

Matroid Prophet Inequalities and Bayesian Mechanism Design

by

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B.A. in Mathematics, Cornell University (2010)

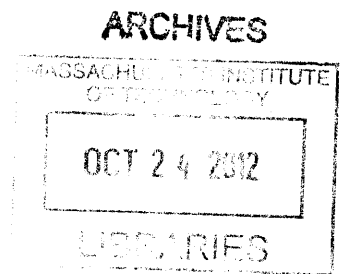
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Abstract

Consider a gambler who observes a sequence of independent, non-negative random numbers and is allowed to stop the sequence at any time, claiming a reward equal to the most recent observation. The famous prophet inequality of Krengel, Sucheston, and Garling asserts that a gambler who knows the distribution of each random variable can achieve at least half as much reward, in expectation, as a “prophet” who knows the sampled values of each random variable and can choose the largest one. We generalize this result to the setting in which the gambler and the prophet are allowed to make more than one selection, subject to a matroid constraint. We show that the gambler can still achieve at least half as much reward as the prophet; this result is the best possible, since it is known that the ratio cannot be improved even in the original prophet inequality, which corresponds to the special case of rank-one matroids. Generalizing the result still further, we show that under an intersection of p matroid constraints, the prophet’s reward exceeds the gambler’s by a factor of at most $O(p)$, and this factor is also tight.

Beyond their interest as theorems about pure online algorithms or optimal stopping rules, these results also have applications to mechanism design. Our results imply improved bounds on the ability of sequential posted-price mechanisms to approximate optimal mechanisms in both single-parameter and multi-parameter Bayesian settings. In particular, our results imply the first efficiently computable constant-factor approximations to the Bayesian optimal revenue in certain multi-parameter settings.

This work was done in collaboration with Robert Kleinberg.

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Chapter 1

Introduction

Online Optimization is an increasingly important area of theoretical computer science. In this thesis, we study an *online selection problem*. That is, weighted elements of a set are revealed one at a time to the user, who must immediately decide to accept or reject the element upon learning its weight. Some set of rules governs which elements may feasibly be accepted simultaneously. The goal of the online algorithm is accept a feasible set whose weight is competitive with the max-weight feasible set (i.e. the offline optimum). Mechanism Design is a seemingly unrelated field, where the a central agent tries to achieve some goal through the selfish behavior of rational participants. In other words, the agent wants to achieve some outcome involving the participants, who do not necessarily want the same outcome. The agent's goal is to find a way to achieve this outcome nonetheless by appealing to their selfish, but rational behavior. We also show that solutions to the online selection problem we study can be leveraged to design mechanisms in previously unsolved settings.

1.1 Bayesian Online Selection Problems

In 1978, Krengel, Sucheston and Garling [18] proved a surprising and fundamental result about the relative power of online and offline algorithms in Bayesian settings. They showed that if X_1, X_2, \dots, X_n is a sequence of independent, non-negative, real-valued random variables and $\mathbb{E}[\max_i X_i] <$

∞ , then there exists a stopping rule τ such that

$$2 \cdot \mathbb{E}[X_\tau] \geq \mathbb{E}\left[\max_i X_i\right]. \quad (1.1)$$

In other words, if we consider a game in which a player observes the sequence X_1, X_2, \dots, X_n and is allowed to terminate the game at any time, collecting the most recently observed reward, then a prophet who can foretell the entire sequence and stop at its maximum value can gain at most twice as much payoff as a player who must choose the stopping time based only on the current and past observations. The inequality (1.1) became the first¹ of many “prophet inequalities” in optimal stopping theory. Expressed in computer science terms, these inequalities compare the performance of online algorithms versus the offline optimum for problems that involve selecting one or more elements from a random sequence, in a Bayesian setting where the algorithm knows the distribution from which the sequence will be sampled whereas the offline optimum knows the values of the samples themselves and chooses among them optimally. Not surprisingly, these inequalities have important applications in the design and analysis of algorithms, especially in algorithmic mechanism design, a connection that we discuss further below.

An instance of the Bayesian online selection problem (BOSP) is specified by a ground set \mathcal{U} , a downward-closed set system $\mathcal{I} \subseteq 2^{\mathcal{U}}$, and for each $x \in \mathcal{U}$ a probability distribution F_x supported on the set \mathbb{R}_+ of non-negative real numbers. These data determine a probability distribution over functions $w : \mathcal{U} \rightarrow \mathbb{R}_+$, in which the random variables $\{w(x) \mid x \in \mathcal{U}\}$ are independent and $w(x)$ has distribution F_x . We refer to $w(x)$ as the *weight of x* , and we extend w to an additive set function defined on $2^{\mathcal{U}}$ by $w(A) = \sum_{x \in A} w(x)$. Elements of \mathcal{I} are called *feasible sets*. For a given assignment of weights, w , we let $\text{MAX}(w)$ denote the maximum-weight feasible set and $\text{OPT}(w)$ denotes its weight; we will abbreviate these to MAX and OPT when the weights w are clear from context.

An *input sequence* is a sequence σ of ordered pairs (x_i, w_i) $i = 1, \dots, n$, each belonging to $\mathcal{U} \times \mathbb{R}_+$, such that every element of \mathcal{U} occurs exactly once in the sequence x_1, \dots, x_n . A *deterministic online selection algorithm* is a function A mapping every input sequence σ to a set $A(\sigma) \in \mathcal{I}$

¹More precisely, it was the second prophet inequality. The same inequality with a factor of 4, instead of 2, was discovered a year earlier by Krengel and Sucheston [17].

such that for any two input sequences σ, σ' that match on the first i pairs $(x_1, w_1), \dots, (x_i, w_i)$, the sets $A_i(\sigma) = A(\sigma) \cap \{1, \dots, i\}$ and $A_i(\sigma') = A(\sigma') \cap \{1, \dots, i\}$ are identical. A *randomized online selection algorithm* is a probability distribution over deterministic ones. The algorithm's choices define *decision variables* $b_i(\sigma)$ which are indicator functions of the events $x_i \in A(\sigma)$. An algorithm is *monotone* if increasing the value of w_i (while leaving the rest of σ unchanged) cannot decrease the value of $\mathbb{E}[b_i(\sigma)]$, where the expectation is over the algorithm's internal randomness but not the randomness of σ (if any). A monotone deterministic online selection algorithm can be completely described by a sequence of thresholds $T_1(\sigma), \dots, T_n(\sigma)$, where $T_i(\sigma) \in \mathbb{R}_+ \cup \{\infty\}$ is the infimum of the set of weights w such that $i \in A(\sigma')$ when σ' is obtained from σ by modifying w_i to w . Conversely, for any sequence of threshold functions T_1, \dots, T_n such that $T_i(\sigma)$ depends only on the first $i - 1$ elements of σ and $T_i(\sigma) = \infty$ whenever $A_{i-1}(\sigma) \cup \{i\} \notin \mathcal{I}$, there is a corresponding monotone deterministic online selection algorithm that selects x_i whenever $w_i \geq T_i(\sigma)$.

1.1.1 Matroids

In this paper, we prove a prophet inequality for matroids, generalizing the original inequality (1.1) which corresponds to the special case of rank-one matroids. Matroids are a commonly-studied notion in computer science that generalize the notion of linear independence in a vector space without assuming that addition is well-defined. Specifically, a *matroid* \mathcal{M} consists of a ground set \mathcal{U} and a non-empty downward-closed set system $\mathcal{I} \subseteq 2^{\mathcal{U}}$ satisfying the *matroid exchange axiom*: for all pairs of sets $I, J \in \mathcal{I}$ such that $|I| < |J|$, there exists an element $x \in J$ such that $I \cup \{x\} \in \mathcal{I}$. Elements of \mathcal{I} are called *independent sets* when $(\mathcal{U}, \mathcal{I})$ is a matroid. A maximal independent set is called a *basis*. If A is a subset of \mathcal{U} , its *rank*, denoted by $\text{rank}(A)$, is the maximum cardinality of an independent subset of A . Its *closure* or *span*, denoted by $\text{cl}(A)$, is the set of all $x \in \mathcal{U}$ such that $\text{rank}(A \cup \{x\}) = \text{rank}(A)$. It is well known that the following greedy algorithm selects a maximum-weight basis of a matroid: number the elements of \mathcal{U} as x_1, \dots, x_n in decreasing order of weight, and select the set of all x_i such that $x_i \notin \text{cl}(\{x_1, \dots, x_{i-1}\})$. We want to design algorithms to solve the BOSP when \mathcal{I} is restricted to be a matroid. Specifically we wish to solve the following problem:

Matroid Prophet Inequality Problem (MPIP): Given as input a matroid $(\mathcal{U}, \mathcal{I})$, and a distri-

bution F_x for all $x \in \mathcal{U}$, design a stopping rule whose accepted set, ALG, is always in \mathcal{I} and such that for some constant c , $c\mathbb{E}[w(\text{ALG})] \geq \mathbb{E}[\text{OPT}]$.

1.1.2 Results

We prove in this thesis that for every matroid, there is a solution to the MPIP with $c = 2$. It is well known that the factor 2 in Krenzel, Sucheston, and Garling’s inequality (1.1) cannot be improved (see Section 4 for a lower bound example) and therefore our result for matroids is the best possible, even in the rank-one case.

Our algorithm is quite simple. At its heart lies a new algorithm for achieving the optimal factor 2 in rank-one matroids: compute a threshold value $T = \mathbb{E}[\max_i X_i]/2$ and accept the first element whose weight exceeds this threshold. This is very similar to the algorithm of Samuel-Cahn [20], which uses a threshold T such that $\Pr(\max_i X_i > T) = \frac{1}{2}$ but is otherwise the same, and which also achieves the optimal factor 2. It is hard to surpass the elegance of Samuel-Cahn’s proof, and indeed our proof, though short and simple, is not as elegant. On the other hand, our algorithm for rank-one matroids has a crucial advantage over Samuel-Cahn’s: it generalizes to arbitrary matroids without weakening its approximation factor. The generalization is as follows. After having selected some set A and considering adding the next element x_i , the algorithm asks, “How much reward would a prophet expect if he were forced to start with the set A (not counting the weight of A)? And how much would this decrease if he were forced to start with $A \cup \{x_i\}$ instead (not counting the weight of A and x_i)?” If the weight of x_i exceeds half this expected decrease, then we accept it. Otherwise we reject. Note that this algorithm, specialized to rank-one matroids, is precisely the one proposed at the start of this paragraph: the expected value of a prophet who accepts nothing is exactly $\mathbb{E}[\max_i X_i]$, and the expected value of a prophet who is forced to start already having accepted an element is 0.

1.1.3 Matroid Intersections

We next extend our algorithm to the case in which the feasibility constraint is given by a matroid intersection rather than a single matroid. The intersection of p matroids $(\mathcal{U}, \mathcal{I}_1), \dots, (\mathcal{U}, \mathcal{I}_p)$ is the

set system $(\mathcal{U}, \cap_j \mathcal{I}_j)$. In other words, a set is “independent” if and only if it is independent in all p matroids. It is easy to extend the MPIP to ask about feasibility constraints that are the intersection of p matroids, and we will denote the problem as p -MPIP. In this thesis, we also show that the p -MPIP admits a solution with $c = 4p - 2$. The algorithm is a natural extension of the one described earlier, but slightly more complex to account for the change in setting. Having already accepted A and considering x_i , the algorithm again considers the expected reward of a prophet forced to start with A versus one forced to start with $A \cup \{x_i\}$. With matroid intersections, the expected reward of a prophet doesn’t behave as nicely as with a single matroid, so we make two modifications. First, we further restrict the prophet so that the set he accepts is not only feasible when combined with A , but is also a subset of the true max-weight feasible set. This restriction is vacuous with a single matroid, as the prophet will always choose such a subset anyway. Second, rather than using the correct feasibility constraints, we pretend that one matroid will be chosen uniformly at random and the constraints from the remaining ones will be ignored. Again, this is vacuous with a single matroid as it will always be chosen. After these modifications to the “prophet,” the algorithm is the same: if the weight of x_i exceeds half the expected decrease in the prophet’s reward, we accept it.

In Section 4 we show that our result for matroid intersections is almost tight: we present a lower bound demonstrating that the ratio $4p - 2$ cannot be improved by more than a constant factor.

1.2 Mechanism Design

Mechanism Design and online selection problems are seemingly unrelated. However, Chawla, Hartline, Malec and Sivan discovered a connection between designing certain types of mechanisms and proving prophet inequalities [5]. Specifically, imagine the following scenario: you are a seller (the central agent) with n distinguishable items for sale. There are m buyers interested in the items. As a salesman, your objective is to maximize your revenue by auctioning the items to the bidders in a way that exploits their selfish, rational behavior. Formally, here are the standard economic assumptions behind the problem considered in [5] that can be traced all the way back to Myerson’s seminal paper on mechanism design, where he solves this problem for $n = 1$ [19].

Bayesian Multi-dimensional Unit-demand Mechanism Design (BMUMD):

1. Each bidder i has a private value v_{ij} for receiving item j . This is unknown to the seller.
2. Each bidder i is *unit-demand*. That is, the value of bidder i for a set of goods, J , is just $\max_{j \in J} v_{ij}$.
3. Each bidder i 's value for item j is sampled independently from a known distribution F_{ij} . F_{ij} is known to the seller.
4. Some set system $([m] \times [n], \mathcal{I})$ determines which items can simultaneously be awarded to which bidders, which includes the w.l.o.g. assumption that each bidder receives at most one item.
5. Bidders are *quasi-linear*. That is, their utility is just their value minus the price they pay.
6. The mechanism must be truthful, or *dominant strategy incentive compatible* (DSIC) to ensure predictable behavior from selfish, rational participants. That is, for all i , no matter what bids are made by the remaining bidders, it is in bidder i 's best interest to report her true bid (i.e. their utility is higher by telling the truth than telling any possible lie).
7. The goal is to design a truthful mechanism whose expected revenue is competitive with that of the highest achievable expected revenue by any truthful mechanism.

Chawla et al. [5] pioneered the study of approximation guarantees for *sequential posted pricings* (SPMs), a very simple class of mechanisms in which the seller makes a sequence of take-it-or-leave-it offers to the agents, with each offer specifying an item and a price that the agent must pay in order to win the item. Despite their simplicity, sequential posted pricings were shown in [5] to approximate the optimal revenue in many different settings. Prophet inequalities constitute a key technique underlying these results; instead of directly analyzing the revenue of the SPM, one analyzes the so-called *virtual values* of the winning bids, proving via prophet inequalities that the combined expected virtual value accumulated by the SPM approximates the offline optimum. Translating this virtual-value approximation guarantee into a revenue guarantee is an application

of standard Bayesian mechanism design techniques introduced by Roger Myerson [19]. In the course of developing these results, Chawla et al. prove a type of prophet inequality for matroids that is of considerable interest in its own right: they show that if the algorithm is allowed to specify the order in which the matroid elements are observed, then it can guarantee an expected payoff at least half as large as the prophet's. Our result can be seen as a strengthening of theirs, achieving the same approximation bound without allowing the algorithm to reorder the elements. Unlike our setting, in which the factor 2 is known to be tight, the best known lower bound for algorithms that may reorder the elements is $\sqrt{\pi/2} \cong 1.25$.

Extending the aforementioned results from single-parameter to multi-parameter domains, Chawla et al. define in [5] a general class of multi-parameter mechanism design problems called Bayesian multi-parameter unit-demand (BMUMD). SPMs in this setting are not truthful but can be modified to yield mechanisms that approximate the Bayesian optimal revenue with respect to a weaker solution concept: implementation in undominated strategies. A narrower class of mechanisms called *oblivious posted pricings* (OPMs) yields truthful mechanisms, but typically with weaker approximation guarantees; for example, it is not known whether OPMs can yield constant-factor approximations to the Bayesian optimal revenue in matroid settings, except for special cases such as graphic matroids. Without resolving this question, our results lead to an equally strong positive result for BMUMD: truthful mechanisms that 2-approximate the Bayesian optimal revenue in matroid settings and $(4p - 2)$ -approximate it in settings defined by an intersection of p matroid constraints.

To extend our results to multi-dimensional settings, one must distinguish between adversaries with respect to their power to choose the ordering of the sequence. The original BOSP treated in previous work [17, 18] considers a *fixed-order* adversary. That is, the adversary chooses an ordering (or distribution over orderings) for revealing the elements of \mathcal{U} *without knowing any of the weights* $w(x)$. Our main result is an algorithm that achieves $\frac{1}{2}\text{OPT}$ (or $\frac{1}{4p-2}\text{OPT}$) against a fixed-order adversary. This result combined with the techniques of [5] immediately yields OPMs for single-parameter mechanism design. To extend our results to BMUMD, we must consider a stronger type of adversary. There are many ways that an adversary could adapt to the sam-

pled weights and/or the algorithm’s decisions, some more powerful than others. The type of adaptivity that is relevant to our paper will be called an *online weight-adaptive* adversary. An online weight-adaptive adversary chooses the next element of \mathcal{U} to reveal one at a time. After choosing x_1, \dots, x_{i-1} and learning $w(x_1), \dots, w(x_{i-1})$, the online weight-adaptive adversary chooses the next x_i to reveal *without knowing the weight* $w(x_i)$ (or any weights besides $w(x_1)$ through $w(x_{i-1})$). Fortunately, the same exact proof shows that our algorithm, without any modification, also achieves $\frac{1}{2}\text{OPT}$ (or $\frac{1}{4p-2}\text{OPT}$) against an online weight-adaptive adversary. The connection between BMUMD and online weight-adaptive adversaries is not trivial, and is explained in Section 5.

1.3 Related Work

The genesis of prophet inequalities in the work of Krengel, Sucheston, and Garling [17, 18] was discussed earlier. It would be impossible to do justice to the extensive literature on prophet inequalities. Of particular relevance to our work are the so-called *multiple-choice prophet inequalities* in which either the gambler, the prophet, or both are given the power to choose more than one element. While several papers have been written on this topic, e.g. [13, 14, 15], the near-optimal solution of the most natural case, in which both the gambler and the prophet have $k > 1$ choices, was not completed until the work of Alaei [1], who gave a factor- $(1 - 1/\sqrt{k+3})^{-1}$ prophet inequality for k -choice optimal stopping; a nearly-matching lower bound of $1 + \Omega(k^{-1/2})$ was already known from prior work.

Research investigating the relationship between algorithmic mechanism design and prophet inequalities was initiated by Hajiaghayi, Kleinberg, and Sandholm [12], who observed that algorithms used in the derivation of prophet inequalities, owing to their monotonicity properties, could be interpreted as truthful online auction mechanisms and that the prophet inequality in turn could be interpreted as the mechanism’s approximation guarantee. Chawla et al. [5] discovered a much subtler relation between the two subjects: questions about the approximability of offline Bayesian optimal mechanisms by sequential posted-price mechanisms could be translated into questions about prophet inequalities, via the use of virtual valuation functions. A fuller discussion of their

contributions appears earlier in this section. Recent work by Alaei [1] deepens still further the connections between these two research areas, obtaining a near-optimal k -choice prophet inequality and applying it to a much more general Bayesian combinatorial auction framework than that studied in [5].

While not directly related to our work, the matroid secretary problem [3] also concerns relations between optimal stopping and matroids, this time under the assumption of a randomly ordered input, rather than independent random numbers in a fixed order. In fact, the “hard examples” for many natural examples in the matroid-secretary setting also translate into hard examples for the prophet inequality setting. In light of this relation, it is intriguing that our work solves the matroid prophet inequality problem whereas the matroid secretary problem remains unsolved, despite intriguing progress on special cases [7, 16], general matroids [4], and relaxed versions of the problem [22].

Finally, the Bayesian online selection problem that we consider here can be formulated as an exponential-sized Markov decision process, whose state reflects the entire set of decisions made prior to a specified point during the algorithm’s execution. Thus, our paper can be interpreted as a contribution to the growing CS literature on approximate solutions of exponential-sized Markov decision processes, e.g. [6, 10, 11]. Most of these papers use LP-based techniques. Combinatorial algorithms based on simple thresholding rules, such as ours, are comparatively rare although there are some other examples in the literature on such problems, for example [9].

Chapter 2

Algorithms for Matroids

In this section we prove our main theorem, asserting the existence of algorithms whose expected reward is at least $\frac{1}{2}\text{OPT}$ when playing against any online weight-adaptive adversary. Here is some intuition as to the considerations guiding the design of our algorithm. Imagine a prophet that is forced to start by accepting the set A , and let the *remainder* of A (denoted $R(A)$, defined formally in the following section) denote the subset that the restricted prophet adds to A . Let the *cost* of A (denoted $C(A)$, defined formally in the following section) denote the subset that the unrestricted prophet selected in place of A . Then the restricted prophet makes $w(A) + \mathbb{E}[w(R(A))]$ in expectation, while the unrestricted prophet makes $\mathbb{E}[w(C(A))] + \mathbb{E}[w(R(A))]$. So if A satisfies $w(A) \geq \frac{1}{\alpha}\mathbb{E}[w(C(A))]$ for a small constant α , it is not so bad to get stuck holding set A . However, just because A is not a bad set to start with does not mean we shouldn't accept anything that comes later. After all, the empty set is not a bad set to start with. If we can choose A in a way such that for any V we reject with $A \cup V \in \mathcal{I}$, $w(V) \leq \frac{1}{\alpha}\mathbb{E}[w(R(A))]$, then A is not a bad set to finish with. Simply put, we want to choose thresholds that are large enough to guarantee that $w(A)$ compares well to $\mathbb{E}[w(C(A))]$, but small enough to guarantee that everything we reject is not too heavy. Indeed, the first step in our analysis is to define this property formally and show that an algorithm with this property obtains a $\frac{1}{\alpha}$ -approximation.

2.1 Detour: The rank-one case

To introduce the ideas underlying our algorithm and its analysis, we start with a very simple analysis of the case of rank-one matroids. This is the special case of the problem in which the algorithm is only allowed to make one selection, i.e. the same setting as the original prophet inequality (1.1). Thus, the algorithm given in this section can be regarded as providing a new and simple proof of that inequality.

Let the random weights of the elements be denoted by X_1, \dots, X_n , and let $T = \mathbb{E}[\max_i X_i]/2$. We will show that an algorithm that stops at the first time τ such that $X_\tau \geq T$ makes at least T in expectation. Let $p = \Pr[\max_i X_i \geq T]$. Then we get the following inequality, for any $x > T$:

$$\Pr[X_\tau > x] \geq (1 - p) \sum_{i=1}^n \Pr[X_i > x]$$

This is true because with probability $1 - p$ the algorithm accepts nothing, so with probability at least $(1 - p)$ it has accepted nothing by the time it processes X_i . So the probability that the algorithm accepts X_i and that $X_i > x$ is at least $(1 - p) \Pr[X_i > x]$. It is also clear, by the union bound, that

$$\sum_{i=1}^n \Pr[X_i > x] \geq \Pr[\max_i X_i > x]$$

and therefore, for all $x > T$,

$$\Pr[X_\tau > x] \geq (1 - p) \Pr[\max_i X_i > x].$$

Now, observe that $\mathbb{E}[\max_i X_i] = \int_0^T \Pr[\max_i X_i > x] dx + \int_T^\infty \Pr[\max_i X_i > x] dx = 2T$. As the first term is clearly at most T , the second term must be at least T . So finally, we write:

$$\begin{aligned} \mathbb{E}[X_\tau] &= \int_0^T \Pr[X_\tau > x] dx + \int_T^\infty \Pr[X_\tau > x] dx \\ &\geq pT + (1 - p) \int_T^\infty \Pr[\max_i X_i > x] dx \\ &\geq pT + (1 - p)T = T = \frac{1}{2} \mathbb{E}[\max_i X_i] \end{aligned}$$

which completes the proof of (1.1).

2.2 Defining α -balanced thresholds

To design and analyze algorithms for general matroids, we begin by defining a property of a deterministic monotone algorithm that we refer to as α -balanced thresholds. In this section we prove that the expected reward of any such algorithm is at least $\frac{1}{\alpha}$ OPT. In the following section we construct an algorithm with 2-balanced thresholds, completing the proof of the main theorem.

To define α -balanced thresholds, we must first define some notation. Let $w, w' : \mathcal{U} \rightarrow \mathbb{R}_+$ denote two assignments of weights to \mathcal{U} , both sampled independently from the given distribution. We consider running the algorithm on an input sequence σ , with $\sigma = (x_1, w(x_1)), \dots, (x_n, w(x_n))$ and comparing the value of its selected set, $A = A(\sigma)$, with that of the basis B that maximizes $w'(B)$. The matroid exchange axiom ensures that there is at least one way to partition B into disjoint subsets C, R such that $A \cup R$ is also a basis of \mathcal{M} . (Consider adding elements of B one-by-one to A , preserving membership in \mathcal{I} , until the two sets have equal cardinality, and let R be the set of elements added to A .) Among all such partitions, let $C(A), R(A)$ denote the one that maximizes $w'(R)$.

Definition 1. For a parameter $\alpha > 1$, a deterministic monotone algorithm has α -balanced thresholds if it has the following property. For every input sequence σ , if $A = A(\sigma)$ and V is a set disjoint from A such that $A \cup V \in \mathcal{I}$, then

$$\sum_{x_i \in A} T_i(\sigma) \geq \left(\frac{1}{\alpha}\right) \cdot \mathbb{E}[w'(C(A))] \quad (2.1)$$

$$\sum_{x_i \in V} T_i(\sigma) \leq \left(1 - \frac{1}{\alpha}\right) \cdot \mathbb{E}[w'(R(A))], \quad (2.2)$$

where the expectation is over the random choice of w' .

Intuitively, Equation 2.1 guarantees that the set we accept is not too bad. The idea is that by selecting A , we have stopped the prophet from accepting $C(A)$, so we should make sure that $w(A)$

is comparable to $\mathbb{E}[w'(C(A))]$. Equation 2.2 guarantees that any set we reject is not too heavy. A prophet expects reward equal to $\mathbb{E}[w'(R(A))]$ when adding elements to A , so if there is no way for the gambler to make even $(1 - \frac{1}{\alpha})\mathbb{E}[w'(R(A))]$, he is not missing out on much by rejecting everything. Proposition 1 below states formally that this intuition holds.

Proposition 1. *If a monotone algorithm has α -balanced thresholds, then it satisfies the following approximation guarantee against online weight-adaptive adversaries:*

$$\mathbb{E}[w(A)] \geq \frac{1}{\alpha} \text{OPT}. \quad (2.3)$$

Proof. We have

$$\text{OPT} = \mathbb{E}[w'(C(A)) + w'(R(A))] \quad (2.4)$$

because $C(A) \cup R(A)$ is a maximum-weight basis with respect to w' , and w' has the same distribution as w . For any real number z , we will use the notation $(z)^+$ to denote $\max\{z, 0\}$. The proof will consist of deriving the following three inequalities, in which w_i stands for $w(x_i)$.

$$\mathbb{E} \left[\sum_{x_i \in A} T_i \right] \geq \frac{1}{\alpha} \mathbb{E}[w'(C(A))] \quad (2.5)$$

$$\mathbb{E} \left[\sum_{x_i \in A} (w_i - T_i)^+ \right] \geq \mathbb{E} \left[\sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right] \quad (2.6)$$

$$\mathbb{E} \left[\sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right] \geq \frac{1}{\alpha} \mathbb{E}[w'(R(A))]. \quad (2.7)$$

Summing (2.5)-(2.7) and using the fact that $T_i + (w_i - T_i)^+ = w_i$ for all $x_i \in A$, we obtain

$$\mathbb{E}[w(A)] \geq \frac{1}{\alpha} \mathbb{E}[w'(C(A))] + \frac{1}{\alpha} \mathbb{E}[w'(R(A))].$$

Inequality (2.5) is a restatement of the definition of α -balanced thresholds. Inequality (2.6) is deduced from the following observations. First, the algorithm selects every i such that $w_i > T_i$, so $\sum_{x_i \in A} (w_i - T_i)^+ = \sum_{i=1}^n (w_i - T_i)^+$. Second, the online property of the algorithm and the fact

that weight-adaptive adversaries do not learn w_i before choosing to reveal x_i imply that T_i depends only on $(x_1, w_1), \dots, (x_{i-1}, w_{i-1})$ and that the random variables $w(x_i), w'(x_i), T_i$ are independent. As $w_i = w(x_i)$ and $w'(x_i)$ are identically distributed, it follows that

$$\begin{aligned} & \mathbb{E} \left[\sum_{i=1}^n (w_i - T_i)^+ \right] \\ &= \mathbb{E} \left[\sum_{i=1}^n (w'(x_i) - T_i)^+ \right] \\ &\geq \mathbb{E} \left[\sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right], \end{aligned}$$

and (2.6) is established. Finally, we apply Property (2.2) of α -balanced thresholds, using the set $V = R(A)$, to deduce that

$$\begin{aligned} & \mathbb{E} \left[\sum_{x_i \in R(A)} w'(x_i) \right] \\ &\leq \mathbb{E} \left[\sum_{x_i \in R(A)} T_i \right] + \mathbb{E} \left[\sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right] \\ &\leq \left(1 - \frac{1}{\alpha}\right) \mathbb{E} \left[\sum_{x_i \in R(A)} w'(x_i) \right] \\ &\quad + \mathbb{E} \left[\sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right] \end{aligned}$$

$$\frac{1}{\alpha} \mathbb{E} \left[\sum_{x_i \in R(A)} w'(x_i) \right] \leq \mathbb{E} \left[\sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right]$$

Consequently (2.7) holds, which concludes the proof. □

2.3 Achieving 2-balanced thresholds

This section presents an algorithm with 2-balanced thresholds. The algorithm is quite simple. In step i , having already selected the (possibly empty) set A_{i-1} , we set threshold $T_i = \infty$ if $A_{i-1} \cup \{x_i\} \notin \mathcal{I}$. Otherwise, imagine that a prophet resamples the sequence and starts over having already accepted A_{i-1} . Such a prophet would obtain expected reward exactly equal to $\mathbb{E}[w'(R(A_{i-1}))]$. Imagine also a prophet resampling and starting over having already accepted $A_{i-1} \cup \{x_i\}$. Such a prophet obtains expected reward exactly equal to $\mathbb{E}[w'(R(A_{i-1} \cup \{x_i\}))]$. We set T_i equal to exactly half of what a prophet would lose in expectation by accepting $\{x_i\}$, or formally:

$$T_i = \frac{1}{2} \mathbb{E}[w'(R(A_{i-1})) - w'(R(A_{i-1} \cup \{x_i\}))] \quad (2.8)$$

$$= \frac{1}{2} \mathbb{E}[w'(C(A_{i-1} \cup \{x_i\})) - w'(C(A_{i-1}))] \quad (2.9)$$

The algorithm selects element x_i if and only if $w_i \geq T_i$. The fact that both (2.8) and (2.9) define the same value of T_i is easy to verify. Let B denote the maximum weight basis of \mathcal{M} with weights w' .

$$\begin{aligned} w'(C(A_{i-1})) + w'(R(A_{i-1})) &= w'(B) \\ &= w'(C(A_{i-1} \cup \{x_i\})) + w'(R(A_{i-1} \cup \{x_i\})) \\ w'(R(A_{i-1})) - w'(R(A_{i-1} \cup \{x_i\})) & \\ &= w'(C(A_{i-1} \cup \{x_i\})) - w'(C(A_{i-1})) \end{aligned}$$

Property (2.1) in the definition of α -balanced thresholds follows from a telescoping sum.

$$\begin{aligned} \sum_{x_i \in A} T_i &= \frac{1}{2} \sum_{x_i \in A} \mathbb{E}[w'(C(A_{i-1} \cup \{x_i\})) - w'(C(A_{i-1}))] \\ &= \frac{1}{2} \sum_{x_i \in A} \mathbb{E}[w'(C(A_i)) - w'(C(A_{i-1}))] \\ &= \frac{1}{2} \mathbb{E}[w'(C(A_n)) - w'(C(A_0))] \\ &= \frac{1}{2} \mathbb{E}[w'(C(A))]. \end{aligned}$$

The remainder of this section is devoted to proving Property (2.2) in the definition of α -balanced thresholds. In the present context, with $\alpha = 2$ and thresholds T_i defined by (2.8), the property simply asserts that for every pair of disjoint sets A, V such that $A \cup V \in \mathcal{I}$,

$$\begin{aligned} & \mathbb{E} \left[\sum_{x_i \in V} w'(R(A_{i-1})) - w'(R(A_{i-1} \cup \{x_i\})) \right] \\ &= 2 \sum_{x_i \in V} T_i(\sigma) \leq \mathbb{E} [w'(R(A))] \end{aligned}$$

We will show, in fact, that this inequality holds for every non-negative weight assignment w' and not merely in expectation. The proof appears in Proposition 2 below. To establish it, we will need some basic properties of matroids.

Definition 2 ([21], Section 39.3). *If \mathcal{M} is a matroid and S is a subset of its ground set, the deletion $\mathcal{M} - S$ and the contraction \mathcal{M}/S are two matroids with ground set $\mathcal{U} - S$. A set T is independent in $\mathcal{M} - S$ if T is independent in \mathcal{M} , whereas T is independent in \mathcal{M}/S if $T \cup S_0$ is independent in \mathcal{M} , where S_0 is any maximal independent subset of S .*

Lemma 1. *Suppose $\mathcal{M} = (\mathcal{U}, \mathcal{I})$ is a matroid and $V, R \in \mathcal{I}$ are two independent sets of equal cardinality.*

1. *There is a bijection $\phi : V \rightarrow R$ such that for every $v \in V$, $(R - \{\phi(v)\}) \cup \{v\}$ is an independent set.*
2. *For a weight function $w' : \mathcal{U} \rightarrow \mathbb{R}$, suppose that R has the maximum weight of all $|R|$ -element independent subsets of $V \cup R$. Then the bijection ϕ in part 1 also satisfies $w'(\phi(v)) \geq w'(v)$.*

Proof. Part 1 is Corollary 39.12a in [21]. To prove part 2, simply observe that the weight of $(R - \{\phi(v)\}) \cup \{v\}$ cannot be greater than the weight of R , by our assumptions on R and ϕ . \square

The next two lemmas establish basic properties of the function $S \mapsto R(S)$.

Lemma 2. *For any independent set A , the set $R(A)$ is equal to the maximum weight basis of \mathcal{M}/A .*

Proof. Let B be the maximum-weight basis of \mathcal{M} . Among all bases of \mathcal{M}/A that are contained in B , the set $R(A)$ is, by definition, the one of maximum weight. Therefore, if it is not the maximum-weight basis of \mathcal{M}/A , the only reason can be that there is another basis of \mathcal{M}/A , not contained in B , having strictly greater weight. But we know that the maximum-weight basis of \mathcal{M}/A is selected by the greedy algorithm, which iterates through the list y_1, \dots, y_k of elements of $\mathcal{U} - A$ sorted in order of decreasing weight, and picks each element y_i that is not contained in $\text{cl}(A \cup \{y_1, \dots, y_{i-1}\})$. In particular, every y_i chosen by the greedy algorithm on \mathcal{M}/A satisfies $y_i \notin \text{cl}(\{y_1, \dots, y_{i-1}\})$ and therefore belongs to B . Thus the maximum-weight basis of \mathcal{M}/A is contained in B and must equal $R(A)$. \square

Lemma 3. *For any independent set J , the function $f(S) = w'(R(S))$ is a submodular set function on subsets of J .*

Proof. For notational convenience, in this proof we will denote the union of two sets by ‘+’ rather than ‘ \cup ’. Also, we will not distinguish between an element x and the singleton set $\{x\}$.

To prove submodularity it suffices to consider an independent set $S + x + y$ and to prove that $f(S) - f(S + x) \leq f(S + y) - f(S + x + y)$. Replacing \mathcal{M} by \mathcal{M}/S , we can reduce to the case that $S = \emptyset$ and prove that $f(\emptyset) - f(x) \leq f(y) - f(x + y)$ whenever $\{x, y\}$ is a two-element independent set.

What is the interpretation of $f(\emptyset) - f(x)$? Recall that $f(\emptyset) = w'(R(\emptyset))$ is the weight of the maximum-weight basis B of \mathcal{M} . Similarly, $f(x)$ is the weight of the maximum-weight basis B_x of $\mathcal{M}/\{x\}$. Let b_1, b_2, \dots, b_r denote the elements of B in decreasing order of weight. Consider running two executions of the greedy algorithm to select B and B_x in parallel. The only step in which the algorithms make differing decisions is the first step i in which $\{b_1, \dots, b_i\} \cup \{x\}$ contains a circuit. In this step, b_i is included in B but excluded from B_x . Similarly, when we run two executions of the greedy algorithm to select B_y and B_{xy} — the maximum-weight bases of $\mathcal{M}/\{y\}$ and $\mathcal{M}/\{x, y\}$, respectively — the only step in which differing decisions are made is the earliest step j in which $\{b_1, \dots, b_j\} \cup \{x, y\}$ contains a circuit. But j certainly cannot be later than i , since $\{b_1, \dots, b_i\} \cup \{x, y\}$

is a superset of $\{b_1, \dots, b_i\} \cup \{x\}$ and hence contains a circuit. We may conclude that

$$f(\emptyset) - f(x) = b_i \leq b_j = f(y) - f(x + y),$$

and hence f is submodular as claimed. \square

Proposition 2. *For any disjoint sets A, V such that $A \cup V \in \mathcal{I}$,*

$$\sum_{x_i \in V} w'(R(A_{i-1})) - w'(R(A_{i-1} \cup \{x_i\})) \leq w'(R(A)).$$

Proof. The function $f(S) = w'(R(S))$ is submodular on subsets $S \subseteq A \cup V$, by Lemma 3. Hence

$$\sum_{x_i \in V} w'(R(A_{i-1})) - w'(R(A_{i-1} \cup \{x_i\})) \tag{2.10}$$

$$\leq \sum_{x \in V} w'(R(A)) - w'(R(A \cup \{x\})). \tag{2.11}$$

Apply Lemma 1 to the independent sets $V, R(A)$ in \mathcal{M}/A to obtain a bijection ϕ such that $w'(\phi(x)) \geq w'(x)$ and $A \cup (R(A) - \phi(x)) \cup \{x\} \in \mathcal{I}$ for all $x \in V$. By definition of $R(\cdot)$, we know that $A \cup \{x\} \cup R(A \cup \{x\})$ is the maximum weight independent subset of $A \cup \{x\} \cup B$ that contains $A \cup \{x\}$. One such set is $A \cup (R(A) - \phi(x)) \cup \{x\}$, so

$$\begin{aligned} w'(A) + w'(R(A)) - w'(\phi(x)) + w'(x) \\ \leq w'(A) + w'(R(A \cup \{x\})) + w'(x) \end{aligned} \tag{2.12}$$

$$\begin{aligned} w'(R(A)) - w'(R(A \cup \{x\})) \\ \leq w'(\phi(x)) \end{aligned} \tag{2.13}$$

$$\begin{aligned} \sum_{x \in V} w'(R(A)) - w'(R(A \cup \{x\})) \\ \leq \sum_{x \in V} w'(\phi(x)) = w'(R). \end{aligned} \tag{2.14}$$

The proposition follows by combining (2.10) and (2.14). \square

One might notice that our choice of thresholds involves computing the expectation of a random variable. In practice, computing our thresholds exactly may be unrealistic. However, it is not hard to see that the work in this section also provides a proof of the following proposition:

Proposition 3. *Let T_i denote the thresholds used by our algorithm to obtain 2-balanced thresholds, and let T'_i denote thresholds used by some other monotone algorithm. Then if $T'_i \leq T_i \leq (1 + \epsilon)T'_i$ for all i , the thresholds T'_i are $2(1 + \epsilon)$ -balanced thresholds.*

Proposition 3 can be deduced by observing that decreasing the thresholds cannot possibly hurt the ratio of Property (2.2), but it might hurt the ratio of Property (2.1). We proved Property (2.1) by simply observing a telescoping sum. It is clear that lower bounding the telescoping sum instead of computing it exactly provides the bound $\sum_{x_i \in A} T_i(\sigma) \geq \frac{1}{\alpha(1+\epsilon)} \mathbb{E}[w'(C(A))]$.

Chapter 3

Matroid intersections

Our algorithm and proof for matroid intersections is quite similar. We need to modify some definitions and extend some proofs, but the spirit is the same.

3.1 Generalizing α -balanced thresholds

We first have to extend our notation a bit. Denote the independent sets for the p matroids as $\mathcal{I}_1, \dots, \mathcal{I}_p$. Denote the “truly independent” sets as $\mathcal{I} = \bigcap_j \mathcal{I}_j$. Still let $w, w' : \mathcal{U} \rightarrow \mathbb{R}_+$ denote two assignments of weights to \mathcal{U} , both sampled independently from the given distribution. We consider running the algorithm on an input sequence $\sigma = (x_1, w(x_1)), \dots, (x_n, w(x_n))$ and comparing the value of its selected set, $A = A(\sigma)$, with that of the $B \in \mathcal{I}$ that maximizes $w'(B)$. For all j , the matroid exchange axiom ensures that there is at least one way to partition B into disjoint subsets C_j, R_j such that $A \cup R_j \in \mathcal{I}_j$, and $B \subseteq \text{cl}_j(A \cup R_j)$. Among all such partitions, let $C_j(A), R_j(A)$ denote the one that maximizes $w'(R_j)$ (greedily add elements from B to R_j unless it creates a dependency in \mathcal{I}_j). We denote by $R(A) = \bigcap_j R_j(A)$ and $C(A) = \bigcup_j C_j(A)$.

Definition 3. *For a parameter $\alpha > 0$, a deterministic monotone algorithm has α -balanced thresholds if it has the following property. For every input sequence σ , if $A = A(\sigma)$ and V is a set disjoint*

from A such that $A \cup V \in I$, then

$$\sum_{x_i \in A} T_i(\sigma) \geq \left(\frac{1}{\alpha}\right) \cdot \mathbb{E} \left[\sum_j w'(C_j(A)) \right] \quad (3.1)$$

$$\sum_{x_i \in V} T_i(\sigma) \leq \left(\frac{1}{\alpha}\right) \cdot \mathbb{E} \left[\sum_j w'(R_j(A)) \right], \quad (3.2)$$

where the expectation is over the random choice of w' .

Proposition 4. *If a monotone algorithm has α -balanced thresholds for $\alpha \geq 2$, then it satisfies the following approximation guarantee against weight-adaptive adversaries when I is the intersection of p matroids:*

$$\mathbb{E}[w(A)] \geq \frac{\alpha - p}{\alpha(\alpha - 1)} \text{OPT}. \quad (3.3)$$

Proof. We have

$$\text{OPT} = \mathbb{E} \left[w'(C_j(A)) + w'(R_j(A)) \right] \quad \forall j \quad (3.4)$$

$$\text{OPT} = \mathbb{E} \left[w'(C(A)) + w'(R(A)) \right] \quad (3.5)$$

because $C_j(A) \cup R_j(A)$ is a maximum-weight independent set with respect to w' for all j , as is $C(A) \cup R(A)$, and w' has the same distribution as w . The proof will again consist of deriving the following three inequalities.

$$\mathbb{E} \left[\sum_{x_i \in A} T_i \right] \geq \frac{1}{\alpha} \mathbb{E} \left[\sum_j w'(C_j(A)) \right] \quad (3.6)$$

$$\begin{aligned} & \mathbb{E} \left[\sum_{x_i \in A} (w_i - T_i)^+ \right] \\ & \geq \mathbb{E} \left[\sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right] \end{aligned} \quad (3.7)$$

$$\begin{aligned} & \mathbb{E} \left[\sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right] \\ & \geq \mathbb{E} [w'(R(A))] - \frac{1}{\alpha} \mathbb{E} \left[\sum_j w'(R_j(A)) \right]. \end{aligned} \quad (3.8)$$

Summing (3.6) + (3.7) + $\frac{1}{\alpha-1}$ (3.8) and using the fact that $T_i + (w_i - T_i)^+ = w_i$ for all $x_i \in A$, we obtain

$$\begin{aligned} \mathbb{E}[w(A)] &\geq \left(\frac{1}{\alpha-1} - \frac{1}{\alpha(\alpha-1)} \right) \mathbb{E} \left[\sum_j w'(C_j(A)) \right] \\ &\quad + \frac{\alpha-2}{\alpha-1} \mathbb{E} \left[\sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right] \\ &\quad + \frac{1}{\alpha-1} \mathbb{E}[w'(R(A))] \\ &\quad - \frac{1}{\alpha(\alpha-1)} \mathbb{E} \left[\sum_j w'(R_j(A)) \right]. \end{aligned}$$

Substituting in Equations (3.4) and (3.5) (and observing that $\frac{\alpha-2}{\alpha-1} \geq 0$ whenever $\alpha \geq 2$), we get:

$$\mathbb{E}[w(A)] \geq \frac{1}{\alpha-1} \text{OPT} - \frac{p}{\alpha(\alpha-1)} \text{OPT} = \frac{\alpha-p}{\alpha(\alpha-1)} \text{OPT}$$

It remains to show that Equations (3.6) - (3.8) hold for any α -balanced thresholds. Equation (3.6) is again a restatement of the definition of α -balanced thresholds. Inequality (3.7) is deduced from the same observations as Equation (2.6). Finally, as in Proposition 1, we apply Property (3.2)

of α -balanced thresholds, using the set $V = R(A)$, to deduce that

$$\begin{aligned}
\mathbb{E} \left[\sum_{x_i \in R(A)} w'(x_i) \right] &\leq \mathbb{E} \left[\sum_{x_i \in R(A)} T_i \right] \\
&\quad + \mathbb{E} \left[\sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right] \\
&\leq \frac{1}{\alpha} \mathbb{E} \left[\sum_j w'(R_j(A)) \right] \\
&\quad + \mathbb{E} \left[\sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right] \\
\Rightarrow \mathbb{E} [w'(R(A))] - \frac{1}{\alpha} \mathbb{E} \left[\sum_j w'(R_j(A)) \right] & \\
&\leq \mathbb{E} \left[\sum_{x_i \in R(A)} (w'(x_i) - T_i)^+ \right]
\end{aligned}$$

Consequently (3.8) holds, which concludes the proof. □

3.2 Obtaining α -balanced thresholds

This section presents an algorithm obtaining α -balanced thresholds for any $\alpha > 1$. One can take a derivative to see that the optimal choice of α for the intersection of p matroids is $\alpha_p = p + \sqrt{p(p-1)}$. For simplicity, we will instead just use $\alpha = 2p$, as this is nearly optimal and always at least 2. When $\alpha = 2p$, the approximation guarantee from Proposition 4 is $\frac{1}{4p-2}$.

We now define our thresholds. Let

$$\begin{aligned}
T(A, i, j) &= \frac{1}{\alpha} \mathbb{E} \left[w'(R_j(A)) - w'(R_j(A \cup \{x_i\})) \right] \\
&= \frac{1}{\alpha} \mathbb{E} \left[w'(C_j(A \cup \{x_i\})) - w'(C_j(A)) \right] \\
T(A, i) &= \sum_j T(A, i, j).
\end{aligned}$$

In step i , having already selected the (possibly empty) set A_{i-1} , we set threshold $T_i = \infty$ if $A_{i-1} \cup \{i\} \notin \mathcal{I}$, and $T_i = T(A_{i-1}, i)$ otherwise. In other words, each $T(A, i, j)$ is basically the same as the threshold used for the single matroid algorithm if \mathcal{I}_j was the only matroid constraint. It is not exactly the same, because $R(A)$ when \mathcal{I}_j is the only matroid is not the same as $R_j(A)$ in the presence of other matroid constraints. $T(A, i)$ just sums $T(A, i, j)$ over all matroids.

The proof of Equation (3.1) follows exactly the proof of Equation (2.1).

The proof of Equation (3.2) follows from Proposition 2, although perhaps not obviously. As $A \cup V \in \mathcal{I}$, we clearly have $A \cup V \in \mathcal{I}_j$ for all j . So the hypotheses of Proposition 2 are satisfied for all j . Summing the bound we get in Proposition 2 over all j gives us Equation (3.2).

Similar to our thresholds for matroids, our thresholds for matroid intersections are robust to small errors in computation. The same reasoning provides us with the following proposition:

Proposition 5. *Let T_i denote the thresholds used by our algorithm to obtain α -balanced thresholds, and let T'_i denote thresholds used by some other monotone algorithm. Then if $T'_i \leq T_i \leq (1 + \epsilon)T'_i$ for all i , the T'_i are $\alpha(1 + \epsilon)$ -balanced thresholds.*

Chapter 4

Lower Bounds

Here we provide two examples. The first is the well-known example of [17] showing that the factor of 2 is tight for matroids. We present their construction here for completeness. The second shows that the ratio $O(p)$ is tight for the intersection of p matroids.

We start with the well-known example of [17]. Consider the 1-uniform matroid over 2 elements. We have $w(1) = 1$ with probability 1, $w(2) = n$ with probability $1/n$ and 0 otherwise. Then the prophet obtains $2 - 1/n$ in expectation, but the gambler obtains at most 1, as his optimal strategy is just to take the first element always.

The example for the intersection of p matroids has appeared in other forms in [3, 5]. Let q be a prime between $p/2$ and p . Then let $\mathcal{U} = \{(i, j) \mid 0 \leq i \leq q^q - 1, 0 \leq j \leq q - 1\}$. Then let \mathcal{I} contain all sets of the form $\{(i, j_1), \dots, (i, j_x)\}$. Now let $w(i, j) = 1$ with probability $1/q$, and $w(i, j) = 0$ otherwise, for all i, j . Reveal the elements in any order. No matter what strategy the gambler uses to pick the first element, his optimal strategy from that point on is to just accept every remaining element with the same first coordinate. However the gambler winds up with his first element, he makes at most $1 - 1/q$ in expectation from the remaining elements he is allowed to pick (as there are at most $q - 1$ remaining elements, and each has $\mathbb{E}[w(i, j)] = 1/q$). Therefore, the expected payoff to the gambler is less than 2. However, with probability at least $(1 - 1/e)$, there exists an i such that $w(i, j) = 1$ for all j (as the probability that this occurs for a fixed i is $1/q^q$ and there are q^q different i 's). So the expected payoff to the prophet is $\Theta(q)$.

Finally, we just have to show that \mathcal{I} can be written as the intersection of q matroids. Let \mathcal{I}_x be the partition matroid that partitions \mathcal{U} into $\sqcup_j S_j = \sqcup_j \cup_i \{(i, xi + j \pmod{q})\}$, and requires that only one element of each S_j be chosen. Then clearly, $\mathcal{I} \subseteq \cap_{x \in \mathbb{Z}_q} \mathcal{I}_x$ as any two elements with the same first coordinate lie in different partitions in each of the \mathcal{I}_x . In addition, $\cap_{x \in \mathbb{Z}_q} \mathcal{I}_x \subseteq \mathcal{I}$. Consider any (i, j) and (i', j') with $i \neq i'$. Then when $(j - j') \pmod{q} = x(i - i') \pmod{q}$, (i, j) and (i', j') are in the same partition of \mathcal{I}_x . As q is prime, this equation always has a solution. Therefore, we have shown that $\mathcal{I} = \cap_{x \in \mathbb{Z}_q} \mathcal{I}_x$, and \mathcal{I} can be written as the intersection of $q \leq p$ matroids. As the prophet obtains $\Theta(p)$ in expectation, and the gambler obtains less than 2 in expectation, no algorithm can achieve an approximation factor better than $O(p)$.

Chapter 5

Interpretation as OPM's

Here, we describe how to use our algorithm to design OPMs for unit-demand multi-parameter bidders under matroid and matroid intersection feasibility constraints. We begin by recalling the definition of Bayesian multi-parameter unit-demand mechanism design (BMUMD) from [5]. In any such mechanism design problem, there is a set of services, \mathcal{U} , partitioned into disjoint subsets J_1, \dots, J_n , one for each bidder. The mechanism must allocate a set of services, subject to downward-closed feasibility constraints given by a collection \mathcal{I} of feasible subsets. We assume that the feasibility constraints guarantee that no bidder receives more than a single service, i.e. that the intersection of any feasible set with one of the sets J_i contains no more than one element. (If this property is not already implied by the given feasibility constraints, it can be ensured by intersecting the given constraints with one additional partition matroid constraint.)

As in the work of Chawla et al. [5], we assume that each bidder i 's values for the services in set J_i are independent random variables, and we analyze BMUMD mechanisms for any such distribution by exploring a closely-related single-parameter domain that we denote by $\mathcal{I}^{\text{copies}}$. In $\mathcal{I}^{\text{copies}}$ there are $|\mathcal{U}|$ bidders, each of whom wants just a single service x and has a value v_x for receiving that service. The feasibility constraints are the same in both domains — the mechanism may select any set of services that belongs to \mathcal{I} — and the joint distribution of the values v_x ($x \in \mathcal{I}$) is the same as well; the only difference between the two domains is that an individual bidder i in the BMUMD problem becomes a set of competing bidders (corresponding to the elements of J_i)

in the domain I^{copies} . As might be expected, the increase in competition between bidders results in an increase in revenue for the optimal mechanism; indeed, the following lemma from [5] will be a key step in our analysis.

Lemma 4. *Let \mathcal{A} be any individually rational, truthful deterministic mechanism for instance I of BMUMD. Then the expected revenue of \mathcal{A} is no more than the expected revenue of the optimal mechanism for I^{copies} .*

A second technique that we will borrow from [5] (and, ultimately, from Myerson’s original paper on optimal mechanism design [19]), is the technique of analyzing the expected revenue of mechanisms indirectly via their virtual surplus. We begin by reviewing the definitions of virtual valuations and virtual surplus. Assume that v_x , the value of bidder i for item $x \in J_i$, has cumulative distribution function F_x whose density f_x is well-defined and positive on the interval on which v_x is supported. Then the virtual valuation function ϕ_x is defined by

$$\phi_x(v) = v - \frac{1 - F_x(v)}{f_x(v)},$$

and the *virtual surplus* of an allocation $A \in \mathcal{I}$ is defined to be the sum $\sum_{x \in A} \phi_x(v_x)$. Myerson [19] proved the following:

Lemma 5. *In single-parameter domains whose bidders have independent valuations with monotone increasing virtual valuation functions, the expected revenue of any mechanism in Bayes-Nash equilibrium is equal to its expected virtual surplus.*

The distribution of v_x is said to be *regular* when the virtual valuation function ϕ_x is monotonically increasing. We will assume throughout the rest of this section that bidders’ values have regular distributions, in order to apply Lemma 5. To deal with non-regular distributions, it is necessary to use a technique known as *ironing*, also due to Myerson [19], which in our context translates into randomized pricing via a recipe described in Lemma 2 of [5].

Our plan is now to design truthful mechanisms \mathcal{M} and $\mathcal{M}^{\text{copies}}$ for the BMUMD domain I and the associated single-parameter domain I^{copies} , respectively, and to relate them to the optimal

mechanisms for those domains via the following chain of inequalities.

$$\begin{aligned}
R(\mathcal{M}) &\geq R(\mathcal{M}^{\text{copies}}) \\
R(\mathcal{M}^{\text{copies}}) &= \Phi(\mathcal{M}^{\text{copies}}) \\
\Phi(\mathcal{M}^{\text{copies}}) &\geq \frac{1}{\alpha} \Phi(\text{OPT}^{\text{copies}}) \\
\frac{1}{\alpha} \Phi(\text{OPT}^{\text{copies}}) &= \frac{1}{\alpha} R(\text{OPT}^{\text{copies}}) \\
\frac{1}{\alpha} R(\text{OPT}^{\text{copies}}) &\geq \frac{1}{\alpha} R(\text{OPT}). \tag{5.1}
\end{aligned}$$

Here, $R(\cdot)$ and $\Phi(\cdot)$ denote the expected revenue and expected virtual surplus of a mechanism, respectively, and α denotes the approximation guarantee of a prophet inequality algorithm embedded in our mechanism. Thus, $\alpha = 2$ when I is a matroid, and more generally $\alpha = 4p - 2$ when I is given by an intersection of p matroid constraints.

Most of the steps in line (5.1) are already justified by the lemmas from prior work discussed above. The relation $R = \Phi$ for mechanisms $\mathcal{M}^{\text{copies}}$ and $\text{OPT}^{\text{copies}}$ is a consequence of Lemma 5, while Lemma 4 provides $R(\text{OPT}^{\text{copies}}) \geq R(\text{OPT})$. We will naturally derive the relation $\Phi(\mathcal{M}^{\text{copies}}) \geq \frac{1}{\alpha} \Phi(\text{OPT}^{\text{copies}})$ as a consequence of the prophet inequality. To do so, it suffices to define mechanism $\mathcal{M}^{\text{copies}}$ such that its allocation decisions result from running the prophet inequality algorithm on an input sequence consisting of the virtual valuations $\phi_x(v_x)$, presented in an order determined by an online weight-adaptive adversary. The crux of our proof will consist of designing said adversary to ensure that the relation $R(\mathcal{M}) \geq R(\mathcal{M}^{\text{copies}})$ also holds.

Given these preliminaries, we now describe the mechanisms \mathcal{M} and $\mathcal{M}^{\text{copies}}$. Central to both mechanisms is a pricing scheme using thresholds $T(A, x)$, defined as the threshold T_s that our online algorithm would use at step s when $x_s = x$ and the algorithm has accepted the set A so far. (Contrary to previous sections of the paper in which steps of the online algorithm's execution were denoted by i , here we reserve the variable i to refer to bidders in the mechanism, using s instead to denote a step of the online algorithm. Note that the thresholds assigned by our algorithm depend only on A and x , not on s , hence the notation $T(A, x)$ is justified.) Mechanism \mathcal{M} , described by the pseudocode in Algorithm 1, simply makes posted-price offers to bidders $1, 2, \dots, n$ in that order,

Algorithm 1: Mechanism \mathcal{M} for unit-demand multi-dimensional bidders

```
1: Initialize  $A = \emptyset$ .
2: for  $i = 1, 2, \dots, n$  do
3:   for all  $x \in J_i$  do
4:     Set  $p_x = \begin{cases} \phi_x^{-1}(T(A, x)) & \text{if } A \cup \{x\} \in \mathcal{I} \\ \infty & \text{otherwise.} \end{cases}$ 
5:   end for
6:   Post price vector  $(p_x)_{x \in J_i}$ .
7:   Bidder  $i$  chooses an element  $x \in J_i$ , or possibly nothing at these posted prices.
8:   if  $x$  is chosen then
9:     Allocate  $x$  to bidder  $i$  and charge price  $p_x$ .
10:     $A \leftarrow A \cup \{x\}$ 
11:   else
12:     Allocate nothing to bidder  $i$  and charge price 0.
13:   end if
14: end for
```

defining the posted price for each item by applying its inverse-virtual-valuation function to the threshold that the prophet inequality algorithm sets for that item.

To define mechanism $\mathcal{M}^{\text{copies}}$, we first define an online weight-adaptive adversary and then run the prophet inequality algorithm on the input sequence presented by this adversary, using its thresholds to define posted prices exactly as in mechanism \mathcal{M} above. The adversary is designed to minimize the mechanism's revenue, subject to the constraint that the elements are presented in an order that runs through all of the elements of J_1 , then the elements of J_2 , and so on. In fact, it is easy to compute this worst-case ordering by backward induction, which yields a dynamic program presented in pseudocode as Algorithm 2. The dynamic programming table consists of entries $V(A, i)$ denoting the expected revenue that $\mathcal{M}^{\text{copies}}$ will gain from selling elements of the set $J_{i+1} \cup \dots \cup J_n$, given that it has already allocated the elements of A . Computing and storing these values requires exponential time and space, but we are not concerned with making $\mathcal{M}^{\text{copies}}$ into a computationally efficient mechanism because its role in this paper is merely to provide an intermediate step in the analysis of mechanism \mathcal{M} .

The formula for $V(A, i)$ is guided by the following considerations. Since $\mathcal{M}^{\text{copies}}$ will post

prices $p_x = \phi_x^{-1}(T(A, x))$ for all $x \in J_{i+1}$ given that it has already allocated A , it will not allocate any element of J_{i+1} if $v_x < p_x$ for all $x \in J_{i+1}$, and otherwise it will allocate some element $x \in J_{i+1}$. In the former case, its expected revenue from the remaining elements will be $V(A, i + 1)$. In the latter case, it extracts revenue p_x from bidder $i + 1$ and expected revenue $V(A \cup \{x\}, i + 1)$ from the remaining bidders. Thus, an adversary who wishes to minimize the revenue obtained by the mechanism will order the elements $x \in J_{i+1}$ in increasing order of $p_x + V(A \cup \{x\}, i + 1)$. Denoting the elements of J_{i+1} in this order by x_1, x_2, \dots, x_k , we obtain the formula

$$\begin{aligned}
V(A, i) &= \left(\prod_{j=1}^k F_{x_j}(p_{x_j}) \right) \cdot V(A, i + 1) \\
&\quad + \sum_{\ell=1}^k P_1(\ell) \cdot P_2(\ell) \cdot P_3(\ell) \\
P_1(\ell) &= \prod_{j=1}^{\ell-1} F_{x_j}(p_{x_j}) \\
P_2(\ell) &= 1 - F_{x_\ell}(p_{x_\ell}) \\
P_3(\ell) &= p_{x_\ell} + V(A \cup \{x_\ell\}, i + 1).
\end{aligned} \tag{5.2}$$

The first term on the right side accounts for the possibility that bidder $i + 1$ buys nothing, while the sum accounts for the possibility that bidder $i + 1$ buys x_ℓ , for each $\ell = 1, \dots, k$: $P_1(\ell)$ is the probability that bidder $i + 1$ did not buy any of $\{x_1, \dots, x_\ell\}$, $P_2(\ell)$ is the probability that bidder $i + 1$ is willing to purchase x_ℓ , and $P_3(\ell)$ is the expected value of $V(A, i)$ given that bidder $i + 1$ purchases x_ℓ .

Mechanism $\mathcal{M}^{\text{copies}}$ has already been described above, and is specified by pseudocode in Algorithm 3. We note that $\mathcal{M}^{\text{copies}}$ does not satisfy the definition of an OPM in [5], since the price p_x for $x \in J_i$ may depend on the bids b_y for $y \in J_1 \cup \dots \cup J_{i-1}$. However, it retains a key property of OPMs that make them suitable for analyzing multi-parameter mechanisms: the prices of elements of J_i are predetermined before any of the bids in J_i are revealed.

Theorem 1. *Mechanism \mathcal{M} for BMUMD settings with independent regular valuations obtains*

Algorithm 2: Online weight-adaptive adversary for $\mathcal{I}^{\text{copies}}$

```
1: for  $i = n, n - 1, \dots, 1$  do                                // Preprocessing loop: fill in DP table
2:   for all feasible sets  $A \subseteq J_1 \cup \dots \cup J_i$  do
3:     if  $i = n$  then
4:        $V(A, i) = 0$ 
5:     else
6:        $p_x = \phi_x^{-1}(T(A, x))$  for all  $x \in J_{i+1}$ .
7:       Sort  $J_{i+1}$  in order of increasing  $p_x + V(A \cup \{x\}, i + 1)$ .
8:       Denote this sorted list by  $x_1, \dots, x_k$ .
9:       Compute  $V(A, i)$  using formula (5.2).
10:    end if
11:  end for
12: end for
13:                                     // Main loop: choose the ordering of each set  $J_i$ 
14: Initialize  $A = \emptyset$ .
15: for  $i = 1, \dots, n$  do
16:    $p_x = \phi_x^{-1}(T(A, x))$  for all  $x \in J_i$ .
17:   Sort the elements of  $J_i$  in order of increasing  $p_x + V(A \cup \{x\}, i - 1)$ .
18:   Present the elements of  $J_i$  to the online algorithm in this order.
19:   if  $\exists x \in J_i$  s.t.  $v_x \geq p_x$  then
20:     Find the first such  $x$  in the ordering of  $J_i$ ,
     and insert  $x$  into  $A$ .
21:   end if
22: end for
```

a 2-approximation to the revenue of the optimal deterministic mechanism for matroid feasibility constraints, and a $(4p - 2)$ -approximation to the revenue of the optimal deterministic mechanism for feasibility constraints that are the intersection of p matroids.

Proof. Both \mathcal{M} and $\mathcal{M}^{\text{copies}}$ are posted-price (hence, truthful) mechanisms that always output a feasible allocation. To prove that the allocation is always feasible, one can argue by contradiction: if not, there must be a step in which the set A becomes infeasible through adding an element x . However, in both \mathcal{M} and $\mathcal{M}^{\text{copies}}$, we can see that the price p_x is infinite in that case, while bid b_x is greater than or equal to p_x , a contradiction.

The proof of the approximate revenue guarantee follows the outline given by equation (5.1) above. As explained earlier, the only two steps in that equation that do not follow from prior work

Algorithm 3: Mechanism $\mathcal{M}^{\text{copies}}$ for single-parameter domain $\mathcal{I}^{\text{copies}}$.

- 1: *// Set prices using adversary and with BOSP*
 - 2: Obtain bids b_x for all bidders $x \in \mathcal{U}$.
 - 3: Run Algorithm 2, using $v_x = b_x$ for all x ,
to obtain an ordering of \mathcal{U} .
 - 4: Set $w(x) = \phi_x(b_x)$ for all $x \in \mathcal{U}$.
 - 5: Present the pairs $(x, w(x))$ to the prophet inequality algorithm, in the order computed
above.
 - 6: Obtain thresholds $T(A, x)$ from the prophet inequality algorithm.
 - 7: Set price $p_x = \phi_x^{-1}(T(A, x))$ for all $x \in \mathcal{U}$.
 - 8: *// Determine allocation and payments*
 - 9: Initialize $A = \emptyset$
 - 10: **for** $i = 1, \dots, n$ **do**
 - 11: **for all** $x \in J_i$ **do**
 - 12: **if** $b_x \geq p_x$ and $b_y < p_y$ for all $y \in J_i$ that precede x in the ordering **then**
 - 13: Add x to the set A .
 - 14: Charge price p_x to bidder x .
 - 15: **end if**
 - 16: **end for**
 - 17: **end for**
-

are the relations

$$R(\mathcal{M}) \geq R(\mathcal{M}^{\text{copies}}) \tag{5.3}$$

$$\Phi(\mathcal{M}^{\text{copies}}) \geq \frac{1}{\alpha} \Phi(\text{OPT}^{\text{copies}}). \tag{5.4}$$

To justify the second line, observe that the ‘‘adversary’’ (Algorithm 2) that computes the ordering of the bids is an online weight-adaptive adversary. This is because the adversary does not need to observe the values v_x ($x \in J_i$) in order to sort the elements of J_i in order of increasing $p_x + V(A \cup \{x\}, i - 1)$. Thus, the prophet inequality algorithm running on the input sequence specified by the adversary achieves an expected virtual surplus that is at least $\frac{1}{\alpha} \Phi(\text{OPT}^{\text{copies}})$. Furthermore, the set of elements selected by $\mathcal{M}^{\text{copies}}$ is exactly the same as the set of elements selected by the prophet inequality algorithm — the criterion $b_x \geq p_x$ is equivalent to the criterion $w(x) \geq T(A, x)$ because $w(x) = \phi_x(b_x)$, $T(A, x) = \phi_x(p_x)$, and ϕ is monotone increasing. This completes the proof of (5.4).

To prove (5.3) we use an argument that, in effect, justifies our claim that Algorithm 2 is a worst-

case adversary for mechanism $\mathcal{M}^{\text{copies}}$. Specifically, for each $i = 0, \dots, n$ and each feasible set $A \subseteq J_1 \cup \dots \cup J_i$, let $R(\mathcal{M}, A, i)$ and $R(\mathcal{M}^{\text{copies}}, A, i)$ denote the expected revenue that \mathcal{M} (respectively, $\mathcal{M}^{\text{copies}}$) obtains from selling items in $J_{i+1} \cup \dots \cup J_n$ conditional on having allocated set A while processing the bids in $J_1 \cup \dots \cup J_i$. (In evaluating the expected revenue of the two mechanisms, we assume that the bidders are presented to \mathcal{M} in the order $i = 1, \dots, n$, and that they are presented to $\mathcal{M}^{\text{copies}}$ in the order determined by the adversary, Algorithm 2.) We will prove, by downward induction on i , that

$$\forall i, A \quad R(\mathcal{M}, A, i) \geq R(\mathcal{M}^{\text{copies}}, A, i) = V(A, i)$$

and then (5.3) follows by specializing to $i = 0, A = \emptyset$. When $i = n$, we have $R(\mathcal{M}, A, i) = R(\mathcal{M}^{\text{copies}}, A, i) = V(A, i) = 0$ so the base case of the induction is trivial. The relation $R(\mathcal{M}^{\text{copies}}, A, i) = V(A, i)$ for $i < n$ is justified by the discussion preceding equation (5.2). To prove $R(\mathcal{M}, A, i) \geq R(\mathcal{M}^{\text{copies}}, A, i)$, suppose that both mechanisms have allocated set A while processing the bids in $J_1 \cup \dots \cup J_i$. Conditional on the set of $x \in J_{i+1}$ such that $v_x \geq p_x$ being equal to any specified set K , we will prove that \mathcal{M} obtains at least as much expected revenue as $\mathcal{M}^{\text{copies}}$ from selling the elements of $J_{i+1} \cup \dots \cup J_n$. If K is empty, then the two mechanisms will obtain expected revenue $R(\mathcal{M}, A, i + 1)$ and $R(\mathcal{M}^{\text{copies}}, A, i + 1)$, respectively, from elements of $J_{i+1} \cup \dots \cup J_n$, and the claim follows from the induction hypothesis. Otherwise, $\mathcal{M}^{\text{copies}}$ obtains expected revenue $\min\{p_x + V(A \cup \{x\}, i + 1) \mid x \in K\}$ while \mathcal{M} obtains expected revenue $p_y + R(\mathcal{M}, A \cup \{y\}, i + 1)$ where $y \in K$ is the element of K chosen by bidder $i + 1$ when presented with the menu of posted prices for the elements of J_{i+1} . The induction hypothesis implies

$$\begin{aligned} & p_y + R(\mathcal{M}, A \cup \{y\}, i + 1) \\ & \geq p_y + V(A \cup \{y\}, i + 1) \\ & \geq \min\{p_x + V(A \cup \{x\}, i + 1) \mid x \in K\}, \end{aligned}$$

and this completes the proof. □

Chapter 6

Simple Thresholding Rules

Most existing prophet inequalities for the rank-one case (including ours) are based on especially simple algorithms: they compute a single threshold T , and accept any element whose weight exceeds T . We'll call such an algorithm a *simple thresholding rule*. In this section, we show that the algorithm of [8] for the matroid secretary problem implies the existence of a constant-competitive simple thresholding rule for the BOSP when \mathcal{M} is uniformly dense¹ and each F_x is identical. However, we also show that even if we restrict \mathcal{M} to be uniformly dense, that no simple thresholding rule can achieve a constant competitive ratio when the F_x are non-identical. It is worth noting the algorithm of [8] combined with the reduction of [22] also implies a constant competitive algorithm for all matroids when each F_x is identical, although it is no longer a simple thresholding rule.

The problem solved in [8] is called the AO-RA-MK version of the matroid secretary problem. That is, a matroid $\mathcal{M} = (\mathcal{U}, \mathcal{I})$ is given as input. An adversary chooses a fixed ordering of the elements of \mathcal{U} , x_1, \dots, x_n and n non-negative weights. Then, the weights are randomly permuted into w_1, \dots, w_n . One at a time, the pair (x_i, w_i) is revealed to the algorithm, which must decide to accept or reject x_i , and must maintain that the set of accepted elements is always in \mathcal{I} . Like in the BOSP, the goal is to have the expected weight of the accepted set be large relative to the weight of the maximum weight basis. The following theorem is shown in [8]:

¹A matroid is uniformly dense if $\mathcal{U} \in \operatorname{argmax}_{S \subseteq \mathcal{U}} \frac{|S|}{\operatorname{rank}(S)}$.

Theorem 2. ([8]) *A 40-competitive solution to the AO-RA-MK version of the matroid secretary problem exists when \mathcal{M} is uniformly dense. The algorithm observes the first half of the revealed weights without accepting anything, sets a single threshold T equal to the $(\lfloor \text{rank}(\mathcal{M})/4 \rfloor + 1)^{\text{st}}$ largest observed weight, and accepts every remaining element whose weight exceeds T (without creating a dependency in \mathcal{M}).*

Corollary 1. *A 20-competitive simple thresholding rule exists for the BOSP when each F_x is identical, and \mathcal{M} is uniformly dense.*

Proof. We claim that the following simple thresholding rule gets a competitive ratio of 20 for the BOSP: Take n independent samples from F_x (which is the same for all x) and call them w'_1, \dots, w'_n . Let T denote the $(\lfloor \text{rank}(\mathcal{M})/4 \rfloor + 1)^{\text{st}}$ largest weight and use threshold T . Clearly, if this method for choosing T achieves a competitive ratio of 20, then there exists a deterministic choice of T that achieves a competitive ratio of 20 as well.

Because each F_x is identical, we may sample each w_i and w'_i in the following manner: First, take $2n$ independent samples from F_x and call the collection of weights \vec{w} . Then, pick a permutation uniformly at random and permute the weights accordingly (i.e. weights may be swapped between the samples used to set T as well as the online samples). Fixing any collection of weights \vec{w} and taking expectations with respect to the random permutation of the weights, if $V_P(\vec{w})$ denotes the expected value of the prophet and $V_G(\vec{w})$ denotes the expected value of the gambler using the simple thresholding rule, we will show that $V_G(\vec{w}) \geq V_P(\vec{w})/20$. We will prove this by showing that this is exactly an execution of the algorithm from [8] when the weights are \vec{w} .

Let \mathcal{M}' be a copy of \mathcal{M} on a disjoint ground set \mathcal{U}' , and let \mathcal{M}'' be the direct sum $\mathcal{M} \cup \mathcal{M}'$. Then because \mathcal{M} is uniformly dense, so is \mathcal{M}'' . Consider any fixed ordering that reveals every element of \mathcal{U}' before revealing any element of \mathcal{U} . If we run the algorithm of [8], it is clear that we will only accept elements of \mathcal{U} , as their algorithm does not accept anything in the first half of the input. It is also clear that their algorithm will choose T in the same manner as our simple thresholding rule. Therefore, the expected weight achieved by their algorithm in this setting is exactly $V_G(\vec{w})$. Finally, it is clear that the expected weight of the maximum-weight basis of \mathcal{M}'' is exactly $2V_P(\vec{w})$. Because their algorithm achieves a competitive ratio of 40, we must have $V_G(\vec{w}) \geq V_P(\vec{w})/20$. \square

Proposition 6. *There is no constant-competitive*

simple thresholding rule for the BOSP if the F_x are non-identical, even if \mathcal{M} is uniformly dense.

Proof. Let \mathcal{M} be the rank-one matroid on two elements, and let \mathcal{M}' be the partition matroid with $n = \sum_{i=1}^k k^{i-1}$ disjoint copies of \mathcal{M} . Then \mathcal{M}' is uniformly dense. For $i = 0, \dots, k$, define F_i to be the distribution that samples k^i with probability 1. For all $1 \leq i \leq k$, pick k^{k-i} copies of \mathcal{M} and let the distribution for one element be F_{i-1} , and the other be F_i . Order the elements in increasing order of weight.

It is clear that the prophet can obtain $\sum_i k^{k-i} k^i = k^{k+1}$ by accepting the heavier element from every partition. It is also clear that whenever we have $k^i \leq T < k^{i+1}$ that the value obtained by the gambler using threshold T is $k^k + \sum_{j>i} k^{k-j} k^{j-1} \leq 2k^k$. So no simple thresholding rule can obtain a competitive ratio better than $k/2$. □

Bibliography

- [1] Saeed Alaei. Bayesian combinatorial auctions: Expanding single buyer mechanisms to many buyers. In *Proc. 52nd IEEE Symp. on Foundations of Computer Science*, pages 512–521, 2011.
- [2] Susanne Albers, Alberto Marchetti-Spaccamela, Yossi Matias, Sotiris E. Nikolettseas, and Wolfgang Thomas, editors. *Automata, Languages and Programming, 36th International Colloquium, ICALP 2009, Rhodes, Greece, July 5-12, 2009, Proceedings, Part II*, volume 5556 of *Lecture Notes in Computer Science*. Springer, 2009.
- [3] Moshe Babaioff, Nicole Immorlica, and Robert Kleinberg. Matroids, secretary problems, and online mechanisms. In *Proc. 18th ACM Symp. on Discrete Algorithms*, pages 434–443, 2007.
- [4] Sourav Chakraborty and Oded Lachish. Improved competitive ratio for the matroid secretary problem. In *SODA12*. to appear.
- [5] Shuchi Chawla, Jason D. Hartline, David L. Malec, and Balasubramanian Sivan. Multi-parameter mechanism design and sequential posted pricing. In *Proc. 41st ACM Symp. on Theory of Computing*, pages 311–320, 2010.
- [6] Brian C. Dean, Michel X. Goemans, and Jan Vondrák. Approximating the stochastic knapsack problem: The benefit of adaptivity. In *FOCS*, pages 208–217. IEEE Computer Society, 2004.
- [7] Nediálko B. Dimitrov and C. Greg Plaxton. Competitive weighted matching in transversal matroids. In Luca Aceto, Ivan Damgård, Leslie Ann Goldberg, Magnús M. Halldórsson,

- Anna Ingólfssdóttir, and Igor Walukiewicz, editors, *ICALP (1)*, volume 5125 of *Lecture Notes in Computer Science*, pages 397–408. Springer, 2008.
- [8] Shayan Oveis Gharan and Jan Vondrak. On variants of the matroid secretary problem. In *Proc. 19th Annual European Symposium on Algorithms*, pages 225–346, 2011.
- [9] Ashish Goel, Sanjeev Khanna, and Brad Null. The ratio index for budgeted learning, with applications. In Claire Mathieu, editor, *SODA*, pages 18–27. SIAM, 2009.
- [10] Sudipto Guha and Kamesh Munagala. Multi-armed bandits with metric switching costs. In Albers et al. [2], pages 496–507.
- [11] Sudipto Guha, Kamesh Munagala, and Peng Shi. Approximation algorithms for restless bandit problems. *J. ACM*, 58(1):3, 2010.
- [12] MohammadTaghi Hajiaghayi, Robert Kleinberg, and Tuomas W. Sandholm. Automated mechanism design and prophet inequalities. In *Proc. 22nd AAAI Conference on Artificial Intelligence*, pages 58–65, 2007.
- [13] D. P. Kennedy. Optimal stopping of independent random variables and maximization prophets. *Ann. Prob.*, 13:566–571, 1985.
- [14] D. P. Kennedy. Prophet-type inequalities for multi-choice optimal stopping. *Stoch. Proc. Appl.*, 24:77–88, 1987.
- [15] R. P. Kertz. Comparison of optimal value and constrained maxima expectations for independent random variables. *Adv. Appl. Prob.*, 18:311–340, 1986.
- [16] Nitish Korula and Martin Pál. Algorithms for secretary problems on graphs and hypergraphs. In Albers et al. [2], pages 508–520.
- [17] Ulrich Krengel and Louis Sucheston. Semiamarts and finite values. *Bull. Amer. Math. Soc.*, 83:745–747, 1977.

- [18] Ulrich Krengel and Louis Sucheston. On semiamarts, amarts, and processes with finite value. *Adv. in Prob. Related Topics*, 4:197–266, 1978.
- [19] Roger B. Myerson. Optimal Auction Design. *Mathematics of Operations Research*, 6:58–73, 1981.
- [20] Ester Samuel-Cahn. Comparison of threshold stop rules and maximum for independent non-negative random variables. *Annals of Probability*, 12(4):1213–1216, 1984.
- [21] Alexander Schrijver. *Combinatorial Optimization*, volume B. Springer, 2003.
- [22] José A. Soto. Matroid secretary problem in the random assignment model. In *SODA11*, pages 1275–1284, 2011.