Exponential stability of switched linear hyperbolic initial-boundary value problems

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***Abstract*—We consider the initial-boundary value problem governed by systems of linear hyperbolic partial differential equations in the canonical diagonal form and study conditions for exponential stability when the system discontinuously switches between a finite set of modes. The switching system is fairly general in that the system matrix functions as well as the boundary conditions may switch in time. We show how the stability mechanism developed for classical solutions of hyperbolic initial boundary value problems can be generalized to the case in which weaker solutions become necessary due to arbitrary switching. We also provide an explicit dwell-time bound for guaranteeing exponential stability of the switching system when, for each mode, the system is exponentially stable. Our stability conditions only depend on the system parameters and boundary data. These conditions easily generalize to switching systems in the non-diagonal form under a simple commutativity assumption. We present tutorial examples to illustrate the instabilities that can result from switching.**

***Index Terms*—Distributed parameter systems; stability of hybrid systems; switched systems.**

I. Introduction

Switched systems are a convenient modeling paradigm for a variety of control applications in which evolution processes involve logical decisions. However, in contrast to their simplicity on modeling grounds, the stability analysis of switched systems is often non-trivial. An extensive body of literature now exists for the case of switched (linear and non-linear) ordinary differential equations (ODEs) and more generally for differential algebraic equations (DAEs) in finite dimensional spaces. As surveyed in [1] and [2], two different approaches have been mainly considered in the literature: Either one designs switching signals such that solutions of the switched system decay exponentially (or otherwise behave 'optimally'), or one tries to identify conditions which guarantee exponential stability of the switched system for arbitrary switching signals. The later approach is of particular interest when the switching mechanism is either unknown or too complicated for a more careful stability analysis [3], [4]. Stability under arbitrary switching is mainly achieved by constructing common Lyapunov functions or, more directly, by identifying algebraic/geometric conditions on the involved parameters.

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During the past years, several attempts have been made to also consider switched systems in the context of infinite dimensional control theory. Mostly, the problem of designing (optimal or stabilizing) switching control is considered for problems in which the state equation is fixed and just the controller is switched. For example, in [5], model reduction together with control synthesis for the reduced finite dimensional model is used to construct switching control for quasi-linear parabolic equations. The design of boundary switching control actions for semi-linear hyperbolic balance equations using switching time sensitivities is considered in [6]. An algorithm to construct optimal switching control for abstract linear systems on Hilbert spaces with switching control operator at fixed switching times is proposed in [7]. Moreover, for the heat equation, a systematic way of building switching control based on variational methods is described in [8] and, in a similar context, [9] gives conditions under which such switching controls exist for the one dimensional wave equation.

Despite the aforementioned developments, much less is known for problems when not only the controller, but also the state equation is switched. Some general ideas are sketched in [10] and, for semi-linear hyperbolic equations with application to transport networks, optimal open-loop and closed-loop switching control is addressed in [11] and [12]. For problems concerning the stability of switched infinite dimensional systems, the construction of common Lyapunov functions gets very difficult when the state equation is switched, even for abstract switched linear systems on Hilbert spaces. The only available result appears to be [13], in which a common quadratic Lyapunov function is provided for the case when the semigroup generators commute. This condition is, however, too restrictive for some applications. Nevertheless, it is interesting to note that without further restrictions on the generators, common (not necessarily quadratic) Lyapunov functions exist, even more generally for switched linear systems on Banach spaces [14]. Under constrained switching, some algebraic conditions for stability of switched non-linear systems on Banach spaces utilizing Lyapunov functions in each mode are provided in [15].

In this article we are interested in the stability properties of solutions to switched linear hyperbolic systems with reflecting boundary conditions when the boundary conditions and the state equation are switched arbitrarily. Let us first introduce the following (unswitched) system of n linear hyperbolic partial differential equations (PDEs) defined for some interval $[a, b] \subset \mathbb{R}$:

$$\frac{\partial \xi}{\partial t} + \Lambda(s)\frac{\partial \xi}{\partial s} + B(s)\xi = 0, \quad s \in (a, b),\ t > 0, \qquad (1)$$

where $\Lambda(s) = \mathrm{diag}(\lambda_1(s), \ldots, \lambda_n(s))$ is a diagonal real matrix function and $B(s)$ is a $n \times n$ real matrix function on $[a, b]$. Assuming appropriate regularity of the matrix functions $\Lambda(\cdot)$ and $B(\cdot)$ and under the hyperbolicity assumption that for some $1 < m < n$

$$\lambda_1(s), \ldots, \lambda_m(s) < 0 \text{ and } \lambda_{m+1}(s), \ldots, \lambda_n(s) > 0 \quad (2)$$

uniformly in $s \in [a, b]$, a n-dimensional vector solution $\xi(t, s)$ of the system (1) with components $\xi_i(t, s)$ for $i = 1, \ldots, n$, arrayed as

$$\xi_I(t, s) = (\xi_1(t, s), \ldots, \xi_m(t, s))^\top$$

and

$$\xi_{II}(t, s) = (\xi_{m+1}(t, s), \ldots, \xi_n(t, s))^\top,$$

is uniquely determined on the time-space strip $\mathbb{R}_+ \times (a, b)$ with the initial condition

$$\xi(0, s) = \bar{\xi}(s), \quad s \in (a, b), \quad (3)$$

for specified $\mathbb{R}^n$-valued initial data $\bar{\xi}(s)$ and boundary conditions

$$\xi_{II}(t, a) = G_L \xi_I(t, a), \ \xi_I(t, b) = G_R \xi_{II}(t, b), \quad t \geq 0 \quad (4)$$

where G_L, G_R are constant matrices of dimensions $(n - m) \times m$ and $m \times (n - m)$, respectively. A common class of problems studied for initial-boundary value problems (1)–(4) is the stability and stabilization under boundary control actions specified by the matrices G_L and G_R. These problems are of interest because hyperbolic PDE systems can model flows in networks that are monitored and controlled at the boundary nodes [16]. Examples include transportation systems [17], [18], canal systems [19], and gas distribution systems [20]. The available results for this class of problems for linear hyperbolic systems can be found in [21], [22], and more generally for quasilinear hyperbolic systems in [23], [24], [25] and [26].

Here we are interested in the stability properties of the hyperbolic initial boundary value problem (1)–(4) when $\Lambda(\cdot)$, $B(\cdot)$, G_L and G_R are not fixed, but are known to satisfy

$$(\Lambda(\cdot), B(\cdot), G_L, G_R) \in \{(\Lambda^j(\cdot), B^j(\cdot), G_L^j, G_R^j) : j \in Q\}$$

at any time $t > 0$, where $Q = \{1, \ldots, N\}$ is a finite set of modes and, for all $j \in Q$, the data $\Lambda^j(\cdot), B^j(\cdot), G_L^j, G_R^j$ is given. This is equivalent to studying the stability of the switching system

$$\begin{cases} \dfrac{\partial \xi}{\partial t} + \Lambda^{\sigma(t)}(s) \dfrac{\partial \xi}{\partial s} + B^{\sigma(t)}(s) \xi = 0, \\ \xi_{II}(t, a) = G_L^{\sigma(t)} \xi_I(t, a), \ \xi_I(t, b) = G_R^{\sigma(t)} \xi_{II}(t, b), \\ \xi(0, s) = \bar{\xi}(s), \end{cases} \quad (5)$$

for the time-space strip $[0, \infty) \times [a, b]$ where switching occurs according to a piecewise-constant switching signal $\sigma(\cdot) \colon \mathbb{R}_+ \to Q$. Preliminaries and wellposedness of the switched system (5) will be discussed in Section II. Then, recalling the classical observation in the finite dimensional control theory of switched systems that exponential stability of all subsystems does not necessarily guarantee an exponential decay of the solution when the system is switched [3], we study, motivated by a simple PDE counterpart to this observation, the following two specific problems for the switched system (5) in Section III:

(A) Find conditions on the matrix functions $\Lambda^j(\cdot)$, $B^j(\cdot)$ and the matrices G_L^j and G_R^j that guarantee exponential stability for arbitrary switching signals.
(B) Alternatively, characterize a (preferably large) class of switching signals for which exponential stability of all subsystems is sufficient for exponential stability of the switched system.

Our contribution here is twofold. Firstly, we show how the techniques mainly developed for classical solutions (with C^1 data) can be used for weaker solutions (with L^∞ data) based on the geometric picture of propagation along characteristics. This is necessary because switching boundary conditions may introduce discontinuities into the solution. Secondly, we show how the switching enters the known stability mechanism such that the decay rate obtained in this way is independent of the switching signal (Theorem 1). Following from our analysis, we also obtain an explicit dwell-time bound guaranteeing exponential stability of the system under constrained switching when all subsystems satisfy the known stability condition individually (Corollary 1). In Section IV, we discuss how our results for switched diagonal system (5) generalize to switched hyperbolic systems in non-diagonal form under a commutativity assumption (Proposition 1). In Sections III and IV, we also provide illustrative examples of instabilities which can result from switching. Some final remarks are mentioned in Section V.

II. Preliminaries

For an interval $(a, b) \subset \mathbb{R}$ and a measurable function $f \colon (a, b) \to \mathbb{R}^n$, let

$$\|f\|_\infty := \operatorname*{ess\,sup}_{\substack{s \in (a,b) \\ i = 1, \ldots, n}} |f_i(s)|.$$

We call $L^\infty((a, b); \mathbb{R}^n)$ the space of all measurable functions $f \colon (a, b) \to \mathbb{R}^n$ for which $\|f\|_\infty < \infty$. For an $n \times n$ real matrix $M = (m_{ij})$, we define

$$\|M\|_\infty := \max_{1 \leq i \leq n} \sum_{j=1}^{n} |m_{ij}|.$$

Also define the non-negative matrix of M as $|M| := (|m_{ij}|)$ and for eigenvalues $\lambda_1, \ldots, \lambda_n$ of $|M|$ define the spectral radius of $|M|$ as $\rho(|M|) = \max_{1 \leq i \leq n} |\lambda_i|$.

A *switching signal* $\sigma(\cdot)$ is a piecewise-constant function $\sigma(\cdot) \colon \mathbb{R}_+ \to Q$. Here, we restrict admissible piecewise-constant signals to those for which during each finite time interval of $\mathbb{R}_+$, there are only finitely many switches $j \curvearrowright j'$ to avoid *Zeno behavior*. This assumption anticipated with the accumulation of switching times is commonly made in the field of switched and hybrid systems to obtain global existence results; see for e. g. [27]. Thus, necessarily, $\sigma(\cdot)$ has *switching times* $\tau_k \in \mathbb{R}_+$ ($k \in \mathbb{N}$) at which $\sigma(\cdot)$ switches discontinuously from one mode $j_{k-1} \in Q$ to another mode $j_k \in Q$. We denote $\mathcal{S}(\mathbb{R}_+, Q)$ for the set of all such switching signals $\sigma(\cdot)$.

We say that for a given $\sigma(\cdot) \in \mathcal{S}(\mathbb{R}_+, Q)$ the system (5) is *exponentially stable* (with respect to the norm $\|\cdot\|_\infty$) if there exist constants $c \geq 1$ and $\beta > 0$ such that the solution $\xi(t,\cdot)$ satisfies

$$\|\xi(t,\cdot)\|_\infty \leq c\exp(-\beta t)\|\xi(0,\cdot)\|_\infty,\ t \geq 0. \tag{6}$$

In view of problem (A), we say that the switched system (5) is *absolutely exponentially stable* (with respect to a norm $\|\cdot\|_\infty$) if (6) holds for all $\sigma(\cdot) \in \mathcal{S}(\mathbb{R}_+, Q)$ with constants $c \geq 1$ and $\beta > 0$ independently of $\sigma(\cdot)$. In view of problem (B), we say that a value $\tau > 0$ is a *dwell-time* of a switching signal $\sigma(\cdot)$, if the intervals between consecutive switches are no shorter than τ, that is, $\tau_{k+1} - \tau_k \geq \tau$ for all $k > 0$ and we let $\mathcal{S}_\tau(\mathbb{R}_+; Q) \subset \mathcal{S}(\mathbb{R}_+; Q)$ denote the subset of switching signals with dwell-time τ.

III. Diagonal Switching System

For each $j \in Q$, we have the diagonal subsystem

$$\begin{cases} \dfrac{\partial \xi^j}{\partial t} + \Lambda^j(s)\dfrac{\partial \xi^j}{\partial s} + B^j(s)\xi^j = 0,\ s \in (a,b),\ t > 0 \\ \xi^j_{II}(t,a) = G^j_L \xi^j_I(t,a),\ \xi^j_I(t,b) = G^j_R \xi^j_{II}(t,b),\ t \geq 0 \end{cases} \tag{7}$$

for which we impose the following assumptions:

- (A_1) The matrix function $\Lambda^j(s) = \mathrm{diag}(\lambda^j_1(s), \ldots, \lambda^j_n(s))$ is such that the characteristic speeds $\lambda^j_i(\cdot)$ are uniformly bounded, Lipschitz-continuous functions of $s \in [a,b]$ for $i = 1, \ldots, n$, and there exists m_j such that for some $0 < m_j < n$, $\lambda^j_r(s) < 0$ ($r = 1, \ldots, m_j$) and $\lambda^j_l(s) > 0$ ($l = m_j + 1, \ldots, n$); the matrix function $B^j(s)$ is such that $B^j(\cdot) : [a,b] \mapsto \mathbb{R}^{n\times n}$ is bounded measurable with respect to s.
- (A_2) For all $j, j' \in Q$, $m_j = m_{j'} =: m$.

It is well-known that under the hyperbolicity assumption (A_1) for any $j \in Q$, $T > 0$, and initial data $\xi^j(0,\cdot) = \bar{\xi}^j(\cdot)$ where $\bar{\xi}^j : (a,b) \mapsto \mathbb{R}^n$ is bounded measurable with respect to s, a solution ξ^j_i of (7) in the *broad sense* can be defined by the method of characteristics [28], [29]. In this method, for each i and each point (t^*, s^*), one uses that the ODE

$$\frac{d}{dt} z^j_i(t) = \lambda^j_i(z^j_i(t)), \quad z^j_i(t^*) = s^* \tag{8}$$

has a unique Carathéodory solution, defined for all t. As usual, we say that this solution $t \mapsto z^j_i(t; t^*, s^*)$ passing through (t^*, s^*) is the i-th *characteristic curve* for the j-th subsystem. The broad solution $\xi^j(\cdot,\cdot)$ is then defined as a vector function with components ξ^j_i, $i = 1, \ldots, n$, that are absolutely continuous and satisfy

$$\begin{aligned} \frac{d}{dt}\xi^j_i(t, z^j_i(t; t^*, s^*)) = \\ -\sum_{k=1}^n b^j_{ik}(z^j_i(t; t^*, s^*))\xi^j_k(t, z^j_i(t; t^*, s^*)) \end{aligned} \tag{9}$$

along almost every characteristic curve $z^j_i(t; t^*, s^*)$. Here $b^j_{ik}(\cdot)$ corresponds to the i-th row and k-th column of $B^j(\cdot)$.

Existence and uniqueness of such broad solutions $\xi^j(\cdot,\cdot)$ with initial data and boundary conditions for the subsystems (7) with $\xi^j(t,\cdot) \in L^\infty((a,b);\mathbb{R}^n)$ for all t can be obtained on arbitrary finite time horizons using Banach's fixed point theorem. Uniqueness then has to be understood within the usual Lebesgue almost everywhere equivalence class. For further details on the existence and uniqueness of broad solutions, we refer to the iteration method of [28], pages 470–475, and to the text of Bressan [30], pages 46–50, though noting that the latter does not treat boundary conditions. For treatment of the boundary conditions see, instead, [29].

We now justify the existence and uniqueness of solutions for the switching system (5), which we need in deriving the main stability result in Section III. Any switching signal $\sigma(\cdot) \in \mathcal{S}(\mathbb{R}_+, Q)$ defines a mode $j_k \in Q$ for each interval $[\tau_k, \tau_{k+1})$. For an initial condition, $\bar{\boldsymbol{\xi}} := \bar{\xi}(\cdot) \in L^\infty((a,b);\mathbb{R}^n)$, we define $\boldsymbol{\xi}(t) = \xi(t,\cdot)$ where

$$\xi(t,\cdot) = \xi^{j_k}(t,\cdot), \quad \text{for } t \in [\tau_k, \tau_{k+1}]$$

and $\xi^{j_k}(t,\cdot)$ is a solution of the subsystem corresponding to mode $j = j_k$ in (7) with the initial condition

$$\xi^{j_k}(\tau_k,\cdot) = \begin{cases} \xi^{j_{k-1}}(\tau_k,\cdot) & \text{if } k > 0, \\ \bar{\xi}(\cdot) & \text{if } k = 0. \end{cases}$$

Thus, under Hypothesis (A_1), for every $\sigma(\cdot) \in \mathcal{S}(\mathbb{R}_+, Q)$, by construction there exists a unique broad solution $\boldsymbol{\xi}(\cdot)$ with data $\boldsymbol{\xi}(t) \in L^\infty((a,b);\mathbb{R}^n))$ for all $t \in \mathbb{R}_+$ of the switching system (5). Again, uniqueness then has to be understood within the usual Lebesgue almost everywhere equivalence class.

In the following, we denote by $z_i^{\sigma(t)}(t; t^*, s^*)$ the i-th characteristic path that passes through a point $(t^*, s^*) \in [0,\infty) \times [a,b]$ and is the concatenation of the characteristic curves $z^j_i(t)$ through switching times defined by the switching signal $\sigma(\cdot)$. When needed, we omit the dependence of $z_i^{\sigma(t)}(t; t^*, s^*)$ on $\sigma(t)$ for notational convenience and simply write $z_i(t; t^*, s^*)$.

Observe that, if (A_2) holds in addition to (A_1), each characteristic path can be classified into left- and right-going depending on the sign of the corresponding characteristic speeds $\lambda^j_i(s)$, independently of the switching signal $\sigma(\cdot)$. Although (A_2) is not required for the existence and uniqueness of the solution, it is crucial for the kind of stabilizing mechanisms that we consider here. This is further discussed in Example 3.

Furthermore, for the switching system (5) we define

$$\bar{\tau} := \frac{b-a}{\min\limits_{\substack{i=1,\ldots,m_j \\ s\in[a,b], j\in Q}} |\lambda^j_i(s)|} + \frac{b-a}{\min\limits_{\substack{i=m_j+1,\ldots,n \\ s\in[a,b], j\in Q}} |\lambda^j_i(s)|} \tag{10}$$

Geometrically, $\bar{\tau}$ is an upper bound of the time in which the slowest of all possible characteristic paths will have undergone reflections at both boundaries.

Our motivation to study the stability of the diagonal switching system (5) is inspired a simple PDE counterpart to the classical ODE observation [3] that exponential stability of all subsystems is *not* sufficient for the exponential stability of the switching system.

Example 1: Let $Q = \{1,2\}$, $[a,b] = [0,1]$, $\Lambda^j = \mathrm{diag}(-1,1)$, $B^j = \mathrm{diag}(0,0)$, $G^j_L = 1.5(j-1)$, $G^j_R = 1.5(2-j)$, and consider $\bar{\xi}(s) = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ for $s \in (0,1)$. For the case of no switching, that is when $\sigma(t) = 1$ or $\sigma(t) = 2$ for all $t \in \mathbb{R}_+$, the solution $\boldsymbol{\xi}(\cdot)$ of the system (5) is zero after $t > 2$,

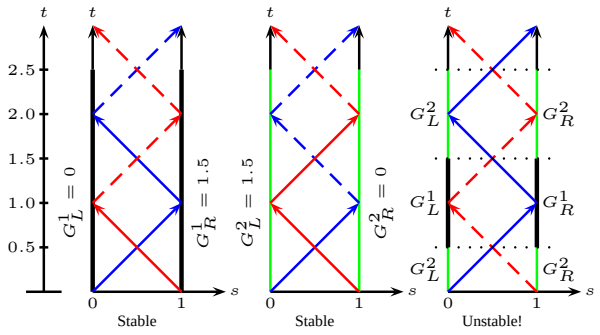

Fig. 1. Instability by switching according to Example 1.

but the solution of the system with a switching signal $\sigma(t)$ that is defined over the switching times $\tau_k = 0.5, 1.5, 2.5, \ldots$ and alternates between modes in Q starting with $\sigma(0) = 2$ is not exponentially stable. Indeed, $\|\boldsymbol{\xi}(t)\|_\infty$ is not bounded as $t \to \infty$, because the values on the right-going characteristic emerging from $s \in (0, 0.5)$ always increase by reflection of the characteristics along the boundary; see Figure 1. Thus, we can conclude that the instability due to switching can occur for certain combinations between the characteristic speeds and the switching times. (Note, however, that with a switching signal $\sigma(t)$ that is defined over the switching times $\tau_k = 0.5, 1.0, 1.5, 2.0, \ldots$ the system is exponentially stable.) □

We now focus on conditions on the matrix functions $\Lambda(s)^j$, $B^j(s)$ and the boundary data G_L^j, G_R^j under which the switching system is absolutely exponentially stable. Our main result, presented next, shows that if a spectral radius condition is jointly satisfied for the left and right boundary data and all pairs of modes $j, j' \in Q$ then a sufficiently small bound on $\|B^j(s)\|_\infty$ exists such that the switching system is absolutely exponentially stable with respect to the norm $\|\cdot\|_\infty$.

Theorem 1: Assume Hypotheses (A_1) and (A_2) and suppose that for $j, j' \in Q$ the following condition holds:

$$\rho\left(\begin{bmatrix} 0 & |G_R^{j'}| \\ |G_L^j| & 0 \end{bmatrix}\right) < 1. \tag{11}$$

Then there exists an $\epsilon > 0$ such that if $\|B^j(s)\|_\infty \le \epsilon$ for all $s \in [a, b]$ and $j \in Q$, the switching system (5) is absolutely exponentially stable with respect to the norm $\|\cdot\|_\infty$.

Proof: We define the following constants in terms of boundary data

$$\begin{gathered} K_1 := \max\{1, \tilde{K}_1\},\ K_2 := \max\{1, \tilde{K}_2\} \\ K := \max\{K_1, K_2\} \end{gathered} \tag{12}$$

where

$$\tilde{K}_1 = \max_{\substack{r=1,\ldots,m \\ j\in Q}} \sum_{l=m+1}^{n} |g_{rl}^{R,j}|$$

and

$$\tilde{K}_2 = \max_{\substack{l=m+1,\ldots,n \\ j\in Q}} \left\{ \sum_{p=1}^{m} |g_{lp}^{L,j}| \right\}.$$

From the Lemma 2.1 of Li [23], we note that the condition (11) implies

$$\begin{aligned} \theta &:= \max_{j,j'\in Q} \left\{ \left\| |G_L^j||G_R^{j'}| \right\|_\infty, \left\| |G_R^{j'}||G_L^j| \right\|_\infty \right\} \\ &= \max_{\substack{r=1,\ldots,m \\ l=m+1,\ldots,n \\ j,j'\in Q}} \left\{ \sum_{p=1}^{m} \sum_{k=m+1}^{n} |g_{rk}^{R,j'}||g_{kp}^{L,j}|, \right. \\ &\qquad \left. \sum_{k=m+1}^{n} \sum_{p=1}^{m} |g_{lp}^{L,j}||g_{pk}^{R,j'}| \right\} < 1, \end{aligned} \tag{13}$$

where $G_L^j = (g_{pq}^{L,j})$ and $G_R^{j'} = (g_{pq}^{R,j'})$. Let us define

$$T_{\min} := \frac{b-a}{\max\limits_{\substack{a\le s\le b \\ i=1,\ldots,n \\ j=1,\ldots,N}} |\lambda_i^j(s)|}, \quad T_{\max} := \frac{b-a}{\min\limits_{\substack{a\le s\le b \\ i=1,\ldots,n \\ j=1,\ldots,N}} |\lambda_i^j(s)|}.$$

Thus, $T_{\min}$ (resp. $T_{\max}$) is the time in which the fastest (resp. slowest) of all possible characteristic paths will have traveled the domain (a, b).

Under the assumption of the theorem, we choose a $c \ge 1$ such that

$$c := \frac{K}{\theta}, \tag{14}$$

and we choose an ω such that $\theta < \omega < 1$, and select a $\beta > 0$ such that

$$\beta := \frac{1}{2T_{\max}} \ln\left(\frac{\omega}{\theta}\right). \tag{15}$$

We also choose an $\eta > 0$ such that $\theta < \omega < \eta < 1$, and select an $\epsilon > 0$ such that

$$\epsilon := \min\{\epsilon_1, \epsilon_2\} \tag{16}$$

where

$$\epsilon_1 = \frac{\theta}{T_{\min} K \omega} \ln\left(\frac{\eta}{\omega}\right), \quad \epsilon_2 = \frac{\theta(1-\eta)}{2T_{\max} K \eta} \ln\left(\frac{\omega}{\theta}\right).$$

We will show that under the aforementioned assumptions and the choice of constants, if the bound

$$\|B^j(s)\|_\infty \le \epsilon \tag{17}$$

holds for $s \in [a, b]$ and for all $j \in Q$, then

$$\|\boldsymbol{\xi}(t)\|_\infty \le c \exp(-\beta t) \|\bar{\xi}\|_\infty,\ t \ge 0 \tag{18}$$

uniformly for all switching signals $\sigma(\cdot) \in \mathcal{S}(\mathbb{R}_+, Q)$. Note that the chosen c, β, and ϵ are independent of $\sigma(\cdot)$ and only depend on the boundary data and system parameters. We will prove (18) using the method of characteristics and induction. To this end, we will first prove the induction basis in *Part A* and the induction step in *Part B*. We define

$$\widehat{\|\boldsymbol{\xi}(t)\|}_\infty := \exp(\beta t) \|\boldsymbol{\xi}(t)\|_\infty. \tag{19}$$

Part A (Proof of the induction basis): We show that under the chosen constants $\beta > 0, c \ge 1, \epsilon > 0$, (18) holds on the domain $[0, \delta] \times (a, b)$ when δ satisfies $0 \le \delta < T_{\min}$. For any $\sigma(\cdot)$, let $z_i(t; t^*, s^*)$ denote the i-th characteristic path passing

through the point $(t^*, s^*) \in [0,\delta]\times(a,b)$, $(i = 1,\dots,n)$. Then, we have

$$\frac{dz_i(t;t^*,s^*)}{dt} = \lambda_i^{\sigma(t)}(z_i(t;t^*,s^*))$$
$$z_i(t^*;t^*,s^*) = s^*.$$

For any fixed $r = 1,\dots,m$, consider the r-th characteristic path $z_r(t;t^*,s^*)$ passing through (t^*,s^*). Under the assumptions (A_1) and (A_2), backwards in time, $z_r(t;t^*,s^*)$ either intersects $t = 0$ within the interval $[a,b]$ before hitting any boundary (case A.1) or it intersects the line $s = b$ (case A.2). See Figure 2 for an illustration of both possible cases. The point of intersection of the characteristic path with the boundary of the domain is denoted by $(0, z_r(0;t^*,s^*))$ for case A.1 and $(t_r(t^*,s^*),b)$ for case A.2 with $z_r(t_r(t^*,s^*);t^*,s^*) = b$. Furthermore, let $z_l(t;t_r(t^*,s^*),b)$ denote the l-th characteristic path passing through $(t_r(t^*,s^*),b)$ $(l = m+1,\dots,n)$. Then, since $\delta < T_{\min}$, $z_l(t;t_r(t^*,s^*),b)$ intersects the line $t = 0$ before hitting the line $s = a$. We denote the point of intersection by $(0, z_l(0;t_r(t^*,s^*),b))$. For the ease of notation, we will use t_r for $t_r(t^*,s^*)$.

Estimate for paths with negative slope: We first obtain an estimate of $e^{\beta t^*}|\xi_r(t^*,s^*)|$ for any $(t^*,s^*)\in[0,\delta]\times(a,b)$ by considering cases A.1 and A.2 for the r-th characteristic path $z_r(t;t^*,s^*)$ passing through (t^*,s^*) $(r = 1,\dots,m)$.

For case A.1: Using $j = \sigma(t)$ in (9), and integrating the r-th equation from 0 to t^* for any $r = 1,\dots,m$ we get

$$\xi_r(t^*,s^*) = \xi_r(0,\tilde{s}_1) - \int_0^{t^*}\sum_{k=1}^{n} b_{rk}^{\sigma(t)}(z_r(t))\xi_k(t,z_r(t))dt$$

where we use the notation $\tilde{s}_1$ for $z_r(0;t^*,s^*)$ and $z_r(t)$ for $z_r(t;t^*,s^*)$. Using the bound (17), we obtain

$$|\xi_r(t^*,s^*)| \le \|\bar{\xi}\|_\infty + \epsilon\int_0^{t^*}\|\boldsymbol{\xi}(t)\|_\infty dt.$$

Multiplying both sides by $e^{\beta t^*}$, and noting that $t^* \le \delta < T_{\min}$, we obtain

$$e^{\beta t^*}|\xi_r(t^*,s^*)| \le e^{\beta t^*}\|\bar{\xi}\|_\infty + \epsilon\int_0^{t^*} e^{\beta(t^*-t)}\widehat{\|\boldsymbol{\xi}(t)\|}_\infty dt$$
$$\le C_1\|\bar{\xi}\|_\infty + C_2\int_0^{t^*}\widehat{\|\boldsymbol{\xi}(t)\|}_\infty dt \qquad (20)$$

where $C_1 = e^{\beta T_{\min}}$ and $C_2 = e^{\beta T_{\min}}\epsilon$.

For case A.2: Integrating the r-th equation from t_r to t^* we get

$$\xi_r(t^*,s^*) = \xi_r(t_r,b) - \int_{t_r}^{t^*}\sum_{k=1}^{n} b_{rk}^{\sigma(t)}(z_r(t))\xi_k(t,z_t(t))dt.$$

Using $\xi_r(t_r,b) = \sum_{l=m+1}^{n} g_{rl}^{R,j}\xi_l(t_r,b)$ with $j = \sigma(t_r)$,

$$|\xi_r(t^*,s^*)| \le \sum_{l=m+1}^{n}|g_{rl}^{R,j}||\xi_l(t_r,b)| + \epsilon\int_{t_r}^{t^*}\|\boldsymbol{\xi}(t)\|_\infty dt. \quad (21)$$

Integrating l-th equation from 0 to t_r we get

$$\xi_l(t_r,b) = \xi_l(0,\tilde{s}_2) - \int_0^{t_r}\sum_{k=1}^{n} b_{lk}^{\sigma(t)}(z_l(t))\xi_k(t,z_l(t))dt,$$

where we use the notation $\tilde{s}_2$ for $z_l(0;t_r(t^*,s^*),b)$ and $z_l(t)$ for $z_l(t;t_r,b)$. Again using the bound (17),

$$|\xi_l(t_r,b)| \le \|\bar{\xi}\|_\infty + \epsilon\int_0^{t_r}\|\boldsymbol{\xi}(t)\|_\infty dt$$

Substituting this bound in equation (21), we obtain

$$|\xi_r(t^*,s^*)|$$
$$\le \tilde{K}_1\|\bar{\xi}\|_\infty + \epsilon\tilde{K}_1\int_0^{t_r}\|\boldsymbol{\xi}(t)\|_\infty dt + \epsilon\int_{t_r}^{t^*}\|\boldsymbol{\xi}(t)\|_\infty dt$$
$$\le K_1\|\bar{\xi}\|_\infty + K_1\epsilon\int_0^{t^*}\|\boldsymbol{\xi}(t)\|_\infty dt$$

where $\tilde{K}_1$ and K_1 are defined in (12). Multiplying by $e^{\beta t^*}$ and noting again that since $t^* \le \delta < T_{\min}$, we have

$$e^{\beta t^*}|\xi_r(t^*,s^*)| \le C_3\|\bar{\xi}\|_\infty + C_4\int_0^{t^*}\widehat{\|\boldsymbol{\xi}(t)\|}_\infty dt \qquad (22)$$

with $C_3 = K_1 e^{\beta T_{\min}}$ and $C_4 = K_1\epsilon e^{\beta T_{\min}}$.

Combination of cases A.1, A.2.: From inequalities (20) and (22) we obtain a combined estimate

$$e^{\beta t^*}|\xi_r(t^*,s^*)| \le C_5\|\bar{\xi}\|_\infty + C_6\int_0^{t^*}\widehat{\|\boldsymbol{\xi}(t)\|}_\infty dt \qquad (23)$$

with $C_5 = K_1 e^{\beta T_{\min}}$ and $C_6 = K_1\epsilon e^{\beta T_{\min}}$.

Estimate for paths with positive slope: Similarly, we can write an estimate of $e^{\beta t^*}|\xi_l(t^*,s^*)|$ $(l = m+1,\dots,n)$ for $(t^*,s^*)\in[0,\delta]\times(a,b)$ by considering the corresponding cases for l-th characteristic path $z_l(t;t^*,s^*)$ passing through (t^*,s^*) $(l = m+1,\dots,n)$. We have

$$e^{\beta t^*}|\xi_l(t^*,s^*)| \le C_7\|\bar{\xi}\|_\infty + C_8\int_0^{t^*}\widehat{\|\boldsymbol{\xi}(t)\|}_\infty dt \qquad (24)$$

with $C_7 = K_2 e^{\beta T_{\min}}$ and $C_8 = K_2\epsilon e^{\beta T_{\min}}$, where K_2 is defined in (12).

Estimate for all paths: From (23) and (24), by taking the maximum over r-th and l-th characteristic paths $(r = 1,\dots,m$ and $l = m+1,\dots,n)$ respectively, and taking the essential supremum over $s^*\in(a,b)$ we obtain the estimate

$$\|\widehat{\boldsymbol{\xi}(t^*)}\|_\infty \le C_9\|\bar{\xi}\|_\infty + C_{10}\int_0^{t^*}\widehat{\|\boldsymbol{\xi}(t)\|}_\infty dt \qquad (25)$$

with $C_9 = Ke^{\beta T_{\min}}$ and $C_{10} = K\epsilon e^{\beta T_{\min}}$, where K is defined in (12).

Now, by using c as defined in (14) and noting that $T_{\min} < 2T_{\max}$, we can write

$$\|\widehat{\boldsymbol{\xi}(t^*)}\|_\infty \le C_{11}\|\bar{\xi}\|_\infty + C_{12}\int_0^{t^*}\widehat{\|\boldsymbol{\xi}(t)\|}_\infty dt \qquad (26)$$

with $C_{11} = c\theta e^{2\beta T_{\max}}$ and $C_{12} = K\epsilon e^{\beta T_{\min}}$. By applying Gronwall's lemma, we obtain the inequality for any $(t^*,s^*)\in[0,\delta]\times(a,b)$

$$\widehat{\|\boldsymbol{\xi}(t^*)\|}_\infty \le C_{11}\exp(C_{12}T_{\min})\|\bar{\xi}\|_\infty, \qquad (27)$$

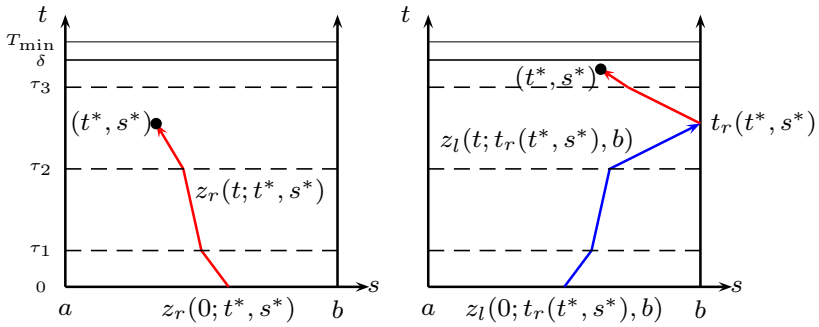

Fig. 2. Illustration of case A.1 (left) and case A.2 (right) for the proof of induction basis.

for all $\sigma(\cdot)$. With the $\beta > 0$ and $\epsilon > 0$ chosen according to (15) and (16) respectively, we note that

$$\theta \exp(2\beta T_{\max}) \exp\left(K\epsilon T_{\min} \exp(\beta T_{\min})\right) \leq \omega \exp\left(K T_{\min} \epsilon_1 \frac{\omega}{\theta}\right) = \eta < 1.$$

Then by expanding the right-hand-side of inequality (27) we obtain that

$$\begin{aligned} &\widehat{\|\boldsymbol{\xi}(t^*)\|}_\infty \\ &\quad \leq c\left[\theta \exp(2\beta T_{\max}) \exp\left(K\epsilon T_{\min} \exp(\beta T_{\min})\right)\right] \|\bar{\xi}\|_\infty \\ &\quad < c\|\bar{\xi}\|_\infty \end{aligned}$$

holds on $(t^*, s^*) \in [0, \delta] \times (a, b)$ for all switching signals $\sigma(\cdot) \in \mathcal{S}(\mathbb{R}_+, Q)$. Finally, using the definition (19), we obtain that

$$\|\boldsymbol{\xi}(t)\|_\infty \leq c \exp(-\beta t)\|\bar{\xi}\|_\infty, \quad 0 \leq t \leq \delta < T_{\min}.$$

This completes the proof of the induction basis.

Part B (Proof of the induction step): We will now show that under the chosen constants $\beta > 0, c \geq 1, \epsilon > 0$, if (18) holds on the domain $[0, T] \times (a, b)$, then it still holds on domain $[0, T + T_{\min}] \times (a, b)$. Let $T > 0$ and assume that (18) holds on $[0, T] \times (a, b)$. In this case we have to distinguish three cases as illustrated in Figure 3.

Proceeding as before, for any fixed $r = 1, \ldots, m$, the r-th characteristic path $z_r(t; t^*, s^*)$ passing through (t^*, s^*) considered backward in time, either intersects $t = 0$ within the interval $[a, b]$ before hitting any boundary (case B.1) or it intersects the line $s = b$ (case B.2); the points of intersection with the boundary of the domain are denoted by $(0, z_r(0; t^*, s^*))$ and $(t_r(t^*, s^*), b)$ respectively, where $z_r(t_r(t^*, s^*); t^*, s^*) = b$. Furthermore, the l-th characteristic path $z_l(t; t_r(t^*, s^*), b)$ passing through $(t_r(t^*, s^*), b)$ $(l = m + 1, \ldots, n)$ either $z_l(t; t_r(t^*, s^*), b)$ intersects the line $t = 0$ before hitting the line $s = a$ (case B.2(i)) or it hits $s = a$ (case B.2(ii)). The point of intersection is denoted by $(0, z_l(0; t_r(t^*, s^*), b))$ for case B.2(i) and $(t_{rl}(t^*, s^*), a)$ for case B.2(ii). We will again use t_r for $t_r(t^*, s^*)$ and t_{rl} for $t_{rl}(t^*, s^*)$.

Estimate for paths with negative slope: We first obtain an estimate of $e^{\beta t^*}|\xi_r(t^*, s^*)|$ for any $(t^*, s^*) \in [T + T_{\min}] \times (a, b)$ by considering the above three cases for the r-th characteristic path $z_r(t; t^*, s^*)$ passing through (t^*, s^*) $(r = 1, \ldots, m)$.

For case B.1: Using $j = \sigma(t)$ in (9), and integrating the r-th equation from 0 to t^* for any $r = 1, \ldots, m$, and using the bound (17),

$$\begin{aligned} |\xi_r(t^*, s^*)| &\leq \|\bar{\xi}\|_\infty + \epsilon \int_0^{t^*} \|\boldsymbol{\xi}(t)\|_\infty dt \\ &\leq \left(1 + \frac{\epsilon c}{\beta}\right) \|\bar{\xi}\|_\infty + \epsilon \int_T^{t^*} \|\boldsymbol{\xi}(t)\|_\infty dt \end{aligned}$$

where the second inequality is obtained using the assumption that (18) holds on $[0, T] \times (a, b)$. Multiplying both sides by $e^{\beta t^*}$, using definition (19); and noting that for the present situation (case B.1), we have $t^* \leq T_{\max}$, then $T \leq T_{\max}$, and for $t^* \in [T, T + T_{\min}]$, $t \in (T, t^*)$ then $(t^* - t) \leq T_{\min}$, we obtain

$$e^{\beta t^*}|\xi_r(t^*, s^*)| \leq C_{13}\|\bar{\xi}\|_\infty + C_{14} \int_T^{t^*} \widehat{\|\boldsymbol{\xi}(t)\|}_\infty dt \quad (28)$$

with $C_{13} = (1 + \frac{\epsilon c}{\beta}) e^{\beta T_{\max}}$ and $C_{14} = \epsilon e^{\beta T_{\min}}$.

For case B.2: Again integrating the r-th equation from t_r to t^*, and using that $\xi_r(t_r, b) = \sum_{l=m+1}^{n} g_{rl}^{R,j} \xi_l(t_r, b)$ with $j = \sigma(t_r)$,

$$|\xi_r(t^*, s^*)| \leq \sum_{l=m+1}^{n} |g_{rl}^{R,j}||\xi_l(t_r, b)| + \epsilon \int_{t_r}^{t^*} \|\boldsymbol{\xi}(t)\|_\infty dt. \quad (29)$$

For case B.2(i): Integrating l-th equation from 0 to t_r and using the bound (17) we have

$$|\xi_l(t_r, b)| \leq \|\bar{\xi}\|_\infty + \epsilon \int_0^{t_r} \|\boldsymbol{\xi}(t)\|_\infty dt$$

Substituting this bound in equation (29), we obtain

$$\begin{aligned} &|\xi_r(t^*, s^*)| \\ &\quad \leq \tilde{K}_1 \|\bar{\xi}\|_\infty + \epsilon \tilde{K}_1 \int_0^{t_r} \|\boldsymbol{\xi}(t)\|_\infty dt + \epsilon \int_{t_r}^{t^*} \|\boldsymbol{\xi}(t)\|_\infty dt \\ &\quad \leq K_1 \|\bar{\xi}\|_\infty + K_1 \epsilon \int_0^{t^*} \|\boldsymbol{\xi}(t)\|_\infty dt \\ &\quad \leq K_1 \left(1 + \frac{\epsilon c}{\beta}\right) \|\bar{\xi}\|_\infty + K_1 \epsilon \int_T^{t^*} \|\boldsymbol{\xi}(t)\|_\infty dt, \end{aligned}$$

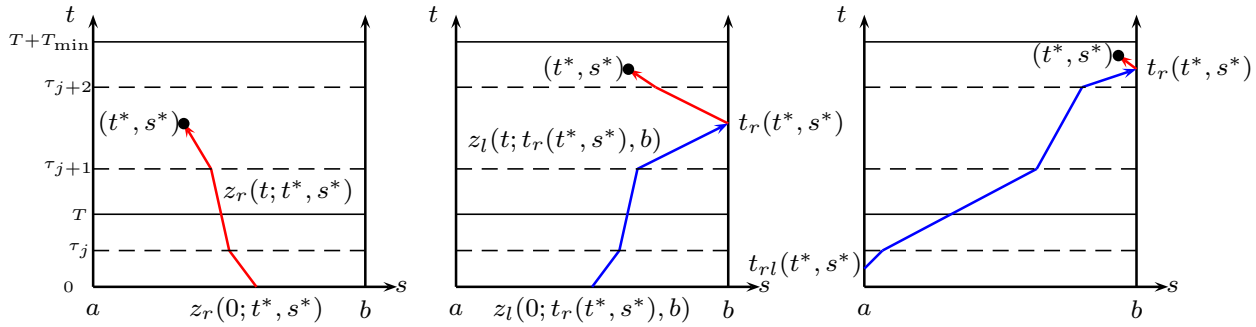

Fig. 3. Illustration of case B.1 (left), case B.2(i) (center) and case B.2(ii) (right) for the proof of induction step.

where the last inequality is obtained using the assumption that (18) holds on $[0,T]\times(a,b)$. Noting that for the present situation (case B.2(i)), $t^*\le 2T_{\max}$ then $T\le 2T_{\max}$, and for $t^*\in[T,T+T_{\min}]$, $t\in(T,t^*)$ then $(t^*-t)\le T_{\min}$ we obtain

$$e^{\beta t^*}|\xi_r(t^*,s^*)|\le C_{15}\|\bar\xi\|_\infty+C_{16}\int_T^{t^*}\widehat{\|\boldsymbol{\xi}(t)\|}_\infty dt \qquad (30)$$

with $C_{15}=K_1\left(1+\frac{\epsilon c}{\beta}\right)e^{2\beta T_{\max}}$ and $C_{16}=K_1\epsilon e^{\beta T_{\min}}$.

For case B.2(ii): We have

$$\xi_l(t_r,b)=\xi_l(t_{rl},a)-\int_{t_{rl}}^{t_r}\sum_{k=1}^{n}b_{lk}^{\sigma(t)}(z_l(t))\xi_k(t,z_l(t))dt.$$

Using $\xi_l(t_{rl},a)=\sum_{p=1}^m g_{lp}^{L,j'}\xi_p(t_{rl},a)$ with $j'=\sigma(t_{rl})$, we have

$$|\xi_l(t_r,b)|\le\sum_{p=1}^{m}|g_{lp}^{L,j'}||\xi_p(t_{rl},a)|+\epsilon\int_{t_{rl}}^{t_r}\|\boldsymbol{\xi}(t)\|_\infty dt$$

Substituting this bound in equation (29), and using the induction hypothesis, we obtain

$$\begin{aligned}|\xi_r(t^*,s^*)|&\le\theta ce^{-\beta t_{rl}}\|\bar\xi\|_\infty+K_1\epsilon\int_{t_{rl}}^{t^*}\|\boldsymbol{\xi}(t)\|_\infty dt\\&\le\left(\theta+\frac{K_1\epsilon}{\beta}\right)ce^{-\beta t_{rl}}\|\bar\xi\|_\infty+K_1\epsilon\int_T^{t^*}\|\boldsymbol{\xi}(t)\|_\infty dt\end{aligned}$$

with θ as in (13). Again, noting that for $t^*\in[T,T+T_{\min}]$, in the present situation (case B.2(ii)), $0\le t_{rl}\le T$ and $T-t_{rl}\le 2T_{\max}$, we obtain

$$e^{\beta t^*}|\xi_r(t^*,s^*)|\le C_{17}\|\bar\xi\|_\infty+C_{18}\int_T^{t^*}\widehat{\|\boldsymbol{\xi}(t)\|}_\infty dt \qquad (31)$$

with $C_{17}=c\left(\theta+\frac{K_1\epsilon}{\beta}\right)e^{2\beta T_{\max}}$ and $C_{18}=K_1\epsilon e^{\beta T_{\min}}$.

Combination of cases B.1, B.2(i) and B.2(ii): From inequalities (28), (30) and (31) and the K defined in (12), we obtain

$$e^{\beta t^*}|\xi_r(t^*,s^*)|\le C_{19}\|\bar\xi\|_\infty+C_{20}\int_T^{t^*}\widehat{\|\boldsymbol{\xi}(t)\|}_\infty dt, \qquad (32)$$

with $C_{19}=c\left(\theta+\frac{K\epsilon}{\beta}\right)e^{2\beta T_{\max}}$ and $C_{20}=K\epsilon e^{\beta T_{\min}}$.

Estimate for paths with positive slope: By using similar arguments, we also obtain an estimate of $e^{\beta t^*}|\xi_l(t^*,s^*)|$ for any $(t^*,s^*)\in[T+T_{\min}]\times(a,b)$ by considering the corresponding cases for the l-th characteristic path $z_l(t;t^*,s^*)$ passing through (t^*,s^*) $(l=m+1,\dots,n)$, for c chosen according to (14), and K defined in (12)

$$e^{\beta t^*}|\xi_l(t^*,s^*)|\le C_{21}\|\bar\xi\|_\infty+C_{22}\int_T^{t^*}\widehat{\|\boldsymbol{\xi}(t)\|}_\infty dt, \qquad (33)$$

with $C_{21}=c\left(\theta+\frac{K\epsilon}{\beta}\right)e^{2\beta T_{\max}}$, $C_{22}=K\epsilon e^{\beta T_{\min}}$.

Estimate for all paths: We now combine (32) and (33) by taking the maximum over r-th and l-th characteristic paths ($r=1,\dots,m$ and $l=m+1,\dots,n$) respectively, taking the essential supremum over $s^*\in(a,b)$ to obtain the estimate

$$\widehat{\|\boldsymbol{\xi}(t^*)\|}_\infty\le C_{23}\|\bar\xi\|_\infty+C_{24}\int_T^{t^*}\widehat{\|\boldsymbol{\xi}(t)\|}_\infty dt \qquad (34)$$

where $C_{23}=c\left(\theta+\frac{K\epsilon}{\beta}\right)e^{2\beta T_{\max}}$, $C_{24}=K\epsilon e^{\beta T_{\min}}$. By applying Gronwall's lemma, we obtain the inequality

$$\widehat{\|\boldsymbol{\xi}(t^*)\|}_\infty\le C_{23}\exp(C_{24}(t^*-T))\|\bar\xi\|_\infty \qquad (35)$$

for any $(s^*,t^*)\in[T,T+T_{\min}]\times(a,b)$ and thus

$$\widehat{\|\boldsymbol{\xi}(t^*)\|}_\infty\le C_{23}\exp(C_{24}T_{\min}))\|\bar\xi\|_\infty$$

for all $\sigma(\cdot)$. Using (34), plugging in the expressions for C_{23} and C_{24}, given the expression of β in (15) we obtain the inequality

$$\widehat{\|\boldsymbol{\xi}(t^*)\|}_\infty\le c\left(\theta+\frac{K\epsilon}{\beta}\right)\frac{\omega}{\theta}\exp\left(K\epsilon\frac{\omega}{\theta}T_{\min}\right)\|\bar\xi\|_\infty. \qquad (36)$$

With β and ϵ given by (15) and (16) respectively, we have

$$\begin{aligned}&\left(\theta+\frac{K\epsilon}{\beta}\right)\frac{\omega}{\theta}\exp\left(K\epsilon\frac{\omega}{\theta}T_{\min}\right)\\&\qquad\le\left(\theta+\frac{K\epsilon_2}{\beta}\right)\frac{\omega}{\theta}\exp\left(K\epsilon_1\frac{\omega}{\theta}T_{\min}\right)=1,\end{aligned}$$

and using this in the right hand side of (36) we obtain

$$\widehat{\|\boldsymbol{\xi}(t^*)\|}_\infty\le c\|\bar\xi\|_\infty$$

holds on $(t^*,s^*)\in[T,T+T_{\min}]\times(a,b)$ for all switching signals $\sigma(\cdot)\in\mathcal{S}(\mathbb{R}_+,Q)$. Finally, from (19) we obtain that

$$\|\boldsymbol{\xi}(t)\|_\infty\le c\exp(-\beta t)\|\bar\xi\|_\infty,\quad 0\le t\le T+T_{\min}.$$

This completes the proof of the induction step. ■

Remark 1: From the proof of Theorem 1 we see that with K and θ given by (12) and (13) respectively, and the constants ω and η chosen such that $\theta < \omega < \eta < 1$, equation (16) gives a concrete value of ϵ for which the conditions of Theorem 1 guarantee exponential stability for all switching signals. That is, (18) holds uniformly for all switching signals $\sigma(\cdot) \in \mathcal{S}(\mathbb{R}_+, Q)$ with c and β given by (14) and (15) respectively. We then see that the so obtained bound on $\|B^j(s)\|_\infty$ satisfies $\epsilon \to 0$ as $\theta \to 1$. Similar conditions are known for the unswitched case, where such systems with sufficiently small inhomogeneities are called 'almost conservative' [31].

For an illustration of the decay estimate and the size of ϵ obtained by Theorem 1 and Remark 1 we provide the following example.

Example 2: Consider a switched system of the form (5) with two modes ($Q = \{1, 2\}$) and $[a, b] = [0, 1]$. The parameters and boundary data are specified as

$$\begin{aligned} \Lambda^1 &= \begin{bmatrix} -1.2 & 0 \\ 0 & 1.8 \end{bmatrix}, \quad B^1 = \begin{bmatrix} -0.005 & 0 \\ 0 & -0.005 \end{bmatrix}, \\ \Lambda^2 &= \begin{bmatrix} -0.8 & 0 \\ 0 & 1.4 \end{bmatrix}, \quad B^2 = \begin{bmatrix} 0 & 0.005 \\ 0.005 & 0 \end{bmatrix}, \\ G_L^1 &= 0.61, \; G_R^1 = 1.15, \; G_L^2 = 0.42, \; G_R^2 = 1.21. \end{aligned} \tag{37}$$

In this example the hypotheses (A_1) and (A_2) of Theorem 1 are clearly satisfied. We have $K = 1.21$ and

$$\theta = \max_{j,j' \in Q} \rho \left(\begin{bmatrix} 0 & |G_R^{j'}| \\ |G_L^j| & 0 \end{bmatrix} \right) = 0.7381 < 1. \tag{38}$$

Following Remark 1, we choose $\omega = 0.87$ and $\eta = 0.88$ to obtain that $\|B^{1,2}\|_\infty = 0.0050 < \epsilon = 0.0054$. Therefore, according to Theorem 1, the switched system is absolutely exponentially stable. Moreover, for equation (18), we obtain $c = 1.6393$ and $\beta = 0.0658$ from equations (14) and (15) respectively. For initial data $\bar{\xi}(s) = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$ on $s \in (0, 1)$, the exponential bound in (18) is plotted together with the observed decay of $\|\boldsymbol{\xi}(t)\|_\infty$ for three different switching signals $\sigma(\cdot)$ in Figure 4. The solution approximations are computed using the two-step Lax-Friedrichs finite difference scheme from [32].□

In general, assumption (A_2) is necessary for exponential stability under arbitrary switching as evident from the following example.

Example 3: Let $Q = \{1, 2\}$, $[a, b] = [0, 1]$, $\Lambda^1 = \operatorname{diag}(-1, 1, 1)$, $\Lambda^2 = \operatorname{diag}(-1, -1, 1)$, $B^j = \operatorname{diag}(0, 0, 0)$ and let G_L^1, G_R^2, G_L^2, and G_R^1 be any boundary data of appropriate dimensions. It is clear that this example satisfies assumption (A_1) but does not satisfy (A_2). Now consider initial data $\bar{\xi}(s) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$ on $s \in (0, 1)$, and a switching signal $\sigma(t)$ defined over the switching times $\tau_k = 0.5k$, where $k = 0, 1, 2, \ldots$ and $\sigma(\tau_0) = 1$, $\sigma(\tau_1) = 2$, $\sigma(\tau_2) = 1$ and so on. For the second component of the solution $\boldsymbol{\xi}(t)$, we then have $\xi_2(t, s) = 1$ for s almost everywhere on the interval $(0, 0.5)$ and $t = 1, 2, 3, \ldots$. Hence, the solution $\|\boldsymbol{\xi}(t)\|_\infty$ cannot decay exponentially irrespective of the decay that might be imposed on $\xi_1(t, s)$, $\xi_2(t, s)$ and $\xi_3(t, s)$ by the boundary data. □

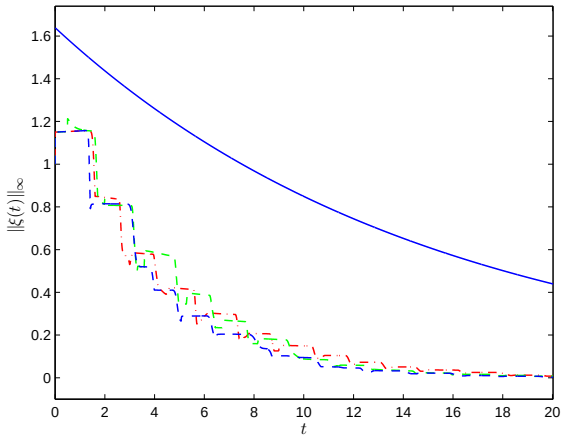

Fig. 4. The exponential bound for $\|\boldsymbol{\xi}(t)\|_\infty$ obtained in Theorem 1 (solid line) and the observed decay for three different switching signals (dashed lines) of solutions for the system considered in Example 2.

A consequence of our results is that, when the only stabilizing mechanism is at the boundary and arbitrary changes of sign of the eigenvalues of Λ cannot be ruled out a-priori, the decay of the solution can in general not be concluded from the rate of decay at the boundary (for e. g., in terms of condition (11) of Theorem 1).

Remark 2: The condition (11) implies the following spectral radius condition to hold for the subsystems (7) with $j \in Q$ fixed:

$$\rho \left(\begin{bmatrix} 0 & |G_R^j| \\ |G_L^j| & 0 \end{bmatrix} \right) < 1. \tag{39}$$

Under this assumption, classical solutions of (7) are known to be exponentially stable [23]. However, assumption (39) for all $j \in Q$ is not sufficient for the switching system to be exponentially stable. Note that G_L^j, G_R^j in Example 1 satisfy (39) but not (11) for $j = 1, 2$, i. e.,

$$\rho \left(\begin{bmatrix} 0 & 1.5 \\ 0 & 0 \end{bmatrix} \right) = \rho \left(\begin{bmatrix} 0 & 0 \\ 1.5 & 0 \end{bmatrix} \right) = \rho \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) = 0,$$

but

$$\rho \left(\begin{bmatrix} 0 & 1.5 \\ 1.5 & 0 \end{bmatrix} \right) = 1.5.$$

Nevertheless, as shown next in Corollary 1, the switched system satisfying (39) in every mode j can be stabilized by switching slow enough. Note that Corollary 1 does not require assumption (A_2) to hold.

Corollary 1: (Dwell-Time) Under the hypotheses (A_1), there exists an $\epsilon > 0$ such that if $\|B^j(s)\|_\infty < \epsilon$ for all $s \in [a, b]$ and $j \in Q$, the switching system in diagonal form (5) is exponentially stable with respect to the norm $\|\cdot\|_\infty$ for all switching signals in $\mathcal{S}_\tau(\mathbb{R}_+; Q)$ for which the dwell-time $\tau > \bar{\tau}$ ($\bar{\tau}$ given by (10)) if the condition (39) holds for all $j \in Q$.

Proof: From the definition of $\bar{\tau}$ in (10) it is easy to see that if $\tau > \bar{\tau}$, then in case B.2(ii), t_r and t_{rl} lie in the same inter switching interval and all the required estimates can be made using a $\bar{\theta}$ defined similar to (13) but where the maximum is only taken over $j \in Q$. ■

IV. Non-diagonal Switching System

We now focus on non-diagonal systems. Suppose that the system switches among non-diagonal subsystems

$$\begin{cases} \dfrac{\partial u^j}{\partial t} + A^j(s)\dfrac{\partial u^j}{\partial s} + \tilde{B}^j(s)u^j = 0,\ s \in (a,b),\ t > 0 \\ D_L^j u^j(t,a) = 0, \quad D_R^j u^j(t,b) = 0,\ t \geq 0 \end{cases} \tag{40}$$

where, for each $j \in Q$, $A^j(s)$, $B^j(s)$ are $n \times n$ dimensional matrix functions on (a,b) and D_L^j, D_R^j are constant matrices of appropriate dimensions. Each subsystem can be written in the diagonal form (7) under certain assumptions. For instance, if we impose that for each $j \in Q$,

(A_1^*) The matrix function $A^j(\cdot)\colon [a,b] \mapsto \mathbb{R}^{n\times n}$ is Lipschitz-continuous such that for all $s \in [a,b]$, there exists m_j such that $0 < m_j < n$ and $A^j(s)$ has m_j negative and $(n-m_j)$ positive eigenvalues $\lambda_i^j(s)$ with n corresponding linearly independent left (resp. right) eigenvectors $l_i^j(s)$ (resp. $r_i^j(s)$), $i = 1,\dots,n$ all Lipschitz-continuous functions of s. The matrix function $\tilde{B}^j(\cdot)\colon [a,b] \mapsto \mathbb{R}^{n\times n}$ is bounded measurable with respect to s. Furthermore, the following two rank conditions hold for $D_L^j \in \mathbb{R}^{(n-m_j)\times n}$ and $D_R^j \in \mathbb{R}^{m_j \times n}$

$$\operatorname{rank}\left[(D_L^j)^\top \middle| l_1^j(a)\middle| \cdots \middle| l_{m_j}^j(a)\right] = n$$
$$\operatorname{rank}\left[(D_R^j)^\top \middle| l_{m_j+1}^j(b)\middle| \cdots \middle| l_n^j(b)\right] = n.$$

Under the assumption (A_1^*) the matrix functions $S_j(\cdot) = [l_1^j(\cdot)\big|\dots\big|l_n^j(\cdot)]^\top$ and $S_j^{-1}(\cdot) = [r_1^j(\cdot)\big|\dots\big|r_n^j(\cdot)]^\top$ are Lipschitz-continuous functions with partial derivatives defined a. e. We refer the reader to the text by Bressan [30], pages $46-50$, for the details about assumption (A_1)*.

For all $s \in [a,b]$, we have

$$S_j(s)A^j(s)S_j^{-1}(s) = \Lambda^j(s). \tag{41}$$

with $\Lambda^j(s)$ as in (A_1). By applying a transformation $u^j(t,s) = S_j^{-1}(s)\xi^j(t,s)$, $\tilde{D}_L^j = D_L^j S_j^{-1}(a)$ and $\tilde{D}_R^j = D_R^j S_j^{-1}(b)$ and using the representation

$$\begin{aligned} &B^j(s) = S_j(s)\left(A^j(s)\frac{\partial}{\partial s}S_j^{-1}(s) + \tilde{B}^j(s)S_j^{-1}(s)\right), \\ &\tilde{D}_L^j = [\tilde{D}_{L,I}^j\big|\tilde{D}_{L,II}^j],\ \tilde{D}_R^j = [\tilde{D}_{R,I}^j\big|\tilde{D}_{R,II}^j], \\ &G_L^j = -(\tilde{D}_{L,II}^j)^{-1}\tilde{D}_{L,I}^j,\ G_R^j = -(\tilde{D}_{R,I}^j)^{-1}\tilde{D}_{R,II}^j, \end{aligned} \tag{42}$$

with $\tilde{D}_{L,I}^j \in \mathbb{R}^{(n-m_j)\times m_j}$, $\tilde{D}_{L,II}^j \in \mathbb{R}^{(n-m_j)\times(n-m_j)}$, $\tilde{D}_{R,I}^j \in \mathbb{R}^{m_j\times m_j}$, $\tilde{D}_{R,II}^j \in \mathbb{R}^{m_j\times(n-m_j)}$, $G_L^j \in \mathbb{R}^{(n-m_j)\times m_j}$ and $G_R^j \in \mathbb{R}^{m_j\times(n-m_j)}$, the system corresponding to (40) and initial data $\bar{u}(s)$ corresponding to mode j becomes (7) with initial data $\bar{\xi}^j(s) = S_j(s)\bar{u}(s)$.

Now observing that the switching system in the non-diagonal form for a switching signal $\sigma(\cdot) \in \mathcal{S}(\mathbb{R}_+, Q)$

$$\begin{cases} \dfrac{\partial u}{\partial t} + A^{\sigma(t)}(s)\dfrac{\partial u}{\partial s} + \tilde{B}^{\sigma(t)}(s)u = 0,\ s \in (a,b),\ t > 0 \\ D_L^{\sigma(t)}u(t,a) = 0, \quad D_R^{\sigma(t)}u(t,b) = 0,\ t \geq 0 \\ u(0,s) = \bar{u}(s), \quad s \in (a,b) \end{cases} \tag{43}$$

can be written as a switching system in the diagonal form with discontinuous resets at the switching times τ_k for $k = 1,2,\dots$ and $\tau_0 = 0$, i. e.,

$$\begin{cases} \dfrac{\partial \xi}{\partial t} + \Lambda^{\sigma(t)}(s)\dfrac{\partial \xi}{\partial s} + B^{\sigma(t)}(s)\xi = 0,\ t \in [\tau_k, \tau_{k+1}] \\ \xi_{II}(t,a) = G_L^{\sigma(t)}\xi_I(t,a),\ \xi_I(t,b) = G_R^{\sigma(t)}\xi_{II}(t,b), \\ \xi(0,\cdot) = \bar{\xi}(\cdot) = S_{\sigma(\tau_0)}(\cdot)\bar{u}(\cdot), \\ \xi(\tau_k,\cdot) = S_{j_k}(\cdot)S_{j_{k-1}}^{-1}(\cdot)\lim\limits_{t\to\tau_k, t<\tau_k}\xi(\tau_k,\cdot),\ k > 0, \end{cases} \tag{44}$$

the existence and uniqueness of solutions can be argued as before.

Our next proposition is a very simple consequence of simultaneous diagonalization.

Proposition 1: Under hypotheses (A_1^*)-(A_2) and under the pairwise commutativity assumption that for all $s \in [a,b]$ and for all $j, j' \in Q$

$$A^j(s)A^{j'}(s) = A^{j'}(s)A^j(s), \tag{45}$$

and let G_L^j, G_R^j and $B^j(s)$ are given by (42). Then, if condition (11) holds for all $j, j \in Q$, there exists an $\epsilon > 0$ such that if $\|B^j(s)\|_\infty < \epsilon$ for all $s \in [a,b]$ and $j \in Q$, the switching system in non-diagonal form (43) is absolutely exponentially stable in $\|\cdot\|_\infty$.

Furthermore, if the condition (39) holds for all $j \in Q$, there exists an $\epsilon > 0$ such that if $\|B^j(s)\|_\infty < \epsilon$ for all $s \in [a,b]$ and $j \in Q$, the system (43) is exponentially stable in $\|\cdot\|_\infty$ for all switching signals in $\mathcal{S}_\tau(\mathbb{R}_+; Q)$ for which the dwell-time $\tau > \bar{\tau}$ and $\bar{\tau}$ given by (10).

Proof: Recall that a set of diagonalizable matrices are simultaneously diagonalizable if (and only if) they commute. Thus, system (43) can be transformed into a switching system in diagonal form (44) with a common diagonalizing matrix function $S^j(\cdot) \equiv S(\cdot)$. The assertion then follows. ∎

Though the commutativity assumption in Proposition 1 seems very strong, we include an example showing that it is in general necessary for conditions such as in Section III to be sufficient for absolutely exponential stability.

Example 4: Consider a non-diagonal switching system of form (43) with two modes ($Q = \{1,2\}$) and initial data $\bar{u}(s) = \begin{bmatrix}1 & 1\end{bmatrix}^\top$ on $s \in (a,b)$, for an alternating switching signal $\sigma(\cdot)$ with switching times $\tau_k = 0.5k$ where $k = 0,1,2,3,\dots$ and $\sigma(\tau_0) = 1$, $\sigma(\tau_1) = 2$, $\sigma(\tau_2) = 1$ and so on. The parameters and boundary data are specified as

$$\begin{aligned} A^1 &= \begin{bmatrix}-1 & 0\\ 0 & +1\end{bmatrix}, \quad A^2 = \begin{bmatrix}-1 & -4\\ 0 & +1\end{bmatrix}, \quad B^{1,2} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \\ D_L^1 &= \begin{bmatrix}-\frac{3}{2} & 1\end{bmatrix}, \quad D_L^2 = \begin{bmatrix}-\frac{3}{4} & -1\end{bmatrix}, \\ D_R^1 &= \begin{bmatrix}1 & -\frac{1}{4}\end{bmatrix}, \quad D_R^2 = \begin{bmatrix}1 & \frac{7}{4}\end{bmatrix} \end{aligned}$$

The non-diagonal system so specified satisfies (A_1^*)-(A_2) but does not satisfy the commutativity condition (45) ($A_1A_2 \neq A_2A_1$). With

$$S_1 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad S_2 = \begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix},$$

and doing a change of variables by this transformation, both the constituting subsystems of the non-diagonal switching

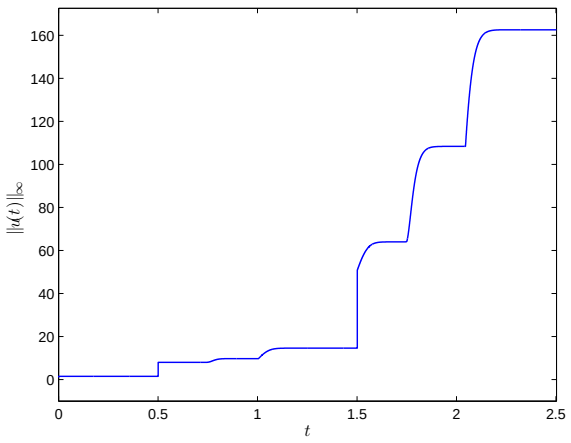

Fig. 5. The observed blowup of $\|\mathbf{u}(t)\|_\infty$ for the system considered in Example 4 under switching.

system reduce to the same diagonal subsystem

$$\begin{aligned} &\frac{\partial \xi}{\partial t} + \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial \xi}{\partial s} = 0, \\ &\xi^2(t,0) = \frac{3}{2}\xi^1(t,0), \quad \xi^1(t,1) = \frac{1}{4}\xi^2(t,1), \end{aligned} \tag{46}$$

which satisfied the spectral radius condition

$$\rho\left(\begin{bmatrix} 0 & \frac{1}{4} \\ \frac{3}{2} & 0 \end{bmatrix}\right) = 0.6124 < 1,$$

implying that the solution of the subsystem (46) starting with initial condition for $\bar{\xi}(s) = \begin{bmatrix} 1 & 1 \end{bmatrix}^\top$, $s \in (0,1)$ decays exponentially for $t \to \infty$. However, following the representation (44), we observe that for the non-diagonal switching system $\|\mathbf{u}(t)\|_\infty$ is not bounded as $t \to \infty$. See Figure 5 for the growth of $\|\mathbf{u}(t)\|_\infty$ where the solution $\mathbf{u}$ is again obtained by using a two-step Lax-Friedrichs scheme as in Example 2. □

V. Final remarks

We present a generalization of a well-known mechanism for stability of hyperbolic PDE systems [23] to the case in which the switching occurs among a set of systems that may differ in the system matrix function and/or boundary conditions. When constituent PDEs are in the canonical diagonal form, we derive a sufficient condition for exponential stability under arbitrary switching signals. For the case in which the system matrix functions are not diagonal, the result holds when they are jointly diagonalizable. This results in a commutativity condition that has a counterpart in the switched ODE literature [3].

It is also clear that, although the switching signal represents joint switching of the boundary conditions and system matrices, the results apply for switching the boundary conditions or system matrices individually by introducing appropriate auxiliary modes, which is just a matter of notational convenience. Thus, the treatment presented in this article might be of interest in control settings under abruptly changing boundary conditions and operating regimes such as the opening and closing of gates in a cascade of open-canal pools, the dynamics of which are classically modeled by the linearized Saint-Venant equations [31].

A limitation of the results obtained here is that they are valid only for almost conservative systems (see Remark 1). Thus, it will be interesting to investigate if, possibly by using different methods, other conditions can be found that guarantee absolute exponential stability for less conservative systems. In particular, our results motivate a Lyapunov theory for switching infinite dimensional systems.

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