

# Extending and Characterizing Quantum Magic Games

by

Alex Arkhipov

Submitted to the Department of Electrical Engineering and Computer Science

in partial fulfillment of the requirements for the degree of

Master of Science in Computer Science and Engineering

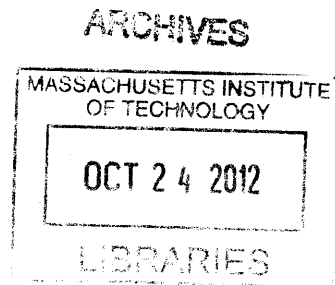
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## Abstract

The Mermin-Peres magic square game is a cooperative two-player nonlocal game in which shared quantum entanglement allows the players to win with certainty, while players limited to classical operations cannot do so, a phenomenon dubbed “quantum pseudo-telepathy”. The game has a referee separately ask each player to color a subset of a  $3 \times 3$  grid. The referee checks that their colorings satisfy certain parity constraints that can’t all be simultaneously realized.

We define a generalization of these games to be played on an arbitrary arrangement of intersecting sets of elements. We characterize exactly which of these games exhibit quantum pseudo-telepathy, and give quantum winning strategies for those that do. In doing so, we show that it suffices for the players to share three Bell pairs of entanglement even for games on arbitrarily larger arrangements. Moreover, it suffices for Alice and Bob to use measurements from the three-qubit Pauli group. The proof technique uses a novel connection of Mermin-style games to graph planarity.

Thesis Supervisor: Scott Aaronson  
Title: Associate Professor

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## I. INTRODUCTION

Our goal is to generalize the Mermin-Peres magic square and magic pentagram game [2, 3, 5] to be played on general configurations of points and lines that we will call *arrangements*. We characterize exactly which of these games allow two quantum players to win with certainty while two classical players cannot do so.

We will begin by introducing the Mermin magic square game and related prior work. Afterwards, we define more general terms for talking about magic games that will let us extend the proofs for the magic square games to a more general class of games. Our main result is to characterize which of these magic games are winnable by a quantum strategy, and then to construct the winning strategy. The construction will imply a bound on the resources needed to win such a game.

### A. Quantum Nonlocal Games

The Mermin-Peres magic-square game and the generalizations we will study are examples of *nonlocal games*. See [1] for an overview including formal definitions. Nonlocal games are multiplayer games played by a referee and some number of cooperating players. We will look at only two-player examples. The game protocol is as follows: the referee randomly selects questions to ask each player, the players respond to the referee independently without communicating, and the referee decides whether the players win or lose based on the players' answers.

For some nonlocal games, players sharing quantum entanglement can win with higher probability than is possible in the classical world. Examples include the CHSH game and the Kochen-Specker Game [1]. Such results allow empirical demonstrations of the power of quantum operations to correlate outcomes of causally separated events in a classically impossible way, thereby violating locality. Quantum nonlocal games demonstrate the power of quantum operations in a context without the quantitative resource bounds that commonly appear in quantum information and quantum computing.

## B. Mermin-Peres Magic Square Game

The Mermin-Peres magic square game [2–4] is a two-player nonlocal game in which shared quantum entanglement allows the players to win with certainty, while players limited to classical operations cannot do so, a phenomenon dubbed “quantum pseudo-telepathy”.

**Definition 1.** The **Mermin-Peres magic square game** is a two-player cooperative game played by the protocol described below. The players Alice and Bob may agree on a prior strategy in advance, but cannot communicate once the game starts.

1. The referee picks a random row  $r \in \{1, 2, 3\}$  and a random column  $c \in \{1, 2, 3\}$
2. The referee sends  $r$  to Alice and  $c$  to Bob.
3. Alice colors each of three cells in row  $r$  in a  $3 \times 3$  grid either red or green, and sends this coloring to the referee.
4. Bob colors each of three cells in column  $c$  in a  $3 \times 3$  grid either red or green, and sends this coloring to the referee.
5. The referee checks that Alice has colored red an even number of cells.
6. The referee checks that Bob has colored red an odd number of cells.
7. The referee checks that Alice and Bob have assigned the same color to the cell in row  $r$  and column  $c$ .
8. If all these checks succeed, then Alice and Bob win the game; otherwise they lose.

**Theorem 2.** *There is a quantum strategy by which Alice and Bob win the magic square game with certainty, but no such classical strategy.*

We will not prove this here and instead wait to prove the result for a more general class of games. A proof can be found in [4]. The key to winning the game comes from a construction Mermin gave to prove a version of the Kochen-Specker Theorem[2], which has the following properties.

**Theorem 3.** *There is a labeling of each cell of the  $3 \times 3$  squares with a quantum observables such that*

- The eigenvalues of each observable are all  $+1$  or  $-1$ .
- The observables in each row commute and multiply to  $+I$ .
- The observables in each row commute and multiply to  $-I$ .

Moreover, there is no way to do this “classically” using observables of the form  $\pm I$ .

### C. Generalizations of the Magic Square Construction

Mermin constructed another example of a quantum telepathy game called the magic pentagram game [5]. This game is played on an arrangement of ten points joined by five lines of four points each, arranged like a five-sided star. A recent result [8] looked at a different arrangement, a subset of the Fano plane, and proved that there is no such quantum winning strategy for this game. These examples suggest the generalized notion of magic games that is explored in this paper.

Prior research into generalizing Mermin’s constructions focused on understanding the observables used to win the magic square and magic pentagram games, and finding all possible sets of such observables. The results of [6, 7] interpret Mermin’s construction in terms of geometrical structures on finite rings. Our work, however, seeks to determine when there exist winning strategies for generalized arrangement rather than to classify all winning strategies for existing arrangements.

## II. PRELIMINARIES AND DEFINITIONS

### A. Arrangements and Realizations

The magic square and pentagram are examples of configurations on which Mermin-style games may be played. We will call these *arrangements*.

**Definition 4.** An **arrangement**  $A = (V, E)$  is a finite connected hypergraph with vertex set  $V$  and hyperedge set  $E$ , where a hyperedge is a nonempty subset of  $V$ , such that each vertex lies in exactly two hyperedges (*connected* means the hypergraph can’t be split into two smaller disjoint hypergraphs). A **signed arrangement**  $A = (V, E, l)$  also contains a labelling  $l: E \rightarrow \{+1, -1\}$  of each hyperedge in  $E$  with a sign of  $+1$  or  $-1$ .

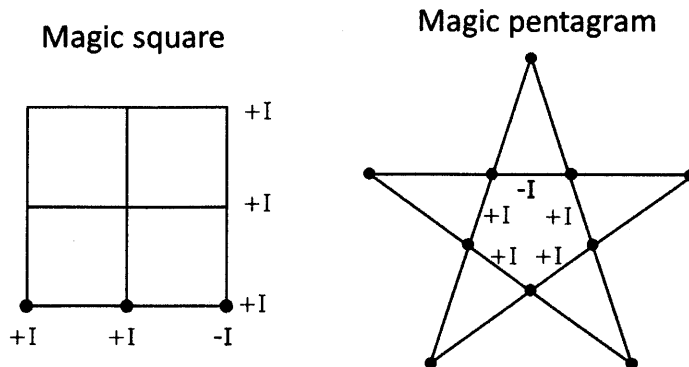


Figure II.1:

We'll often represent an arrangement by drawing each vertex as a *point* and each hyperedge as a line, arc, or circle that passes through the points it contains, which we'll generally call a *line*. In a signed arrangement, each line is labelled by  $+1$  or  $-1$ . Note that where the points are drawn and how they are ordered on a line is immaterial; it only matters which points share a line.

**Example 5.** Two well-studied arrangements, the magic square and magic pentagram are shown in Figure II.1 with signs. Note that while in both every hyperedge contains an equal number of vertices, this need not be the case in general.

**Definition 6.** A **classical realization** of a signed arrangement  $A = (V, E, l)$  is a labelling  $c : V \rightarrow \{+1, -1\}$  of the vertices so that the product of the labels on vertices within any hyperedge equals the label of that hyperedge:

$$\prod_{u \in e} c(u) = l(e) \text{ for each } e \in E$$

**Definition 7.** A **quantum realization** of a labelled arrangement  $A = (V, E, l)$  is a labelling  $c : V \rightarrow GL(\mathcal{H})$  of the vertices with observables on a fixed finite-dimensional Hilbert-space  $\mathcal{H}$  such that:

- The observable  $M$  assigned to any vertex is Hermitian and squares to the identity ( $M^2 = I$ ), or equivalently, each observables orthogonally diagonalizes with eigenvalues of  $+1$  and  $-1$ .
- For each hyperedge, the observables assigned to its vertices pairwise commute.

- For each hyperedge, the product of the observables assigned to its vertices equals either the identity in  $\mathcal{H}$  or its negation according to the sign of that hyperedge

$$\prod_{u \in e} c(u) = l(e) I \text{ for each } e \in E$$

We'll say that an arrangement is **classically realizable** if it has a classical realization and likewise **quantum realizable** if it has a quantum realization. We note that a classical realization is simply a quantum realization in which  $\mathcal{H} = \mathbb{R}$ ; therefore, every classically realizable arrangement is quantumly realizable. An example of a quantum realization is given in Figure II.2.

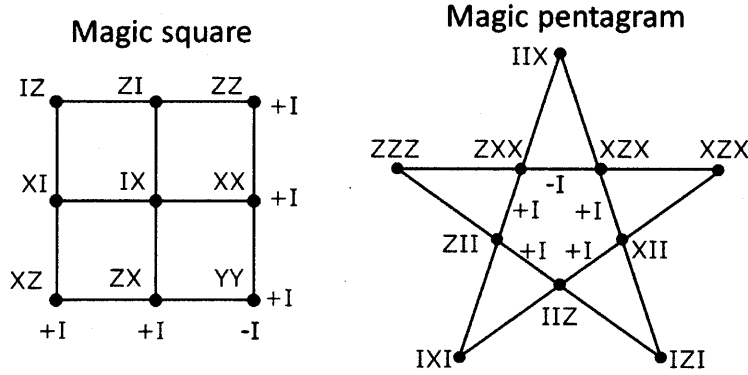


Figure II.2: Quantum realization of the magic square arrangement and magic pentagram arrangements. The strings of symbols  $I, X, Y, Z$  represent tensor product of Pauli matrices.

In order to take advantage of the extra freedom in constructing a quantum realization, one must take advantage of noncommuting observables, since using commuting observables gives no extra power beyond classical realizability.

**Proposition 8.** *If a signed arrangement is quantumly realizable with observables that all mutually commute, then it is classically realizable.*

*Proof.* Commuting observables are mutually diagonalizable. Replacing each observable by its diagonalization via conjugation gives a new quantum realization where the observables are diagonal matrices with diagonal entries of  $\pm 1$ . For any basis index  $i$ , the  $(i, i)$  entries of each diagonalized observable give a classical realization of the arrangement.  $\square$



## B. Sign Parities

It turns out that whether an signed arrangement is classically realizable or quantumly realizable depend on the hyperedge signs only in a limited manner, as one can adjust the signs of the realization operators to achieve different signs hyperedges signs. The only salient feature of the hyperedge signs is whether the number of  $-1$  labels is odd or even.

**Definition 9.** The **parity**  $p(l)$  of the signing  $l$  of an arrangement  $A = (V, E, l)$  is

$$p(l) = \prod_{e \in E} l(e),$$

which is  $-1$  if there's an odd number of  $-1$  labels, and  $+1$  if there's an even number.

**Proposition 10.** *The classical realizability of a signed arrangement  $A = (V, E, l)$  depends on  $l$  only via its parity  $p(l)$ . In other words,  $A' = (V, E, l')$  has the same classical realizability as  $A' = (V, E, l')$  if  $p(l') = p(l)$ .*

*Proof.* Suppose  $A$  is classically realizable, and let  $c$  be its classical realization. We will construct a corresponding classical realization of  $c'$ .

First, we show that the result holds when  $l'$  is achieved by flipping the signs  $l$  assigns to two hyperedges,  $a$  and  $b$ . We note that flipping the label of a vertex flips the parity products of the two hyperedges containing it. Since an arrangement is finite and connected, there must be a path  $e_0, e_1, \dots, e_n$  of distinct edges starting at  $e_0 = a$  and ending at  $e_n = b$  such that any pair of hyperedges  $e_i, e_{i+1}$  adjacent in the sequence intersects at a vertex  $v_i$ . Then, negating the label of every vertex  $v_i$  in the path

$$l'(v) = \begin{cases} -l(v), & \text{if } v \in \{v_0, \dots, v_{n+1}\} \\ l(v), & \text{otherwise} \end{cases}$$

achieves the desired result: The product of the labels of the vertices on edge  $e_i$  is unaffected, as two vertices within it have flipped labels, except the edges  $e_0 = a$  and ending at  $e_n = b$  at the ends of the chain.

By repeatedly changing the realization to flip pairs of signs in  $l$ , one can go from any labelling to any other labelling of equal parity.  $\square$

**Proposition 11.** *A classical arrangement  $A = (V, E, l)$  is realizable if and only if the parity  $p(l)$  is  $+1$ .*

*Proof.* The even-parity signing where each hyperedge has sign +1 is realized by assigning +1 to each vertex. Then, by Proposition 10, any even-parity signing is classically realizable.

No odd-parity signing is realizable, since each vertex lies in two hyperedges, so the product of the vertex labels of each edge will contain each vertex label twice.

$$\prod_{e \in E} l(e) = \prod_{e \in E} \prod_{v \in e} c(e) = \prod_{v \in V} c(e)^2 = \prod_{v \in V} 1 = 1$$

□

A similar result to Proposition 10 follows for quantum realizability.

**Proposition 12.** *The quantum realizability of a signed arrangement  $A = (V, E, l)$  depends on  $l$  only via its parity  $p(l)$ .*

*Proof.* The proof is the same as that of Proposition 10, except we also check that negating the quantum observables does not change them having order two or mutually commuting within each hyperedge. □

In light of the results of Propositions 10 and Proposition 12, we should think of quantum realizability as a property of an unsigned arrangement.

**Definition 13.** An arrangement is **magic** if it has an odd-parity signing that is quantumly realizable.

So, a signed arrangement is magic if its underlying arrangement is magic and the signing has odd parity. Note that by Proposition 11, any magic signed arrangement is not classically realizable, and therefore represents a gap in what's classically possible and what's quantumly possible.

**Theorem 14.** *[Mermin, Peres] The magic square and magic pentagram pentagram are magic arrangements.*

*Proof.* Example quantum realizations for the magic square and pentagram are pictured in Figure II.2, using odd-parity signings for both of them. The magic square uses Hilbert space  $(\mathbb{C}^2)^{\otimes 2}$  and measurement operators from the two-qubit Pauli group, and the magic pentagram does likewise with  $(\mathbb{C}^2)^{\otimes 3}$  and the three-qubit Pauli group. □

### C. Parity telepathy games

We extend the Mermin magic square game to be played on an arbitrary arrangement.

**Definition 15.** The **parity telepathy game** on a signed arrangement  $A = (V, E, l)$  is a game played by two cooperative players (call them Alice and Bob) and a referee. Alice and Bob may agree on a prior strategy but cannot communicate once the game starts. They both know the signed arrangement  $A$  that the game takes place on.

1. The referee picks a random vertex  $v$  in  $V$  and one of the two hyperedges containing it at random.
2. The referee sends  $v$  to Alice and  $e$  to Bob.
3. Alice colors  $v$  with one of two “colors”,  $+1$  and  $-1$ , and sends the color  $f(v)$  to the referee.
4. Bob colors each vertex of  $e$  with one of two “colors”,  $+1$  and  $-1$ , and send this coloring  $c: e \rightarrow \{+1, -1\}$  to the referee.
5. The referee confirms that Alice or Bob have given valid colorings, and that each label is either  $+1$  or  $-1$ .
6. The referee checks that the parity of Bob’s coloring matches the sign of the edge in the arrangement, that  $\prod_{u \in e} c(u) = l(u)$ .
7. The referee checks that Bob’s coloring is consistent with Alice’s coloring of  $v$ , meaning that  $c(v) = f(v)$ .
8. If both the parity and consistency checks succeed, then Alice and Bob have won the game, otherwise they have lost.

Note that this protocol differs from the one in the magic square where both Alice and Bob colored a hyperedge, with Alice coloring rows and Bob coloring columns. We use this modification, which is also used for the magic pentagram game [5], because it generalizes to arrangements that do not share the magic square game’s property that its lines can be divided into two sets (rows and columns) that do not intersect within each set.

**Proposition 16.** *When limited to classical strategies, Alice and Bob win with certainty in the parity telepathy game on an arrangement only if it is classically realizable.*

*Proof.* First, we consider only deterministic strategies. Let  $f(v)$  be the color Alice assigns to vertex  $v$ . In order to always pass the consistency check, Bob must color each vertex as per  $f(v)$ . Then, Bob passing every parity check is equivalent to  $f(v)$  being a classical realization of the arrangement.

Since Alice and Bob cannot communicate after the protocol starts, we may assume that any randomized strategy has Alice and Bob perform all coin flips before the game. After the flips, the randomized winning strategy would become a deterministic winning strategy.  $\square$

Quantum realizations give rise to quantum winning strategies, provided an awkward technical caveat.

**Theorem 17.** *On any magic signed arrangement that has a quantum realization in which all the operators have all real eigenvectors, if Alice and Bob may share quantum entanglement in advance and perform quantum operations, then they have a strategy that wins with certainty.*

*Proof.* We begin by stating the winning strategy. Let  $c$  be the quantum realization of the arrangement on the finite-dimensional Hilbert-space  $\mathcal{H}$  and let  $n = \dim \mathcal{H}$ . Let  $|1\rangle, \dots, |n\rangle$  be a basis for  $H$ . Alice and Bob share between them the maximally entangled state:

$$\Psi_{AB} = \sum_{i=1}^n |i\rangle |i\rangle$$

Then, to obtain the coloring of a vertex  $v$ , Alice or Bob performs the indicated measurement  $c(v)$  on their half of  $\Psi_{AB}$  to obtain  $+1$  or  $-1$ . Note that the order of Bob's measurements doesn't matter, since all the measurements he must make commute.

We first check that this strategy always passes the parity check. Within any hyperedge  $e$ , the assigned measurements commute, and therefore are mutually diagonalize. In the diagonal basis, these measurements are basis measurements with each basis element resulting in a  $+1$  or  $-1$  as labelled. Since the product of the measurements is  $l(e)I$ , then the product of values corresponding to each basis element is  $l(e)$ . So, the measured values satisfy the indicated parity constraint.

Next, we check consistency. We use the well-known fact that if two parties each rotate their halves of a Bell state by an arbitrary real orthogonal matrix, the Bell state remains

fixed. When Alice and Bob perform equal measurements that have real sets of eigenvectors, it is equivalent to both performing an orthogonal followed by a standard basis measurement that determines the outcome. The rotations leaves the Bell pair invariant, after which the standard basis measurements produce equal outcomes for Alice and Bob.  $\square$

Note that the Pauli operators in the quantum realization of the magic square and pentagram that we provided in Figure II.2 satisfy the real eigenvectors property and therefore suffice to win those magic games. This will be the case for all quantum realizations that we give. Also, note that the result of Theorem 17 does not imply that a nonmagic game has no winning quantum strategy – even if no quantum realization exists of the corresponding arrangement, this does not rule out a different quantum strategy.

#### D. Intersection graphs

We have define realized arrangements to have label vertices with measurements, with hyperedges encoding constraints on these measurements. For our main result, it will be convenient to switch to a dual representation in which measurements label edges and vertices encode constraints. In our drawings of arrangements, this corresponds to interchanging the roles of points and lines.

**Definition 18.** The **intersection graph** to an arrangement  $A = (V, E)$  is the undirected graph  $(V', E')$  where  $V' = E$ , and there's an edge between  $e_1, e_2 \in V'$  for each vertex in the intersection  $e_1 \cap e_2$ .

Signed intersection graph also include a sign  $\pm 1$  on each vertex. Applying a quantum realization of an arrangement to its intersection graph, we obtain a labelling of the edges of the intersection graph such that the labels on edges sharing a vertex commute and multiply to the identity times the vertex's sign.

An intersection graph is equivalent to the hypergraph dual of  $A$ , obtained by interchanging the roles of vertices and hyperedges. In other words, we think of each vertex as the “hyperedge” that is the set of all the hyperedges that contain it. Since in an arrangement each vertex lies on exactly two hyperedges, this dual hypergraph contains only two-element hyperedges and is simply an undirected graph (technically a multigraph).

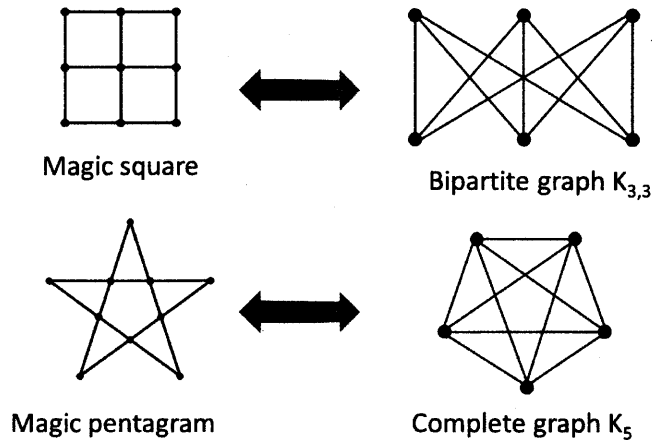


Figure II.3: The magic square has intersection graph  $K_{3,3}$  and the magic pentagram has intersection graph  $K_5$ .

**Example 19.** The intersection graphs of the magic square is the complete bipartite graph on six vertices  $K_{3,3}$ , and the intersection graph of the magic pentagram is the complete graph on five vertices  $K_5$  (see Figure II.3).

### III. MAIN RESULT

We will prove our main result. Recall that a graph is planar if it can be embedded into the Euclidian plane so that no two edges intersect.

**Theorem 20.** *An arrangement is magic if and only if its intersection graph is not planar.*

For ease of terminology, we'll call an intersection graph *magic* if its associated arrangement is magic. We will also talk about quantum realizations of intersection graphs; these can be produced by taking the operator vertex labels of an arrangement and transferring them to the corresponding edges of the intersection graph.

We'll prove the two directions of Theorem 20 separately.

### A. Planar implies not magic

**Theorem 21.** *If the intersection graph of an arrangement is planar, then that arrangement is not magic.*

*Proof.* We show that any realization of a signed version of this arrangement must have even parity, and therefore is not magic. Our strategy will be to collapse the algebraic constraints on the vertex quantum operators by repeatedly cancelling variable terms until we reach a contradiction.

Recall that a quantum realization on a signed intersection graph labels each edge with a measurement operator such that edges sharing a vertex commute and multiply to the identity times a sign equal to that vertex's label. Consider a planar embedding of this quantum realization of this signed intersection graph. To contract an edge, delete that edge and merge the two endpoint vertices into one, with edges that pointed to one of these two vertices now pointing to the new vertex, and set the new vertex's sign to be the product of the signs of the two merged vertices. Merging may cause self-loops and multiple parallel edges.

We observe that contraction preserves the following properties:

- The product of the labels of edges around any vertex, in cyclic order (say, without loss of generality, counterclockwise), equals identity times the vertex's sign (a self-loop on a vertex will have its edge label appear twice in the product.) We prove this here. This is only nontrivial to check for the newly formed vertex. First, note that if label the operators  $M_1, \dots, M_n$  going in a circle from any starting point, if  $M_1 M_2 \dots M_n = I$ , then for any starting point,  $M_k M_{k+1} \dots M_n M_1 M_2 M_{k-1} = I$ , since

$$\begin{aligned} M_k M_{k+1} \dots M_n M_1 M_2 M_{k-1} &= (M_{k-1} \dots M_2 M_1) (M_1 M_2 \dots M_n) (M_1 \dots M_{k-1}) \\ &= (M_{k-1} \dots M_2 M_1) I (M_1 \dots M_{k-1}) \\ &= I \end{aligned}$$

A similar result follows with  $-I$  in place of  $I$ . Now, consider an edge labelled by an operator  $X$ , and let the labels around its endpoint vertices be  $M_1, M_2, \dots, M_m, X$  and  $X, N_1, N_2, \dots, N_n$  respectively going counterclockwise, and let the signs of the two

vertices be  $\alpha_M$  and  $\alpha_N$ . Then,

$$M_1 \dots M_n X = \alpha_M I$$

$$X N_1 \dots N_n = \alpha_N I$$

Multiplying these gives

$$M_1 \dots M_n N_1 \dots N_n = \alpha_M \alpha_N I$$

The left hand side is the cyclically ordered product of edge labels around the newly formed vertex, and the right hand side is the identity with the sign of the new vertex, so the invariant remains. Note that it may no longer be true that operators whose edges share a vertex commute.

- The graph embedding remains planar.
- The sign parity (product of all the vertex labels) remains the same.

Since each contraction reduces the number of vertices by 1, contracting any sequence of edges eventually produces a graph with a single vertex. The sign of this vertex equals the product of the labels of all vertices of the original intersection graph, and therefore equals its parity. We will show that this parity is  $+1$ .

It is easy to check that removing any self-loop that does not enclose anything in the planar embedding also preserves the stated invariants; such a self-loop contributes its operator twice in sequence to a cyclic product, which cancels. Since there's always an innermost self-loop, we may repeatedly remove such self-loops until none remain. So, the sign of this vertex must equal the empty product, or  $+I$ . But, since the sign parity has been preserved throughout the process, this implies that the original arrangement has sign parity  $+1$  and therefore has a classical realization and is not magic.  $\square$

## B. Nonplanar implies magic

In this section, we prove the forward direction of the main result (Theorem 20).

**Theorem 22.** *If the intersection graph of an arrangement is nonplanar, then the arrangement is magic.*



This proof will come in two pieces. First, we'll show that if an intersection graph contains a magic intersection graph as a topological minor, then it is magic. (Recall that we're saying an intersection graph is magic as a shorthand for its associated arrangement being magic.) We then use the well-known theorem of Pontyagin and Kuratowski that any nonplanar graph contains either the complete graph  $K_5$  or the bipartite complete graph  $K_{3,3}$  as a topological minor, and that both of these intersection graphs are magic.

**Definition 23.** A graph  $H$  is a **topological minor** of  $G$  if  $G$  has a subgraph that is isomorphic to a subdivision of  $H$ , where a subdivision is obtained by replacing each edge by a simple path of one or more edges.

Note that "subgraph" as used in the above definition allows both deleting edges and deleting vertices. By considering the isomorphism explicitly, we obtain the following equivalent definition.

**Definition 24.** A graph  $H$  is a topological minor of  $G$  if there is an embedding of  $H$  in  $G$  that consists of an injective map  $\phi$  that takes each vertex  $v$  of  $H$  to a vertex  $\phi(v)$  of  $G$ , and a map from each edge  $(u, v)$  of  $H$  to a simple path from  $\phi(u)$  to  $\phi(v)$  in  $G$ , such that these paths are disjoint except on their endpoints.

**Theorem 25.** *If an intersection graph  $H$  is a topological minor of an intersection graph  $G$ , then  $H$  being magic implies that  $G$  is magic.*

*Proof.* We will give a construction to turn a quantum realization of  $H$  into one of  $G$ . Choose arbitrarily some odd-parity signing of  $H$ , and let  $c$  be a quantum realization of the corresponding arrangement over some Hilbert space. We will use the topological minor inclusion map to assign a corresponding signing and quantum realization on  $G$  on the same Hilbert space, as follows:

1. For each vertex of  $H$ , label the corresponding vertex of  $G$  with the same sign. Label all other vertices of  $G$  as  $+1$ .
2. For each edge of  $H$ , label each edge of the corresponding path in  $G$  with the same quantum operator. Label the remaining edges of  $G$  as  $I$ .

We now show that this gives a quantum realization of  $G$ . We check each of the required properties of a quantum realization, as interpreted in the language of intersection graphs.

- Each measurement  $M$  assigned to  $G$  is either one in  $H$  or the identity, and therefore is Hermitian and has order 2.
- Each vertex of  $G$  corresponding to a vertex of  $H$  touches the same measurement operators on its edges plus copies of the identity. Therefore, these operators commute and have the same product as for the vertex in  $H$ , which is labelled with the same sign.
- Each vertex of  $G$  that lies on a path that is the image on a edge in  $H$  touches two edges labelled with the same operator from that edge in  $H$ , and possibly copies of the identity. These clearly commute and multiply to  $+I$ , this vertex's label.
- Each other vertex of  $G$  only touches edges labelled as  $I$ , which commute and have the correct product  $+I$ .

□

**Theorem 26.** (*Pontryagin-Kuratowski*) *A graph is nonplanar if and only if it contains  $K_5$  or  $K_{3,3}$  as a topological minor.*

Other statements of this theorem use the stronger notion of graph minors rather than topological minors, but in this case the notions are equivalent. Note that we will only be using one direction of this result, that every nonplanar graph contains  $K_5$  or  $K_{3,3}$ .

**Proposition 27.** *The intersection graphs  $K_5$  and  $K_{3,3}$  are both magic.*

*Proof.* The intersection graph  $K_5$  corresponds to the magic pentagram arrangement and  $K_{3,3}$  to the magic square arrangement, which we showed to be magic in Theorem 14. □

Combining Theorems 25 and 26 and Proposition 27 gives the result stated at the start of the section.

**Theorem 28.** *If the intersection graph of an arrangement is nonplanar, then the arrangement is magic.*

## IV. DISCUSSION

The construction of quantum realizations for magic games suggests that the magic square and magic pentagram are “universal” for magic games – their quantum realizations give quantum realizations for any magic arrangement. The fact the magic square uses only two-qubit Pauli measurements and the magic pentagram uses three-qubit Paulis (Theorem 14) therefore gives a bound on the measurements necessary to win a magic game.

**Corollary 29.** *Any magic arrangement has a quantum realization with operators taken from the three-qubit Pauli group.*

**Corollary 30.** *Any magic game can be won with certainty by players that share only three Bell pairs of entanglement, and only use measurements from the three-qubit Pauli group.*

## V. FUTURE WORK

We have defined the notion of a magic game based on whether it can be won by using quantum strategies based on quantum realizations of magic arrangements. We conjecture that no quantum strategy can win the quantum telepathy game on a nonmagic arrangement.

We have given one way to construct quantum realizations for magic arrangements based on the magic square and pentagram as excluded minors. We would like to understand the set of all possible quantum realizations, along the vein of [6, 7] for the magic square and pentagram. We conjecture that any quantum realization is isomorphic to one that uses Pauli operators.

Finally, we would like to extend our results to arrangements in which a point may lie on more than two hyperedges.

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