

Refining the Multi-field Effects of Higgs Inflation

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Ross Norman Greenwood

Submitted to the Department of Physics
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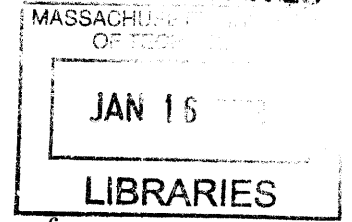
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Abstract

In this thesis, I investigated the extent to which the inclusion of multiple scalar fields in models of Higgs inflation produces observable departures from the single field case. I designed simulations to numerically solve the differential equations of motion in the Einstein frame governing the evolution of two nonminimally coupled fields in a quartic potential. Determining the behavior of the Hubble parameter, I found that successful inflation results in this model from a range of initial conditions. Based on calculations of the evolution of field perturbations by Courtney Peterson and Max Tegmark [6], the results suggest that the turn rate of the fields' combined velocity through field space has been sufficiently small since Hubble crossing that it may be excluded as a source of the amplification of perturbations in the Cosmic Microwave Background to their present size.

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Chapter 1

Inflationary Cosmology

Cosmologists aim to improve our understanding of the universe as a whole - the global properties of spacetime and the matter and energy that occupy it. The inspiration and final test of cosmological models are the results of observational research, but the study is also guided by a set of key assumptions, namely the principle of relativity and the weak cosmological principle.

According to the principle of relativity, the laws of physics can be formulated in a way that is independent of coordinate systems. Absolute adherence to this principle is the foundation of Einstein's general theory of relativity. Within cosmology, we work whenever possible with relativistically invariant quantities and generally covariant equations, and operate within general relativity or departures thereof that preserve the principle to a high degree.

The weak cosmological principle asserts that there are no privileged locations in space; one set of physical laws applies everywhere. Combined with observational evidence of large-scale structure and the Cosmic Microwave Background (CMB), this forms the basis for the assumption that at large enough scales the universe is the same in all places (homogeneous) and looks the same in all directions (isotropic), and its contents can be approximated as a perfect fluid, with a stress-energy tensor $T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}$, where ρ and p are the energy density and pressure, respectively; u^μ is the four-velocity of the fluid; and $g^{\mu\nu}$ is the metric tensor.

1.1 Friedmann-Robertson-Walker Metric

The Friedmann-Robertson-Walker (FRW) metric defines the line element of the most general spacetime that is both homogeneous and isotropic in its spatial dimensions. In natural units ($c = \hbar = 1$), the equation of the FRW line element takes the form:

$$ds^2 = -dt^2 + a^2(t) \left(\frac{1}{1 - \kappa r^2} dr^2 + r^2 d\theta^2 + \sin^2 \theta d\phi^2 \right). \quad (1.1.1)$$

The parameter $a(t)$, known as the *scale factor*, characterizes the distance scale between objects separated in space, such that the ratio of the separations between the same two objects at times t and t_0 is given by $a(t)/a(t_0)$. All of the information regarding the geometry of the spacetime is contained here in the curvature constant κ , which can be normalized to take the value 0 or ± 1 . A value of $\kappa = 0$ corresponds to a spatially flat universe. A closed FRW spacetime is analogous to a 3-sphere in its spatial dimensions; it has positive curvature, with $\kappa > 0$. In such a universe, the sum of the angles of a triangle would be greater than π . An open universe has negative curvature, with $\kappa < 0$.

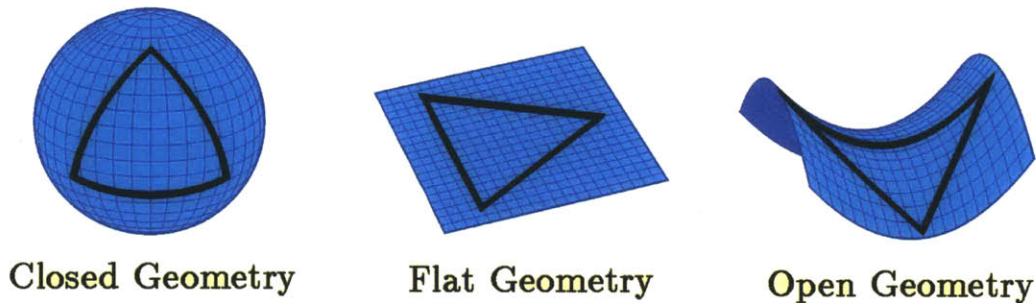


Figure 1-1: Triangles composed of three points joined by geodesics on the three possibilities of homogeneous, isotropic spaces in two (spatial) dimensions. Source: [3]

The FRW universe seems most in line with the assumptions and observations that underlie cosmological theory. However, there are several fundamental problems arising from the model of the Big Bang that conflict with our understanding - namely the horizon, monopole, and flatness problems. The prospect of solving these problems

is largely what motivates exploration of modified theories like inflation. (For a review of inflation, see [3] and references therein.)

1.2 The Flatness Problem

The flatness problem addresses the remarkably small degree of curvature of the present day universe, in the context of a prediction of the standard cosmological model that any departure from the spatially flat universe should grow at an increasing rate with time. To illustrate this prediction, we first need an equation relating the curvature constant to functions of cosmic time t , namely the scale factor $a(t)$ and time-dependent elements of the stress-energy tensor. Evaluating the components of Einstein's field equations for a perfect fluid yields the two Friedmann equations

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3M_{pl}^2} - \frac{\kappa}{a^2} \quad \frac{\ddot{a}}{a} = -\frac{1}{6M_{pl}^2}(\rho + 3p) \quad (1.2.1)$$

where M_{pl} is the reduced Planck mass, defined in natural units as

$$M_{pl} \equiv \frac{1}{\sqrt{8\pi G}} = 2.43 \cdot 10^{18} \text{ GeV.}$$

The Friedmann equation identifies a critical density $\rho_{cr} = 3M_{pl}^2 H^2$ for which the curvature constant vanishes, corresponding to a flat spacetime. We define a density parameter Ω to characterize the measured density in relation to this critical density:

$$\Omega \equiv \frac{\rho}{\rho_{cr}} = \frac{\rho}{3M_{pl}^2 H^2} = 1 + \frac{\kappa}{a^2 H^2}. \quad (1.2.2)$$

The condition $\Omega = 1$ corresponds to a universe that is spatially flat, with open and closed universes defined by $\Omega < 1$ and $\Omega > 1$, respectively. Through further manipulation of the Friedmann equation, we can express the departure from flatness as a function of the scale factor and energy density:

$$\frac{\Omega - 1}{\Omega} = 3M_{pl}^2 \frac{\kappa}{a^2 \rho}. \quad (1.2.3)$$

For non-relativistic matter, the density scales inversely with the volume element like $\rho \propto a^{-3}(t)$. In the case of relativistic matter, an additional factor of $a^{-1}(t)$, analogous to the cosmological redshift of light, makes $\rho \propto a^{-4}(t)$. We may now express the density parameter Ω in terms of its proportionality to pure functions of the scale factor:

$$\frac{\Omega - 1}{\Omega} \propto \frac{1}{a^2 \rho} \propto \begin{cases} a(t), & \text{non-relativistic matter} \\ a^2(t), & \text{relativistic matter} \end{cases} \quad (1.2.4)$$

According to Equation (1.2.4), we would expect any deviation of the density parameter from the flatness condition, quantified as $(\Omega - 1)/\Omega = 0$, to be magnified with the growth of the scale factor: $\Omega = 1$ is an unstable solution. In models like the Big Bang for which the scale factor has grown by many orders of magnitude since some initial time, a universe that began as anything but infinitesimally close to flat would be incredibly positively or negatively curved today. Likewise, an initially flat universe would stay flat as $a(t)$ evolved. In order to produce the present, measured degree of flatness, corresponding to a density parameter of $\Omega_0 = 1 \pm 10^{-3}$ [2], the universe at some initial time would have to be *fine-tuned* to be incredibly, but not perfectly, flat. In addition to being aesthetically questionable, the need for fine tuning often predicts the surfacing of more problems that can spell demise for a theory of physics.

In order for energy-momentum to be conserved locally, the contracted covariant derivative of the stress-energy tensor must vanish for all x^μ . Evaluating the time component of the covariant derivative yields the conservation equation for the energy density ρ :

$$\mathcal{D}_\mu T^{\mu 0} = \dot{\rho} + 3H(\rho + p) = 0 \quad (1.2.5)$$

We can solve this differential equation if we assume that the energy density is proportional to the pressure - let's say by a factor of some constant w . The density

becomes

$$\rho(t) = \rho_0 a^{-3(1+w)} \text{ for } p = w\rho. \quad (1.2.6)$$

For normal matter, $w \simeq 1/3$ or $w = 0$ for the relativistic and non-relativistic cases, respectively. Plugging this density function into Equation (1.2.4), we have

$$\frac{\Omega - 1}{\Omega} \propto \frac{1}{a^2 \rho} \propto a^{1+3w} \quad (1.2.7)$$

One can see that for $w < -\frac{1}{3}$, the expression on the right-hand side actually shrinks with the growth of the scale factor! Referring back to Equation (1.2.1), we note that $p < -1/3\rho$ corresponds to $\ddot{a} > 0$. That is, a period of accelerated cosmic expansion can solve the flatness problem.

1.3 Dynamics of Matter as Scalar Fields

We can begin our search for a type of matter that satisfies this equation of state by deriving the behavior of a scalar field describing a perfect fluid in FRW spacetime. A scalar field is represented by a single value at every place and time, and the covariant derivative of a scalar field is equivalent to an ordinary partial derivative: $\mathcal{D}_\mu \phi = \partial_\mu \phi$. Because we are working in a homogeneous, isotropic space, the scalar fields in our equations will take the same value at all locations in space and are thus purely functions of time, with $\phi(x^\mu) = \phi(t)$.

Following the procedure from classical mechanics, we may derive the equations of motion for the field ϕ by specifying the Lagrangian density \mathcal{L} and extremizing the action - the integral of \mathcal{L} over the four-dimensional volume element. The Lagrangian density describing a field ϕ subject to general relativity and an external potential $V(\phi)$ is

$$\mathcal{L} = \frac{1}{16\pi G} R(x_\mu) + T(\partial_\mu \phi) - V(\phi), \quad (1.3.1)$$

where $T(\partial_\mu \phi)$ is the field's kinetic energy, given by $T(\partial_\mu \phi) = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$. But

because the field varies only in time, $T(\partial_\mu\phi)$ becomes $T(\dot{\phi}) = -\frac{1}{2}g^{00}\partial_0\phi\partial_0\phi = \frac{1}{2}\dot{\phi}^2$.

To find the equations of motion for ϕ , we calculate the variation of the action when $\phi \rightarrow \phi + \delta\phi$ and find the conditions under which it vanishes. Taking advantage of the product rule for partial derivatives and integrating by parts, we can factor out $\delta\phi$ from the integrand.

$$S \equiv \int d^4x \sqrt{-g} \mathcal{L}$$

$$\delta S = \int d^4x \sqrt{-g} \left[\left(\frac{\partial \mathcal{L}}{\partial \phi} \right) \delta\phi + \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \right) \delta\dot{\phi} \right]$$

$$0 = \int d^4x \left[\frac{\partial(\sqrt{-g} \mathcal{L})}{\partial \phi} - \partial_t \left(\frac{\partial(\sqrt{-g} \mathcal{L})}{\partial \dot{\phi}} \right) \right] \delta\phi$$

Since $g = \text{Det}(g_{\mu\nu})$ has no ϕ dependence, we are free to include it as a constant factor in the partial derivatives. The action is to be extremized for any function $\delta\phi(x^\mu)$, so in order for the integral to vanish, the bracketed term must be identically zero. Evaluating this equality yields the equation of motion for the field

$$\square\phi - V_{,\phi}(\phi) = 0, \tag{1.3.2}$$

where $\square \equiv g^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu$ is the d'Alembert operator. From Equation (1.1.1) we have $\square\phi = -\ddot{\phi} - 3H\dot{\phi}$, where $H \equiv \dot{a}/a$ is the Hubble parameter. Equation (1.3.2) now becomes

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi}(\phi) = 0. \tag{1.3.3}$$

Note that in addition to the acceleration and potential terms, there appears a term dependent on $\dot{\phi}$ and proportional to the Hubble constant. This term is known as *Hubble drag*, and is analogous to the frictional damping term in a harmonic oscillator system.

1.4 Inflation Solves the Flatness Problem

Modeling matter as a scalar field yields a promising solution to the flatness problem. The stress-energy tensor can be computed by manipulating the action using the calculus of variations. The tensor is given by

$$\frac{1}{2}\sqrt{-g}T_{\mu\nu} = \frac{\delta S^{(M)}}{\delta g^{\mu\nu}}, \quad (1.4.1)$$

where $S^{(M)}$ is the action associated with the matter field(s) and $\delta g_{\mu\nu}$ is the variation of the metric. Evaluating the above equation for elements $T_{00} = \rho$ and $T_{ij} = a^2 p g_{ij}$ yields expressions for ρ and p in terms of the time derivative of ϕ and the potential:

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi) \quad p = \frac{1}{2}\dot{\phi}^2 - V(\phi) \quad (1.4.2)$$

We can see that if the potential dominates the kinetic term above such that $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$, we will end up with the equation of state

$$p \simeq -\rho, \quad (1.4.3)$$

which satisfies our requirement that $p = < -\frac{\rho}{3}$ for a satisfactory solution to the flatness problem. An epoch in which matter obeys $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$ is said to be in *slow roll*, a term adopted by comparison to a ball rolling slowly (low kinetic energy) down a tall slope (large potential). Recalling the second Friedmann equation (1.2.1) and noting that $\rho \simeq -p \simeq V(\phi)$ for $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$, we have

$$\frac{\ddot{a}}{a} = -\frac{1}{6M_{pl}^2}(\rho + 3p) \simeq \frac{1}{3M_{pl}^2}V(\phi).$$

For a universe that is nearly flat, such that $M_{pl}^2 \kappa \ll a^2 \rho$, Equation (1.2.1) becomes

$$H^2 \simeq \frac{1}{3M_{pl}^2} \left[\frac{1}{2}\dot{\phi}^2 + V(\phi) \right] \simeq \frac{1}{3M_{pl}^2} V(\phi). \quad (1.4.4)$$

Using the definition $H \equiv \dot{a}/a$, one can show that $\ddot{a}/a = \dot{H} + H^2$. With \ddot{a}/a and H^2 equal in the slow roll approximation, \dot{H} vanishes. This means that H is approximately

constant as long as slow roll is in effect. The definition of the Hubble constant yields a solution for the scale factor that is an exponential function of time:

$$a(t) = a(t_0) e^{H(t-t_0)} \quad (1.4.5)$$

1.5 Worked Example: $V(\phi) \propto \phi^4$

To demonstrate application of this model, let us suppose that the universe is governed by the following potential:

$$V(\phi) = \frac{\lambda}{4} \phi^4 \Rightarrow V_{,\phi} = \lambda \phi^3 \quad (1.5.1)$$

In this case, Equation (1.3.3) describing the evolution of the field becomes

$$\ddot{\phi} + 3H\dot{\phi} + \lambda\phi^3 = 0. \quad (1.5.2)$$

In order for this potential to produce sufficient inflation given appropriate initial conditions, the following conditions must be met:

1. $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$ Slow roll condition
2. $|\dot{H}| \ll H^2$ H remains nearly constant (follows from **1**)
3. $|\ddot{\phi}| \ll |3H\dot{\phi}|$ Hubble drag keeps the field changing slowly

Asserting condition **3**, the field equation of motion becomes $3H\dot{\phi} \approx -\lambda\phi^3$. Substituting the Hubble constant using Equation (1.4.4), we have

$$3\sqrt{\frac{1}{3M_{pl}^2} \left(\frac{\lambda}{4}\phi^4\right)} \dot{\phi} \approx -\lambda\phi^3 \Rightarrow \dot{\phi} \approx -2M_{pl}\sqrt{\frac{\lambda}{3}} \phi$$

The resulting differential equation has an exponential solution:

$$\phi(t) = \phi(t_0) \text{Exp} \left[-2M_{pl}\sqrt{\frac{\lambda}{3}} (t - t_0) \right]. \quad (1.5.3)$$

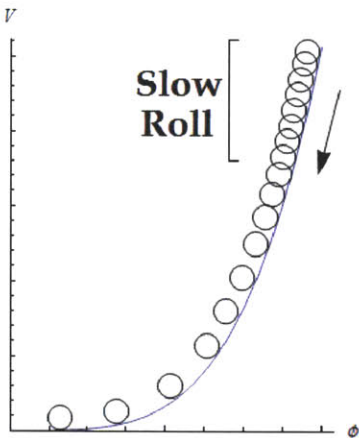


Figure 1-2: An illustration of a ball rolling down our ϕ^4 potential as an analogy to the field. The portion of the trajectory during which the ball is high up on the slope with a small velocity constitutes slow roll.

This solution will be consistent with the requirements of slow-roll for sufficiently small values of the dimensionless coupling constant, λ .

Chapter 2

Higgs Inflation

2.1 Nonminimal Coupling

The Lagrangian for a set of N time-dependent scalar fields $\varphi^I(t)$ under the influence of unmodified general relativity is

$$\mathcal{L} = \frac{M_{pl}^2}{2} R + \mathcal{L}^{(M)} = \frac{M_{pl}^2}{2} R - \frac{1}{2} \delta_{IJ} g^{\mu\nu} \partial_\mu \varphi^I \partial_\nu \varphi^J - V(\varphi^I), \quad (2.1.1)$$

In this case, we say that φ is *minimally coupled* to the Ricci scalar because the Lagrangian's dependence on R is unaffected by the addition of the field. As we explore models that adopt deviations from general relativity, we may expand our scope to consider *nonminimally coupled* fields with Lagrangians whose dependence on the Ricci scalar is governed by a nonminimal coupling function $f(\varphi^I)$:

$$\mathcal{L} = f(\varphi^I) R - \frac{1}{2} \delta_{IJ} g^{\mu\nu} \partial_\mu \varphi^I \partial_\nu \varphi^J - V(\varphi^I) \quad (2.1.2)$$

where $\dot{\varphi}^2 = \delta_{IJ} \dot{\varphi}^I \dot{\varphi}^J$. We may once again use the Principle of Extremal Action to derive the equations of motion for the fields. We find that the stress tensor now takes the form

$$T_{\mu\nu} = \frac{M_{pl}^2}{f} \left[\frac{1}{2} g_{\mu\nu} \mathcal{L}^{(M)} - \frac{\partial \mathcal{L}^{(M)}}{\partial g^{\mu\nu}} + f_{;\mu;\nu} - g_{\mu\nu} \square f \right], \quad \text{where} \quad (2.1.3)$$

$$\square f(\varphi^I) = g^{\mu\nu} f(\varphi^I)_{;\mu;\nu} = -\ddot{f} - 3H\dot{f}.$$

Evaluating this expression, we have for the components of the tensor

$$T_{00} = \frac{M_{pl}^2}{2f} \left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi^I) - 6H\dot{f} \right] \quad T_{ij} = \frac{M_{pl}^2}{2f} a^2 \delta_{ij} \left[\frac{1}{2}\dot{\varphi}^2 - V(\varphi^I) + 2\ddot{f} + 4H\dot{f} \right]$$

Plugging into the Friedmann equations yields the following relations for the Hubble constant and its time derivative. And varying the action with respect to φ^I yields the equation of motion for the field.

$$H^2 = \frac{1}{6f} \left[\frac{1}{2}\dot{\varphi}^2 + V(\varphi^I) - 6H\dot{f} \right] \quad \dot{H} = -\frac{1}{4f} \left(\dot{\varphi}^2 + 2\ddot{f} - 2H\dot{f} \right) \quad (2.1.4)$$

$$\ddot{\varphi}^I + 3H\dot{\varphi}^I + \delta^{IJ}(V_{,J} - f_{,J}R) = 0 \quad (2.1.5)$$

2.2 Jordan and Einstein Frames

We could solve Equations (2.1.4) - (2.1.5) numerically for the evolution of φ and H , but the presence of R in particular in the equations of motion makes this a difficult task. We can simplify these equations and the solution by making a conformal transformation to the Einstein frame.

Up until now we have been working in the Jordan frame, which is defined as the frame in which the Ricci tensor is multiplied by the non-minimal coupling function $f(\phi)$ in the Lagrangian, and the action is given by

$$S_J = \int d^4x \sqrt{-\hat{g}} \left[f(\varphi^I) \hat{R}(x_\mu) - \frac{1}{2} \delta_{IJ} \hat{g}^{\mu\nu} \partial_\mu \varphi^I \partial_\nu \varphi^J - \hat{V}(\varphi^I) \right]. \quad (2.2.1)$$

Here we have added a hat (i.e. $V \rightarrow \hat{V}$) to quantities evaluated in the Jordan frame. It is possible to make a conformal transformation from the Jordan frame to the Einstein frame by defining a scaled metric tensor

$$g_{\mu\nu} = \frac{2f}{M_{pl}^2} \hat{g}_{\mu\nu}, \quad (2.2.2)$$

which in turn allows us to express \hat{R} and $\sqrt{-\hat{g}}$ in terms of transformed quantities. The action in the Einstein frame is given by

$$S_E = \int d^4x \sqrt{-g} \left[\frac{M_{pl}^2}{2} R(x_\mu) - \frac{1}{2} \mathcal{G}_{IJ} g^{\mu\nu} \partial_\mu \varphi^I \partial_\nu \varphi^J - V(\varphi^I) \right] \quad (2.2.3)$$

where $V(\varphi^I) = \frac{M_{pl}^2}{4f^2} \hat{V}(\varphi^I)$ is the potential in the Einstein frame and \mathcal{G}_{IJ} is the *field space metric*, with [4]

$$\mathcal{G}_{IJ} = \frac{M_{pl}^2}{2f} \left(\delta_{IJ} + \frac{3}{f} f_{,I} f_{,J} \right). \quad (2.2.4)$$

In exchange for adopting the nontrivial field space metric, the conformal transformation has restored the gravitational term to its minimally-coupled form. The field space metric is analogous to the spacetime metric $g_{\mu\nu}$; it tells us how the fields' kinetic contribution to the Lagrangian density is calculated in the Einstein frame from the individual components of $\dot{\varphi}$. We may in turn define an analogous affine connection, which tells us how to calculate derivatives in a curved field space, and derive a new set of equations of motion:

$$\Gamma_{JK}^I \equiv \frac{1}{2} \mathcal{G}^{IL} (\mathcal{G}_{LJ,K} + \mathcal{G}_{LK,J} - \mathcal{G}_{JK,L}) \quad (2.2.5)$$

$$H^2 = \frac{1}{3M_{pl}^2} \left[\frac{1}{2} \mathcal{G}_{IJ} \dot{\varphi}^I \dot{\varphi}^J + V(\varphi^I) \right] \quad \dot{H} = -\frac{1}{2M_{pl}^2} \mathcal{G}_{IJ} \dot{\varphi}^I \dot{\varphi}^J \quad (2.2.6)$$

$$\ddot{\varphi}^I + \Gamma_{JK}^I \dot{\varphi}^J \dot{\varphi}^K + 3H \dot{\varphi}^I + \mathcal{G}^{IJ} V_{,J} = 0 \quad (2.2.7)$$

2.3 The Higgs Mechanism

The Higgs mechanism plays a critical role in our understanding of particle physics, providing the means to break electroweak gauge symmetry and generate mass terms for the W^\pm and Z^0 bosons [5]. While there is no evidence yet of the existence of a Higgs field or the corresponding Higgs boson, the theoretical success of this mechanism is substantial, and its study has generated much of our understanding of scalar fields. If the Higgs field exists, it may have had the properties necessary to drive inflation [1].

The fields that comprise the Higgs sector form a complex doublet h with components h^+ and h^0 . These complex components can be decomposed in terms of four real scalar fields $\varphi \equiv (\phi, \chi^1, \chi^2, \chi^3)$, of which χ^a are Goldstone fields.

$$h = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi^1 + i\chi^2 \\ \phi + i\chi^3 \end{pmatrix} \quad (2.3.1)$$

Any potential that governs the Higgs couples to the whole doublet, rather than to the individual field components. As such, the potential may only depend on a particular combination of fields given by $\varphi^2 \equiv \delta_{IJ}\varphi^I\varphi^J = \phi^2 + \delta_{ij}\chi^i\chi^j$. (In our notation, uppercase Roman indices are used to label the N field components of the pseudo-vector φ and later, the N corresponding directions in field space.) The potential that breaks electroweak symmetry, which is at the heart of the Higgs mechanism, is

$$V(\varphi^2) = -\mu^2\varphi^2 + \lambda(\varphi^2)^2. \quad (2.3.2)$$

The first term on the right hand side in Equation (2.3.2) acts like a mass term for the four-field system with its sign reversed, while the second is a self-interaction term. As shown in Figure 2-1, the true vacuum for this potential is displaced from $\varphi^2 = 0$ by some value v . One can show that v relates to the potential's parameters by the relation $2\mu^2 = \lambda v^2$. The potential can now be rewritten (up to a constant) as

$$V(\varphi^2) = \frac{\lambda}{4}(\varphi^2 - v^2)^2, \quad (2.3.3)$$

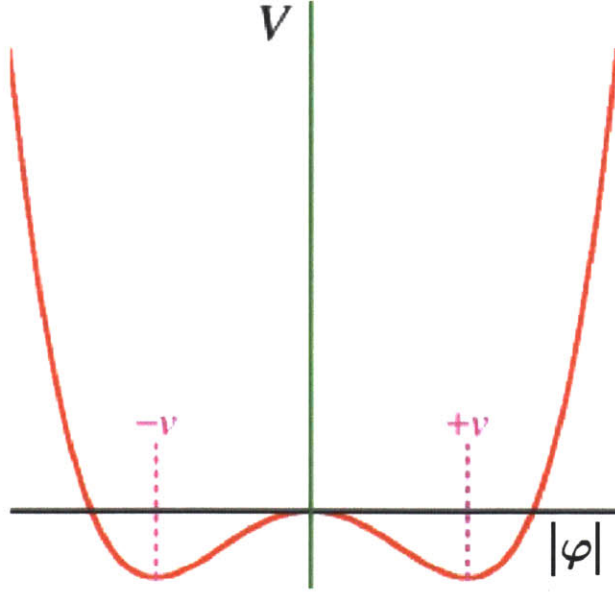


Figure 2-1: The Higgs potential as a function of the magnitude $|\varphi|$ of the collection of scalar fields, plotted for a range of values close to v . The local maximum at zero is a false vacuum; the true vacuum states are at $|\varphi| = \pm v$.

For the Higgs inflation model, the nonminimal coupling function takes the form

$$f(\varphi^I) = \frac{1}{2} [M_{pl}^2 + \xi (\varphi^2 - v^2)] = \frac{1}{2} (M_0^2 + \xi \varphi^2). \quad (2.3.4)$$

This nonminimal coupling function is *maximally symmetric* in that the nonminimal coupling constant ξ is the same for all fields ϕ and χ^a , as opposed to resembling $\xi_\phi \phi^2 + \xi_\chi (\chi^1)^2 + \dots$.

During inflation, we have $\xi \varphi^2 \gg M_{pl}^2 \gg v^2$, for $\lambda \sim 0.1$ and $\log_{10} \xi < 17$. For this condition, it is safe to say that $M_0^2 \equiv M_{pl}^2 - v^2 \simeq M_{pl}^2$ and that the potential is strongly dominated by the self-interaction term, becoming

$$V(\varphi^2) \simeq \frac{\lambda}{4} \varphi^4. \quad (2.3.5)$$

If φ were replaced by a single field ϕ , we would have the simple potential from our single field model worked out above. In later sections we will analyze a two field model of Higgs inflation, for which φ will become (ϕ, χ) , with $\varphi^4 = (\phi^2 + \chi^2)^2$.

2.4 Dynamics of Higgs Inflation

Here we have calculated the field space metric and affine connection for the Higgs' nonminimal coupling function given in Equation (2.3.4). A new quantity C was introduced in order to make our equations more compact; it is defined below.

$$\mathcal{G}_{IJ} = \frac{M_{pl}^2}{2f} \left(\delta_{IJ} + \frac{3\xi^2}{f} \varphi_I \varphi_J \right) \quad \mathcal{G}^{IJ} = \frac{2f}{M_{pl}^2} \left(\delta^{IJ} - \frac{6\xi^2}{C} \varphi^I \varphi^J \right) \quad (2.4.1)$$

$$\Gamma_{JK}^I = \frac{\xi(1+6\xi)}{C} \varphi^I \delta_{JK} - \frac{\xi}{2f} (\delta_J^I \varphi_K + \delta_K^I \varphi_J) \quad (2.4.2)$$

$$C \equiv M_0^2 + \xi(1+6\xi)\varphi^2 = 2f + 6\xi^2\varphi^2 \quad (2.4.3)$$

From Equations (2.2.6) and (2.2.7), we have

$$H^2 = \frac{1}{12f} \left[\dot{\varphi}^2 + \frac{3\xi^2}{f} (\varphi \cdot \dot{\varphi})^2 + \frac{\lambda M_{pl}^2}{4f} \varphi^4 \right] \quad \dot{H} = -\frac{1}{4f} \left[\dot{\varphi}^2 + \frac{3\xi^2}{f} (\varphi \cdot \dot{\varphi})^2 \right] \quad (2.4.4)$$

$$\ddot{\varphi}^I + \frac{\xi(1+6\xi)}{C} \dot{\varphi}^2 \varphi^I - \frac{\xi}{f} (\varphi \cdot \dot{\varphi}) \dot{\varphi}^I + 3H\dot{\varphi}^I + \lambda M_{pl}^4 \frac{\varphi^2}{2fC} \varphi^I = 0 \quad (2.4.5)$$

Chapter 3

Two Field Model and Simulation

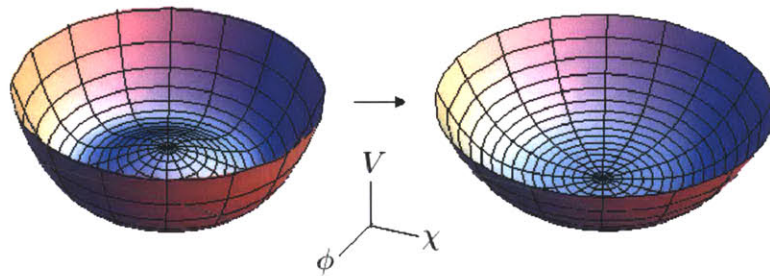


Figure 3-1: The two field Higgs potential with $\xi\varphi^2 \sim v^2$ (left) and $\xi\varphi^2 \gg v^2$ (right).

Let us now turn our attention to a Higgs inflation model with two fields $\varphi = (\phi, \chi)$. Our goal is to understand the effect that the addition of scalar fields on the prediction of Higgs inflation, and whether those predictions are substantially different from those we get by analyzing the single field model.

Rendered in Figure 3-1, the potential resembles a bowl that is symmetric about $|\varphi| = \sqrt{\phi^2 + \chi^2}$, and we can imagine the system of fields as a ball rolling around inside. By choosing the initial values of the fields and their time rates of change, we determine the trajectory that the system takes toward the bottom of the bowl. For example, setting $\chi = \dot{\chi} = \dot{\phi} = 0$ and $\phi = \phi_0$, one would expect a trajectory for which the ball rolls straight down the edge of the bowl, losing energy due to Hubble drag, and oscillates around the bottom; this scenario, along with any for which $\chi = \dot{\chi} = 0$, constitute the single field limit. Due to the bowl's symmetry, we can always rotate

the initial values of ϕ and χ into a convenient configuration. To introduce two-field effects, we add a kick in the χ direction by giving $\dot{\chi}$ a nonzero value. The new trajectory following a kick might be one for which the ball spirals toward the center, much like a coin in the spiral wishing well arcade game.

3.1 Two Field Dynamics

Setting $\varphi = (\phi, \chi)$, the equation of motion (2.4.5) for ϕ becomes

$$\ddot{\phi} + \frac{\xi(1+6\xi)}{C} \phi(\dot{\phi}^2 + \dot{\chi}^2) - \frac{\xi}{f} \dot{\phi}(\phi\dot{\phi} + \chi\dot{\chi}) + 3H\dot{\phi} + \lambda M_{pl}^4 \frac{\phi(\phi^2 + \chi^2)}{2fC} = 0,$$

with the corresponding equation for χ obtained by exchanging $\phi \leftrightarrow \chi$. And the equations expressed in (2.4.4) can be combined to produce

$$H^2 + \dot{H} = -\frac{1}{6f} \left[(\dot{\phi}^2 + \dot{\chi}^2) + \frac{3\xi^2}{f} (\phi\dot{\phi} + \chi\dot{\chi})^2 - \frac{\lambda M_{pl}^4}{8f} (\phi^2 + \chi^2)^2 \right] \quad (3.1.1)$$

Because we are working in a computer environment that does not understand or manipulate units, it is in our interest to define dimensionless versions of the quantities that appear in our equations. The fields ϕ and χ have dimensions of mass, so it is convenient for us to scale their dimensionless forms Φ and X , respectively, by the reduced Planck mass M_{pl} . Cosmic time t , with derivatives denoted by an overdot ($\partial_t \phi = \dot{\phi}$), is replaced with a dimensionless time coordinate $\tau \equiv \sqrt{\lambda} M_{pl} t$. (Time has dimensions of inverse mass in our unit system.) Derivatives with respect to τ are denoted by a prime symbol ($\partial_\tau \phi = \phi'$).

$$\begin{aligned}
\phi &= M_{pl}\Phi & \chi &= M_{pl}X \\
\dot{\phi} &= \sqrt{\lambda}M_{pl}^2\Phi' & \dot{\chi} &= \sqrt{\lambda}M_{pl}^2X' \\
\ddot{\phi} &= \lambda M_{pl}^3\Phi'' & \ddot{\chi} &= \lambda M_{pl}^3X''
\end{aligned}$$

We go on to define a dimensionless analogue to the Hubble parameter:

$$\mathcal{H} \equiv \frac{a'}{a} = \frac{H}{\sqrt{\lambda} M_{pl}}$$

Finally, the above definitions identify dimensionless forms of the nonminimal coupling function and our compacting function C :

$$\mathcal{F} \equiv \frac{f}{M_{pl}^2} = \frac{1}{2} [1 + \xi (\Phi^2 + X^2)] \quad C \equiv \frac{C}{M_{pl}^2} = 1 + \xi (1 + 6\xi) (\Phi^2 + X^2)$$

We now have all of the ingredients for our simulation in terms of these dimensionless quantities:

SIMULATION EQUATIONS

$$\begin{aligned}
\Phi'' + \frac{\xi(1+6\xi)}{C}\Phi(\Phi'^2 + X'^2) - \frac{\xi}{\mathcal{F}}\Phi'(\Phi\Phi' + XX') + 3\mathcal{H}\Phi' + \frac{\Phi(\Phi^2 + X^2)}{2\mathcal{F}C} &= 0 \\
X'' + \frac{\xi(1+6\xi)}{C}X(\Phi'^2 + X'^2) - \frac{\xi}{\mathcal{F}}X'(\Phi\Phi' + XX') + 3\mathcal{H}X' + \frac{X(\Phi^2 + X^2)}{2\mathcal{F}C} &= 0 \\
\dot{\mathcal{H}} + \mathcal{H}^2 = -\frac{1}{6\mathcal{F}} \left[(\Phi'^2 + X'^2) + \frac{3\xi^2}{\mathcal{F}} (\Phi\Phi' + XX')^2 - \frac{1}{8\mathcal{F}} (\Phi^2 + X^2) \right] &
\end{aligned}$$

In Wolfram Mathematica, we use the **NDSolve** function to solve our system of differential equations numerically with a set of initial conditions, generating \mathcal{H} , Φ , and X as functions of τ . Figures 3-2 – 3-3 plot the resulting functions for initial conditions

$$X[0] = 0 \quad \Phi'[0] = \frac{-1}{3\sqrt{3}\xi^2\Phi'[0]} \quad X'[0] = 0.05 \quad (3.1.2)$$

with $\xi = 10^2$, $\Phi[0] = 0.95$ and $\xi = 10^3$, $\Phi[0] = 0.3$. So our scalar fields have together an initial value of $0.95 M_{pl}$ or $0.3 M_{pl}$ concentrated in the ϕ direction in field space. The initial velocity of the system along the direction that runs straight from the bottom of the bowl has a negative value (toward the bottom) on the order of $10^{-4} M_{pl}^2$. There is a sizable kick in the perpendicular direction along the equipotential with a value on the order of $10^{-3} M_{pl}^2$.

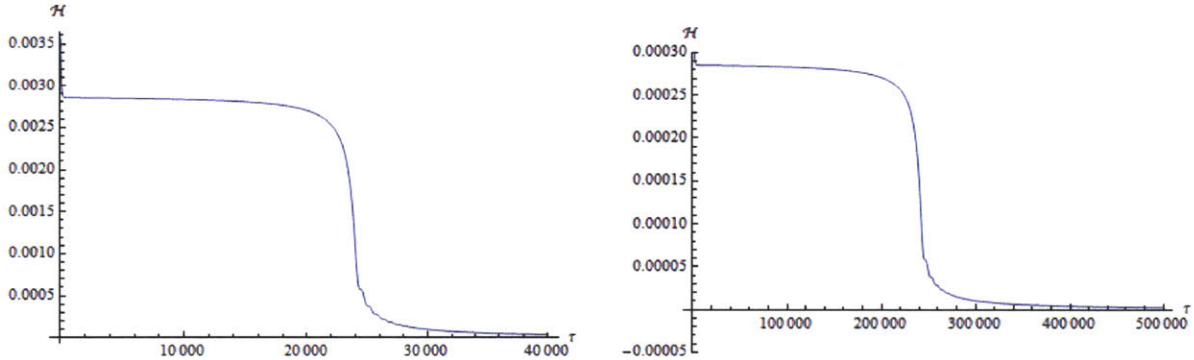


Figure 3-2: Numerical results for the dimensionless Hubble parameter \mathcal{H} as a function of τ , with $\xi = 10^2$, $\Phi[0] = 0.95$ (left) and $\xi = 10^3$, $\Phi[0] = 0.3$ (right).

We see that the Hubble parameter behaves as expected, remaining fairly constant up until the end of inflation at $\tau \simeq 17,000$ in the case of $\xi \simeq 10^2$, or $\tau \simeq 250,000$ in the case of $\xi = 10^3$. The fields show abrupt changes in magnitude corresponding to the initial kick, and then fall toward the bottom of the bowl with Hubble drag over the course of inflation, ending with small oscillations at the bottom.

3.2 Field Rotation

In the previous example, it is easy to characterize the initial configuration of the field velocities due to our choice to begin with $X[0]=0$. At later times, the system will not evolve only in the Φ direction in field space, but instead along some combination of the Φ and X directions. In essence, Φ and X form a Cartesian coordinate system,

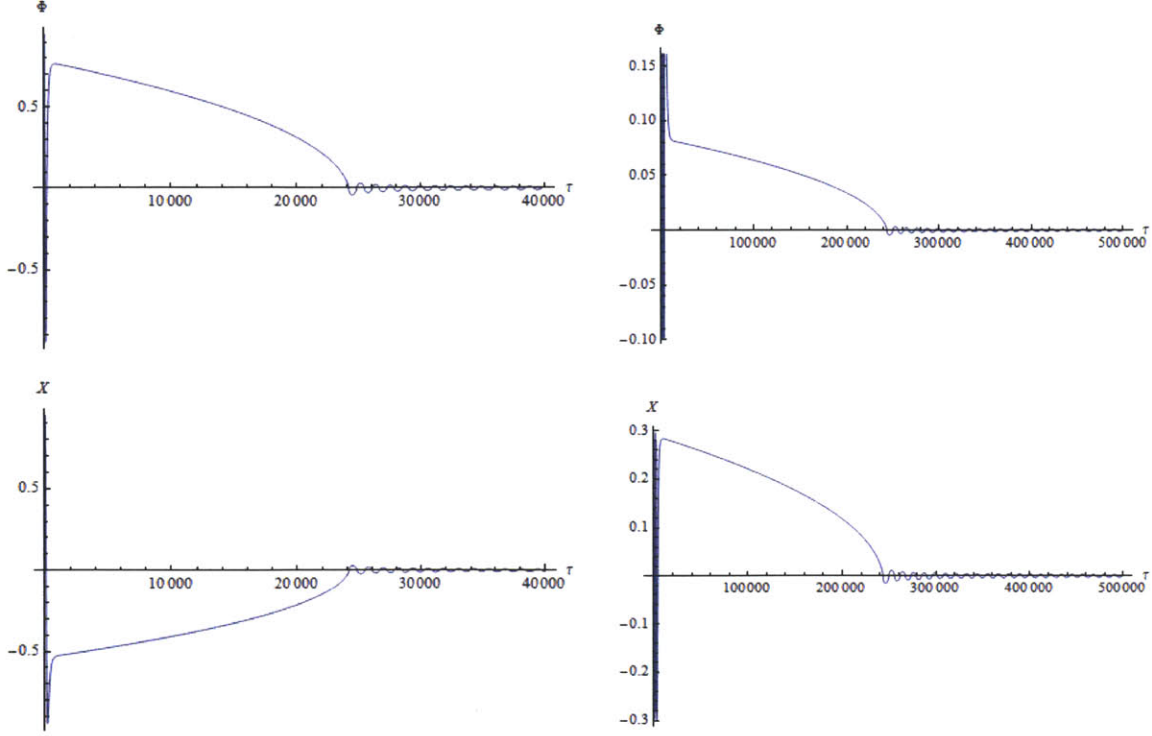


Figure 3-3: Numerical results for the dimensionless field values as functions of τ , with $\xi = 10^2$, $\Phi[0] = 0.95$ (left) and $\xi = 10^3$, $\Phi[0] = 0.3$ (right).

while the symmetries of our potential favor polar coordinates. We would like a way to project the total velocity onto our familiar Φ and X directions in field space.

For this it is helpful to define a vector quantity to characterize the magnitude and direction of the system's velocity. In a flat field space, an ordinary dot product $\delta_{IJ}\dot{\varphi}^I\dot{\varphi}^J$ would do to define the magnitude. But our field space has curvature defined by the field space metric \mathcal{G}_{IJ} . So the magnitude of the field velocity becomes

$$\dot{\sigma} \equiv \sqrt{\mathcal{G}_{IJ}\dot{\varphi}^I\dot{\varphi}^J}, \quad (3.2.1)$$

while the unit vector components specifying its direction can be calculated as

$$\hat{\sigma}^I \equiv \frac{\dot{\varphi}^I}{\dot{\sigma}}. \quad (3.2.2)$$

The vector $\hat{\sigma}$ encodes the evolution of the system of fields, and describes the background onto which perturbations are added to, for example, explore the origin of

inhomogeneities in the Cosmic Microwave Background. We label the direction of the evolution of the background, $\hat{\sigma}$, as the adiabatic direction. Directions perpendicular to the adiabatic direction are the entropic directions, defined as

$$\hat{s}_J^I = \delta_J^I - \hat{\sigma}^I \hat{\sigma}_J \quad (3.2.3)$$

In the two field case, there is only one entropic direction. The covariant derivative on field space of the unit vector σ^I , quantifying the turn rate of system in field space, is expressed as

$$\hat{\alpha}^I \equiv \mathcal{D}_t \hat{\sigma}^I = \frac{1}{\dot{\sigma}} [\mathcal{D}_t \dot{\varphi}^I - \hat{\sigma}^I \ddot{\sigma}] = -\frac{1}{\dot{\sigma}} V_{,K} \hat{s}^{IK}. \quad (3.2.4)$$

In their recent paper [6], Courtney Peterson and Max Tegmark worked out the dynamics of field perturbations for multi-field models of inflation. The gauge-invariant field perturbations may be decomposed into components along the classical field trajectory, $\hat{\sigma}$, and along directions orthogonal to it in field space, \mathbf{s}^I .

$$\delta\sigma_\psi \equiv \hat{\sigma}_I \mathcal{Q}^I \quad \delta s^I \equiv \hat{s}_J^I \mathcal{Q}^J \quad (3.2.5)$$

where $\mathcal{Q} \equiv \delta\varphi^I + (\varphi^I/H)\psi$ are the gauge-invariant Mukhanov-Sasaki variables. Perturbations in the adiabatic direction evolve according to the following equation of motion:

$$\delta\ddot{\sigma}_\psi^I + 3H\delta\dot{\sigma}_\psi^I + \left[\frac{\kappa^2}{a^2} + V_{,\sigma\sigma} - \hat{\alpha}^2 - \frac{1}{M_{pl}^2 a^3} \frac{d}{dt} \left(\frac{a^3 \dot{\sigma}^2}{H} \right) \right] \delta\sigma_\psi^I = 2 \left(\frac{d}{dt} - \frac{V_{,\sigma}}{\dot{\sigma}} - \frac{\dot{H}}{H} \right) (\hat{\alpha}_I \delta s^I). \quad (3.2.6)$$

Note that the term on the right hand side, the source term of the equation of motion, is dependent on α^I and the entropic field perturbations δs^I . Thus if α^I is nonzero, we end up with a nonzero source term driving the evolution of perturbations in the adiabatic direction. If we find the conditions under which α^I does not vanish, we should see effects that depart from the single field case.

In simulating field rotation dynamics, we define dimensionless forms for $\dot{\sigma}$ and α^I :

$$\dot{\sigma} = \sqrt{\lambda} M_{pl}^2 \Sigma' \quad \alpha^I = \lambda M_{pl}^3 A^I \quad (3.2.7)$$

Using the initial conditions expressed in (3.1.2), we generated plots of these dimensionless quantities, shown in Figures 3-4 – 3-5.

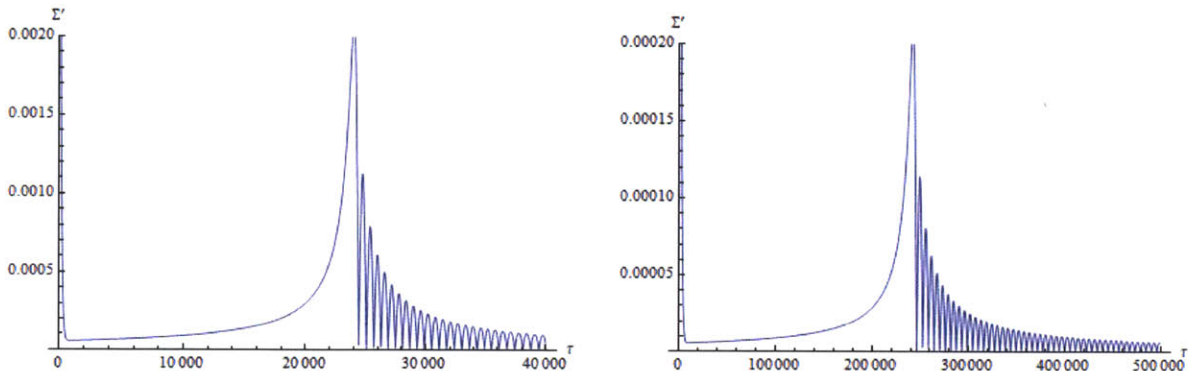


Figure 3-4: Numerical results for the dimensionless magnitude of the velocity in field space as a function of τ , with $\xi = 10^2$, $\Phi[0] = 0.95$ (left) and $\xi = 10^3$, $\Phi[0] = 0.3$ (right).

With attention to Figure 3-6, note that the turn-rate, α^I , only departs from zero for a short period after the start of inflation, and damps out to zero within a few e-folds. In particular, α^I becomes negligible well before the cosmologically-relevant length scales cross the Hubble radius during inflation. That means that the wavelength of the modes of the primordial perturbations whose evolution is affected by multi-field effects is exponentially larger than those visible in the cosmic microwave background radiation – such modes remain exponentially longer, even today, than the observable horizon. Thus, after an early, transient phase, multi-field Higgs inflation should behave just like a single-field model, and its observational signature in the CMB should be indistinguishable from single-field models.

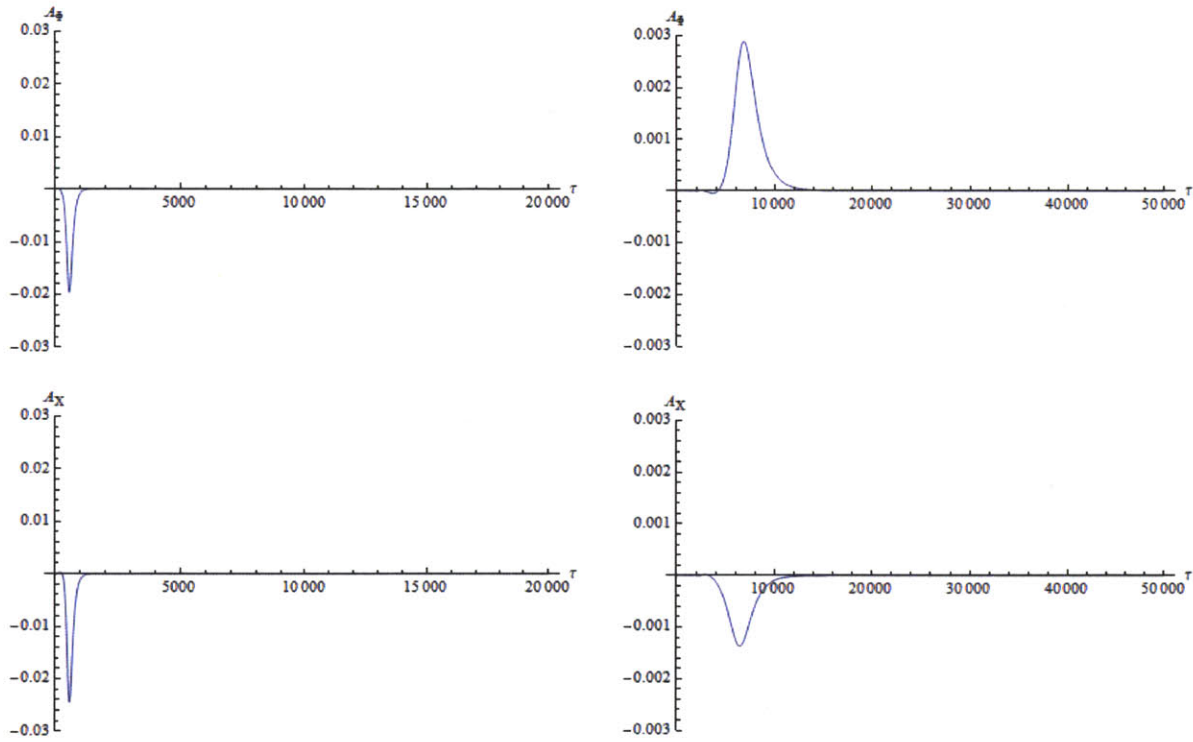


Figure 3-5: Numerical results for the dimensionless turn rates A^I as functions of τ , with $\xi = 10^2$, $\Phi[0] = 0.95$ (left) and $\xi = 10^3$, $\Phi[0] = 0.3$ (right).

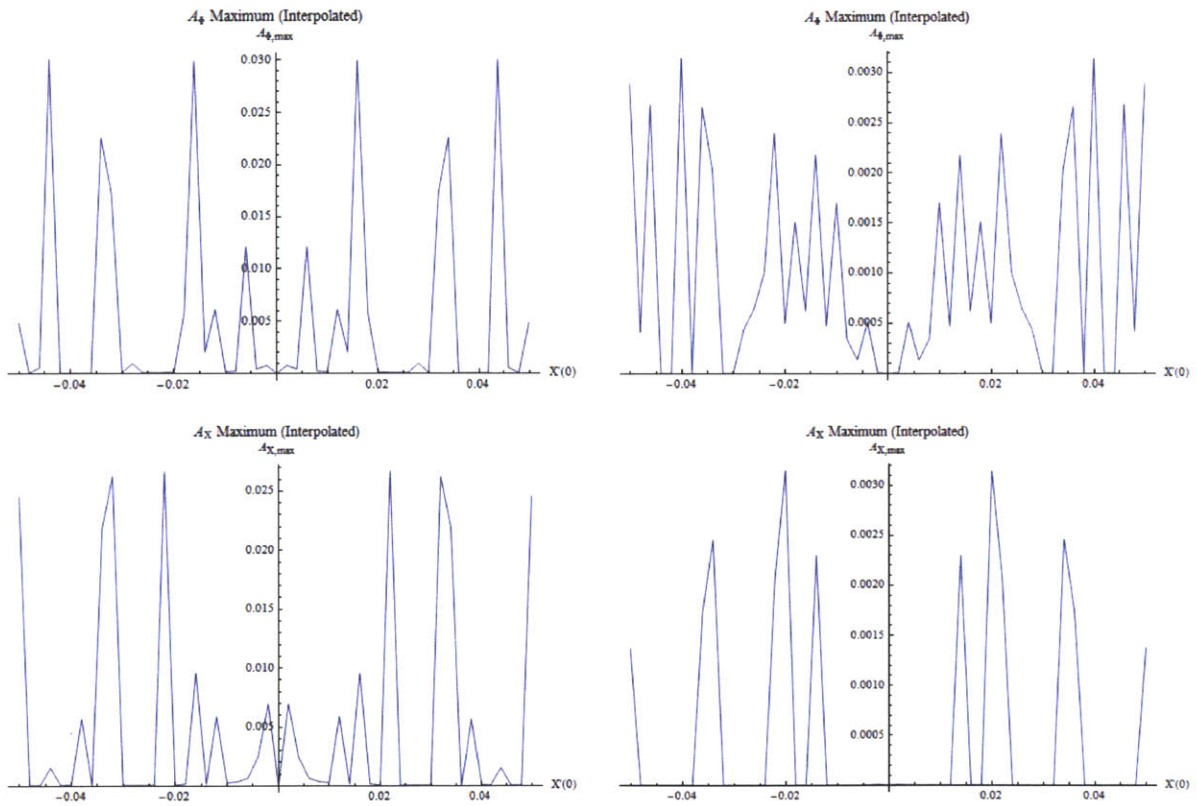


Figure 3-6: Maximum values of A^I obtained for a range of initial values $X[0]$, with $\xi = 10^2$, $\Phi[0] = 0.95$ (left) and $\xi = 10^3$, $\Phi[0] = 0.3$ (right).

Appendix A

Indexed Quantities for Two Field Model

$$\mathcal{G}_{\phi\phi} = \frac{M_{pl}^2}{2f} \left(1 + \frac{3\xi^2\phi^2}{f} \right)$$

$$\mathcal{G}^{\phi\phi} = \frac{4f^2}{M_{pl}^2 C} \left(1 + \frac{3\xi^2\chi^2}{f} \right)$$

$$\mathcal{G}_{\phi\chi} = \mathcal{G}_{\chi\phi} = \frac{M_{pl}^2}{2f} \frac{3\xi^2\phi\chi}{f}$$

$$\mathcal{G}^{\phi\chi} = \mathcal{G}^{\chi\phi} = -\frac{4f^2}{M_{pl}^2 C} \frac{3\xi^2\phi\chi}{f}$$

$$\mathcal{G}_{\chi\chi} = \frac{M_{pl}^2}{2f} \left(1 + \frac{3\xi^2\chi^2}{f} \right)$$

$$\mathcal{G}^{\chi\chi} = \frac{4f^2}{M_{pl}^2 C} \left(1 + \frac{3\xi^2\phi^2}{f} \right)$$

$$\Gamma_{\phi\phi}^{\phi} = \frac{\xi(1+6\xi)\phi}{C} - \frac{\xi\phi}{f}$$

$$\Gamma_{\chi\chi}^{\chi} = \frac{\xi(1+6\xi)\chi}{C} - \frac{\xi\chi}{f}$$

$$\Gamma_{\phi\chi}^{\phi} = \Gamma_{\chi\phi}^{\phi} = -\frac{\xi\chi}{2f}$$

$$\Gamma_{\phi\chi}^{\chi} = \Gamma_{\chi\phi}^{\chi} = -\frac{\xi\phi}{2f}$$

$$\Gamma_{\chi\chi}^{\phi} = \frac{\xi(1+6\xi)\phi}{C}$$

$$\Gamma_{\phi\phi}^{\chi} = \frac{\xi(1+6\xi)\chi}{C}$$

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