

# Control of Hidden Mode Hybrid Systems: Algorithm Termination

Rajeev Verma and Domitilla Del Vecchio

**Abstract**— We consider the problem of safety control in Hidden Mode Hybrid Systems (HMHS) that arises in the development of a semi-autonomous cooperative active safety system for collision avoidance at an intersection. We utilize the approach of constructing a new hybrid automaton whose discrete state is an estimate of the HMHS mode. A dynamic feedback map can then be designed that guarantees safety on the basis of the current mode estimate and the concept of the capture set. In this work, we relax the conditions for the termination of the algorithm that computes the capture set by constructing an abstraction of the new hybrid automaton. We present a relation to compute the capture set for the abstraction and show that this capture set is equal to the one for the new hybrid automaton.

## I. INTRODUCTION

The continuous advances of embedded computing and communication technologies are pushing most engineering systems toward increased levels of autonomy. One such example is vehicles that can drive autonomously or semi-autonomously interacting with drivers and other human-driven vehicles. These technologies fall under Intelligent Transportation System initiatives of government and industry consortia. The availability of vehicle-to-vehicle (V2V) and vehicle-to-infrastructure (V2I) wireless communication will bring these technologies one step closer to reality in the near future. For example, collision avoidance among multiple cars merging on an intersection is studied in [10]. The scheme employs a centralized control scheme that resides on the intersection and acts as a scheduler that assigns a safe time slot to each car for crossing the intersection.

While newer vehicles will be equipped with wireless radio to communicate and cooperate with other vehicles and the infrastructure, there will still be vehicles that will not be able to communicate. The control algorithms developed for guaranteeing safety must be able to operate in this *partially autonomous* real world scenario as long as road-side infrastructure (e.g., cameras, radar, and magnetic-induction loops) is employed to measure the approximate position of the non-communicating vehicles. This approach can be elegantly formulated as a safety control problem for hidden mode hybrid systems [20,21].

The safety control problem for hybrid systems has been extensively considered in the literature when both the

continuous and discrete state are available for measurement [14,16,18,19]. These measurements are required to compute a safe control input. In [1,15], hybrid systems whose continuous dynamics is linear time-invariant and discrete state switching is due to transition guards are considered. An over approximation of the reachable set is computed using simulation techniques over bounded time in [15] and by using zonotopes in [1]. In [17], a hybrid system is considered whose discrete state can switch due to discrete control, discrete disturbance and discrete human input. Hybrid reachability results are then utilized to create an invariance-preserving discrete event system abstraction of the so called hybrid human-automation system. The knowledge of discrete input and perfect state information is assumed.

A number of works have addressed the control problem for special classes of hybrid systems with imperfect state information [5,6,24]. In [24], a controller that relies on a state estimator is proposed for finite state systems. The results are then extended to control a class of rectangular hybrid automata with imperfect state information, which can be abstracted by a finite state system. In [5–7,11], computationally efficient state estimation and control algorithms were proposed for special classes of hybrid system with order-preserving dynamics. The problem of safety control for hidden mode hybrid systems has been addressed in [20,21]. A perfect state information control problem is obtained by constructing a new hybrid automaton, whose discrete state is an estimate of the HMHS mode and is known. This problem is solved by computing the capture set and the least restrictive control map for the new hybrid automaton. Sufficient conditions for the termination of the algorithm that computes the capture set are provided in [20,21]. It has also been shown that the solved perfect state information control problem is equivalent to the original problem with imperfect state information under suitable assumptions. The main contribution of this paper is to show that in the case where the termination conditions for the algorithm that computes the capture set are not satisfied, an abstraction of the new hybrid automaton can be constructed for which the algorithm is guaranteed to terminate and such that the fixed point gives the capture set for the new hybrid automaton.

This paper is organized as follows. We recall some results from [20,21] in Section II, the construction of the abstraction is shown in Section III and Section IV presents an application example.

R. Verma is with Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, USA {rajverma}@umich.edu

D. Del Vecchio is with the Department of Mechanical Engineering, MIT, Cambridge, USA dddv@mit.edu Supported in part by NSF CAREER Award Number CNS-0642719

## II. SAFETY CONTROL PROBLEM FOR HIDDEN MODE HYBRID SYSTEMS

In this section, we summarize the results on safety control of HMHS from [20,21]. We first present the general hybrid automaton model.

**Definition 1.** A *Hybrid Automaton with Uncontrolled Mode Transitions*  $H$  is a tuple  $H = (Q, X, U, D, \Sigma, R, f)$ , in which  $Q$  is the set of modes;  $X$  is the continuous state space;  $U$  is the continuous set of control inputs;  $D$  is the continuous set of disturbance inputs;  $\Sigma$  is the set of disturbance events that trigger transitions among modes;  $\epsilon \in \Sigma$  is the silent event, which correspond to no transition occurring;  $R : Q \times \Sigma \rightarrow Q$  is the mode update map and  $f : X \times Q \times U \times D \rightarrow X$  is the vector field, which is allowed to be piecewise continuous with its arguments.

For a hybrid automaton  $H$ , a hybrid time trajectory [16] is denoted by  $\mathcal{T} = \bigcup_{i=0}^N [\tau_i, \tau'_i]$  with  $\sigma(\tau'_i) \in \Sigma/\epsilon$  and  $\sigma(t) = \epsilon$  for  $t \in [\tau_i, \tau'_i)$  for all  $i$  such that  $\tau_i < \tau'_i$ . Since the last interval may be open or closed (if  $N < \infty$ , a “)”) parenthesis is used. We thus represent  $H$  by  $q(\tau_{i+1}) = R(q(\tau'_i), \sigma(\tau'_i))$ ,  $\sigma(\tau'_i) \in \Sigma/\epsilon$  and  $\dot{x}(t) = f(x(t), q(t), u(t), d(t))$ ,  $d(t) \in D$ ,  $\sigma(t) = \epsilon$ , where  $\tau_i$  for  $i \in \{0, \dots, N\}$  are the times at which a discrete transition takes place and satisfy  $\tau_i \leq \tau'_i = \tau_{i+1}$ , the value of  $q$  after the  $i^{\text{th}}$  transition is denoted by  $q(\tau_{i+1})$ ,  $q(t) := q(\sup_{\tau_i \leq t} \tau_i)$  for  $t \in \mathcal{T}$  and  $\sigma(t) = \epsilon$ ,  $x(0) = x_0 \in X$ , and  $q(\tau_0) = q_0 \in Q$ . The initial state  $x_0$  is known (the case where  $x_0$  is subject to uncertainty is considered in [11]). We assume without loss of generality that  $\tau_0 = 0$ . The continuous state remains the same after the discrete transition, i.e.,  $x(\tau_{i+1}) = x(\tau'_i)$  for all  $i$ . For input signals  $\mathbf{u} : \mathcal{T} \rightarrow U$ ,  $\mathbf{d} : \mathcal{T} \rightarrow D$ ,  $\sigma : \mathcal{T} \rightarrow \Sigma$ , we denote the continuous trajectory of the system by  $x(t) = \phi_x(t, (q_0, x_0), \mathbf{u}, \mathbf{d}, \sigma)$  for all  $t \geq 0$ , in which  $x(0) = \phi_x(0, (q_0, x_0), \mathbf{u}, \mathbf{d}, \sigma) := x_0$  and the discrete state trajectory by  $\phi_q(t, q_0, \sigma) = q(t)$  with  $q(0) = \phi_q(0, q_0, \sigma)$ . The set of reachable modes from any initial set of modes  $\bar{q} \subseteq Q$  is denoted  $\text{Rch}(\bar{q}) := \bigcup_{q_0 \in \bar{q}} \bigcup_{t \geq 0} \bigcup_{\sigma} \phi_q(t, q_0, \sigma)$ .

**Definition 2.** A *Hidden Mode Hybrid System* (HMHS) is a hybrid automaton with uncontrolled mode transitions in which the discrete state  $q(t)$  is not measured and  $q_0$  is only known to belong to a set  $\bar{q}_0 \subseteq Q$ .

Thus, the mode of a HMHS is not known, the only measured state is  $x(t)$  and its evolution is driven by hidden mode transitions. In the remainder of the paper, we denote a HMHS by  $H$ . Let  $\text{Bad} \subseteq X$  be a bad set of states. The control task is to keep the continuous state  $x(t)$  outside  $\text{Bad}$  for all time using all the available information. The available information at any time is the initial mode uncertainty, denoted  $\bar{q}_0 \subseteq Q$ , the measured signals  $x(t)$  and the control signal  $u(t)$ .

**Definition 3.** A *discrete state estimate* is a time-dependent set, denoted  $\hat{q}(t) \in \hat{Q} \subseteq 2^Q$ , with the properties that (i)

$q(t) \in \hat{q}(t)$  for all  $t \geq 0$ ; (ii) For  $t_2 \geq t_1$ , we have that  $\hat{q}(t_2) \subseteq \text{Rch}(\hat{q}(t_1))$ .

Define the new hybrid automaton  $\hat{H} = (\hat{Q}, X, U, D, Y, \hat{R}, f)$ , in which  $\hat{Q} \subseteq 2^Q$  is a new set of discrete states,  $Y$  is a set of discrete events,  $\epsilon \in Y$  is the silent event,  $\hat{R} : \hat{Q} \times Y \rightarrow \hat{Q}$  is a discrete state transition map. Let  $\hat{\mathcal{T}} = \bigcup_{i=0}^N [\hat{\tau}_i, \hat{\tau}'_i]$  be a hybrid time trajectory such that  $\hat{\tau}_0 = \tau_0$ ,  $y(\hat{\tau}'_i) \in Y/\epsilon$  and  $y(t) = \epsilon$  for  $t \in [\hat{\tau}_i, \hat{\tau}'_i)$  for all  $i$  such that  $\hat{\tau}_i < \hat{\tau}'_i$ . We represent  $\hat{H}$  by  $\hat{q}(\hat{\tau}_{i+1}) = \hat{R}(\hat{q}(\hat{\tau}'_i), y(\hat{\tau}'_i))$ ,  $y(\hat{\tau}'_i) \in Y/\epsilon$  and  $\hat{x}(t) \in f(\hat{x}(t), \hat{q}(t), u(t), d(t))$ ,  $d(t) \in D$ ,  $y(t) = \epsilon$ , where we have defined  $\hat{q}(t) := \hat{q}(\sup_{\hat{\tau}_i \leq t} \hat{\tau}_i)$  for all  $t \in \hat{\mathcal{T}}$ . Let the map  $\hat{R}$  be such that  $\hat{q}(t)$  is a discrete state estimate,  $\hat{x}(0) = x_0$  and  $\hat{q}(\hat{\tau}_0) = \bar{q}_0$ . Then, we refer to system  $\hat{H}$  as an *estimator*. This in turn implies that (a)  $\hat{R}(\hat{q}, y) \subseteq \text{Rch}(\hat{q})$  for all  $y \in Y$  and  $\hat{q} \in \hat{Q}$  and that (b)  $\hat{\tau}'_0 = \hat{\tau}_0 = 0$  and  $y(\hat{\tau}'_0)$  is such that  $\hat{R}(\hat{q}(\hat{\tau}'_0), y(\hat{\tau}'_0)) := \text{Rch}(\hat{q}(\hat{\tau}_0)) = \text{Rch}(\bar{q}_0)$ . The discrete input  $y(t)$  derives information from the measured continuous state signal about the values of  $\hat{x}(\tau)$  for  $\tau < t$  and utilizes this information to determine the current set of modes compatible with such a derivative (see [3,8,9] for more information on mode estimators).

Since for system  $\hat{H}$ , the state  $\hat{q}(t)$  and  $\hat{x}(t) = x(t)$  is measured, a safety control problem now becomes a problem with perfect state information. Specifically, given a feedback map  $u(t) = \hat{\pi}(\hat{q}(t), \hat{x}(t))$  for system  $\hat{H}$ , we denote the *closed loop system* by  $\hat{H}^{\hat{\pi}}$ . The flow of  $\hat{H}^{\hat{\pi}}$  is denoted by  $\hat{\phi}^{\hat{\pi}}(t, (\bar{q}_0, x_0), \mathbf{d}, \mathbf{y})$  and the continuous flow by  $\phi_x^{\hat{\pi}}(t, (\bar{q}_0, x_0), \mathbf{d}, \mathbf{y})$ . Also, a feedback map that guarantees safety for  $\hat{H}$  also guarantees safety for  $H$  as the set of trajectories of  $\hat{H}$  contain also those of  $H$ . For more details on the relations between the solutions to the imperfect and the perfect information control problem, the reader is referred to [23]. The capture set for system  $\hat{H}$  is given by  $\hat{C} := \bigcup_{\hat{q} \in \hat{Q}} (\hat{q} \times \hat{C}_{\hat{q}})$ , in which  $\hat{C}_{\hat{q}} := \{x_0 \in X \mid \forall \hat{\pi}, \exists \mathbf{d}, \mathbf{y}, t \geq 0 \text{ s.t. some } \phi_x^{\hat{\pi}}(t, (\hat{q}, x_0), \mathbf{d}, \mathbf{y}) \in \text{Bad}\}$  is called the mode-dependent capture set. It represents the set of all continuous states that are taken to *Bad* for all feedback maps when the initial mode estimate is equal to  $\hat{q}$ .

**Problem 1.** (Control Problem with Perfect State Information) Determine the set  $\hat{C}$  and a feedback map  $\hat{\pi}$  that keeps any initial condition  $(\bar{q}_0, x_0) \notin \hat{C}$  outside  $\hat{C}$ .

The solution to Problem 1 can be obtained by leveraging results available for control of hybrid automata with perfect state information [20,21]. For this purpose, for any  $\hat{q} \in \hat{Q}$  and  $S \subseteq X$  define the operator  $\text{Pre}$  as  $\text{Pre}(\hat{q}, S) := \{x \in X \mid \forall \hat{\pi}, \exists \mathbf{d}, t \geq 0 \text{ s.t. some } \phi_x^{\hat{\pi}}(t, (\hat{q}, x), \mathbf{d}, \epsilon) \in S\}$ . The set  $\text{Pre}(\hat{q}, S)$  is the set of all continuous states that are taken to  $S$  for all feedback maps when the mode estimate is kept constant to  $\hat{q}$ .

### A. Computation of the capture set

An algorithmic procedure is defined in [20,21] for obtaining set  $\hat{C}_{\hat{q}}$ . We recall this procedure here. We use for all  $\hat{q} \in \hat{Q}$  the notation  $\hat{R}(\hat{q}, Y) := \{\hat{q}' \in \hat{R}(\hat{q}, y) \mid y \in Y\}$ , in which we set  $\hat{R}(\hat{q}, y) := \emptyset$  if  $\hat{R}(\hat{q}, y)$  is not defined for some  $y \in Y$ .

**Definition 4.** A set  $\hat{W} \subseteq \hat{Q} \times X$  is termed a *controlled invariant set* for  $\hat{H}$  if there is a feedback map  $\hat{\pi}$  such that for all  $(\bar{q}_0, x_0) \in \hat{W}$ , we have that all flows  $\hat{\phi}^{\hat{\pi}}(t, (\bar{q}_0, x_0), \mathbf{d}, \mathbf{y}) \in \hat{W}$  for all  $t, \mathbf{d}$ , and  $\mathbf{y}$ . A set  $\hat{W} \subseteq \hat{Q} \times X$  is the *maximal controlled invariant set* for  $\hat{H}$  provided it is a controlled invariant set for  $\hat{H}$  and any other controlled invariant set for  $\hat{H}$  is a subset of  $\hat{W}$ .

The next result (Proposition 1 of [20]) states that the complement of the capture set is the maximal controlled invariant set for  $\hat{H}$ .

**Proposition 1.** *The set  $\hat{W} := (\hat{Q} \times X) \setminus \hat{C}$  is the maximal controlled invariant set for  $\hat{H}$  contained in  $(\hat{Q} \times X) \setminus (\hat{Q} \times \text{Bad})$ .*

Let  $\hat{Q} = \{\hat{q}_1, \dots, \hat{q}_M\}$  with  $\hat{q}_i \in 2^Q$  for  $i \in \{1, \dots, M\}$ ,  $S_i \in 2^X$  for  $i \in \{1, \dots, M\}$ , and define  $S := (S_1, \dots, S_M) \subseteq (2^X)^M$ . We define the map  $G : (2^X)^M \rightarrow (2^X)^M$  as

$$G(S) := \begin{bmatrix} \text{Pre}(\hat{q}_1, \bigcup_{\{j \mid \hat{q}_j \in \hat{R}(\hat{q}_1, Y)\}} S_j \cup \text{Bad}) \\ \vdots \\ \text{Pre}(\hat{q}_M, \bigcup_{\{j \mid \hat{q}_j \in \hat{R}(\hat{q}_M, Y)\}} S_j \cup \text{Bad}) \end{bmatrix}.$$

**Algorithm 1.**  $S^0 := (S_1^0, S_2^0, \dots, S_M^0) := (\emptyset, \dots, \emptyset)$ ,  
 $S^1 = G(S^0)$   
*while*  $S^{k-1} \neq S^k$   
 $S^{k+1} = G(S^k)$   
*end.*

If Algorithm 1 terminates, that is, if there is a  $K^*$  such that  $S^{K^*} = (S_1^{K^*}, \dots, S_M^{K^*}) = (S_1^{K^*+1}, \dots, S_M^{K^*+1}) = S^{K^*+1}$ , we denote the fixed point by  $S^*$ . It can be shown that if Algorithm 1 terminates, the fixed point  $S^*$  is such that  $S^* = (\hat{C}_{\hat{q}_1}, \dots, \hat{C}_{\hat{q}_M})$  (see Theorem 1 of [20]).

### B. The control map

To determine the set of feedback maps that keep the complement of  $\hat{C}$  invariant, we employ notions from viability theory [2].

**Definition 5.** A set-valued map  $F : X \rightarrow 2^X$  is said to be *piecewise Lipschitz continuous* on  $X$  if it is Lipschitz continuous on a finite number of sets  $X_i \subset X$  for  $i = 1, \dots, N$  that cover  $X$ , that is,  $\bigcup_{i=1}^N X_i = X$ , and  $X_i \cap X_j = \emptyset$  for  $i \neq j$ .

Let  $X$  be a normed space and let  $S \subset X$  be nonempty. The *contingent cone* to  $S$  at  $x \in S$  is the set given by  $T_S(x) := \{v \in S \mid \liminf_{h \rightarrow 0^+} \frac{d_S(x+hv)}{h} = 0\}$ , in which  $d_S(y)$  denotes the distance of  $y$  from set  $S$ , that is,  $d_S(y) := \inf_{z \in K} \|y - z\|$ . The next result (Proposition 6 of

[20]) extends conditions for set invariance as found in [2] to the case of piece-wise Lipschitz continuous set-valued maps. This extension is required in our case because the vector field  $f$  is allowed to be piece-wise continuous.

**Proposition 2.** *Let  $F : X \rightarrow 2^X$  be a set-valued Marchaud map. Assume that  $F$  is piecewise Lipschitz continuous on  $X$ . A closed set  $S \subseteq X$  is invariant under  $F$  if and only if  $F(x) \subseteq T_S(x)$  for all  $x \in S$ .*

For simplifying notation, for each mode  $\hat{q} \in \hat{Q}$  define the set-valued map  $\bar{f} : X \times \hat{Q} \times U \rightarrow 2^X$  as  $\bar{f}(\hat{x}, \hat{q}, u) = \{\hat{f}(\hat{x}, \hat{q}, u, d), d \in D\}$  for all  $(\hat{x}, \hat{q}, u) \in X \times \hat{Q} \times U$ . Define  $L_{\hat{q}} := X \setminus \hat{C}_{\hat{q}}$  for all  $\hat{q} \in \hat{Q}$  and consider the set-valued map defined as

$$\Pi(\hat{q}, \hat{x}) := \{u \in U \mid \bar{f}(\hat{x}, \hat{q}, u) \subset T_{L_{\hat{q}}}(\hat{x})\}. \quad (1)$$

The following theorem (Theorem 3 of [20]) states that a control map can be selected that makes the complement of the capture set controlled invariant.

**Theorem 1.** *Assume that  $\hat{\pi} : \hat{Q} \times X \rightarrow U$  is such that for all  $\hat{q} \in \hat{Q}$  the set-valued map  $F(\hat{x}, \hat{q}) := \bar{f}(\hat{x}, \hat{q}, \hat{\pi}(\hat{x}, \hat{q}))$  is Marchaud and piecewise Lipschitz continuous on  $X$ . Then, the set  $(\hat{Q} \times X) \setminus \hat{C}$  is invariant for  $\hat{H}^{\hat{\pi}}$  if and only if  $\hat{\pi}(\hat{q}, \hat{x}) \in \Pi(\hat{q}, \hat{x})$ .*

### III. TERMINATION OF ALGORITHM 1

For the termination of Algorithm 1, sufficient conditions on  $\hat{H}$  are provided in [20]. For the systems that do not satisfy these conditions, we show that one can construct an abstraction of  $\hat{H}$  for which Algorithm 1 always terminates and such that the fixed point gives the mode-dependent capture sets of  $\hat{H}$ . In order to proceed, we introduce the notion of kernel sets for  $\hat{H}$ .

**Definition 6.** (Kernel set) The *kernel set* corresponding to a mode  $\hat{q}^* \in \hat{Q}$  is defined as  $\ker(\hat{q}^*) := \{\hat{q} \in \hat{Q} \mid \hat{q} \in \text{Rch}(\hat{q}^*) \text{ and } \hat{q}^* \in \text{Rch}(\hat{q})\}$ .

The kernel set for a mode  $\hat{q}^*$  is thus the set of all modes that can be reached from  $\hat{q}^*$  and from which  $\hat{q}^*$  can be reached. One can verify that for all pairs of modes  $\hat{q}_i, \hat{q}_j \in \hat{Q}$ , we have that  $\hat{q}_i \in \text{Rch}(\hat{q}_j)$  and  $\hat{q}_j \in \text{Rch}(\hat{q}_i)$  if and only if  $\ker(\hat{q}_i) = \ker(\hat{q}_j)$ . The next result shows that any two modes of  $\hat{H}$  in the same kernel set have the same mode-dependent capture set and hence the same set of safe feedback maps.

**Proposition 3.** *For every kernel set  $\ker \subseteq \hat{Q}$  and for any two modes  $\hat{q}, \hat{q}' \in \ker$ , we have that  $\hat{C}_{\hat{q}} = \hat{C}_{\hat{q}'}$  and hence that  $\Pi(\hat{q}, x) = \Pi(\hat{q}', x)$ .*

*Proof.* Since  $\hat{q}, \hat{q}' \in \ker$ , we have that  $\hat{q}' \in \text{Rch}(\hat{q})$  and that  $\hat{q} \in \text{Rch}(\hat{q}')$ . By Proposition 4 of [20], the first inclusion implies that  $\hat{C}_{\hat{q}'} \subseteq \hat{C}_{\hat{q}}$ , while the second inclusion implies that  $\hat{C}_{\hat{q}} \subseteq \hat{C}_{\hat{q}'}$ . Hence, we must have that  $\hat{C}_{\hat{q}} = \hat{C}_{\hat{q}'}$ . By equation (1), this in turn implies also that  $\Pi(\hat{q}, x) = \Pi(\hat{q}', x)$ .  $\square$

Let  $\mathcal{K} := \{ker(\hat{q}_1), \dots, ker(\hat{q}_M)\}$ . Let there be  $p$  distinct elements in  $\mathcal{K}$  denoted  $ker_1, \dots, ker_p$ . Note that  $ker_i \cap ker_j = \emptyset$ , for  $i \neq j$ . If each of the kernel sets is just one element in  $\hat{Q}$ , it means that there are no discrete transitions possible in  $\hat{R}$  that bring a discrete state  $\hat{q}$  back to itself. That is, there is no loop in any of the trajectories of  $\hat{q}$ . In this case, one can verify that Algorithm 1 terminates in a finite number of steps (see [20]). If there are loops, then the existence of a maximal element in each kernel set guarantees termination, as has been shown in Theorem 2 of [20]. However, when not all kernel sets have a maximal element, this result does not hold. Hence, we propose a different approach based on constructing an abstraction of  $\hat{H}$  that merges all the modes that belong to the same kernel set in a unique new mode.

**Definition 7.** Given hybrid system  $\hat{H} = (\hat{Q}, X, U, D, Y, \hat{R}, \hat{f})$ , the abstraction  $\tilde{H} = (\tilde{Q}, X, U, D, \tilde{Y}, \tilde{R}, \tilde{f})$  is a hybrid system with uncontrolled mode transitions such that

- (i)  $\tilde{Q} = \{\tilde{q}_1, \dots, \tilde{q}_p\}$ ,  $\tilde{Y}$  is such that  $\epsilon \in \tilde{Y}$  and  $\tilde{R}(\tilde{q}, \epsilon) = \tilde{q}$  for all  $\tilde{q} \in \tilde{Q}$ ;
- (ii) for all  $i, j \in \{1, \dots, p\}$  there is  $\tilde{y} \in \tilde{Y}$  such that  $\tilde{q}_i = \tilde{R}(\tilde{q}_j, \tilde{y})$  if and only if there are  $\hat{q}' \in ker_i$ ,  $\hat{q} \in ker_j$ , and  $y \in Y$  such that  $\hat{q}' = \hat{R}(\hat{q}, y)$ ;
- (iii) for all  $i \in \{1, \dots, p\}$ ,  $x \in X$ ,  $d \in D$ , and  $v \in U$ , we have that  $\tilde{f}(x, \tilde{q}_i, v, d) := \bigcup_{\hat{q} \in ker_i} \hat{f}(x, \hat{q}, v, d)$ .

Since  $p \leq m$ , the number of discrete states in system  $\tilde{H}$  is always less than or equal to that of system  $\hat{H}$ . For a feedback map  $\tilde{\pi} : \tilde{Q} \times X \rightarrow U$ , initial states  $x_0 \in X$  and  $\tilde{q}_0 \in \tilde{Q}$ , and signals  $\tilde{\mathbf{y}}, \mathbf{d}$ , we denote the flows of the closed loop system  $\tilde{H}^{\tilde{\pi}}$  by  $\phi_{\tilde{q}}(t, \tilde{q}_0, \tilde{\mathbf{y}})$  and  $\phi_{\tilde{x}}^{\tilde{\pi}}(t, (\tilde{q}_0, x_0), \mathbf{d}, \tilde{\mathbf{y}})$ , in which  $\tilde{x}(t) := \phi_{\tilde{x}}^{\tilde{\pi}}(t, (\tilde{q}_0, x_0), \mathbf{d}, \tilde{\mathbf{y}})$  satisfies  $\dot{\tilde{x}}(t) \in \tilde{f}(\tilde{x}(t), \phi_{\tilde{q}}(t, \tilde{q}_0, \tilde{\mathbf{y}}), \tilde{\pi}(\phi_{\tilde{q}}(t, \tilde{q}_0, \tilde{\mathbf{y}}), \tilde{x}), d(t))$ . We also denote by  $\tilde{C}_{\tilde{q}_i}$  for  $i \in \{1, \dots, p\}$  the mode-dependent capture sets of  $\tilde{H}$ . For any  $\tilde{q} \in \tilde{Q}$ , we define  $ker(\tilde{q}) := ker_i$  provided  $\tilde{q} = \tilde{q}_i$ . Also, for all  $\tilde{q} \in \tilde{Q}$ , we denote the set of reachable modes from  $\tilde{q}$  as  $\text{Rch}(\tilde{q}) := \bigcup_{t \geq 0} \bigcup_{\tilde{\mathbf{y}}} \phi_{\tilde{q}}(t, \tilde{q}, \tilde{\mathbf{y}})$ . In the sequel, we denote  $\tilde{R}(\tilde{q}, \tilde{Y}) := \bigcup_{\tilde{y} \in \tilde{Y}} \tilde{R}(\tilde{q}, \tilde{y})$ , in which we set  $\tilde{R}(\tilde{q}, \tilde{y}) := \tilde{q}$  if  $\tilde{R}(\tilde{q}, \tilde{y})$  is not defined for some  $\tilde{y} \in \tilde{Y}$ . The following proposition is a direct consequence of the fact that all kernel sets of  $\tilde{H}$  are singletons and there is no loop in any of the trajectories of  $\tilde{q}$ .

**Proposition 4.** Algorithm 1 terminates for system  $\tilde{H}$ .

The next result shows that any piece-wise continuous signal, which is continuous from the right and contained in  $ker(\phi_{\tilde{q}}(t, \tilde{q}_0, \tilde{\mathbf{y}}))$  is a possible discrete flow of  $\tilde{H}$  for suitable  $\mathbf{y}$  starting from some  $\hat{q}_0 \in ker(\tilde{q}_0)$ .

**Proposition 5.** For any piece-wise continuous signal  $\alpha$  that is continuous from the right and such that  $\alpha(t) \in ker(\phi_{\tilde{q}}(t, \tilde{q}_0, \tilde{\mathbf{y}}))$ , there are  $\hat{q}_0 \in ker(\tilde{q}_0)$  and  $\mathbf{y}$  such that  $\alpha(t) = \phi_{\hat{q}}(t, \hat{q}_0, \mathbf{y})$  for all  $t$ .

*Proof.* Since  $\alpha(t) \in ker(\phi_{\tilde{q}}(t, \tilde{q}_0, \tilde{\mathbf{y}}))$  for all  $t$ , there are

times  $t_0, \dots, t_N \leq t$  and a sequence  $j_0, \dots, j_N \in \{1, \dots, p\}$  such that  $\alpha(t) \in ker_{j_i}$  for all  $t \in [t_i, t_{i+1})$ . Since any mode in  $ker_{j_i}$  can transit to any other mode in  $ker_{j_i}$  instantaneously under the discrete transitions of  $\hat{H}$ , we have that there are  $\hat{q}_{o,i} \in ker_{j_i}$  and  $\mathbf{y}_i$  such that  $\alpha(t) = \phi_{\hat{q}}(t - t_i, \hat{q}_{o,i}, \mathbf{y}_i)$  for all  $t \in [t_i, t_{i+1})$ . Also, for any two modes  $\alpha_i \in ker_{j_i}$  and  $\alpha_{i+1} \in ker_{j_{i+1}}$  we have that  $\alpha_{i+1} \in \text{Rch}(\alpha_i)$ . Hence, let  $\alpha_i^- := \lim_{t \rightarrow t_{i+1}^-} \phi_{\hat{q}}(t - t_i, \hat{q}_{o,i}, \mathbf{y}_i)$  and  $\alpha_i^+ := \lim_{t \rightarrow t_{i+1}^+} \phi_{\hat{q}}(t - t_{i+1}, \hat{q}_{o,i+1}, \mathbf{y}_{i+1})$ . Then, since multiple transitions are possible in  $\hat{H}$  at the same time, there is a signal  $\mathbf{y}_{i,i+1}$  such that  $\alpha_i^+ = \phi_{\hat{q}}(0, \alpha_i^-, \mathbf{y}_{i,i+1})$ . Hence, there is a signal  $\mathbf{y}$  such that  $\alpha(t) = \phi_{\hat{q}}(t, \hat{q}_{o,0}, \mathbf{y})$  for all  $t$ .  $\square$

**Theorem 2.** For all kernel sets  $ker_i$  with  $i \in \{1, \dots, p\}$  and for all  $\hat{q} \in ker_i$ , we have that  $\hat{C}_{\hat{q}} = \tilde{C}_{\tilde{q}_i}$ .

*Proof.* Let  $\hat{q} \in ker_i$ . We first show that  $\hat{C}_{\hat{q}} \subseteq \tilde{C}_{\tilde{q}_i}$ . Let  $x_0 \in \hat{C}_{\hat{q}}$ , then for all  $\hat{\pi} : \hat{Q} \times X \rightarrow U$ , there are  $\mathbf{y}, \mathbf{d}$ , and  $t > 0$  such that  $\phi_{\hat{x}}^{\hat{\pi}}(t, (\hat{q}, x), \mathbf{d}, \mathbf{y}) \in \text{Bad}$ . This is in particular true for all those feedback maps  $\hat{\pi}$  such that  $\hat{\pi}(\hat{q}, x) = \hat{\pi}(\hat{q}', x)$  whenever  $\hat{q}, \hat{q}' \in ker_j$  for some  $j \in \{1, \dots, p\}$ . Hence, we also have that for all  $\tilde{\pi} : \tilde{Q} \times X \rightarrow U$ , there are  $\mathbf{y}, \mathbf{d}$ , and  $t > 0$  such that  $\hat{x}(t) := \phi_{\hat{x}}^{\tilde{\pi}}(t, (\hat{q}, x), \mathbf{d}, \mathbf{y}) \in \text{Bad}$ , in which  $\hat{x} \in \tilde{f}(\hat{x}(t), \phi_{\tilde{q}}(t, \tilde{q}, \tilde{\mathbf{y}}), \tilde{\pi}(\alpha(t), x(t)), d(t))$  with  $\alpha(t) := \tilde{q}_j$  if  $\phi_{\tilde{q}}(t, \tilde{q}, \tilde{\mathbf{y}}) \in ker_j$ . Such a signal  $\hat{x}(t)$  also satisfies  $\hat{x} \in \tilde{f}(\hat{x}(t), \alpha(t), \tilde{\pi}(\alpha(t), x(t)), d(t))$  by the definition of  $\tilde{f}$ . By the definition of  $\tilde{R}$ , there is  $\tilde{\mathbf{y}}$  such that  $\alpha(t) = \phi_{\tilde{q}}(t, \tilde{q}_i, \tilde{\mathbf{y}})$  for all  $t$ . Hence,  $\hat{x}(t)$  is also a continuous flow of  $\tilde{H}$  starting at  $(\tilde{q}_i, x_0)$  and therefore  $x_0 \in \tilde{C}_{\tilde{q}_i}$ .

We now show that  $\tilde{C}_{\tilde{q}_i} \subseteq \hat{C}_{\hat{q}}$ . If  $x_0 \in \tilde{C}_{\tilde{q}_i}$ , then for all feedback maps  $\tilde{\pi} : \tilde{Q} \times X \rightarrow U$ , there are  $\tilde{\mathbf{y}}, \mathbf{d}$ , and  $t > 0$  such that  $\tilde{x}(t) := \phi_{\tilde{x}}^{\tilde{\pi}}(t, (\tilde{q}_i, x_0), \tilde{\mathbf{y}}, \mathbf{d}) \in \text{Bad}$ . Here, we have that  $\tilde{x}(t)$  satisfies  $\dot{\tilde{x}}(t) \in \tilde{f}(\tilde{x}(t), \phi_{\tilde{q}}(t, \tilde{q}_i, \tilde{\mathbf{y}}), \tilde{\pi}(\phi_{\tilde{q}}(t, \tilde{q}_i, \tilde{\mathbf{y}}), \tilde{x}), d(t))$ , which is equivalent (by the definition of  $\tilde{f}$ ) to  $\dot{\tilde{x}}(t) \in \tilde{f}(\tilde{x}(t), ker(\phi_{\tilde{q}}(t, \tilde{q}_i, \tilde{\mathbf{y}})), \tilde{\pi}(\phi_{\tilde{q}}(t, \tilde{q}_i, \tilde{\mathbf{y}}), \tilde{x}), d(t))$ , which is equivalent to  $\dot{\tilde{x}}(t) = \tilde{f}(\tilde{x}(t), \alpha(t), \tilde{\pi}(\phi_{\tilde{q}}(t, \tilde{q}_i, \tilde{\mathbf{y}}), \tilde{x}), d(t))$  for piece-wise continuous signal  $\alpha$  (continuous from the right) such that  $\alpha(t) \in ker(\phi_{\tilde{q}}(t, \tilde{q}_i, \tilde{\mathbf{y}}))$ . By Proposition 5, any such  $\alpha(t)$  is such that there are  $\mathbf{y}$  and  $\hat{q}_0 \in ker(\tilde{q}_i)$  such that  $\alpha(t) = \phi_{\hat{q}}(t, \hat{q}_0, \mathbf{y})$  for all  $t$ , that is, it is a discrete flow of system  $\hat{H}$ . Hence, for all  $\hat{\pi}' : \hat{Q} \times X \rightarrow U$  with  $\hat{\pi}'(\hat{q}, x) = \hat{\pi}'(\hat{q}', x)$  for all  $\hat{q}, \hat{q}' \in ker_j$  for all  $j$ , there are  $\mathbf{y}, \mathbf{d}, \hat{q}_0 \in ker_i$ , such that  $\phi_{\hat{x}}^{\hat{\pi}'}(t, (\hat{q}_0, x_0), \mathbf{y}, \mathbf{d}) \in \text{Bad}$ . By Proposition 3, this implies that for all  $\hat{\pi} : \hat{Q} \times X \rightarrow U$  there are  $\mathbf{y}, \mathbf{d}, \hat{q}_0 \in ker_i$ , such that  $\phi_{\hat{x}}^{\hat{\pi}}(t, (\hat{q}_0, x_0), \mathbf{y}, \mathbf{d}) \in \text{Bad}$ . Hence,  $x_0 \in \hat{C}_{\hat{q}}$ .  $\square$

The above theorem can be utilized to compute the capture set of  $\hat{H}$  by constructing the abstraction  $\tilde{H}$  and applying Algorithm 1 to it, which is guaranteed to terminate for  $\tilde{H}$ . The next two technical propositions provide a characterization of the Pre operator computed for system  $\tilde{H}$  and the relationship between  $\tilde{R}$  and  $R$ . Specifically, denote the predecessor operator for system  $\tilde{H}$  for some  $S \subseteq X$  as



$\text{Pre}^a(\tilde{q}, S) := \{x_0 \in X \mid \forall \tilde{\pi} \exists t, \mathbf{d}, \text{ s.t. } \phi_{\tilde{x}}^{\tilde{\pi}}(t, (\tilde{q}, x_0), \mathbf{d}, \epsilon) \in S\}$ .

**Proposition 6.** For all  $\tilde{q} \in \tilde{Q}$  and  $S \subseteq X$ , we have that  $\text{Pre}^a(\tilde{q}, S) = \text{Pre}(\bigvee \ker(\tilde{q}), S)$ .

*Proof.* From the definition of  $\text{Pre}^a(\tilde{q}, S)$ , we have that  $x_0 \in \text{Pre}^a(\tilde{q}, S)$  if and only if for all  $\tilde{\pi}$ , there are  $t, \mathbf{d}$  such that  $\tilde{x}(t) = \phi_{\tilde{x}}^{\tilde{\pi}}(t, (\tilde{q}, x_0), \mathbf{d}, \epsilon) \in S$ , in which  $\dot{\tilde{x}}(t) \in \tilde{f}(\tilde{x}(t), \tilde{q}, \tilde{\pi}(\tilde{x}(t)), d(t))$ , which, by the definition of  $\tilde{f}$  and of  $\hat{f}$  is equivalent to  $\dot{\tilde{x}}(t) \in f(\tilde{x}(t), \bigcup_{\hat{q} \in \ker(\tilde{q})} \bigcup_{q \in \hat{q}} q, \tilde{\pi}(\tilde{x}(t)), d(t)) = f(\tilde{x}(t), \bigvee \ker(\tilde{q}), \tilde{\pi}(\tilde{x}(t)), d(t))$ . Hence, by the definition of Pre, we have that  $x_0 \in \text{Pre}^a(\tilde{q}, S)$  if and only if  $x_0 \in \text{Pre}(\bigvee \ker(\tilde{q}), S)$ .  $\square$

**Proposition 7.** Let  $\tilde{q}_{j_1}, \tilde{q}_{j_0} \in \tilde{Q}$ . If  $\tilde{q}_{j_1} \in \tilde{R}(\tilde{q}_{j_0}, \tilde{Y})$  then  $\bigvee \ker(\tilde{q}_{j_1}) \subseteq \text{Rch}(\bigvee \ker(\tilde{q}_{j_0}))$ .

*Proof.* If  $\tilde{q}_{j_1} \in \tilde{R}(\tilde{q}_{j_0}, \tilde{Y})$ , then by the definition of  $\tilde{R}$  there are  $\hat{q} \in \ker(\tilde{q}_{j_0})$  and  $\hat{q}' \in \ker(\tilde{q}_{j_1})$  such that  $\hat{q}' = \hat{R}(\hat{q}, y)$  for some  $y \in \tilde{Y}$ . By the definition of a kernel set, this also implies that for all  $\hat{q} \in \ker(\tilde{q}_{j_0})$  and  $\hat{q}' \in \ker(\tilde{q}_{j_1})$ , there is a sequence of events  $y_1, \dots, y_k$  and of modes  $\hat{q}_{j_0}, \dots, \hat{q}_{j_k} \in \hat{Q}$  such that  $\hat{q}_{j_0} = \hat{q}$ ,  $\hat{q}_{j_k} = \hat{q}'$  and  $\hat{q}_{j_{i+1}} = \hat{R}(\hat{q}_{j_i}, y_{i+1})$  for  $i \in \{0, \dots, k-1\}$ . Since  $\hat{R}(\hat{q}, y) \subseteq \text{Rch}(\hat{q})$  for all  $y \in \tilde{Y}$  and  $\hat{q} \in \hat{Q}$ , this in turn implies that  $\hat{q}_{j_{i+1}} \subseteq \text{Rch}(\hat{q}_{j_i})$  for  $i \in \{0, \dots, k-1\}$ . This leads to  $\hat{q}' \subseteq \text{Rch}(\hat{q})$  for all  $\hat{q} \in \ker(\tilde{q}_{j_0})$  and  $\hat{q}' \in \ker(\tilde{q}_{j_1})$ . This also implies that  $\hat{q}' \subseteq \text{Rch}(\bigvee \ker(\tilde{q}_{j_0}))$  and hence (since this holds for all  $\hat{q}' \in \ker(\tilde{q}_{j_1})$ ) to  $\bigvee \ker(\tilde{q}_{j_1}) \subseteq \text{Rch}(\bigvee \ker(\tilde{q}_{j_0}))$ .  $\square$

**Lemma 1.** For all  $\tilde{q} \in \hat{Q}$ , we have that  $\hat{C}_{\tilde{q}} = \text{Pre}(\text{Rch}(\tilde{q}), \text{Bad})$ .

*Proof.* First, we show that  $\hat{C}_{\tilde{q}} \subseteq \text{Pre}(\text{Rch}(\tilde{q}), \text{Bad})$ . Since Algorithm 1 terminates in a finite number  $n$  of steps for  $\hat{H}$ , we have that  $\hat{C}_{\tilde{q}} = \text{Pre}^a(\tilde{q}, \bigcup_{\tilde{q}_{j_1} \in \tilde{R}(\tilde{q}, \tilde{Y})} \text{Pre}^a(\tilde{q}_{j_1}, \dots, \bigcup_{\tilde{q}_{j_{n-1}} \in \tilde{R}(\tilde{q}_{j_{n-2}}, \tilde{Y})} \text{Pre}^a(\tilde{q}_{j_{n-1}}, \text{Bad}) \dots))$ . By Proposition 6, we also have that  $\hat{C}_{\tilde{q}} = \text{Pre}(\bigvee \ker(\tilde{q}), \bigcup_{\tilde{q}_{j_1} \in \tilde{R}(\tilde{q}, \tilde{Y})} \text{Pre}(\bigvee \ker(\tilde{q}_{j_1}), \bigcup_{\tilde{q}_{j_2} \in \tilde{R}(\tilde{q}_{j_1}, \tilde{Y})} \text{Pre}(\bigvee \ker(\tilde{q}_{j_2}), \dots, \bigcup_{\tilde{q}_{j_{n-1}} \in \tilde{R}(\tilde{q}_{j_{n-2}}, \tilde{Y})} \text{Pre}(\bigvee \ker(\tilde{q}_{j_{n-1}}), \text{Bad}) \dots)))$ . By Proposition 7, we have that  $\bigvee \ker(\tilde{q}_{j_1}) \subseteq \text{Rch}(\bigvee \ker(\tilde{q}))$  and that  $\bigvee \ker(\tilde{q}_{j_{i+1}}) \subseteq \text{Rch}(\bigvee \ker(\tilde{q}_{j_i}))$  for  $i < n$ . Since the Pre operator and Rch preserve the inclusion relation in the first argument, these imply that  $\hat{C}_{\tilde{q}} \subseteq \text{Pre}(\text{Rch}(\bigvee \ker(\tilde{q})), \text{Bad})$ . Since for all  $\tilde{q}_1, \tilde{q}_2 \in \ker(\tilde{q})$  we have that  $\text{Rch}(\tilde{q}_1) = \text{Rch}(\tilde{q}_2)$ , we also have that  $\text{Rch}(\tilde{q}) = \text{Rch}(\bigvee \ker(\tilde{q}))$  for all  $\tilde{q} \in \ker(\tilde{q})$ . Hence,  $\hat{C}_{\tilde{q}} \subseteq \text{Pre}(\text{Rch}(\tilde{q}), \text{Bad})$  for all  $\tilde{q} \in \ker(\tilde{q})$ . This along with Theorem 2 finally imply that for all  $\tilde{q} \in \ker(\tilde{q})$  we have  $\hat{C}_{\tilde{q}} \subseteq \text{Pre}(\text{Rch}(\tilde{q}), \text{Bad})$ .

To show that  $\hat{C}_{\tilde{q}} \supseteq \text{Pre}(\text{Rch}(\tilde{q}), \text{Bad})$ , we employ the properties of the Pre operator and Proposition 4 of [20]. By such a proposition, by the fact that (since  $\hat{H}$  is derived from  $H$ ) for all  $\tilde{q} \in \hat{Q}$  there is  $y \in \tilde{Y}$  such that  $\hat{R}(\tilde{q}, y) =$

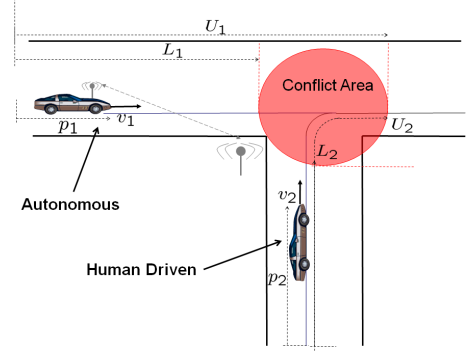


Fig. 1. **Two-vehicle Conflict Scenario.** Vehicle 1 (autonomous) is equipped with a cooperative active safety system and communicates with the infrastructure via wireless. Vehicle 2 (human-driven) is not equipped and does not communicate with the infrastructure. A collision occurs when more than one vehicle occupies the conflict area at one time.

$\text{Rch}(\tilde{q})$ , and by property (iii) of Proposition 2 from [20], it follows that  $\hat{C}_{\tilde{q}} \supseteq \text{Pre}(\tilde{q}, \hat{C}_{\text{Rch}(\tilde{q})})$ . In turn we have that  $\hat{C}_{\text{Rch}(\tilde{q})} \supseteq \text{Pre}(\text{Rch}(\tilde{q}), \text{Bad})$  by Proposition 4 of [20] and property (iii) of Proposition 2 from [20]. Hence, we have that  $\hat{C}_{\tilde{q}} \supseteq \text{Pre}(\tilde{q}, \text{Pre}(\text{Rch}(\tilde{q}), \text{Bad}))$ , which by property (i) of Proposition 2 from [20] leads to  $\hat{C}_{\tilde{q}} \supseteq \text{Pre}(\text{Rch}(\tilde{q}), \text{Bad})$ .  $\square$

This result shows that the mode-dependent capture set  $\hat{C}_{\tilde{q}}$  can be computed by computing the Pre operator only once as opposed to being determined through a (finite, by Theorem 2 and Proposition 4) iteration of Pre operator computations (as was performed in [20, 21]). To illustrate this point, consider as an example a tuple  $(\hat{R}, \hat{Q}, \tilde{Y})$  with  $\hat{Q} = \{\hat{q}_1, \hat{q}_2\}$ ,  $\tilde{Y} = \{\epsilon, y\}$ ,  $\hat{R}(\hat{q}_1, y) = \hat{q}_2$  and  $\hat{R}(\hat{q}_2, y) = \hat{q}_1$  with  $\hat{q}_1 \not\subseteq \hat{q}_2$ . Since there is a loop between  $\hat{q}_1$  and  $\hat{q}_2$  and the kernel set does not contain a maximal element, Theorem 2 of [20] cannot guarantee the termination of Algorithm 1. However, the results presented in this paper show that the desired capture set can be obtained by utilizing Lemma 1, that is,  $\hat{C}_{\hat{q}_1} = \text{Pre}(\text{Rch}(\hat{q}_1), \text{Bad})$  and  $\hat{C}_{\hat{q}_2} = \text{Pre}(\text{Rch}(\hat{q}_2), \text{Bad})$ , in which  $\text{Rch}(\hat{q}_2) = \text{Rch}(\hat{q}_1) = \{\hat{q}_1, \hat{q}_2\}$ . The computation of such a Pre can be efficiently performed if the continuous dynamics for  $q \in \hat{q}_1 \cup \hat{q}_2$  has suitable order preserving properties [23]. We show an application example in the next section.

#### IV. APPLICATION SCENARIO

Referring to Figure 1, vehicle 1 is autonomous and communicates with the infrastructure, while vehicle 2 is human-driven and does not communicate its intent to the infrastructure nor to the other vehicle. We assume that the infrastructure measures the position and speed of vehicle 2 through road-side sensors such as cameras and magnetic-induction loops and that it transmits this information to the on-board controller of vehicle 1. Vehicle 1 has to use this information to avoid a collision. We assume that the



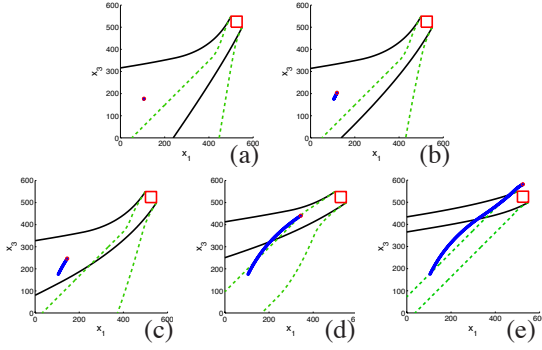


Fig. 3. In each of the plots (a)–(e), the red box represents  $[L_1, U_1] \times [L_2, U_2]$ . We plot the slice of  $\hat{C}_{\hat{q}}$  in the  $(x_1, x_3)$  position plane corresponding to the current speed  $(x_2, x_4)$ . In the  $(x_1, x_3)$  plane and for the current speed values  $(x_2, x_4)$ , the black solid lines delimit the set  $\text{Pre}(\hat{q}, \text{Bad})_H$ , the green dashed lines delimit the set  $\text{Pre}(\hat{q}, \text{Bad})_L$  and the intersection of these two sets is the current mode dependent capture set  $\hat{C}_{\hat{q}}$ . The red circle denotes the projection in the position plane of the continuous trajectory of  $H$ . Plot (a) shows the initial configuration in the position plane. Here, the current mode estimate is  $\hat{q} = \{A, C, B\}$ . Plot (b) shows the mode estimate switching to  $\hat{q} = \{C, B\}$  and the corresponding capture set shrinking. Plot (c) shows the time at which the mode estimate becomes  $\hat{q} = \{B\}$ , so that the current mode is locked and the capture set shrinks further. Plot (d) shows when the continuous state hits the boundary of the current mode-dependent capture set thus resulting in the application of a safe control.

dynamics have suitable order-preserving properties [23]. We introduce an example of a semi-autonomous cooperative active safety system that belongs to this class and present simulation results for collision avoidance between a human-driven and an autonomous vehicle merging at an intersection. In future work, we intend to consider situations with more than two vehicles merging on an intersection, in which some of the vehicles are human-driven and some are autonomous. The approach presented in this paper cannot be directly extended to the multiple vehicle scenario due to the bad set not being convex. Alternative approaches are being investigated, including discrete abstraction techniques exploiting the fact that the vehicles dynamics are differentially flat and order preserving [4].

## REFERENCES

- [1] M. Althoff. *Reachability Analysis and its Application to the Safety Assessment of Autonomous Cars*. PhD thesis, Technische Universitat Munchen, 2010.
- [2] J. Aubin. *Viability Theory*. Birkhäuser, 1991.
- [3] A. Balluchi, L. Benvenuti, M. D. Di Benedetto S, and A. L. Sangiovanni-vincentelli. Design of observers for hybrid systems. In *In Hybrid Systems: Computation and Control, volume 2289 of LNCS*, pages 76–89. Springer-Verlag, 2002.
- [4] A. Colombo and D. Del Vecchio. Supervisory control of differentially flat systems based on abstraction. In *Conf. on Decision and Control*, 2011. to appear.
- [5] D. Del Vecchio. A partial order approach to discrete dynamic feedback in a class of hybrid systems. In *Hybrid Systems: Computation and Control*, Lecture Notes in Computer Science, vol. 4416, A. Bemporad, A. Bicchi, and G. Buttazzo (Eds.), Springer Verlag, pages 159–173, Pisa, Italy, 2007.
- [6] D. Del Vecchio. Observer-based control of block triangular discrete time hybrid automata on a partial order. *International Journal of Robust and Nonlinear Control*, 19(14):1581–1602, 2009.
- [7] D. Del Vecchio, M. Malisoff, and R. Verma. A separation principle for a class of hybrid automata on a partial order. In *American Control Conference*, pages 3638–3643, 2009.
- [8] D. Del Vecchio, R. M. Murray, and E. Klavins. Discrete state estimators for systems on a lattice. *Automatica*, 42(2):271–285, 2006.
- [9] D. Del Vecchio, R. M. Murray, and P. Perona. Decomposition of human motion into dynamics-based primitives with application to drawing tasks. *Automatica*, 39(12):2085–2098, 2003.
- [10] D. Caveney H. Kowshik and P. R. Kumar. Provable systemwide safety in intelligent intersections. *IEEE Transactions on Automatic Control*, 60(3):804–818, March 2007.
- [11] M. Hafner and D. Del Vecchio. Computation of safety control for uncertain piecewise continuous systems on a partial order. In *Conference on Decision and Control*, pages 1671–1677, 2009.
- [12] J. K. Hedrick, Y. Chen, and S. Mahal. Optimized vehicle control/communication interaction in an automated highway system. *Virginia Tech Transportation Institute: Report No. VPI-2006-06.*, 2008.
- [13] Uwe Kiencke and Lars Nielsen. *Automotive Control Systems, For Engine, Driveline, and Vehicle*. Springer Verlag, 2nd edition, 2005.
- [14] A. B. Kurzhanski and P. Varaiya. Ellipsoidal techniques for hybrid dynamics: the reachability problem. In *New Directions and Applications in Control Theory*, Lecture Notes in Control and Information Sciences, vol 321, W.P. Dayawansa, A. Lindquist, and Y. Zhou (Eds.), pages 193–205, 2005.
- [15] C. Le Guernic. *Reachability Analysis of Hybrid Systems with Linear Continuous Dynamics*. PhD thesis, Univerite Joseph Fourier, 2009.
- [16] J. Lygeros, C. J. Tomlin, and S. Sastry. Controllers for reachability specifications for hybrid systems. *Automatica*, 35(3):349–370, 1999.
- [17] Meeko Oishi, Ian Mitchell, Alexandre Bayen, and Claire Tomlin. Invariance-preserving abstractions of hybrid systems: Application to user interface design. *IEEE Transactions on Control Systems Technology*, 16(2):229–244, March 2008.
- [18] O. Shakeria, G. J. Pappas, and Shankar Sastry. Semi-decidable synthesis for triangular hybrid systems. In *Hybrid Systems: Computation and Control*, Lecture Notes in Computer Science, vol. 2034, M. D. Di Benedetto and A. Sangiovanni-Vincentelli (Eds.), Springer Verlag, 2001.
- [19] C. J. Tomlin, I. Mitchell, A. M. Bayen, and M. Oishi. Computational techniques for the verification of hybrid systems. *Proceedings of the IEEE*, 91(7):986–1001, 2003.
- [20] R. Verma and D. Del Vecchio. Continuous control of hybrid automata with imperfect mode information assuming separation between state estimation and control. In *Conference on Decision and Control*, pages 3175–3181, 2009.
- [21] R. Verma and D. Del Vecchio. Control of hybrid automata with hidden modes: translation to a perfect state information problem. In *Conference on Decision and Control*, pages 5768–5774, 2010.
- [22] R. Verma, D. Del Vecchio, and H. Fathy. Development of a scaled vehicle with longitudinal dynamics of a HMMWV for an ITS testbed. *IEEE/ASME Transactions on Mechatronics*, 13:46–57, 2008.
- [23] R. Verma and D. Del Vecchio. Safety control of hidden mode hybrid systems. *IEEE Transactions on Automatic Control*, 2011. to appear.
- [24] M. De Wulf, L. Doyen, and J.-F. Raskin. A lattice theory for solving games of imperfect information. *Hybrid Systems: Computation and Control*, Lecture Notes in Computer Science, vol. 3927, J. Hespanha and A. Tiwari (Eds.), Springer-Verlag, pages 153–168, 2006.