Supplier-Buyer Negotiation Games: Equilibrium Conditions and Supply Chain Efficiency[1]

Abstract

In a decentralized supply chain, supplier-buyer negotiations have a dynamic aspect that requires both players to consider the impact of their decisions on future decisions made by their counterpart. The interaction generally couples strongly the price decision of the supplier and the quantity decision of the buyer. As a result, the outcome of the negotiation may not have an equilibrium. We propose a basic model for a repeated supplier-buyer interaction, during a number of rounds. In each round, the supplier first quotes a price, and the buyer places an order at that price. We find conditions for existence and uniqueness of subgame-perfect equilibrium in the dynamic game. We furthermore identify some demand distributions for which these conditions are met, when costs are stationary and there are no holding costs. In this scenario and for such demands, we examine the efficiency of the equilibrium and in particular show that, as the number of rounds increases, the profits of the supply chain increase towards the supply chain optimum.

1 Introduction

The management of supply chain relationships is an operational lever that can be critical to a firm's profitability. Indeed, supplier-buyer negotiations play a central role in establishing revenues (for suppliers) and costs (for buyers). Well-conducted negotiations are critical in retail for example, where giants like Wal-Mart in the United States or Aldi in Germany strive to offer a low-cost proposition.

The process by which a buyer and a supplier interact to fix price and sales quantity is complex. It is fraught with tensions, as the buyer is interested in obtaining a lower price and the supplier prefers a higher price provided that the sales quantity is sufficient. Generally, the outcome of such process is not necessarily efficient for the supply chain. Indeed, prices are typically higher than the supply chain's preferred one, because the supplier requests a price strictly larger than its cost. As a result, the transacted quantities are lower than what would be best for the chain. This situation is called *double marginalization*, and has been documented and analyzed since the 1950s, see Spengler [18]. A lot of research has been done to propose supply contracts that are beneficial to buyer and supplier, such as buy-back contracts, revenue sharing or quantity discounts. These mechanisms allow the supply chain to move *from local optimization*, where each company takes decisions individually, considering only its own profits, *towards global optimization*, where the decisions of all the companies take into account

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[1]A previous version of this paper was titled "Improving Supply Chain Efficiency Through Wholesale Price Renegotiation."

aggregated supply chain profits. While these have been implemented with great success many times, they presuppose a simple negotiation process that cannot always be taken for granted.

In particular, negotiations have a dynamic aspect that expands the strategy space of both buyer and supplier. This dynamic aspect requires both players to consider the impact of current decisions on future decisions made by their counterpart. In particular, the *strategic use of inventory* by the buyer has been identified as a lever to obtain lower prices from the supplier, see Anand et al. [1]. We can provide an example in the procurement of scrap metal for a steel manufacturer. Even though the manufacturer does not need to carry a high level of scrap metal inventory at any time, it actually stores large piles of it outside the factory, as a way to obtain lower prices from the suppliers. Indeed, this is a credible threat of the buyer: it will only buy more raw materials if the price is low enough. Hence, a dynamic negotiation contains many interesting elements that cannot be revealed in static settings.

The analysis of the dynamic interaction between supplier and buyer is hence complex. It has only been studied under simplistic settings. Namely, most of the academic work has focused on two-period models and/or simple linear demand functions. This is because the interaction generally couples strongly the price decision of the supplier and the quantity decision of the buyer. As a result, the analysis usually becomes intractable. In particular, it is unclear whether the outcome of the negotiation actually has an equilibrium where supplier and buyer have no incentive to unilaterally deviate (as in single-period models). Furthermore, such outcome may not necessarily be unique.

The purpose of this paper is precisely to tackle these questions. We propose a basic model for repeated supplier-buyer interaction, during a number of rounds T. In each stage, the supplier first quotes a price, and the buyer places an order at that price. The costs of delivering the order may vary over time, and the buyer may have to pay for inventory holding charges. At the end of the T rounds, the buyer faces a stochastic demand and fulfills it with its total purchase over the negotiation. With this relatively simple setting, that extends some of the existing models (Anand et al. [1] and Erhun et al. [9]), we determine how the negotiation will proceed, and under which circumstances it will have a well-defined and unique outcome. In other words, we find conditions for existence and uniqueness of subgame-perfect equilibrium in the dynamic game. We furthermore identify some demand distributions for which these conditions are met, when costs are stationary and there are no holding costs. In this scenario and for such demands, we examine the efficiency of the equilibrium and in particular show that, as T increases, the profits of the supply chain increase towards the supply chain optimum. Our paper hence offers a technical contribution: it describes how the negotiation proceeds during multiple periods and a general demand specification.

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Our model reveals that the supplier will in equilibrium propose different prices in each round, which decrease over time. At each one of these prices, the buyer will place an order. Even though prices are decreasing in time, the buyer finds it in its best interest to place a positive order, to force the supplier to reduce its price in the following round. This results suggests that the buyer uses its cumulative purchase to reduce supplier prices, in the same way as strategic inventories. Interestingly, as the length of the negotiation increases, both the supplier's and the buyer's profits increase. Indeed, this simple scheme is equivalent to using a non-linear pricing schedule, which is able to reduce the impact of double marginalization. In other words, the effects of renegotiation are similar to those of volume discounts, which push buyers to place larger orders by promising lower prices for the last units ordered. This insight, that renegotiation has in general the same qualitative effect as a static quantity discount scheme, is another contribution of the paper that echoes that of Erhun et al. [9].

It is worth pointing out that our model extends previous work from the economics literature on *price skimming*, in the case where *the buyer is strategic*, in the context of a supply chain. Strategic customers have been studied before, but this paper considers the market power of buyers as well. That is, in our model, the buyer takes into account the impact of its purchasing decisions on future prices, in contrast with the literature, e.g., Besanko and Winston [3]. In addition, our model can be used for further extensions with many buyers and many suppliers, where buyers are not only strategic but can use their market power.

We start by discussing the literature relevant to this work in §2, and turn to the model in §3. We present our results in §4 and analyze supply chain efficiency improvements in §5. We conclude the paper in §6 with a summary of the insights and further research. All the proofs are contained in the appendix.

2 Literature Review

This paper is related to many models of supplier-buyer interactions. These models are generally included in the supply contracts literature, which focuses on aligning supply chain incentives. Cachon [4] provides an excellent review of the field. Pasternack [15], Cachon and Lariviere [5], Barnes-Schuster et al. [2], Eppen and Iyer [8], among others, present supply contracts that move the supply chain towards better coordination. Our model also considers the effect of the supplier-buyer interaction on supply chain efficiency, and in particular, it shows that extending the negotiation length is beneficial to both parties and the supply chain.

More specifically, the model presented here is directly related to Lariviere and Porteus [13], where the buyer's purchased quantity and the supplier's price are analyzed in a single-

interaction setting. Song et al. [17] examine the equilibrium price and quantity decisions for a price-setting newsvendor, and in particular present the same regularity condition on the demand distribution as in Lariviere and Porteus. Van den Berg [20] discusses the properties of the demand distribution that guarantee a well-behaved solution to the supplier's price decision. Perakis and Roels [16] investigate how serious double marginalization can be in a single-period model. For this purpose, they study the worst-case performance of supply chains, among all possible demand distributions, by considering the *price of anarchy*, i.e., the worst-case ratio between profits achieved by a decentralized supply chain and a centralized one.

The dynamic nature of supplier-buyer interactions has also been explored before. Debo and Sun [6] consider an infinitely-repeated game and investigate when supply chain collaboration can be sustained. They find that, when the discount rate for future profits is high, it is more difficult to achieve supply chain collaboration. Anand et al. [1] coin the term "strategic inventory", and show, in a two-period setting with linear price-dependent demand, that a buyer will find it profitable to carry inventory so as to reduce the price quoted by the supplier. Keskinocak et al. [12] analyze a related problem with capacity constraints. Our model uncovers a similar effect. Namely, the price decision of the supplier is driven by the total purchase made by the buyer up to the date, and hence it can be used strategically by the buyer to reduce future prices. Erhun et al. [9] is probably the work that is most similar to ours. They also analyze a multi-period supplier-buyer interaction, with the difference that in their model the demand is deterministic and linear with price, as in Anand et al. [1]. This can be mapped in our framework to having the buyer face a uniform stochastic demand. They observe, as we do, that supply chain efficiency is improved as the negotiation is extended. In contrast, the focus of our work is to study the general relationship between supplier prices and buyer purchases, when demand is not necessarily uniform (i.e., linear in price for Erhun et al. [9]). In particular, without linearity it is no longer guaranteed that the supplier-buyer game has an equilibrium. We hence focus on providing a set of conditions on the demand, for which this type of games can be analyzed. We thus prove some of the observations made in Erhun et al. [9], that suggest that equilibrium exists when the demand is Pareto or exponentially distributed when $T = 2$. In addition, we extend the efficiency study of Erhun et al. to uncover how it depends on the demand distribution.

Finally, some papers from the revenue management literature are also related to ours, as we study the pricing problem of the supplier. Talluri and van Ryzin [19] provide an overview of the literature, and devote one section to price skimming models, which is one of the features of our equilibrium solution. Elmaghraby and Keskinocak [7] also review the literature: our work falls into their replenishment/strategic-customers category, since we have no capacity constraint, and

the buyer considers its effect on the supplier's pricing strategy. With a myopic buyer, Lazear [14] develops a model where demand is constant equal to one unit, but the buyer's valuation is uncertain and uniformly distributed. The buyer places an order as soon as the price is below its valuation. The price schedule that maximizes the expected revenue extracted by the supplier is characterized and decreases over time. Granot et al. [11] extend Lazear's model by introducing competition between suppliers, and show that the price decrease may be exponential, rather than linear. Closer to our work is the model of Besanko and Winston [3], that consider one supplier and many buyers. They introduce the notion of *strategic customers*, i.e., when the buyers anticipate price decreases before placing their orders. They implicitly assume that the buyers have no market power, i.e., their strategy has no impact on the supplier's price. In contrast, since we consider a single buyer, we take into account how the buyer's ordering strategy influences the supplier's prices.

3   The Model

3.1   The Setting

We consider a firm, that we call the *buyer*, that has a single opportunity to serve a stochastic demand D. In order to fulfill the demand, the buyer must install inventory prior to the demand realization. This inventory can be ordered from a *supplier*. If the total order quantity is lower than the demand, then sales are lost; otherwise there is excess inventory that must be discarded for a low salvage value. We denote by f the p.d.f. of the demand, and by F its c.d.f. F. Let $\overline{F} = 1 - F$.

Upstream on the supply chain, the supplier sells to the buyer, at a price that it must choose appropriately. The details of the interaction between supplier and buyer go as follows. There are T negotiation stages, from $t = 1$ (first) to $t = T$ (last, immediately before demand is realized). In each stage, the supplier proposes a price p_t to the buyer, and the buyer buys $q_t \geq 0$. We denote by x_t be the cumulative order of the buyer from period 1 up to $t - 1$, both included. Thus, we have $x_1 = 0$, and x_{T+1} the total quantity purchased through the entire negotiation.

The per-unit cost for the supplier in period t is denoted $c_t \geq 0$. Hence, the supplier's profit can be expressed as $\sum_{t=1}^{T}(p_t - c_t)q_t$.

For the buyer, the supplier's revenue corresponds to a cost. The buyer must also take into account the cost of holding the inventory purchased: we assume that it pays a per-unit cost of h_t for each unit that has been purchased at t or before. The total holding cost can thus be

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written as $\sum_{t=1}^{T} h_t x_t$. Finally, we must in addition consider the revenue obtained at the end of the negotiation. Without loss of generality, let $r = 1$ be the per-unit sales revenue and $v = 0$ the salvage value, which leads to a revenue of $E \min \{D, x_{T+1}\}$. The buyer's profit is hence

$$E \min \{D, x_{T+1}\} - \sum_{t=1}^{T} h_t x_t - \sum_{t=1}^{T} p_t q_t.$$

Buyer and supplier take decisions so as to maximize their respective expected profits. We are interested in determining the subgame-perfect equilibrium of the game that buyer and supplier play, as defined in Fudenberg and Tirole [10]. For this purpose, we consider that the strategies of each player in period t may depend on the current state of the negotiation (since we focus on subgame-perfect equilibrium, players' decisions can only depend on state variables that can influence the subgame from period t to T). Specifically, for each time period t, for each state of the world (this is captured through the cumulative purchase x_t), the supplier sets the price $p_t(x_t)$ that maximizes its profit-to-go given the buyer's strategy; alternatively, for each t, x_t and p_t, the buyer purchases $q_t(p_t, x_t)$ that maximizes its profit-to-go given the supplier's strategy. When $T = 1$, our model corresponds to Lariviere and Porteus [13].

In order to understand the players' decisions, we denote by $B_t(x_t)$ be the maximum expected profit that the buyer can achieve from period t to T, with a stock of x_t at the beginning of period t. This formulation assumes (for now, we prove it later) that both players follow subgame-perfect equilibrium strategies from $t + 1$ to T. Clearly,

$$B_{T+1}(x_{T+1}) = E \min \{D, x_{T+1}\} = \int_0^{x_{T+1}} \overline{F}(a) da.$$

Similarly, we denote by $S_t(x_t)$ the maximum profit that the supplier receives from period $t + 1$ to T when the buyer has a starting stock of x_t at the end time t. We have that $S_{T+1}(x_{T+1}) = 0$, since, when the negotiation is over, the supplier cannot sell to the buyer anymore.

We can describe the buyer's problem in period t, given p_t, as

$$\max_{q_t \geq 0} \left\{ - p_t q_t - h_t x_t + B_{t+1}(x_t + q_t) \right\}. \tag{1}$$

Let $q_t^*(p_t, x_t)$ be the order that maximizes the buyer's profit at time t. Note that when B_{t+1} is concave, the optimal policy is to order up to x_{t+1}, where $B'_{t+1}(x_{t+1}) = p_t$, i.e., $q_t^*(p_t, x_t) = \max \left\{ \left(B'_{t+1} \right)^{-1} (p_t) - x_t, 0 \right\}$.

Using the optimal quantity from Equation (1), the supplier's problem can simply be expressed as

$$S_t(x_t) = \max_{p_t} \left\{ (p_t - c_t) \, q_t^*(p_t, x_t) + S_{t+1} \left(x_t + q_t^*(p_t, x_t) \right) \right\}. \tag{2}$$

From Equation (2), we obtain $p_t^*(x_t)$ and the corresponding $x_{t+1}^*(x_t) = x_t + q_t^*(p_t^*(x_t), x_t)$. With this notation, we have

$$
\begin{aligned}
S_t(x_t) &= (p_t^*(x_t) - c_t)\left(x_{t+1}^*(x_t) - x_t\right) + S_{t+1}\left(x_{t+1}^*(x_t)\right) \\
B_t(x_t) &= -p_t^*(x_t)\left(x_{t+1}^*(x_t) - x_t\right) - h_t x_t + B_{t+1}\left(x_{t+1}^*(x_t)\right)
\end{aligned}
\tag{3}
$$

We observe that the problem's order and price paths depend only on the parameter x_t, the cumulative amount of orders placed before the negotiation stage t. In particular, the supplier implicitly fixes the buyer's order quantity by setting the right price.

Note that for each negotiation stage,

$$
\begin{aligned}
B_t(x_t) + S_t(x_t) &= B_{t+1}(x_{t+1}) + S_{t+1}(x_{t+1}) - c_t(x_{t+1} - x_t) - h_t x_t \\
&= B_{T+1}(x_{T+1}) - \sum_{\tau=t}^{T} c_\tau(x_{\tau+1} - x_\tau) - \sum_{\tau=t}^{T} h_\tau x_\tau.
\end{aligned}
$$

Obviously, the total supply chain profit does not directly depend on the payments between buyer and supplier.

After formulating the supplier-buyer interaction, several questions arise. First, one must ensure that a subgame-perfect equilibrium in pure strategies exists. As in most dynamic games, it is important that this equilibrium is also unique, in order for the value functions B_t and S_t to be uniquely defined. Second, it is important to understand what drives the equilibrium decisions and profits. It is particularly interesting to understand the impact of the length of the negotiation on profits, as this will drive the incentives for buyer and seller to conduct longer or shorter negotiations.

3.2 Example and Intuition

Consider the case of a buyer that faces a stochastic demand uniformly distributed in $[0, 1]$ and that the production cost is $c = 0$ and there is no holding cost. In that case, a centralized supply chain would install inventory up to the maximum demand, i.e., $x = 1$. The supply chain profits would thus be $\mathbb{E}D = 0.5$.

In the decentralized supply chain, supplier and buyer will sequentially decide p_t and q_t so that their respective expected profits are maximized. In general, their decisions will not coincide with the supply chain optimum, i.e., $x_{T+1} < 1$. This phenomenon, double marginalization, will generally occur in our model.

For example, when there is only one negotiation period, $T = 1$, the supplier would set a wholesale of $w = 0.5$, so that the inventory level installed by the buyer is $x = 0.5$. Consequently the profit of the supplier is $wx = 0.25$, while the expected profit of the buyer is $\mathbb{E}\min\{x, D\} - wx = 0.125$. The total supply chain profits are thus 0.375, only 75% of the centralized case.

Consider now the simplest dynamic problem: the situation where there are two negotiation periods, $T = 2$. We can show that in equilibrium, in the first period, the supplier sets a price of $w_1 = 0.5625$, so that the buyer places an order for $q_1 = 0.25$; in the second period, the supplier lowers the wholesale price to $w_2 = 0.375$, and the buyer places an additional order for $q_2 = 0.375$. Thus the total inventory purchased is $x_2 = 0.625$, which yields profits of $w_1 q_1 + w_2 q_2 = 0.28125 > 0.25$ for the supplier and $\mathbb{E}\min\{x, D\} - w_1 q_1 - w_2 q_2 = 0.1484375 > 0.125$. Thus, both supplier and buyer win. More generally, this example suggests that, as the number of negotiation stages increases, supply chain efficiency increases. This is true provided that there are no holding costs and costs are stationary, as seen in §5.

In this example, one may wonder why the buyer places an order at price $w_1 > 0.5$. Indeed, the buyer can perfectly anticipate the decrease in price at the final period. However, its rational choice is to purchase $q_1 > 0$: by placing a positive-quantity order, it takes into account that this will result in a price decrease even larger than if no order was placed. This improves its overall profits. This dynamic interaction is similar to the one derived in Anand et al. [1], where it is optimal for the buyer to initially carry excess inventory in order to force the supplier to decrease prices.

Through the example above, we can see how extending the negotiation length T can benefit both players. In the next section we develop conditions under which the buyer and supplier problems are well-behaved, so that a unique equilibrium exists. Under these conditions, we can characterize the optimal supplier pricing and buyer purchasing strategies.

4 The T-periods Negotiation

4.1 Existence and Uniqueness of Equilibrium

We first need to guarantee that a multi-period equilibrium exists, and is unique. It is guaranteed when the optimality problems in Equations (1) and (2) have interior unique solutions for all $t = 1, \ldots, T$. This is true if and only if:

- for all t and p_t, $B_t(z) - p_t z$ is pseudo-concave in z: in that case, $p_t^*(x_t) = B_{t+1}'(x_{t+1}^*(x_t))$;

- for all t and x_t, $(B_{t+1}'(z) - c)(z - x_t) + S_{t+1}(z)$ is pseudo-concave in z.

It is not clear that these properties are always satisfied by the recursive Equation (3). Some regularity conditions, involving the demand distribution, are hence necessary for pseudo-concavity to be preserved in the recursion. In the single-period setting with $T = 1$, it has been suggested in Lariviere and Porteus [13], Song et al. [17] or van den Berg [20] that it is sufficient that the demand distribution has the IGFR (increasing generalized failure rate) property, i.e.,

that $\dfrac{zf(z)}{\overline{F}(z)}$ is increasing in z. Interestingly, this statement is accurate provided that the starting inventory is zero. In order to extend it to a multi-period situation, the requirement is somewhat stronger: in the last period, $\dfrac{(z-x)f(z)}{\overline{F}(z)}$ has to be increasing in z for all x. Hence, it seems clear that some strong distributional properties are necessary to extend single-period negotiation into a multi-period one.

In order to simplify the exposition, we define for each x_t, the auxiliary variable x_{fin} that represents the final total order quantity:

$$x_{fin} := x^*_{T+1}\Big(x^*_T(\ldots x^*_{t+1}(x_t)\ldots)\Big).$$

Let y_t such that

$$y_t(x_{fin}) = x_t. \tag{4}$$

y_t relates x_t, the starting inventory level at the beginning of period t, to the total order placed from 1 to T, assuming that supplier and buyer follow their optimal strategies from t to T. In addition, it is clear that $y_{T+1}(x_{fin}) = x_{fin}$. Generally, y_t be defined sequentially for $t = T+1$, then for $t = T$, etc.

It turns out that we can rewrite in relatively simple way B_t and S_t as functions of x_{fin}, using the auxiliary function y_t. Let $b_t(x_{fin}) = B_t\left(y_t(x_{fin})\right)$ and $s_t(x_{fin}) = S_t\left(y_t(x_{fin})\right)$. Working with x_{fin} instead of x_{t+1}, we can rewrite the buyer's problem of Equation (1) as

$$\max_{x_{fin} \geq x_t} \Big\{ -p_t(y_{t+1}(x_{fin}) - x_t) - h_t x_t + b_{t+1}(x_{fin}) \Big\}. \tag{5}$$

For the maximization problem to have a unique interior solution, we must have that $-p_t y'_{t+1}(x_{fin}) + b'_{t+1}(x_{fin}) = 0$ has a unique solution, and is positive before, and negative after that solution. It is thus sufficient that

$$u_t(x_{fin}) := \frac{b'_{t+1}(x_{fin})}{y'_{t+1}(x_{fin})} \tag{6}$$

is decreasing in x_{fin}.

In that case, for each p_t, the buyer selects a unique x^*_{fin} such that $u_t(x_{fin}) = p_t$. In particular, $u_T(x_{fin}) = \overline{F}(x_{fin})$.

Using this observation in Equation (2) allows us to rewrite the equation into

$$S_t(x_t) = \max_{x_{fin} \geq x_t} \Big\{ (u_t(x_{fin}) - c_t)(y_{t+1}(x_{fin}) - x_t) + s_{t+1}(x_{fin}) \Big\}. \tag{7}$$

The theorem below provides the conditions to ensure that both the buyer's and the supplier's problem have a unique optimal solution, and that both y_t and u_t are well-defined. In the theorem, we let $c_{T+1} = h_{T+1} = 0$.

Theorem 1 *Define $y_{T+1}(x) = x$ and $u_T \equiv \overline{F}$. For all $t = T, \ldots, 1$, let*

$$y_t(x) := y_{t+1}(x) - \max\left\{0, \frac{\overline{F}(x) - \sum_{\tau=t+1}^{T+1}(c_{\tau-1} - c_\tau + h_\tau)y_\tau'(x)}{-u_t'(x)}\right\} \tag{8}$$

and

$$u_{t-1}(x) := u_t(x) + \max\left\{0, \frac{\overline{F}(x) - \sum_{\tau=t+1}^{T+1}(c_{\tau-1} - c_\tau + h_\tau)y_\tau'(x)}{y_t'(x_{fin})}\right\} - h_t. \tag{9}$$

When for all $t = 1, \ldots, T$, $u_t(x)$ is decreasing and $y_t(x)$ increasing, then there exists a unique subgame-perfect equilibrium in the T-period game. In this equilibrium, if the current inventory position is x_t in period t, the supplier sets a price equal to $p_t^* = u_t(y_t^{-1}(x_t))$ and the buyer purchases $q_t^* = y_{t+1}(y_t^{-1}(x_t)) - x_t$.

The theorem characterizes recursively y_t, that allows us to retrieve the optimal control from the supplier's point of view, and u_t, that determines the buyer's response to the supplier's price. More importantly, it provides a sufficient condition that guarantees that the multi-period supplier-buyer game has an equilibrium. This condition is non-trivial, and has an implicit formulation. For example, when $T = 1$, the sufficient condition is that $u_T = \overline{F}$ is decreasing and that $y_T(x_{fin}) = x_{fin} - \frac{\overline{F}(x_{fin}) - c_T}{f(x_{fin})}$ is increasing.

For the equilibrium to be well defined and unique, we need the demand distribution (through $\overline{F}$) to satisfy some regularity conditions. As t decreases away from T, it becomes increasingly difficult to verify that y_t is increasing and u_t is decreasing. We investigate next some conditions that lead to these desired regularity conditions. For this purpose, we focus on the scenario where the production cost is constant and there are no inventory costs: $h_t = 0$ and $c_t = c \in [0, 1]$, for $t = 1, \ldots, T$. This simpler setting allows us to derive stronger results analytically.

In that case, Equations (8) and (9) become $y_t(x) = x$, $u_t(x) = 0$ when $\overline{F}(x) \le c$, and when $\overline{F}(x) > c$

$$y_t(x) := y_{t+1}(x) - \frac{\overline{F}(x) - c}{-u_t'(x)} \tag{10}$$

and

$$u_{t-1}(x) := u_t(x) + \frac{\overline{F}(x) - c}{y_t'(x)}. \tag{11}$$

4.2 Conditions on the Demand Distribution

Notice that in Equations (10) and (11), the recursion depends on the shape of the demand distribution, through $\overline{F}(x) - c$. We can transform the problem to identify the demand features that lead to an equilibrium. For this purpose, define

$$g(p) = f\left(\overline{F}^{-1}(p + c)\right). \tag{12}$$

Lemma 1 *Consider y_t, u_t satisfying Equations (10) and (11). Let $z_t(p) := \overline{F}^{-1}(p + c) - y_t\left(\overline{F}^{-1}(p + c)\right)$ and $v_t(p) := u_t\left(\overline{F}^{-1}(p + c)\right)$. Then z_t and v_t satisfy $z_{T+1} \equiv 0$, $v_T(p) = p + c$, and for all $t = T, \ldots, 1$,*

$$z_t(p) = z_{t+1}(p) + \frac{p}{g(p)v_t'(p)} \tag{13}$$

and

$$v_{t-1}(p) = v_t(p) + \frac{p}{1 + g(p)z_t'(p)}. \tag{14}$$

If $z_t - \overline{F}^{-1}(p + c)$ and v_t are increasing for all $t = 1, \ldots, T$, there exists a unique subgame-perfect equilibrium in the T-period game.

This reformulation simplifies the analysis. Indeed, both the cost and the demand distribution have been collapsed into a single parameter, the function $g(p) = f\left(\overline{F}^{-1}(p + c)\right)$. In order to use Theorem 1, g must have some regularity properties. Interestingly, this function is related to the log-concavity of the demand distribution. Indeed, g is concave if and only if f'/f is non-increasing, i.e., f is log-concave, since

$$g'(\overline{F}(x) - c) = -\frac{f'(x)}{f(x)}. \tag{15}$$

Note that the demand distribution is log-concave for uniform, exponential, gamma or normal demands, among many others. Next, we solve the recursion of Equations (13) and (14) for selected demand distributions.

Lemma 2 *Consider $g(p) = ap^b$, with $b \leq 2$. Then the solution to the recursive equations (13) and (14) is given by $z_t(p) = \overline{z}_t p^{1-b}$ and $v_t(p) = \overline{v}_t p + c$, where, for all $t = 1, \ldots, T$,*

$$\overline{z}_t = \frac{1}{a(1-b)}\left(\prod_{k=0}^{T-t} \frac{(2-b)(k+1)}{(2-b)k+1} - 1\right) \tag{16}$$

and

$$\overline{v}_t = (2-b)(T-t+1)\left(\prod_{k=0}^{T-t} \frac{(2-b)k+1}{(2-b)(k+1)}\right). \tag{17}$$

The lemma thus provides a closed-form expression for z_t and v_t when $g = ap^b$. Notice that the case with $b = 0$ corresponds to the case of the uniform distribution. The case $b = 1 + 1/\beta$, with $\beta > 1$ corresponds to a Pareto distribution with finite mean, i.e., $\overline{F}(x) = (1 + x)^{-\beta}$, with $c = 0$. The case $b = 1$ corresponds to the exponential distribution with $c = 0$. This leads to the following corollary.

Corollary 1 *When $h_t = 0$ and $c_t = c$, there exists a unique subgame-perfect equilibrium in the T-period game when*

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- *the demand is uniformly distributed;*
- *the demand is Pareto distributed with finite mean and $c = 0$;*
- *the demand is exponentially distributed and $c = 0$.*

In addition, Lemma 2 can be used to establish the properties around 0 of the solutions to Equations (13) and (14) for *any demand distribution.*

Lemma 3 *Consider g such that $g(0) > 0$. Then the solution to the recursive equations (13) and (14) are such that, for all $t = T, \ldots, 1$,*

$$z_t(0) = 0, \ \frac{dz_t}{dp}(0) = \frac{1}{g(0)} \left(\frac{2^{2(T-t+1)}((T-t+1)!)^2}{(2(T-t+1))!} - 1 \right)$$

and

$$v_t(0) = c, \ \frac{dv_t}{dp}(0) = \frac{(2(T-t)+1)!}{2^{2(T-t)}((T-t)!)^2}.$$

Lemma 3 characterizes the slope of the function z_t around $p = 0$. When a unique subgame-perfect equilibrium exists, this result allows us to derive the asymptotic efficiency of the supply chain for large T, see §5,

Lemma 2 is appropriate when the demand distribution is such that f is decreasing, which results in $g(p)$ being an increasing function from Equation (15). In contrast, when f is unimodal, then $g(p)$ is first increasing and then decreasing. While the general analysis in that case is intractable, the following lemma identifies one family of distributions for which a closed-form solution exists.

Lemma 4 *Consider $g(p) = ap\left(1 - \frac{p}{r}\right)$, with $a \geq 0$, $r \geq 1 - c$. Then the solution to the recursive equations (13) and (14) is given by*

$$z_t = \frac{1}{a} \left(\sum_{k=1}^{T+1-t} \frac{1}{k} \left(1 - \frac{p}{r}\right)^{-k} \right)$$

and

$$v_t = c + r \left(1 - \left(1 - \frac{p}{r}\right)^{T+1-t} \right).$$

This lemma implies that for the unimodal demand distribution such that $g(p) = ap\left(1 - \frac{p}{r}\right)$, a unique subgame-perfect equilibrium in the T-period game. Interestingly, this distribution can be chosen to approximate accurately a normal demand distribution. Indeed, consider a normal distribution of average μ and standard deviation σ, and $c = 0$. As shown in Figure 1, $\overline{F}$ can be approximated well by

$$\overline{F^a}(x) = \frac{1}{1 + e^{\frac{x-\mu}{\sigma^a}}}.$$

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where $\sigma^a = \sigma\sqrt{\dfrac{\Pi}{8}}$. This approximation is very accurate for values around the mean, but has heavier tails than the normal distribution.
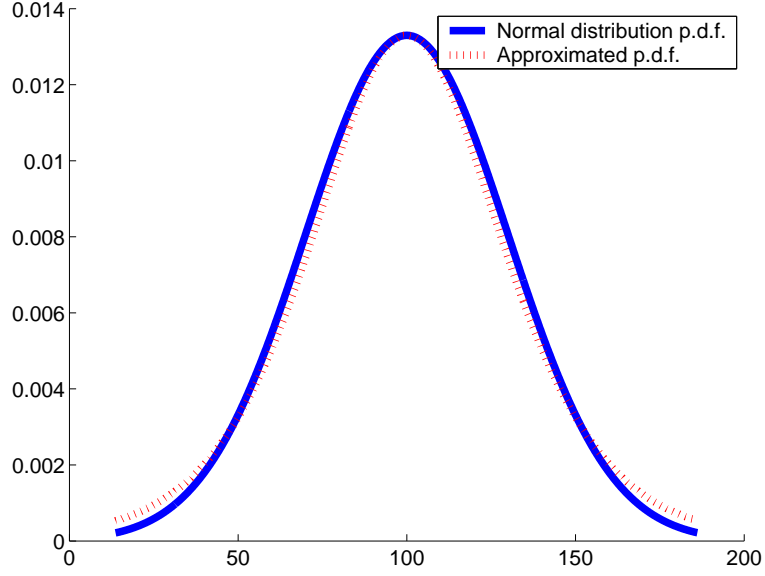

Figure 1: Comparison of the p.d.f. of the normal distribution of mean $\mu = 100$ and standard deviation $\sigma = 30$, with the p.d.f. f^a with $\sigma^a = \sigma\sqrt{\dfrac{\Pi}{8}}$.

For this distribution, $f^a(x) = \dfrac{e^{\frac{x-\mu}{\sigma^a}}}{\sigma^a\left(1 + e^{\frac{x-\mu}{\sigma^a}}\right)^2}$ and when $c = 0$, $g^a(p) = \dfrac{p(1-p)}{\sigma^a}$, for which Lemma 4 can be applied. Hence the result shows that for a demand that is quite similar to the normal distribution, an equilibrium exists.

Finally, to conclude this section, we analyze in detail the solution of the recursion presented in Lemma 1 for the exponential demand. When $\overline{F}(x) = e^{-ax}$ with $a > 0$, then $g(p) = a(p + c)$.

Lemma 5 *Consider $g(p) = a(p + c)$. Then the solution to the recursive equations (13) and (14) is given by*

$$z_t(p) = \frac{POL_t^1\left(\frac{p}{c}\right)}{aPOL_t^2\left(\frac{p}{c}\right)} \quad and \quad v_t(p) = c\left[1 + \frac{POL_t^3\left(\frac{p}{c}\right)}{POL_t^4\left(\frac{p}{c}\right)}\right]$$

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where POL_t^i are polynomials. The sequence of polynomials satisfies the recursion

$$POL_{T+1}^1 = 0, \ POL_{T+1}^2 = 1, \ POL_T^3 = X, \ POL_T^4 = 1$$

$$POL_t^2 = (X+1)POL_{t+1}^2 \left\{ \left(POL_t^3\right)' POL_t^4 - \left(POL_t^4\right)' POL_t^3 \right\}$$

$$POL_t^1 = (X+1)POL_{t+1}^1 \left\{ \left(POL_t^3\right)' POL_t^4 - \left(POL_t^4\right)' POL_t^3 \right\} + XPOL_{t+1}^2 \left(POL_t^4\right)^2$$

$$POL_{t-1}^4 = POL_t^4 \left[\left(POL_t^2\right)^2 + (X+1) \left\{ \left(POL_t^1\right)' POL_t^2 - POL_t^1 \left(POL_t^2\right)' \right\} \right]$$

$$POL_{t-1}^3 = POL_t^3 \left[\left(POL_t^2\right)^2 + (X+1) \left\{ \left(POL_t^1\right)' POL_t^2 - POL_t^1 \left(POL_t^2\right)' \right\} \right] + XPOL_t^4 \left(POL_t^2\right)^2.$$

We present below the first elements of the sequence.

$$POL_T^1 = X,$$
$$POL_T^2 = X + 1,$$
$$POL_{T-1}^3 = X(X+1)(2X+3),$$
$$POL_{T-1}^4 = (X+1)(X+2).$$

$$POL_{T-1}^1 = X(X+1)^3(3X^2 + 12X + 10),$$
$$POL_{T-1}^2 = 2(X+1)^5(X+3),$$
$$POL_{T-2}^3 = 2X(X+1)^9(X+2)^2(6X^3 + 40X^2 + 75X + 45),$$
$$POL_{T-2}^4 = 2(X+1)^9(X+2)^2(2X^3 + 15X^2 + 33X + 24).$$

As t decreases away from T, we obtain a sequence of polynomials with positive coefficients. In addition, we observe that these polynomials are such that $\dfrac{POL_t^1}{POL_t^2}$ and $\dfrac{POL_t^3}{POL_t^4}$ are non-decreasing. These curves are illustrated in Figure 2.

5 Supply Chain Efficiency

In this section, we analyze the gains of supply chain efficiency achieved by extending the length T of the negotiation. For this purpose, we compare the highest supply chain expected profit, achieved by global optimization, to the supply chain expected profit in the decentralized setting, where buyer and supplier have T negotiation periods before facing the demand. We focus again on the case where $c_t = c$, $h_t = 0$ to derive analytical results.

Let Q^* be the optimal centralized quantity, that achieves global optimization: Q^* is such that $\overline{F}(Q^*) = c$. In addition, let SC^* be the corresponding supply chain profit. We compare
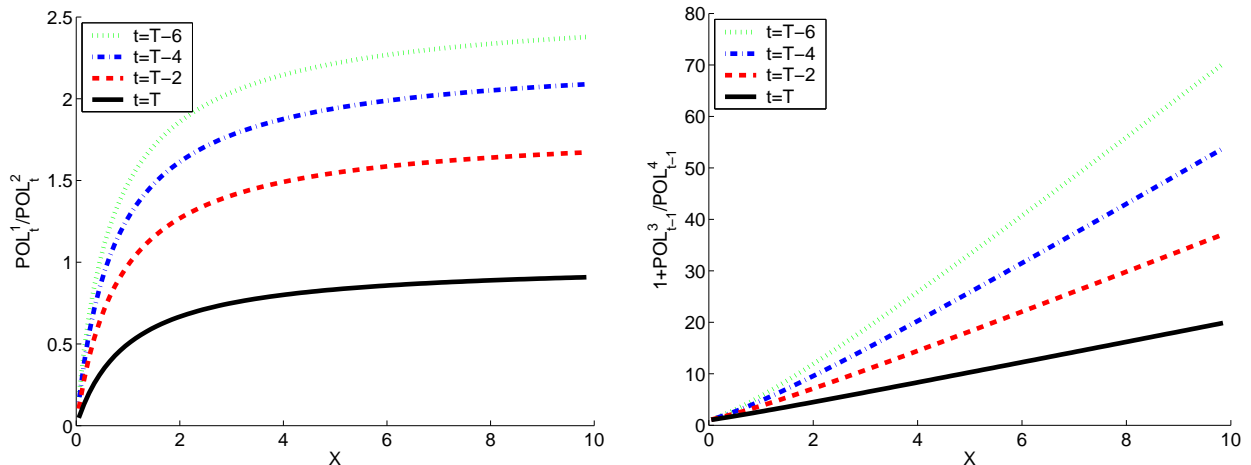
Figure 2: Plot of $\dfrac{POL_t^1}{POL_t^2}$ and $1 + \dfrac{POL_t^3}{POL_t^4}$ for $t = T, T-2, T-4, T-6$. This implies that z_t is increasing concave, and v_t increasing convex.

Q^* and SC^* to Q_T and SC_T, the total ordering quantity and supply chain profit, after T negotiation rounds. Q_T satisfies $y_1(Q_T) = 0$.

Theorem 2 *Consider y_t and u_t defined by Equations (10) and (11) and assume that, for $t = 1, \ldots, T$, $y_t(x)$ is increasing and $u_t(x)$ decreasing for $x < Q^*$. Then Q_T and SC_T are increasing in T. In addition,*

$$\lim_{T \to \infty} Q_T = Q^* \text{ and } \lim_{T \to \infty} SC_T = SC^*.$$

Thus, the efficiency of the supply chain improves with the number of negotiation rounds. In addition, the longer the time horizon, the higher the buyer and the supplier's profits, and hence the higher the supply chain profit. Both players benefit from extending the negotiation. This insight extends the observation made in Anand et al. [1] that a two-period interaction yields higher profits than the single-period scenario.

This result immediately leads to another question: how fast does the ordering quantity Q_T and supply chain profit SC_T converge to the optimal Q^* and SC^*? It turns out that the convergence rate of the ordering quantity is *independent of the demand distribution*, as long some regularity conditions are satisfied, as shown below.

We consider first the uniform distribution in $[D_{min}, D_{max}]$. Applying Lemma 2 with $a = \dfrac{1}{D_{max} - D_{min}}$, $b = 0$, yields that for $z \in [D_{min}, D_{max}]$

$$y_T(x) = x - \left(\frac{2^{2T}(T!)^2}{(2T)!} - 1 \right)(Q^* - x).$$

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where $Q^* = D_{max} - \dfrac{c}{a}$ is the centralized optimal order quantity. The total capacity installed after T negotiation stages Q_T satisfies $y_1(Q_T) = 0$. Solving the algebra yields that $\dfrac{Q^* - Q_T}{Q^*} = \dfrac{(2T)!}{2^{2T}(T!)^2}$. The Stirling factorial approximation allows us to approximate the relative deviation to Q^* for large T, as

$$\frac{Q^* - Q_T}{Q^*} \approx \sqrt{\frac{1}{\Pi T}}, \tag{18}$$

where $\Pi \approx 3.1416$. Furthermore, the supply chain profit can be expressed as $SC_T = \displaystyle\int_0^{Q_T} \overline{F}(t)dt - cQ_T$ while the centralized optimal profit is $SC^* = \displaystyle\int_0^{Q^*} \overline{F}(t)dt - cQ^*$. Thus, we have

$$SC^* - SC_T = \int_{Q_T}^{Q^*} \overline{F}(t)dt - (Q^* - Q_T)\overline{F}(Q^*) = \int_{Q_T}^{Q^*} (t - Q_T)f(t)dt.$$

and since $f(t) = a$ and $SC^* = \dfrac{a(Q^*)^2}{2}$,

$$\frac{SC^* - SC_T}{SC^*} = \left(\frac{(2T)!}{2^{2T}(T!)^2}\right)^2. \tag{19}$$

For $T = 1$, the supply chain inefficiency is thus 25% and for large T, $\dfrac{SC^* - SC_T}{SC^*} \approx \dfrac{1}{\Pi T}$. The supply chain loss of optimality thus decreases with $1/T$.

The split of profit between supplier and buyer can also be calculated. The supplier's profit can be expressed as

$$
\begin{aligned}
s_1(Q_T) &= \sum_{k=1}^T (u_k(Q_T) - c)\left(y_k(Q_T) - y_{k-1}(Q_T)\right) \\
&= a\,(Q^* - Q_T)^2 \sum_{k=1}^T \frac{(2k-1)!}{2^{2k-2}((k-1)!)^2} \frac{2^{2k-2}((k-1)!)^2}{(2k-1)!} \\
&= a\,(Q^* - Q_T)^2\,T \\
&= 2T\left(\frac{Q^* - Q_T}{Q^*}\right)^2 \cdot SC^*
\end{aligned}
$$

As a result, when $T \to \infty$, $s_T(Q_T) \to \dfrac{2}{\Pi}SC^*$, a result contained in Erhun et al. [9]. Also, since $\dfrac{Q^* - Q_1}{Q^*} = \dfrac{1}{2}$, $s_1(Q_1) = \dfrac{1}{2}SC^*$. Thus, the maximum gain achieved by the supplier is $4/\Pi - 1 \approx 27.3\%$. The maximum supply chain gain is $4/3 - 1 = 33.3\%$, while the gain by the buyer is $(4 - 8/\Pi) - 1 \approx 45.6\%$. The extension of the negotiation thus benefits the buyer more than the supplier, and the supply chain share of profit for the supplier goes from $\dfrac{s_1(Q_1)}{SC_1} = \dfrac{2}{3} = 66.6\%$ to $\dfrac{s_T(Q_T)}{SC_T} \to \dfrac{2}{\Pi} \approx 63.7\%$.

Interestingly, the asymptotic behavior of Q_T and SC_T in the general case can be derived from the uniform demand case. Indeed, Lemma 3 shows that, around Q^* (and $p = 0$ by using the transformation proposed in Theorem 1), the functions z_1 and y_1 can be approximated locally by linear functions. This allows us to derive the following result.

Theorem 3 *Consider y_t and u_t defined by Equations (10) and (11). Assume that $f(Q^*) > 0$ and that around Q^*, f is smooth, i.e., infinitely differentiable. Assume also that y_t is increasing and u_t is decreasing for $x < Q^*$. Then, for large T,*

$$\frac{Q^* - Q_T}{Q^*} = \frac{1}{\sqrt{\Pi}} \cdot \frac{1}{\sqrt{T}} + \epsilon^Q \left(\frac{1}{\sqrt{T}} \right)$$

and

$$\frac{SC^* - SC_T}{SC^*} = \frac{f(Q^*)(Q^*)^2}{2SC^*\Pi} \cdot \frac{1}{T} + \epsilon^{SC} \left(\frac{1}{T} \right),$$

where $\epsilon^i(s)/s \to 0$ when $s \to 0$.

This asymptotic result complements the observations of Erhun et al. [9] and Anand et al. [1]. It establishes that not only the outcome of the multi-period negotiation improves supply chain efficiency, but also it provides a technical derivation of the speed of this improvement.

The theorem suggests that Q_T converges to Q^* with the square-root of T. In addition, $1 - \frac{Q_T}{Q^*}$ falls with $\frac{\gamma}{\sqrt{T}}$, where $\gamma = \frac{1}{\sqrt{\Pi}}$, independent of the distribution, and relies only on the fact that the demand p.d.f. is sufficiently smooth near Q^*. Finally, we observe that the sub-optimality gap $1 - \frac{SC_T}{SC^*}$ falls with $\frac{1}{T}$. The convergence coefficient does depend on the demand distribution.

Theorem 3's convergence results are illustrated by the numerical experiments below. We examine the improvement of supply chain efficiency, as a function of the length of the negotiation horizon. We focus on uniform, exponential, normal and Pareto distributions. Interestingly, Perakis and Roels [16] show that, when $T = 1$, the class of Pareto distributions achieves the worst-case sub-optimality gap. As we show below, this gap is rapidly corrected as T increases. Figure 3 (right) shows how, for all four distributions plotted, the sub-optimality gap decreases with $\frac{1}{T}$ approximately. Figure 3 (left) shows the decrease of $1 - \frac{Q_T}{Q^*}$. Figure 4 shows how the sub-optimality gap goes to 0, for several distributions.

Finally, we have compared the share of the supply chain profit going to the buyer. It is relatively stable, as shown in Figure 5. This implies that the additional profit generated by extending the negotiation horizon is shared approximately in a proportional manner, according to the initial split of profit with $T = 1$.
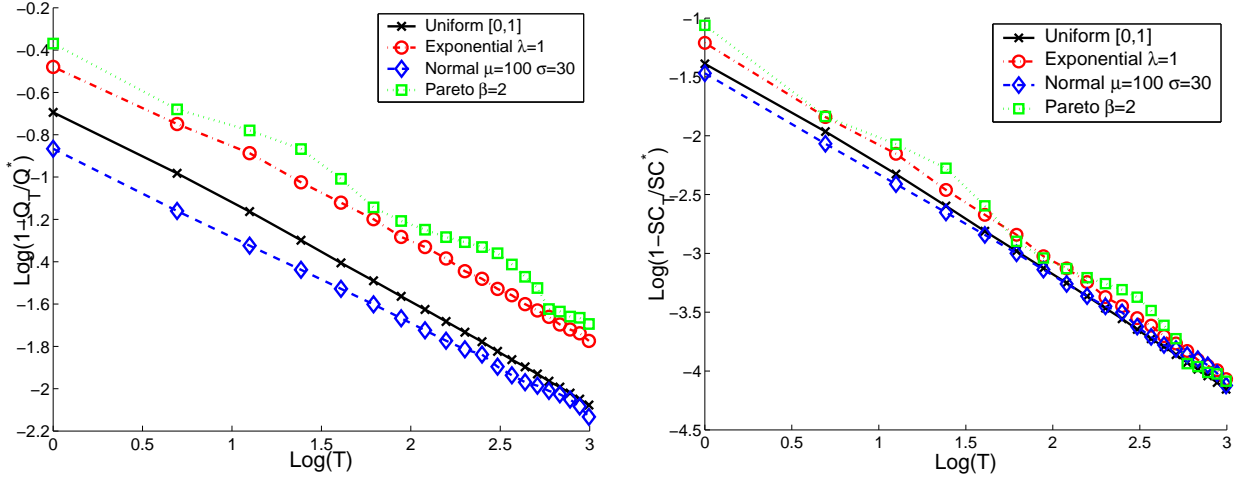
Figure 3: Evolution of $1 - \dfrac{Q_T}{Q^*}$ (left) and $1 - \dfrac{SC_T}{SC^*}$ (right) as a function of T, shown in a log-log scale plot. We show the results for several demand distributions: the uniform [0,1], the exponential of decay rate 1, the normal distribution of mean 100 and standard deviation 30, and the Pareto distribution with $\overline{F}(x) = \dfrac{1}{(1+x)^2}$. We set $c = 0.2$. We observe that the log-log slope is approximately $-1/2$ for the left figure, and -1 for the right figure.

	$T = 1$	$T = 2$	$T = 5$	$T = 20$
Uniform [0,1]	24.9%	14.0%	6.0%	1.6%
Uniform [5,6]	8.8%	3.3%	1.4%	0.4%
Exponential $z = 1$	29.8%	15.9%	6.9%	1.7%
Normal $\mu = 100, \sigma = 30$	23.0%	12.6%	5.8%	1.6%
Normal $\mu = 100, \sigma = 50$	24.4%	15.9%	7.0%	1.7%
Pareto $\beta = 2$	34.6%	15.9%	7.5%	1.7%
Pareto $\beta = 1.1$	29.7%	17.2%	7.4%	1.7%

Figure 4: Optimality gap $1 - \dfrac{SC_T}{SC^*}$, for several demand distributions, and $c = 0.2$. Note that the gap for the uniform [5,6] is much smaller than the rest because there $Q^* = 5.8$, $SC^* = 4.32$, and thus $\dfrac{f(Q^*)(Q^*)^2}{2SC^*} \approx 3.89$, relatively high. This is in contrast with the uniform [0,1], where $\dfrac{f(Q^*)(Q^*)^2}{2SC^*} = 1$.

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	$T = 1$	$T = 2$	$T = 5$	$T = 20$
Uniform [0,1]	33.4%	34.6%	35.6%	36.2%
Uniform [5,6]	0%	2.6%	3.5%	4.2%
Exponential $z = 1$	37.3%	38.4%	39.3%	40.1%
Normal $\mu = 100, \sigma = 30$	19.1%	21.6%	23.2%	23.6%
Normal $\mu = 100, \sigma = 50$	26.9%	25.3%	26.8%	27.4%
Pareto $\beta = 2$	36.6%	39.1%	39.6%	41.2%
Pareto $\beta = 1.1$	45.4%	44.9%	45.1%	45.4%

Figure 5: Share of supply chain profit going to the buyer, for several demand distributions, and $c = 0.2$.

6 Conclusions

In this paper, we have presented a model to analyze repeated supplier-buyer interactions. The buyer faces a stochastic demand, and must purchase inventory to serve this demand before it is realized. The inventory can be ordered from a supplier, over a T-period horizon, where in each period, the supplier chooses the price in its best interest.

We use the concept of subgame perfection to define the equilibrium price (for the supplier) and quantity purchase (for the buyer). We provide sufficient conditions to guarantee that such equilibrium exists and is unique. These conditions are satisfied for several demand distributions including uniform, approximate normal and exponential demand. In the resulting equilibrium, the buyer will place initial orders in order to force the supplier to reduce its prices, a motivation that is similar to the use of strategic inventory in Anand et al. [1].

In addition, we show that supply chain efficiency increases with the length of the negotiation T. Specifically, we show that the sub-optimality gap between the T-periods negotiation and the centralized supply chain falls with $1/T$, regardless of the demand distribution. Thus, for large T, the negotiation situation approaches the highest possible efficiency for the supply chain. Interestingly, our iterative approach provides an asymptotic coordination mechanism with a single profit sharing between buyer and supplier. While it requires a more complex interaction between supplier and buyer, it replicates the effect of a quantity discounts, since the buyer now places orders at different prices with the supplier.

Furthermore, our work presents a number of interesting questions to be explored in the future.

First, our work focuses on the negotiation between one supplier and one buyer, both strategic. The revenue management literature has studied in a different setting the pricing problem

of one supplier pricing against one buyer with probabilistic willingness-to-pay. Since the supplier maximizes its expected profit, this is equivalent to pricing against infinite buyers. This situation has been studied both for myopic buyers, see Lazear [14], and for strategic customers, see Besanko and Wilson [3]. Thus, both the one-buyer situation and the infinite-buyer situation have been studied. The n-buyers situation is an immediate extension of this work.

Second, following Granot et al. [11], the extension to the case of multiple suppliers is also interesting. In that situation, the buyer faces the trade-off between placing orders in the beginning, at a higher price, so that suppliers can offer lower prices, or wait for the suppliers to compete and reduce prices. This new trade-off may change the suppliers' behavior in comparison with the present paper.

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Online Appendix: Proofs

Proof of Theorem 1

Proof. We prove the theorem by recursion for $t = T$ to 1. The induction property at t is the following: if for $\tau = t$ to T, u_τ is decreasing, y_τ is increasing, then there exists a unique subgame-perfect equilibrium for the subgame from t to T that satisfies the properties from the theorem.

The property is clearly true for $t = T$. We assume it is true for $t + 1$, and prove it for t. Let us assume that for $\tau = t$ to T, u_τ is decreasing, y_τ is increasing, and show that there is a unique subgame-perfect equilibrium starting at t.

We first focus on the buyer's decision at t. As mentioned before the theorem, the buyer's problem is well-behaved when u_t is decreasing, which we assumed.

Second, the final order quantity x_{fin} preferred by the supplier is unique when

$$s'_{t+1}(x_{fin}) + b'_{t+1}(x_{fin}) - c_t\Big(y'_{t+1}(x_{fin})\Big) + u'_t(x_{fin})\Big(y_{t+1}(x_{fin}) - x_t\Big) = 0 \qquad (20)$$

has a unique solution no smaller than x_t (or is always negative above x_t).

Noting that $b_t(x_{fin}) + s_t(x_{fin}) = \int_0^{x_{fin}} \overline{F} - \sum_{\tau=t}^{T} c_\tau(y_{\tau+1}(x_{fin}) - y_\tau(x_{fin})) - \sum_{\tau=t}^{T} h_\tau y_\tau(x_{fin})$,

we have that $b'_t(x_{fin}) + s'_t(x_{fin}) = \overline{F}(x_{fin}) - \sum_{\tau=t}^{T} c_\tau(y'_{\tau+1}(x_{fin}) - y'_\tau(x_{fin})) - \sum_{\tau=t}^{T} h_\tau y'_\tau(x_{fin})$. We can rewrite this as

$$b'_t(x_{fin}) + s'_t(x_{fin}) = \overline{F}(x_{fin}) + c_{t-1} y'_t(x_{fin}) - \sum_{\tau=t}^{T+1} (c_{\tau-1} - c_\tau + h_\tau) y'_\tau(x_{fin}).$$

Hence Equation (20) can be expressed as

$$\overline{F}(x_{fin}) - \sum_{\tau=t+1}^{T+1} (c_{\tau-1} - c_\tau + h_\tau) y'_\tau(x_{fin}) + u'_t(x_{fin})\Big(y_{t+1}(x_{fin}) - x_t\Big) = 0$$

Hence, it is sufficient that

$$y_{t+1}(x_{fin}) - \frac{\overline{F}(x_{fin}) - \sum_{\tau=t+1}^{T+1} (c_{\tau-1} - c_\tau + h_\tau) y'_\tau(x_{fin})}{-u'_t(x_{fin})}$$

is increasing, because it is equal to $y_t(x_{fin})$, which was assumed to be increasing. Hence, a unique equilibrium exists in the subgame from t to T.

The characterization of the optimal order quantity is such that

$$x_t = y_t(x_{fin}) = y_{t+1}(x_{fin}) - \frac{\overline{F}(x_{fin}) - \sum_{\tau=t+1}^{T+1} (c_{\tau-1} - c_\tau + h_\tau) y'_\tau(x_{fin})}{-u'_t(x_{fin})}.$$

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when the solution is interior. When the optimal solution is $x_{fin} = x_t$, then $x_t = x_{t+1} = \ldots = x_{fin}$, and hence

$$x_t = y_t(x_{fin}) = y_{t+1}(x_{fin}).$$

This explains Equation (8).

Finally, we can rewrite the buyer's profit, i.e., $b_t(x_{fin})$:

$$b_t(x_{fin}) = -u_t(x_{fin})(y_{t+1}(x_{fin}) - y_t(x_{fin})) - h_t y_t(x_{fin}) + b_{t+1}(x_{fin}) \tag{21}$$

which implies, when the solution is interior, that

$$
\begin{aligned}
b'_t(x_{fin}) = {} & b'_{t+1}(x_{fin}) - u_t(x_{fin})(y'_{t+1}(x_{fin}) - y'_t(x_{fin})) - u'_t(x_{fin})(y_{t+1}(x_{fin}) - y_t(x_{fin})) - h_t y'_t(x_{fin}) \\
= {} & u_t(x_{fin}) y'_{t+1}(x_{fin}) - u_t(x_{fin})(y'_{t+1}(x_{fin}) - y'_t(x_{fin})) \\
& + \overline{F}(x_{fin}) - \sum_{\tau=t+1}^{T+1} (c_{\tau-1} - c_\tau + h_\tau) y'_\tau(x_{fin}) - h_t y'_t(x_{fin})
\end{aligned}
$$

or equivalently

$$u_{t-1}(x_{fin}) = u_t(x_{fin}) + \frac{\overline{F}(x_{fin}) - \sum_{\tau=t+1}^{T+1}(c_{\tau-1} - c_\tau + h_\tau) y'_\tau(x_{fin})}{y'_t(x_{fin})} - h_t.$$

In contrast, when the solution is not interior, $y_t(x_{fin}) = y_{t+1}(x_{fin})$, and hence $b'_t(x_{fin}) = b'_{t+1}(x_{fin}) - h_t y'_t(x_{fin})$, and hence $u_{t-1}(x_{fin}) = u_t(x_{fin}) - h_t$. This provides Equation (9). Note that the recursion can proceed while y_t is increasing and u_t decreasing, the required condition.
∎

Proof of Lemma 1

Proof. This simply involves the change of variables $p = \overline{F}(x) - c$. Thus $z_t(\overline{F}(x) - c) = x - y_t(x)$ and $v_t(\overline{F}(x) - c) = u_t(x)$. z_t and v_t satisfy the recursion stated in the lemma because

$$-f(x) z'_t(\overline{F}(x) - c) = 1 - y'_t(x) \text{ and } -f(x) v'_t(\overline{F}(x) - c) = u'_t$$

and hence

$$z_t(p) = z_{t+1}(p) + \frac{p}{g(p) v'_t(p)} \text{ and } v_{t-1}(p) = v_t(p) + \frac{p}{1 + g(p) z'_t(p)}$$

When v_t and $z_t - \overline{F}^{-1}(p + c)$ functions are increasing in p for $t = 1, \ldots, T$, then y_t is increasing and u_t is decreasing. Applying Theorem 1 yields the existence and uniqueness of equilibrium.
∎

Proof of Lemma 2

Proof. We can verify easily that the recursion given by Equations (13) and (14) is satisfied by $z_t = \overline{z}_t p^{1-b}$ and $v_t = \overline{v}_t p + c$, where $\overline{z}_{T+1} = 0, \overline{v}_T = 1$ and for $t \leq T$,

$$\overline{z}_t = \overline{z}_{t+1} + \frac{1}{a\overline{v}_t}, \quad \overline{v}_{t-1} = \overline{v}_t + \frac{1}{1 + a(1-b)\overline{z}_t}.$$

The coefficients $\overline{z}_t$ and $\overline{v}_t$ can be found observing that

$$\begin{aligned}
\left((1-b)\overline{z}_t + \frac{1}{a}\right)\overline{v}_{t-1} &= \left((1-b)\overline{z}_t + \frac{1}{a}\right)\overline{v}_t + \frac{1}{a} \\
&= \left((1-b)\overline{z}_{t+1} + \frac{1}{a}\right)\overline{v}_t + \frac{2-b}{a}
\end{aligned}$$

Thus, using the initial conditions at $t = T+1$, we have that $\left((1-b)\overline{z}_t + \frac{1}{a}\right)\overline{v}_{t-1} = \frac{(2-b)(T+1-t)+1}{a}$ and hence

$$\overline{v}_{t-1} = \frac{(2-b)(T+1-t)+1}{a(1-b)\overline{z}_t + 1}. \tag{22}$$

In addition, substituting this in the recursion for $\overline{z}_t$ yields $\overline{z}_t = \overline{z}_{t+1} + \frac{(1-b)\overline{z}_{t+1} + \frac{1}{a}}{(2-b)(T-t)+1}$. Thus,

$$(1-b)\overline{z}_t + \frac{1}{a} = \left((1-b)\overline{z}_{t+1} + \frac{1}{a}\right)\left(1 + \frac{1-b}{(2-b)(T-t)+1}\right) = \frac{1}{a}\prod_{k=0}^{T-t}\frac{(2-b)(k+1)}{(2-b)k+1},$$

which implies

$$\overline{z}_t = \frac{1}{a(1-b)}\left(\prod_{k=0}^{T-t}\frac{(2-b)(k+1)}{(2-b)k+1} - 1\right).$$

Substituting this expression in Equation (22) yields for $t \leq T$,

$$\overline{v}_t = [(2-b)(T-t)+1]\left(\prod_{k=0}^{T-t-1}\frac{(2-b)k+1}{(2-b)(k+1)}\right) = (2-b)(T-t+1)\left(\prod_{k=0}^{T-t}\frac{(2-b)k+1}{(2-b)(k+1)}\right)$$

∎

Proof of Lemma 3

Proof. The recursion around $p = 0$ yields

$$\begin{aligned}
z_t(0) &= z_{t+1}(0) + 0 \\
z'_t(0) &= z'_{t+1}(0) + \frac{1}{g(0)v'_t(0)} + 0 \\
v_{t-1}(0) &= v_t(0) + 0 \\
v'_{t-1}(0) &= v'_t(0) + \frac{1}{1 + g(0)z'_t(0)} + 0,
\end{aligned}$$

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which results on the recursion used in Lemma 2 with $b = 0$, i.e., $z_t'(0) = \bar{z}_t$ and $v_t'(0) = \bar{v}_t$. ∎

Proof of Lemma 4

Proof. We prove the expression for z_t, v_{t-1} by induction. It is true for $t = T+1$. If it is true for $t + 1 \leq T+1$, at time t, we have:

$$v_t' = (T + 1 - t)\left(1 - \frac{p}{r}\right)^{T-t}.$$

Hence,

$$z_t = \left(\frac{1}{a}\right)\left(\sum_{k=1}^{T-t} \frac{1}{k}\left(1 - \frac{p}{r}\right)^{-k}\right) + \frac{1}{a\left(1 - \frac{p}{r}\right)(T + 1 - t)\left(1 - \frac{p}{r}\right)^{T-t}} = \left(\frac{1}{a}\right)\left(\sum_{k=1}^{T+1-t} \frac{1}{k}\left(1 - \frac{p}{r}\right)^{-k}\right)$$

which yields

$$
\begin{aligned}
z_t' &= \left(\frac{1}{ar}\right)\left(\sum_{k=1}^{T+1-t}\left(1 - \frac{p}{r}\right)^{-k-1}\right) = \left(\frac{1}{ar}\right)\left(1 - \frac{p}{r}\right)^{-2}\left(\frac{\left(1 - \frac{p}{r}\right)^{-(T+1-t)} - 1}{\left(1 - \frac{p}{r}\right)^{-1} - 1}\right) \\
&= \left(\frac{1}{ap}\right)\left(1 - \frac{p}{r}\right)^{-1}\left[\left(1 - \frac{p}{r}\right)^{-(T+1-t)} - 1\right].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
v_{t-1} &= c + r\left(1 - \left(1 - \frac{p}{r}\right)^{T+1-t}\right) + \frac{p}{1 + ap\left(1 - \frac{p}{r}\right)z_t'} \\
&= c + r\left(1 - \left(1 - \frac{p}{r}\right)^{T+1-t}\right) + \frac{p}{1 + \left(1 - \frac{p}{r}\right)^{-(T+1-t)} - 1} \\
&= c + r\left(1 - \left(1 - \frac{p}{r}\right)^{T+1-t}\right) + r\left(\frac{p}{r} - 1 + 1\right)\left(1 - \frac{p}{r}\right)^{T+1-t} \\
&= c + r\left(1 - \left(1 - \frac{p}{r}\right)^{T+1-(t-1)}\right).
\end{aligned}
$$

This completes the induction. ∎

Proof of Lemma 5

Proof. The result is derived by induction and is quite straightforward: if

$$z_{t+1}(p) = \frac{POL_{t+1}^1\left(\frac{p}{c}\right)}{a POL_{t+1}^2\left(\frac{p}{c}\right)} \quad \text{and} \quad v_t(p) = c\left[1 + \frac{POL_t^3\left(\frac{p}{c}\right)}{POL_t^4\left(\frac{p}{c}\right)}\right]$$

then letting $q = \dfrac{p}{c}$,

$$z_t(p) = \frac{POL^1_{t+1}}{aPOL^2_{t+1}} + \frac{q}{a\left(q+1\right)\dfrac{\left(POL^3_t\right)' POL^4_t - POL^3_t\left(POL^4_t\right)'}{\left(POL^4_t\right)^2}}$$

$$= \frac{(q+1)POL^1_{t+1}\left\{\left(POL^3_t\right)' POL^4_t - POL^3_t\left(POL^4_t\right)'\right\} + qPOL^2_{t+1}\left(POL^4_t\right)^2}{a(q+1)POL^2_{t+1}\left\{\left(POL^3_t\right)' POL^4_t - POL^3_t\left(POL^4_t\right)'\right\}}$$

and

$$v_{t-1}(p)$$
$$= c + \frac{cPOL^3_t}{POL^4_t} + \frac{cq}{1 + a(q+1)\dfrac{\left(POL^1_t\right)' POL^2_t - POL^1_t\left(POL^2_t\right)'}{a\left(POL^2_t\right)^2}}$$

$$= c\left[1 + \frac{POL^3_t\left[\left(POL^2_t\right)^2 + (q+1)\left\{\left(POL^1_t\right)' POL^2_t - POL^1_t\left(POL^2_t\right)'\right\}\right] + qPOL^4_t\left(POL^2_t\right)^2}{POL^4_t\left[\left(POL^2_t\right)^2 + (q+1)\left\{\left(POL^1_t\right)' POL^2_t - POL^1_t\left(POL^2_t\right)'\right\}\right]}\right].$$

■

Proof of Theorem 2

Proof. From Theorem 1, we have that y_t is decreasing in t, and u_t increasing in t. Thus, the solution to $y_1(x) = 0$, that characterizes Q_T after T negotiation rounds, is increasing in T. As a result, $SC_T = b_1(Q_T) + s_1(Q_T) = \int_0^{Q_T} \overline{F} - cQ_T$ also increases in T.

Finally, as a function of T, Q_T increases but cannot grow larger than Q^*, since $p_T > c$ always. As a result, it converges to a finite limit. This limit $\overline{Q}$ can be calculated from the recursion: it satisfies Equation (8) taken for large $T - t$, where $y_{t-1}(\overline{Q}) = y_t(\overline{Q}) = \overline{Q}$:

$$\overline{Q} = \overline{Q} + \frac{\overline{F}(\overline{Q}) - c}{-u'_t(\overline{Q})},$$

which can only hold when $\overline{F}(\overline{Q}) - c = 0$, i.e., $\overline{Q} = Q^*$. ■

Proof of Theorem 3

Proof. Assuming that f is sufficiently smooth around Q^*, e.g., when it is infinitely differentiable near Q^*, the Taylor expansion of z_1 around $p = 0$, using Lemma 3, is

$$z_1(p) = \frac{1}{f(Q^*)} \left(\frac{2^{2T}(T!)^2}{(2T)!} - 1 \right) p + \epsilon^z(p),$$

where ϵ^i denote functions such that $\epsilon^i(p)/p \to 0$ when $p \to 0$. Using the reverse transformation of Lemma 1, around $x = Q^*$, we have

$$y_1(x) = x - \left(\frac{2^{2T}(T!)^2}{(2T)!} - 1 \right)(Q^* - x) + \epsilon^y(Q^* - x).$$

As a result, the solution to $y_1(Q_T) = 0$ can be also approximated, so that when $T \to \infty$:

$$\frac{Q^* - Q_T}{Q_T} \left(\frac{(2T)!}{2^{2T}(T!)^2} \right)^{-1} \to 1.$$

In addition, using the Stirling approximation, we have that

$$\left(\frac{(2T)!}{2^{2T}(T!)^2} \right) \sqrt{\Pi T} \to 1.$$

This yields the result for $Q^* - Q_T$. The approximation of $SC^* - SC_T$ follows from

$$SC^* - SC_T = \int_{Q_T}^{Q^*} (x - Q_T) f(x) dx = \frac{f(Q^*)(Q^* - Q_T)^2}{2} + \epsilon^{SC} \left((Q^* - Q_T)^2 \right).$$

∎