 154-bus list; this induces at each node

[2] Although a process is underway to refine and validate it, the model used in this work is a preliminary one which has not yet been validated.

values for $\underline{p}_i, \overline{p}_i, \underline{q}_i$, and $\overline{q}_i$. For generators, $\underline{p}_i = \underline{q}_i = 0$: the minimum real and reactive power generation levels were zero. We similarly assumed loads that with negative power demands act as generators with minimum generation level of zero. We set the remaining user-input parameter, voltage sag, to $f = 0.95$.

For each trial, we used the following steps, where the set of nodes in the original problem is N:

1. Discard the trial if the sum of generations is inadequate for sum of the loads.
2. Run the model to obtain a relaxed solution L.
3. Discard the trial if any two generators are directly connected by a line in solution L; this case is not suitable for MATPOWER.
4. If $\{N, L\}$ contains multiple islands (i.e., connected components), rerun the model on each such subset of nodes N_i to obtain a corresponding island solution L_i. Continue running the model on each new island until each is the outcome of a model run. *The procedure for Step 6 and beyond is carried out separately for each island.*
5. Discard the trial if, in any island, the topology with line ratings set to the maximum is infeasible.
6. Run MATPOWER; if L_i is feasible, go to Step 10.
7. Deflate the nodal loads uniformly until MATPOWER reports a feasible flow solution F_i. In this work, we deflate the loads by 10% between feasibility checks.
8. Increment the line that is loaded closest to capacity by the flow F_i.
9. Inflate the nodal loads to the original values. Go to Step 6.
10. Implement any finishing heuristics: see description below.

Step 4 reflects the fact that MATPOWER is unreliable in treating multiple islands. Steps 6-9 form an iteration to account for the fact that when MATPOWER encounters an infeasible situation, it does not provide enough information to justify any particular line increments. This iterative process leads to feasibility with the original loads for all the cases we have considered.

Finishing heuristics can be posed at various levels of detail and effort, as desired by the user. In the present cases, we manually checked for line decrements only, i.e., we did not look for line exchanges. Conservative designs can arise from increments made by the load deflation procedure.

B. Results

Power flows in all tables and figures of this section are given in MVA magnitude, i.e., square root of real power squared plus reactive power squared.

The model was tested on systems containing up to fifteen nodes, with overall results in Table IV. The "Trials" column includes in parentheses the number of trials directly feasible without needing the heuristic; in total, 31 of the 36 trials were directly feasible. The five trials that were not directly feasible required at most three iterations of the load deflation heuristic, which corresponds to a 27% reduction in load. The number of adjustments includes both increments (made during deflation iterations) and decrements (made by finishing heuristics).

Two of the five reinforced cases were feasible with one increment, and three of them were feasible with two increments. In the five reinforced cases, three allowed subsequent decrements. Of the 31 cases that were directly feasible, eleven allowed decrements. The greatest number of decrements for a case was two for those that had been incremented, and four for those that had not.

In the four-bus systems, four of the five trials had directly feasible solutions that were confirmed to be optimal by enumeration. In the fifth trial, the model solution was directly feasible but also conservative; after two line decrements from the finishing heuristic, it was confirmed to be optimal. This one trial highlights an interesting point about our model. Usually when one solves a relaxed convex problem, the occurrence of a feasible solution guarantees a global optimum. This would appear to be the case in our approach as well, for the constraints are convex in the lift-and-project variables, and the binary variables are kept explicit and exact. The trial is a counterexample to that intuition, however, and indeed we are not aware of any guarantees that a construction such as ours provides optimality if feasible.

In the single fifteen-bus case, the solution turned out to be directly feasible; one line could be decremented.

We focus our attention now on the mid-size systems with seven to ten buses. Capacity designs and resulting power flows given by MATPOWER are shown in Figure 1 for four test cases at different sizes, that were directly feasible with no decrements possible. These solutions are *not* conservative when we consider that the larger line types are cheap compared to the smaller ones; many of the lines are operating near or at capacity. The data from Figure 1 are also given in Table V, and the nine-bus solution is shown geographically in Figure 2. Several cases that were subject to the load deflation heuristic or line decrementing are listed in Table VI. In these cases also the capacities are apparently not conservative.

We note that all of our solutions are trees instead of meshes, and that a fair number of the solutions include islands. This is not an artifact of our method - which admits a fully connected network - but rather of the example domain. For the dataset and sampling method we used, the load levels are small compared to the line capacities, and a significant fraction of the nodes are generation.

IV. Summary

The formulation we have presented is a binary counterpart to our earlier work with integer discrete variables [12]. This approach achieves better accuracy through maintaining the binary variables explicitly – it is very strong on the Garver benchmark – but does not match the scalability of the integer version.

In our computational experiments with certain realistic problem parameters, the binary model always yields feasible solutions to four-bus systems, which are either optimal or can be easily decremented to be optimal. On larger systems, the model gives many feasible solutions and a few infeasible; in the latter case, the load deflation heuristic we described is effective in identifying a few lines to increment. Referring to Table IV, it is a remarkable fact that neither the level of deflation nor the number of total line adjustments varies significantly with problem size – in fact they stay on a par with the noted four-bus case, which is optimal.

We believe that our relaxation is a powerful tool that can be applied standalone to problems of moderate size, or incorporated into a larger framework, e.g., optimizing subgraphs. While binary programming incurs complexity, the lift-and-project procedure evidently captures the underlying non-convex power equations extremely well.

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TABLE I
User Input Parameters (top) and Key Design Parameters (bottom)

K, L	number of unique line types, number of unique nodal voltage levels
$\mathbb{I}_k, \mathbb{V}_k$	Current and voltage rating of line type k
$\mathbb{R}_{kij}, \mathbb{X}_{kij}$	Resistance and reactance of line type k on Edge $ij \in \mathcal{E}$
$\mathcal{V}_l$	Voltage level l
$\mathbb{C}_{kij}$	Cost of line type k on Edge $ij \in \mathcal{E}$, without transformers
$(\underline{p}_i, \overline{p}_i), (\underline{q}_i, \overline{q}_i)$	Lower and upper real and reactive power limits at node i
R^o_{ij}, X^o_{ij}	Initial configuration resistance and reactance on Edge $ij \in \mathcal{E}^o$
τ_{1ij}, τ_{2ij}	Square root approximation parameters
f	factor: $f \times \overline{v}_i$ is voltage magnitude lower limit $\underline{v}_i$ on node i
$v_i = w_i + jx_i$	Complex voltage at node i
$s_{ij} = p_{ij} + jq_{ij}$	Directed complex power flow on Edge ij
z_{kij}	Binary variable indicating use of Line Type k on Edge ij
e_{li}	Binary variable indicating use of voltage level l at node i

TABLE II
Characteristics of each line type.

Line Type	Rated Power (MVA)	Rated Voltage (kV)	Rated Current (A)	Resistance Ω /mile
1	79	115	970	0.119
2	180	230	1110	0.1
3	1273	500	3600	0.028
4	2812	765	5200	0.019
Line Type	**Reactance Ω /mile**	**Cost $10^6\times$\$/ mile**	**Cost/ VA-mile**	**Relative Cost**
1	0.723	0.94	0.0119	1
2	0.777	1.1	0.0061	0.51
3	0.543	1.8	0.0014	0.12
4	0.548	2.5	0.0009	0.08

TABLE III
Distances between nodes for all example cases, except with fifteen busses.

Node	2	3	4	5	6	7	8	9	10
1	322	148	185	124	317	303	158	259	225
2	-	217	204	202	6	24	404	120	102
3	-	-	38	65	211	201	188	121	143
4	-	-	-	95	198	196	205	94	147
5	-	-	-	-	197	184	218	141	108
6	-	-	-	-	-	20	398	111	97
7	-	-	-	-	-	-	389	115	81
8	-	-	-	-	-	-	-	305	325
9	-	-	-	-	-	-	-	-	112

TABLE IV

Overview of results on four- to fifteen-bus systems. Number of trials directly feasible without the heuristic is given in parentheses in Trials column. Each deflation iteration imposes a 10% reduction in loads. Line adjustments include all increments and decrements (even if they negate).

Nodes	Trials	10% Deflations	Total Line Adjustments	Computation Time (sec)
4	5 (5)	0	0 – 2	6.8 – 7.0
5	5 (5)	0	0 – 3	6.7 – 7.0
6	5 (4)	0 – 1	0 – 3	6.7 – 10.2
7	5 (5)	0	0 – 4	6.9 – 12.8
8	5 (3)	0 – 2	0 – 3	6.9 – 358
9	5 (3)	0 – 3	0 – 3	7.4 – 410
10	5 (5)	0	0 – 4	24 – 197
15	1 (1)	0	1	9163

TABLE V

Real power flows/capacities (MVA) for four- to ten-bus systems that were feasible with no iterations of the heuristic and allowed no decrements; same data as in Figure 1. The four-bus case was confirmed optimal by enumeration.

Nodes: 4	7	8	9	10
352/1273	77/79	180/180	180/180	180/180
11/79	408/1273	118/180	90/180	95/180
348/1273	180/180	205/1273	1272/1273	374/1273
-	180/180	316/1273	1499/2812	172/180
-	712/1273	220/1273	712/1273	1197/1273
-	1179/1273	295/1273	78/79	1895/2812
-	-	-	1006/1273	-
-	-	-	159/180	-

TABLE VI

Real power flows and capacities (MVA) for eight- to ten-bus cases that used some line adjustment to reach final feasible solution.

Nodes: 8	9	10
45/79	1155/1273	1047/1273
374/1273	217/1273	1609/2812
180/180	95/180	154/180
599/1273	189/1273	1426/2812
1202/1273	869/1273	1234/1273
1065/1273	707/1273	-
180/180	532/1273	-

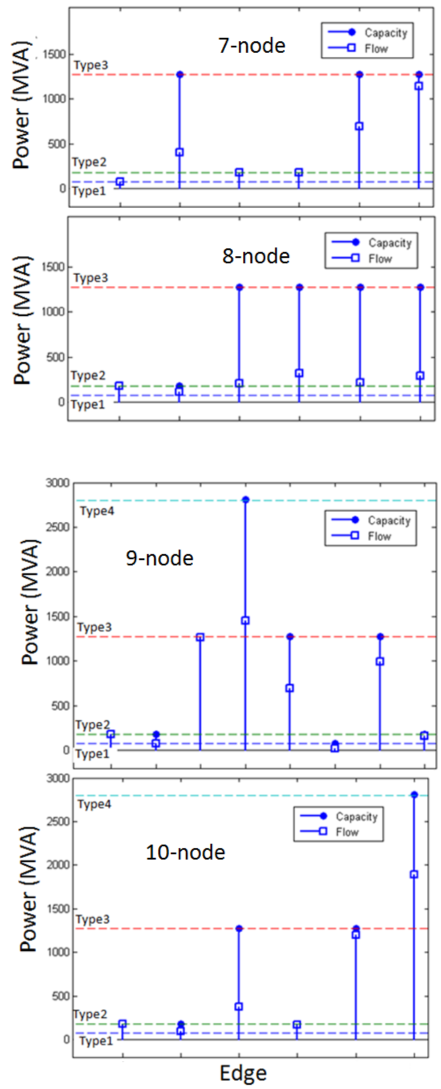

Fig. 1. Line capacities and real power flows (MVA) for 7-,8-,9- and 10-bus systems from our sample set, for which the relaxed design is feasible without iteration and no decrements were possible. Horizontal dashed lines represent the capacities of the three smaller line types.

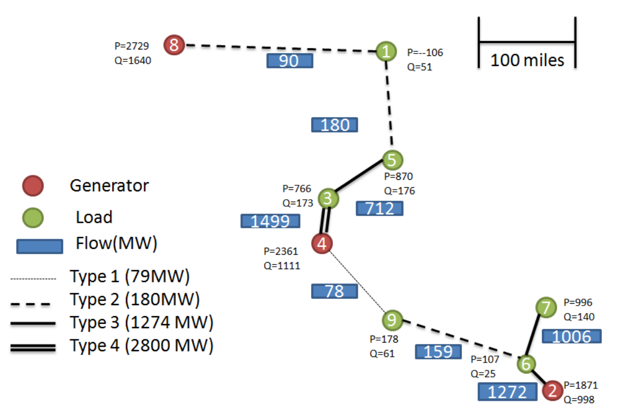

Fig. 2. Example of feasible solution found for a nine-bus system; this is the nine-bus case shown in Figure 1 and tabulated in Table V. This system did not require any iterations of the load deflation heuristic, nor any line decrements. Loads are shown in green with real (P) and reactive (Q) demands, and generators are shown in red with maximum real and reactive generation limits.