

# Linear relaxations for transmission system planning

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## Abstract

We apply a linear relaxation procedure for polynomial optimization problems to transmission system planning. The approach recovers and improves upon existing linear models based on the DC approximation. We then consider the full AC problem, and obtain new linear models with nearly the same efficiency as the linear DC models. The new models are applied to standard test systems, and produce high quality approximate solutions in reasonable computation time.

## 1 Introduction

Static transmission system planning is a network design problem in which lines are selected from a candidate set to meet certain physical requirements while minimizing investment and operational costs [1,2]. Linearized or ‘DC’ power flow is a standard simplification of AC power flow [3], which is usually too computationally intensive a representation of electrical physics for usage in optimization; only recently has this problem been approached in full [4]. For network design problems in which the existence of a line may be a variable, even linearized power flow becomes nonlinear, and furthermore, non-convex.

Current approaches can be divided into metaheuristics [5] and classical algorithms [1,6], as well as combined approaches [7]. This work falls among classical approaches, the primary focus of which is circumventing the non-convexity of DC transmission system planning. This has traditionally been accomplished through linear relaxations, namely the so called transportation and disjunctive models [1].

A relaxation is an approximation to an optimization problem which always bounds the minimum below (or maximum above), and is typically easier to solve than the original problem. Relaxed solutions such as those from the hybrid and disjunctive models are often infeasible for the original problem, but can contain a significant portion of the true optimal solution, thus reducing the size of the original problem, which may be intractable when approached directly [1]. Furthermore, the distance to feasibility is often slight, and thus the chance of obtaining an optimal or near optimal solution through modification to a relaxed solution is higher than when attempting to solve the original problem, which may have many local minima.

In this work, we apply a general linear relaxation procedure for polynomial optimization problems [8] to transmission system planning. The approach is broadly applicable; we employ it here with the intention of both formulating new models and providing a tutorial in its usage for power system optimization. First, we approach the DC power flow case and obtain a hierarchy of linear models, which includes the hybrid transportation [9] and disjunctive [10–13] models, the latter of which we make more flexible. The relaxation is then applied to the full AC case, yielding a linear AC model which to our knowledge is the first of its kind.

The DC and AC models are tested on several standard examples from the transmission system planning literature. The new DC models demonstrate improved efficiency, and the AC models compare well with existing nonlinear approaches while retaining the efficiency of linear DC models.

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## 2 Linear relaxations

Before applying the relaxation of [8] to transmission system planning problems, we briefly describe it through a simple example. Consider the following bilinear optimization problem:

$$\begin{aligned} \min_x \quad & x_1(x_2 - 1) \\ \text{s.t.} \quad & x_1 \geq 1, \quad x_2 \geq 2 \end{aligned}$$

A second-order relaxation is formulated as follows. Add the redundant constraint  $(x_1 - 1)(x_2 - 2) \geq 0$ , and substitute a new variable  $y$  for all instances of  $x_1x_2$ . The relaxation is given by

$$\begin{aligned} \min_{x,y} \quad & y - x_1 \\ \text{s.t.} \quad & x_1 \geq 1, \quad x_2 \geq 2, \quad y - 2x_1 - x_2 + 2 \geq 0 \end{aligned}$$

In this manner we lift polynomial problems with nonlinear constraints and objectives into higher dimensional spaces. Approximate solutions are obtained by then projecting the optimal relaxed solution onto the original space, which for our purposes means simply eliminating the new variables from the relaxed solution. Here, the minima of the original and relaxed problems are both one, with  $x_1 = 1$ ,  $x_2 = 2$ , and  $y = 2$ . Exactness is certified by the factorability of new variables into the original ones; in this example,  $y = x_1x_2$ . Although here the relaxed and actual minima are identical, it is not true in general, and usually will not be the case for transmission system planning problems.

Substitutions of any order can be performed within this framework, and it has been shown that as larger and larger constraint products are formed, the relaxation converges to the true optimum [14]. However, the size of the resulting linear programs grows rapidly, and so a compromise must be made at some point between accuracy and practicality. We exclusively consider second-order relaxations here, but do not dismiss the potential value of higher order formulations.

In the following sections, we use the relaxation to address the nonlinear physical constraints of power flow. We remark that this and other lift-and-project methods were originally developed for handling binary variables, which can be constrained polynomially with the equality  $x^2 = x$  [15, 16]. We leave the discrete aspect to commercial solvers, which have robust, efficient algorithms for solving mixed integer linear programs.

## 3 DC power flow

In the standard DC load flow network design problem, we are given the following parameters: a line investment vector  $c$ , a vector of generation and demand  $p$ , normalized susceptances  $b$ , and normalized flow limits  $\bar{f}$ . Also given are the number of lines present in the existing network,  $\eta^0$ , and the number of additional lines which may be constructed,  $\bar{\eta}$ . Let  $\Gamma$  denote the set of buses,  $\Omega_0$  the set of existing lines, and  $\Omega$  the set of candidate lines. We follow the notational conventions that unless otherwise specified, single subscripts denote members of  $\Gamma$ , double subscripts members of  $\Omega$ , and  $i \sim j$  summation over  $\Omega_0 \cup \Omega$ .

The nonlinear DC load flow network design problem is given by

$$\text{NLDC} \quad \min_{\theta, \eta, f} \quad \sum_{i \sim j} c_{ij} \eta_{ij} \tag{1}$$

$$\text{s.t.} \quad \sum_{j: i \sim j} f_{ij} = p_i \tag{2}$$

$$\begin{aligned} f_{ij} - b_{ij}(\eta_{ij}^0 + \eta_{ij})(\theta_i - \theta_j) &= 0 \\ (i, j) &\in \Omega_0 \cup \Omega \end{aligned} \tag{3}$$

$$|f_{ij}| \leq (\eta_{ij}^0 + \eta_{ij}) \bar{f}_{ij} \tag{4}$$

$$0 \leq \eta_{ij} \leq \bar{\eta}_{ij}, \quad \eta_{ij} \in \mathbb{N} \tag{5}$$

where the variables  $\theta$  are bus angles,  $\eta$  candidate lines, and  $f$  power flows. The main difficulty in using the above formulation is constraint (3): it is bilinear and hence non-convex.

We first reformulate NLDC in a way which, upon application of the relaxation of [8], leads to a class of disjunctive models. Eliminate the variables  $f$  by substitution of constraint (3), and rewrite NLDC

$$\text{NLDCS} \quad \min_{\theta, \eta} \quad \sum_{i \sim j} c_{ij} \eta_{ij} \quad (6)$$

$$\text{s.t.} \quad \sum_{j: i \sim j} b_{ij} (\eta_{ij}^0 + \eta_{ij}) (\theta_i - \theta_j) = p_i \quad (7)$$

$$b_{ij} |\theta_i - \theta_j| \leq \bar{f}_{ij} \quad (i, j) \in \Omega_0 \quad (8)$$

$$b_{ij} \eta_{ij} |\theta_i - \theta_j| \leq \bar{f}_{ij} \eta_{ij} \quad (9)$$

$$0 \leq \eta_{ij} \leq \bar{\eta}_{ij}, \quad \eta_{ij} \in \mathbb{N} \quad (10)$$

(9) is required to preserve the equivalence of NLDCS and NLDC. If (8) alone was enforced over all the lines, artificial constraints on angles would arise between buses that were not directly connected.

We derive an additional constraint set which is implicit in (8), but leads to a tighter relaxation. Consider a line  $(i, j) \in \Omega$ , and let  $s_{ij}$  be a path connecting  $i$  and  $j$  through  $\Omega_0$ . Summing constraint (8) along  $s_{ij}$  and multiplying by  $b_{ij}$  gives

$$b_{ij} |\theta_i - \theta_j| \leq M_{ij} \quad (i, j) \in \Omega. \quad (11)$$

where  $M_{ij} = b_{ij} \sum_{(k,l) \in s_{ij}} \bar{f}_{kl} / x_{kl}$ . Clearly (11) is sharpest when  $s_{ij}$  is the shortest path through the graph induced by  $\Omega_0$  with edge weights  $\bar{f}_{kl} / x_{kl}$ ,  $(k, l) \in \Omega_0$ , which matches the result stated in [13]. If no path between the nodes  $i$  and  $j$  exists in  $\Omega_0$ ,  $s_{ij}$  can be set to the longest path through the existing and candidate networks [13]. This however is an NP-hard calculation that contributes little accuracy;  $M_{ij}$  can instead be set to some sufficiently large number, e.g.  $\sum_{(i,j) \in \Omega \cup \Omega_0} \bar{f}_{ij} / b_{ij}$ .

We now apply the relaxation to NLDCS with constraint (11). We develop a second-order linearization by introducing the variable  $\zeta_{ij} = b_{ij} \eta_{ij} (\theta_i - \theta_j)$ , and then constraining  $\zeta$  with (9) and products of (8), (10), and (11). For example, we obtain constraint (16) by multiplying  $M_{ij} - b_{ij} |\theta_i - \theta_j|$  from (11) with  $\bar{\eta}_{ij} - \eta_{ij}$  from (10), and then substituting  $\zeta_{ij}$  wherever  $b_{ij} \eta_{ij} (\theta_i - \theta_j)$  appears.

We thus have the following relaxation:

$$\text{LDC} \quad \min_{\theta, \eta, \zeta} \quad \sum_{i \sim j} c_{ij} \eta_{ij} \quad (12)$$

$$\text{s.t.} \quad \sum_{j \sim i} \zeta_{ij} + b_{ij} \eta_{ij}^0 (\theta_i - \theta_j) = p_i \quad (13)$$

$$b_{ij} |\theta_i - \theta_j| \leq \bar{f}_{ij} \quad (i, j) \in \Omega_0 \quad (14)$$

$$|\zeta_{ij}| \leq \min\{M_{ij}, \bar{f}_{ij}\} \eta_{ij} \quad (15)$$

$$|\zeta_{ij} - b_{ij} \bar{\eta}_{ij} (\theta_i - \theta_j)| \leq M_{ij} (\bar{\eta}_{ij} - \eta_{ij}) \quad (16)$$

$$0 \leq \eta_{ij} \leq \bar{\eta}_{ij}, \quad \eta_{ij} \in \mathbb{N} \quad (17)$$

For comparison, the original disjunctive model, which only admits a binary formulation, is

$$\text{DM} \quad \min_{\theta, \eta, \zeta} \quad \sum_{i \sim j} c_{ij} \sum_k \eta_{ij}^k \quad (18)$$

$$\text{s.t.} \quad \sum_{j \sim i} b_{ij} \eta_{ij}^0 (\theta_i - \theta_j) + \sum_k \zeta_{ij}^k = p_i \quad (19)$$

$$b_{ij} |\theta_i - \theta_j| \leq \bar{f}_{ij} \quad (i, j) \in \Omega_0 \quad (20)$$

$$|\zeta_{ij}^k| \leq \bar{f}_{ij} \eta_{ij}^k \quad (21)$$

$$|\zeta_{ij}^k - b_{ij} (\theta_i - \theta_j)| \leq M_{ij} (1 - \eta_{ij}^k) \quad (22)$$

$$\eta_{ij}^k \in \{0, 1\} \quad k = 1, \dots, \bar{\eta}_{ij} \quad (23)$$

DM and LDC are quite similar; indeed LDC can be straightforwardly transformed to a binary problem which is identical to DM save constraint (21), which is looser than (15), its counterpart in LDC. Moreover,

line quanta may be aggregated in any fashion, so that a particular discrete variable may represent any number of candidate lines between one and  $\bar{\eta}_{ij}$ . From this perspective, DM and LDC represent opposing ends of a spectrum, which in general becomes less accurate and more efficient as one moves from  $\bar{\eta}_{ij}$  binary variables to a single integer variable per line.

We remark that if we apply the substitution  $\zeta_{ij} = b_{ij}\eta_{ij}(\theta_i - \theta_j)$  to NLDCS without forming any constraint products, the flows are only constrained by line capacities, and we obtain a hybrid model [1] in which flows in existing lines are governed by DC power flow, and in new lines by the network flow [17,18].

## 4 AC power flow

There has been little work to date on transmission system planning using AC power flow. A notable recent approach is [4], in which a full AC model is solved by an interior point method in tandem with a constructive heuristic algorithm. We now derive linear models for AC transmission system planning which are similar in structure and size to the disjunctive models of the previous section. Solutions can be used in the same fashion as those from linear DC models, and hence the new models mark a significant improvement in the overall design procedure via removal of the DC approximation.

Let  $s$ ,  $v$ , and  $y$  respectively denote complex powers, voltages and admittances, and let  $\underline{p}$ ,  $\bar{p}$ ,  $\underline{q}$ , and  $\bar{q}$  respectively be lower and upper real and reactive power limits, e.g. if bus  $i$  is a purely real load,  $\underline{p}$  and  $\bar{p}$  are both equal to the load value, and  $\underline{q} = \bar{q} = 0$ . The remaining notation is identical to the previous section. The basic AC power flow model is then given by

$$\text{NLAC} \quad \min_{\eta, s, v} \quad \sum_{i \sim j} c_{ij} \eta_{ij} \quad (24)$$

$$\text{s.t.} \quad s_{ij} = (\eta_{ij}^0 + \eta_{ij}) (v_i v_i^* y_{ij}^* - v_i v_j^* y_{ij}^*) \quad (25)$$

$$\underline{p}_i \leq \text{Re} \sum_j s_{ij} \leq \bar{p}_i \quad (26)$$

$$\underline{q}_i \leq \text{Im} \sum_j s_{ij} \leq \bar{q}_i \quad (27)$$

$$\underline{v}_i \leq |v_i| \leq \bar{v}_i \quad (28)$$

$$|s_{ij}| \leq (\eta_{ij}^0 + \eta_{ij}) \bar{s}_{ij} \quad (i, j) \in \Omega_0 \cup \Omega \quad (29)$$

$$0 \leq \eta_{ij} \leq \bar{\eta}_{ij}, \quad \eta_{ij} \in \mathbb{N} \quad (30)$$

Note that although line variables and parameters are non-directional, i.e.  $\eta_{ij} = \eta_{ji}$ ,  $\bar{s}_{ij} = \bar{s}_{ji}$  and so on, sending and receiving power flows  $s_{ij}$  and  $s_{ji}$  are not.

We first must rewrite NLAC in terms of real, polynomial constraints so we may begin to build a relaxation. Let  $y = g + jb$ ,  $v = w + jx$ ,  $s = p + jq$ , and let  $b^s = b + b^{sh}$ , where  $b^{sh}$  is the line shunt

susceptance. NLAC is then given by

$$\text{NLACS} \quad \min_{\eta, p, q, w, x} \quad \sum_{i \sim j} c_{ij} \eta_{ij} \quad (31)$$

$$\text{s.t.} \quad p_{ij} = (\eta_{ij}^0 + \eta_{ij}) (b_{ij}(w_i x_j - w_j x_i) - g_{ij}(x_i x_j + w_i w_j) + g_{ij}(w_i^2 + x_i^2)) \quad (32)$$

$$q_{ij} = (\eta_{ij}^0 + \eta_{ij}) (g_{ij}(w_i x_j - w_j x_i) + b_{ij}(x_i x_j + w_i w_j) - b_{ij}^s(w_i^2 + x_i^2)) \quad (33)$$

$$\underline{p}_i \leq \sum_j p_{ij} \leq \bar{p}_i \quad (34)$$

$$\underline{q}_i \leq \sum_j q_{ij} \leq \bar{q}_i \quad (35)$$

$$\underline{v}_i^2 \leq w_i^2 + x_i^2 \leq \bar{v}_i^2 \quad (36)$$

$$\sqrt{p_{ij}^2 + q_{ij}^2} \leq (\eta_{ij}^0 + \eta_{ij}) \bar{s}_{ij} \quad (i, j) \in \Omega_0 \cup \Omega \quad (37)$$

$$0 \leq \eta_{ij} \leq \bar{\eta}_{ij}, \quad \eta_{ij} \in \mathbb{N} \quad (38)$$

Constraint (37) represents a slight obstacle: although it can be expressed polynomially, fourth order products of the original variables are involved, rendering the size of the resulting relaxation impractically large. We instead approximate it so that  $p$  and  $q$  are involved linearly. A few options are apparent; for example, introduce the constants  $\tau^1$  and  $\tau^2$ , and replace (37) with

$$\tau_{ij}^1 |p_{ij}| + \tau_{ij}^2 |q_{ij}| \leq (\eta_{ij}^0 + \eta_{ij}) \bar{s}_{ij} \quad (39)$$

Notice, for example, that by setting  $\tau^1$  and  $\tau^2$  to one, we obtain a more conservative constraint than (37), which is no longer a relaxation, while by setting  $\tau^2$  to zero, we relax (37) by only limiting the flow of active power. Although we have opted to approximate (37) with a single constraint, any piecewise linear approximation can be accommodated.

Define the new variables:

$$\begin{aligned} \alpha_i &= w_i^2 + x_i^2 \\ \delta_{ij} &= \eta_{ij} (w_i^2 + x_i^2) \\ \mu_{ij} &= b_{ij}(w_i x_j - w_j x_i) - g_{ij}(x_i x_j + w_i w_j) \\ &\quad + g_{ij}(w_i^2 + x_i^2) \\ \nu_{ij} &= g_{ij}(w_i x_j - w_j x_i) + b_{ij}(x_i x_j + w_i w_j) \\ &\quad - b_{ij}^s(w_i^2 + x_i^2) \\ \phi_{ij} &= \eta_{ij} (b_{ij}(w_i x_j - w_j x_i) - g_{ij}(x_i x_j + w_i w_j) \\ &\quad + g_{ij}(w_i^2 + x_i^2)) \\ \psi_{ij} &= \eta_{ij} (g_{ij}(w_i x_j - w_j x_i) + b_{ij}(x_i x_j + w_i w_j) \\ &\quad - b_{ij}^s(w_i^2 + x_i^2)) \end{aligned}$$

Certain symmetries are present in these variables, which we use to form additional constraints. Before showing them, we first give a brief example illustrating why they exist; Suppose  $y_{ij}$  is substituted for the product  $x_i x_j$ ; the implicit constraint  $y_{ij} = y_{ji}$  follows from the fact that  $x_i x_j = x_j x_i$ . The following constraints are similarly formed by taking linear combinations of (32) and (33) and performing the above

substitutions to relate new variables from  $i$  to  $j$  and  $j$  to  $i$ :

$$\begin{aligned}
& g_{ij}(\mu_{ij} - \mu_{ji}) - b_{ij}(\nu_{ij} - \nu_{ji}) \\
& \quad = (g_{ij}^2 + b_{ij}b_{ij}^s)(\alpha_i - \alpha_j) \\
& b_{ij}(\mu_{ij} + \mu_{ji}) + g_{ij}(\nu_{ij} + \nu_{ji}) \\
& \quad = (g_{ij}b_{ij} - g_{ij}b_{ij}^s)(\alpha_i + \alpha_j) \\
& g_{ij}(\phi_{ij} - \phi_{ji}) - b_{ij}(\psi_{ij} - \psi_{ji}) \\
& \quad = (g_{ij}^2 + b_{ij}b_{ij}^s)(\delta_{ij} - \delta_{ji}) \\
& b_{ij}(\phi_{ij} + \phi_{ji}) + g_{ij}(\psi_{ij} + \psi_{ji}) \\
& \quad = (g_{ij}b_{ij} - g_{ij}b_{ij}^s)(\delta_{ij} - \delta_{ji})
\end{aligned}$$

Let  $\Phi$  denote the set on which the variables  $\mu$ ,  $\nu$ ,  $\phi$ ,  $\psi$ ,  $\alpha$ , and  $\delta$  satisfy these equalities. Forming constraints containing up to second-order terms and substituting the new variables, we have

$$\text{LAC} \quad \min_{\eta, \mu, \nu, \phi, \psi, \alpha, \delta} \sum_{i \sim j} c_{ij} \eta_{ij} \quad (40)$$

$$\text{s.t.} \quad \{\mu, \nu, \phi, \psi, \alpha, \delta\} \in \Phi \quad (41)$$

$$\underline{p}_i \leq \sum_j \eta_{ij}^0 \mu_{ij} + \phi_{ij} \leq \bar{p}_i \quad (42)$$

$$\underline{q}_i \leq \sum_j \eta_{ij}^0 \nu_{ij} + \psi_{ij} \leq \bar{q}_i \quad (43)$$

$$\underline{v}_i^2 \leq \alpha_i \leq \bar{v}_i^2 \quad (44)$$

$$\underline{v}_i^2 \eta_{ij} \leq \delta_{ij} \leq \bar{v}_i^2 \eta_{ij} \quad (45)$$

$$\underline{v}_i^2 (\bar{\eta}_{ij} - \eta_{ij}) \leq \bar{\eta}_{ij} \alpha_i - \delta_{ij} \leq \bar{v}_i^2 (\bar{\eta}_{ij} - \eta_{ij}) \quad (46)$$

$$\tau_{ij}^1 |\mu_{ij}| + \tau_{ij}^2 |\nu_{ij}| \leq \bar{s}_{ij} \quad (i, j) \in \Omega_0 \quad (47)$$

$$\tau_{ij}^1 |\phi_{ij}| + \tau_{ij}^2 |\psi_{ij}| \leq \bar{s}_{ij} \eta_{ij} \quad (48)$$

$$\begin{aligned}
& \tau_{ij}^1 |\bar{\eta}_{ij} \mu_{ij} - \phi_{ij}| + \tau_{ij}^2 |\bar{\eta}_{ij} \nu_{ij} - \psi_{ij}| \\
& \quad \leq \bar{s}_{ij} (\bar{\eta}_{ij} - \eta_{ij}) \quad (i, j) \in \Omega_0
\end{aligned} \quad (49)$$

$$0 \leq \eta_{ij} \leq \bar{\eta}_{ij}, \quad \eta_{ij} \in \mathbb{N} \quad (50)$$

LAC is quite similar to LDC and the disjunctive model.  $\mu$  and  $\phi$  respectively represent active power flows in the existing and new networks, and  $\nu$  and  $\psi$  similarly represent reactive power flows. Constraint (49) is directly analogous to (16); unfortunately, it cannot be extended to lines not in the preexisting network, because (47) does not reduce to an expression with only  $i$  and  $j$  indices when summed along paths from  $i$  to  $j$  through  $\Omega_0$ .

As with the disjunctive model, we are also able to formulate a binary version which is less efficient but

more accurate:

$$\text{LACB} \quad \min_{\eta, \mu, \nu, \phi, \psi, \alpha, \delta} \quad \sum_{i \sim j} c_{ij} \sum_k \eta_{ij}^k \quad (51)$$

$$\text{s.t.} \quad \underline{p}_i \leq \sum_j \eta_{ij}^0 \mu_{ij} + \sum_k \phi_{ij}^k \leq \bar{p}_i \quad (52)$$

$$\underline{q}_i \leq \sum_j \eta_{ij}^0 \nu_{ij} + \sum_k \psi_{ij}^k \leq \bar{q}_i \quad (53)$$

$$\underline{v}_i^2 \leq \alpha_i \leq \bar{v}_i^2 \quad (54)$$

$$\underline{v}_i^2 \eta_{ij}^k \leq \delta_{ij}^k \leq \bar{v}_i^2 \eta_{ij}^k \quad (55)$$

$$\underline{v}_i^2 (1 - \eta_{ij}^k) \leq \alpha_i - \delta_{ij}^k \leq \bar{v}_i^2 (1 - \eta_{ij}^k) \quad (56)$$

$$\tau_{ij}^1 |\mu_{ij}| + \tau_{ij}^2 |\nu_{ij}| \leq \bar{s}_{ij} \quad (i, j) \in \Omega_0 \quad (57)$$

$$\tau_{ij}^1 |\phi_{ij}^k| + \tau_{ij}^2 |\psi_{ij}^k| \leq \bar{s}_{ij} \eta_{ij}^k \quad (58)$$

$$\tau_{ij}^1 |\mu_{ij} - \phi_{ij}^k| + \tau_{ij}^2 |\nu_{ij} - \psi_{ij}^k| \leq \bar{s}_{ij} (1 - \eta_{ij}^k) \quad (i, j) \in \Omega_0 \quad (59)$$

$$\eta_{ij}^k \in \{0, 1\} \quad k = 1, \dots, \bar{k}_{ij} \quad (60)$$

Note that LAC has roughly two to four times the number of constraints of LDC, depending on the choice of  $\tau^1$  and  $\tau^2$ ; essentially, any system that LDC or the hybrid model is applicable to is within the scope of LAC as well. The same parity exists for DM and LACB.

## 5 Computational results

In this section we compare the performance of our models to existing approaches. The resulting mixed integer linear programs were solved using the modeling language AMPL [19] and solver CPLEX [20] on a desktop computer representative of current standards. Objectives are given in terms of relative (unitless) values to facilitate comparison.

### 5.1 DC models

The main advantage in using LDC over DM is the retention of constraint (16) without the introduction a large number of binary variables. Of course, this constraint only has influence when  $\bar{\eta}_{ij}$  is not too much larger than the optimal  $\eta_{ij}$ . We compare the models on the 46-bus, 79-line Brazilian system of [9, 21] and the 24-bus, 41-line IEEE reliability test system [22]. In the original Brazilian system, line additions are unlimited, effectively nullifying constraint (16) and reducing LDC to the hybrid model of [1]. We modify the Brazil system so as to observe the differences in using LDC by setting  $\bar{\eta}_{ij} = 2$  for all  $(i, j) \in \Omega$ .

We give the objective value and running time of DM and LDC, as well as the hybrid model of [1] in Table 1. In both cases, LDC achieves an objective between the hybrid model and DM, while requiring twice the time of the hybrid model and substantially less time than DM. In practical terms, LDC has similar efficiency to but greater accuracy than the hybrid model, and thus can be applied to much larger problems than the disjunctive model.

Table 1: Efficiency of DC models

Model	DM		LDC		Hybrid	
	Obj.	Time	Obj.	Time	Obj.	Time
IEEE RTS	4.01	1.25	3.61	0.62	3.45	0.35
Brazilian	1.63	10.36	1.45	1.61	1.41	0.71

## 5.2 AC models

We demonstrate LAC on two of the example systems from [4], which are AC versions of the Garver’s six bus system with  $\Omega_0 = \emptyset$  [23] (a variant is studied in [24] using a metaheuristic) and the Brazilian test system of the previous section. Tables 2 and 3 show the objective value and solution reported for the nonlinear approach in [4] (NL), and obtained by the linear model LAC with  $\tau^2 = 1$  and  $\tau^2 = 0$  for all  $(i, j) \in \Omega$ . Running times in seconds are reported for each linear model as well. In the ‘line additions’ section of each table, the left column indicates which line a given row corresponds to, and the other columns how many additions to that line were made by each algorithm; lines not listed were changed by none of the algorithms. LACB performed identically to LAC on these examples, and so is not shown. Note that we do not consider reactive power source allocation, and so our solutions for the latter two examples correspond to slightly different scenarios than those in [4].

Table 2: Garver’s six bus system

Model	NL [4]	LAC, $\tau^2 = 1$	LAC, $\tau^2 = 0$
Obj.	260	190	160
Time	-	0.18	0.13
Line additions			
1 - 5	1	1	0
2 - 3	1	2	0
2 - 6	3	1	2
3 - 5	2	2	2
4 - 6	3	2	2

Table 3: Brazilian system

Model	NL [4]	LAC, $\tau^2 = 1$	LAC, $\tau^2 = 0$
Obj.	10.258	10.800	8.254
Time	-	7.4	3.3
Line additions			
5 - 6	2	2	0
5 - 11	0	0	2
6 - 46	1	1	0
11 - 46	0	0	1
12 - 14	0	1	0
14 - 26	0	1	0
18 - 19	0	0	1
19 - 25	1	0	0
20 - 21	1	2	2
20 - 23	0	2	1
24 - 25	1	0	0
28 - 31	0	1	1
31 - 32	0	1	1
42 - 43	1	2	2
42 - 44	1	0	0

For each system, the linear model solutions are reasonably similar to the nonlinear ones (which are not necessarily optimal). By setting  $\tau^2 = 1$ , a more conservative solution is obtained, which can in fact have a higher objective than the nonlinear solutions, whereas setting  $\tau^2 = 0$  yields relaxed solutions in considerably less time. Note that these results also suggest a certain measure of discretion must be applied when interpreting relaxed solutions: some of the obtained solutions are likely to be infeasible, and would require reinforcement before being implementable.



## 6 Conclusion and future work

We have applied a linear relaxation technique to transmission system planning. We obtain mild improvements over existing linear DC models, and formulate the first linear AC model, which compares well with the more expensive nonlinear approach of [4]. As an alternative approach, the AC model substantially simplifies transmission system design by circumventing DC approximations.

There are multiple venues for future work in this context. The first is the development of a general purpose software tool along the lines of the Gloptipoly [25] semidefinite relaxation suite, but which automatically generates linear relaxations of a specified order and calls linear rather than semidefinite solvers. Along these lines, mixed integer conic optimization is an active area of research [26], and may soon yield algorithms for mixed integer second-order cone and semidefinite programming of similar sophistication to those of mixed integer linear programming. It is a reasonable assumption that more accurate, convex models may be formulated in terms of second-order cone and semidefinite constraints, which are more expressive than their linear counterparts; this will be the subject of future work. Lastly, within power system design and operation there is an abundance of optimization problems complicated by low-order polynomial nonlinearities arising from electrical physics, for example optimal power flow [3] and distribution system reconfiguration [27]. The scale and often discrete nature of these problems calls for mixed integer linear and, pending further advancement, conic programming approaches. It is the authors' opinion that the relaxation procedure employed here is well suited to these tasks, which is supported by the implicit role it has already played through the use of hybrid and disjunctive DC transmission system planning models.

## References

- [1] R. Romero, A. Monticelli, A. Garcia, and S. Haffner, "Test systems and mathematical models for transmission network expansion planning," *IEE Proceedings - Generation, Transmission and Distribution*, vol. 149, no. 1, pp. 27–36, 2002.
- [2] G. Latorre, R. Cruz, J. Areiza, and A. Villegas, "Classification of publications and models on transmission expansion planning," *Power Systems, IEEE Transactions on*, vol. 18, no. 2, pp. 938 – 946, May 2003.
- [3] A. J. Wood and B. F. Wollenberg, *Power generation, operation, and control*, 2nd ed. Knovel, 1996.
- [4] M. Rider, A. Garcia, and R. Romero, "Power system transmission network expansion planning using AC model," *Generation, Transmission Distribution, IET*, vol. 1, no. 5, pp. 731 –742, September 2007.
- [5] I. J. Ramirez-Rosado and J. L. Bernal-Agustin, "Reliability and costs optimization for distribution networks expansion using an evolutionary algorithm," *Power Engineering Review, IEEE*, vol. 21, no. 4, pp. 70–70, April 2001.
- [6] N. Alguacil, A. Motto, and A. Conejo, "Transmission expansion planning: a mixed-integer LP approach," *Power Systems, IEEE Transactions on*, vol. 18, no. 3, pp. 1070 – 1077, Aug. 2003.
- [7] M. Rider, A. Garcia, and R. Romero, "Transmission system expansion planning by a branch-and-bound algorithm," *Generation, Transmission Distribution, IET*, vol. 2, no. 1, pp. 90 –99, January 2008.
- [8] H. D. Sherali and C. H. Tuncbilek, "A global optimization algorithm for polynomial programming problems using a reformulation-linearization technique," *Journal of Global Optimization*, vol. 2, pp. 101–112, 1992.
- [9] S. Haffner, A. Monticelli, A. Garcia, J. Mantovani, and R. Romero, "Branch and bound algorithm for transmission system expansion planning using a transportation model," *IEE Proceedings - Generation, Transmission and Distribution*, vol. 147, no. 3, pp. 149–156, 2000.
- [10] R. Villasana, "Transmission network planning: A method for synthesis of minimum cost secure networks," Ph.D. dissertation, Troy, NY, 1984.

- [11] A. Sharifnia and H. Z. Aashtiani, "Transmission network planning: A method for synthesis of minimum-cost secure networks," *Power Apparatus and Systems, IEEE Transactions on*, vol. PAS-104, no. 8, pp. 2025–2034, August 1985.
- [12] S. Granville and M. V. F. Pereira, "Analysis of the linearized power flow model in Bender's decomposition," Stanford, CA, USA, Tech. Rep., 1985, EPRI-report RP 2473-6.
- [13] S. Binato, M. Pereira, and S. Granville, "A new Benders decomposition approach to solve power transmission network design problems," *Power Systems, IEEE Transactions on*, vol. 16, no. 2, pp. 235–240, May 2001.
- [14] J. B. Lasserre, "Polynomial programming: LP-relaxations also converge," *SIAM Journal on Optimization*, vol. 15, no. 2, pp. 383–393, 2005.
- [15] H. D. Sherali and W. P. Adams, "A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems," *SIAM Journal on Discrete Mathematics*, vol. 3, no. 3, pp. 411–430, 1990.
- [16] L. Lovász and A. Schrijver, "Cones of matrices and set-functions and 0–1 optimization," *SIAM Journal on Optimization*, vol. 1, no. 2, pp. 166–190, 1991.
- [17] L. R. F. Jr. and D. R. Fulkerson, *Flows in Networks*. Princeton University Press, 1962.
- [18] R. K. Ahuja, T. L. Magnanti, and J. B. Orlin, *Network flows: theory, algorithms, and applications*. Upper Saddle River, NJ, USA: Prentice-Hall, Inc., 1993.
- [19] R. Fourer, D. M. Gay, and B. W. Kernighan, *AMPL: A Modeling Language for Mathematical Programming*. Duxbury Press, November 2002.
- [20] "IBM ILOG CPLEX." [Online]. Available: <http://www-01.ibm.com/software/integration/optimization/cplex-optimizer/>
- [21] M. Pereira and N. Balu, "Composite generation/transmission reliability evaluation," *Proceedings of the IEEE*, vol. 80, no. 4, pp. 470–491, Apr. 1992.
- [22] C. Grigg, P. Wong, P. Albrecht, R. Allan, M. Bhavaraju, R. Billinton, Q. Chen, C. Fong, S. Haddad, S. Kuruganty, W. Li, R. Mukerji, D. Patton, N. Rau, D. Reppen, A. Schneider, M. Shahidehpour, and C. Singh, "The IEEE reliability test system-1996," *Power Systems, IEEE Transactions on*, vol. 14, no. 3, pp. 1010–1020, Aug. 1999.
- [23] L. Garver, "Transmission network estimation using linear programming," *Power Apparatus and Systems, IEEE Transactions on*, vol. PAS-89, no. 7, pp. 1688–1697, Sept. 1970.
- [24] L. A. Gallego, M. J. Rider, R. Romero, and A. V. Garcia, "A specialized genetic algorithm to solve the short term transmission network expansion planning," in *PowerTech, 2009 IEEE Bucharest*, June 2009.
- [25] D. Henrion and J. B. Lasserre, "Gloptipoly: Global optimization over polynomials with Matlab and SeDuMi," *ACM Trans. Math. Softw.*, vol. 29, no. 2, pp. 165–194, 2003.
- [26] J. P. Vielma, S. Ahmed, and G. L. Nemhauser, "A lifted linear programming branch-and-bound algorithm for mixed-integer conic quadratic programs," *INFORMS J. on Computing*, vol. 20, pp. 438–450, July 2008.
- [27] M. Baran and F. Wu, "Network reconfiguration in distribution systems for loss reduction and load balancing," *Power Delivery, IEEE Transactions on*, vol. 4, no. 2, pp. 1401–1407, Apr. 1989.