Lessons from the Landscape of Six-dimensional Supergravity Theories

by

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Abstract

Comparing the set of supergravity theories allowed by low-energy consistency conditions with the set of string vacua provides useful insights into quantum gravity and string theory. In fact, such a "landscape analysis" for ten-dimensional supergravity theories was at the core of the exciting series of developments that is now referred to as the first superstring revolution. In this thesis, we discuss the lessons we learn about quantum supergravity and string theory by carrying out such an analysis for the space of six-dimensional supergravity theories with minimal supersymmetry.

We first review six-dimensional supergravity theories and explain why the space of these theories is an ideal place to carry out the landscape analysis. We then describe how anomaly constraints bound the space of consistent theories, *i.e.,* we map the space of theories T that satisfy known low-energy consistency conditions. We then go on to describe string constructions that give six-dimensional string vacua with minimal supersymmetry, *i.e.,* we map the space of theories $S \subset T$ that come from string vacua. Finally, we compare the space of theories T and S and explore its implications.

We first find that there is a large discrepancy between T and S. Among the set $T - S$, we identify some theories that are potentially new string vacua, but also identify many theories that cannot be embedded in any known string vacua. These theories may potentially be ruled out by yet undiscovered low energy constraints. Understanding these theories is an important step in addressing the question of string universality in six dimensions. We also find some surprising equalities that hold for Calabi-Yau threefolds that follow from demanding that F-theory string vacua should be consistent.

Thesis Supervisor: Washington Taylor
Title: Professor of Physics

To my family —

to Abba and Umma, who more than happily gave up

whatever they had at the moment for their Junni,

to Wullbaengee and Jaerongee,

the two brightest stars on my celestial sphere,

and to Bolmu with best wishes on her new journey.

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Chapter 1

Introduction

1.1 Motivation

It is by now well established that our universe can be described using the formalism of quantum field theory[1] if we ignore gravity. Another way to state this is to say that we know how to describe the nature of the three fundamental forces — electromagentic, weak and strong forces — by the standard model[2] in the formalism of quantum field theory. It has been a long-standing problem how to incorporate gravity into this picture. In fact, gaining a full understanding of quantum gravity is one of the principal objectives of theoretical physics research today. This objective turns out to be a surprisingly difficult task, that seems far out of reach at the moment.[3]

Nevertheless, there has been amazing progress in understanding quantum gravity. One framework that has improved our understanding is string theory [13, 14, 15, 16]. String theory is formulated by assuming that the fundamental objects in the universe are strings propagating in space-time. It was first conceived as an effective theory of strong interactions, but was subsequently realized to have much richer structure than

[1]We have listed a far from extensive collection of standard texts on the subject in the bibliography [1, 2, 3, 4, 5, 6, 7, 8].

[2]A standard textbook on the standard model is [9].

[3]Why it is hard to formulate a quantum theory of gravity in and of itself is a fascinating subject, that we will not be able to do justice to in this thesis. We have listed an incomplete subset of some recent discussions in the bibliography. A review of some general challenges that quantum gravity presents can be found in [10, 11, 12].

intended.[4] String theory ultimately found success as an ultraviolet complete theory with quantum gravity, that does not have many of the problems that arise when one attempts to quantize gravity in a more conventional way. For example, string theory does not have issues with regularization that plague quantum gravity theories. Also the theory contains non-perturbative objects that account for the microscopic structure of black holes [18, 19, 20, 21, 22, 23]. Furthermore, string theory provides a framework to compute physical observables in various perturbative limits.

We cannot help but mention that there have been some surprising developments — inspired by string theory — that have revolutionized our understanding of quantum gravity more recently. We now have a sharper picture of how to think about gravity in asymptotically Anti-de Sitter space through the "AdS/CFT (Anti-de Sitter/Conformal Field Theory) correspondence [24, 25, 26, 27]," or rather the "QG/QFT (Quantum Gravity/Quantum Field Theory) correspondence [28]." In this framework, a bulk quantum gravity theory living in Anti-de Sitter space is defined as a boundary quantum field theory. Such a correspondence highlights the holographic nature of quantum gravity [29, 30, 31], which has been, and, to some extent, still is one of the mysteries of quantum gravity. There also are many exciting attempts that aim to understand gravity beyond Anti-de Sitter space inspired by the success of AdS/CFT, that have yet to be as successful as is predecessor.[5]

The goal of this thesis is conservative compared to these daring attempts to develop a new framework for quantum gravity. Here, we build on the simple observation that string theory is — despite the fact that there is much room for improvement — exceptionally successful in accounting for the microscopic structure of quantum gravity. For example, Strominger and Vafa famously reproduced the Bekenstein-Hawking entropy for a class of five-dimensional extremal black-holes by counting the microscopic states in string theory [21]. Also, the most tractable context of the AdS/CFT correspondence is given by the duality between type IIB string theory on $AdS_5 \times S_5$

[4] A nice account on the early history of string theory by one of its principal architects can be found in [17].

[5] We have compiled a far from complete list of some work along these lines in the bibliography [32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52].

and $\mathcal{N} = 4$ super Yang-Mills theory [24, 25, 26, 27]. Such exceptional success of string theory leads one to wonder whether all quantum gravity theories that have a sensible microscopic description are secretly string theories [53, 54]. This is the central question we attempt to address in this thesis. Since this question is so important to us, let us dignify it with a box and a label:

(Q') Are all consistent quantum gravity theories, string theories?

Having written down the question, a little thought reveals that question **(Q')** is not a well-posed one, principally due to our ignorance of fundamental aspects of quantum gravity and string theory. Let us break down the ill-definedness of question:

1. We do not understand how to define "quantum gravity theories."
2. We do not know how to formulate a complete definition of "consistency" in quantum gravity theories.
3. We do not know how to rigorously define "string theory."
4. We do not understand how to define the word "is," *i.e.,* how to define the equivalence of a quantum gravity theory to a string theory.

Despite of the state of our ignorance, it is actually possible to repose and explore this question in a meaningful way. Such efforts were carried out in many different contexts and have helped us gain new insight into quantum gravity and string theory.[6] Arguably the most impactful among such efforts was made in the mid-80's that lead to ground breaking work that we now refer to as the "first superstring revolution." The problem underlying the developments of the first superstring revolution was a version of the unapproachable question **(Q')** modified in the following way:

1. Although a "quantum gravity theory" is not well defined one can ask definite questions about the low-energy data — such as the massless particle spectrum — of a given quantum gravity theory.

[6] We have compiled an incomplete list of such work in the bibliography [53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65].

2. Although the complete set of consistency conditions of a quantum gravity theory could not be formulated, it is sensible to expect that the low-energy data of consistent theories should obey known low-energy consistency conditions.

3. Although we do not understand "string theory" we understand how to define it perturbatively in certain regimes, *i.e.*, we do have a knowledge of various string vacua.

4. Although we do not understand how to show that a string theory and a quantum gravity theory are equivalent, we can ask whether a quantum gravity theory with given low-energy data can be embedded into string vacua.

The modified problem is now phrased:

> **(Q)** Can all low-energy data of quantum gravity theories satisfying known low-energy constraints be embedded into known string vacua?

This question is still a very broad one, in that we do not have a full knowledge of all string vacua. Also, low-energy constraints turn out to be stronger when further constraints are placed on quantum gravity theories that one would like to investigate. The question implicit in the work of the protagonists of the superstring revolution was the following one:

> **($\mathbf{Q}_{10}$)** Can all massless particle spectra of ten-dimensional $\mathcal{N} = 1$ quantum gravity theories that satisfy known low-energy constraints be embedded into known string vacua?

The answer to this seemingly very restricted question had surprisingly deep implications on quantum gravity and string theory. The crucial work that provided the breakthrough addressing this problem was [55]. In this work, Álvarez-Gaumé and Witten computed gravitational/gauge and mixed anomalies for chiral theories in various dimensions, and in particular, showed that any ten-dimensional $\mathcal{N} = 1$

supergravity theory has a non-vanishing anomaly at one-loop. Anomalies [66, 67], which were originally discovered by Adler, Bell and Jackiw, are quantum effects that break symmetries that are present in the classical theory. The existence of gravitational/gauge or mixed anomalies in a quantum theory implies that there is a violation of diffeomorphism/gauge invariance, which is needed for consistency[7].

Ten-dimensional $\mathcal{N} = 1$ supergravity has very restricted structure, and its massless spectrum can be parameterized by its gauge group $\mathcal{G}$. At the time when the results of Álvarez-Gaumé and Witten were presented, it was known that type I string theory, whose massless spectrum has $SO(32)$ gauge symmetry, is a consistent theory. This led Green and Schwarz to discover the Green-Schwarz mechanism [56] at play in type I string theory in which a tree level term cancels the anomalies generated at one-loop computed by Álvarez-Gaumé and Witten. Green and Schwarz also found that in order for this mechanism to work, the gauge algebra could be only one of $SO(32)$, $E_8 \times E_8$, $E_8 \times U(1)^{248}$ and $U(1)^{496}$ [14].

Motivated by this observation, Gross, Harvey, Martinec and Rohm discovered the $E_8 \times E_8$ and $SO(32)$ heterotic string theories [57, 58, 59], whose role in string theory, especially in string phenomenology, is hard to overstate. The $E_8 \times U(1)^{248}$ and $U(1)^{496}$ theories were not an active topic of research for a while, but believed to be pathological.[8] Adams, de Wolfe and Taylor were able to show more recently that these theories are inconsistent by examining the supersymmetry of these theories closely [69].

Now we have an answer for question $(\mathbf{Q}_{10})$, and it is "yes." The impact of this answer is clear by how we refer to the series of events that arrived at this conclusion. For one, this is strong evidence — in fact, as close to a proof as one can expect in quantum gravity — that supergravity in ten-dimensions must be a string theory. Put in more simple terms, if we lived in a ten-dimensional universe with quantum supergravity, we would not be having much of a debate on whether string theory was the underlying theory of our universe.

[7] For more discussion on this point, consult [55].

[8] See, for example, [68].

It would be ideal if we could duplicate the success of this approach on quantum gravity — which we refer to as the "landscape analysis" throughout this thesis — in four dimensions. At the present, this goal is out of reach due to the fact that anomaly constraints — which serve as the principal low-energy constraint in landscape analyses — are much weaker in four dimensions. In fact, pure gravitational anomalies exist only in $(4k+2)$ dimensions [55]. We may, therefore, be less ambitious and ask the question **(Q)** in six dimensions:

> **(Q$_6$)** Can all massless particle spectra of six-dimensional $\mathcal{N}=1$ quantum gravity theories that satisfy known low-energy constraints be embedded into known string vacua?

As we explain in the next section, this turns out to be an interesting question to ask. Not only were we able to learn much about quantum gravity and string theory by pursuing **(Q$_6$)**, but were also able to identify, what we believe to be important puzzles that stand in the way of answering it. The objective of this thesis is to present these results and puzzles.

1.2 The Landscape Analysis and Six Dimensions

Before presenting the results and challenges that we have encountered through studying the space of six-dimensional supergravity theories, it is useful set up a framework in which to understand them. In this section, we set up the language of the "landscape analysis"[9] and explain why the space of six-dimensional supergravity theories with minimal supersymmetry is fertile ground for investigation using this method.

[9]We note that the term "landscape" used in this thesis is different from the conventional use in the string theory literature. Conventionally, the word "string landscape" is used to describe semi-realistic vacua of string theory, and the term "landscape analysis" refers to a statistical analysis of this space. In the context of this thesis, by "landscape" we refer simply to the space of theories under investigation, and the "landscape analysis" involves classifying the individual theories in this space, as we explain further throughout the course of this section. We have included a small sample of the string landscape literature in the bibliography [70, 71, 72, 73, 74, 75]. An extensive review on the topic can be found in [76] and [77].

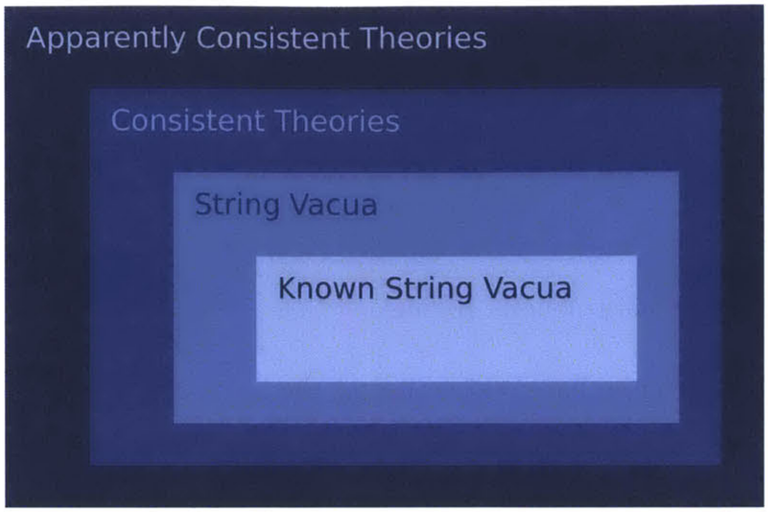


Figure 1-1: The landscape of quantum gravity theories. The crux of the landscape analysis is to identify the "intermediate regime," by which we denote the theories that are apparently consistent, but that are not known string vacua.

A useful picture to keep in mind when addressing a question of the type $(\mathbf{Q}_X)$ on some space of theories[10] of interest X is figure 1-1. One can draw four boundaries in X — the boundary of apparently consistent theories, consistent theories, string vacua and known string vacua. By saying a theory — or rather, low energy data of a given theory — is "apparently consistent" we mean that it satisfies all known consistency conditions. The two intermediate boundaries — the boundary of consistent theories, and the boundary of string vacua — are actual boundaries in "theory space" that are, for most interesting X, boundaries that we do not have access to at the moment. Meanwhile, the outermost boundary and the innermost boundary are artificial boundaries that are defined by our present knowledge, but are accessible to us.

The landscape analysis can be summarized in two steps:

1. For given theories X, identify the theories in the "intermediate regime" of the diagram in figure 1-1, *i.e.,* theories that are apparently consistent, but not known string vacua. This process itself also involves two steps.

[10]The correct term to use in place of "theory" would be "low-energy data of the theory" or to be more specific, "massless spectrum and low-energy parameters of the theory," as elaborated previously. For most instances in this thesis, we have chosen concision over precision and have used the word "theory" in most places these other terms could have been used.

- First, use currently known consistency conditions to identify theories that are apparently consistent.
- Second, identify the known string vacua among these apparently consistent theories.

2. Understand the intermediate regime.

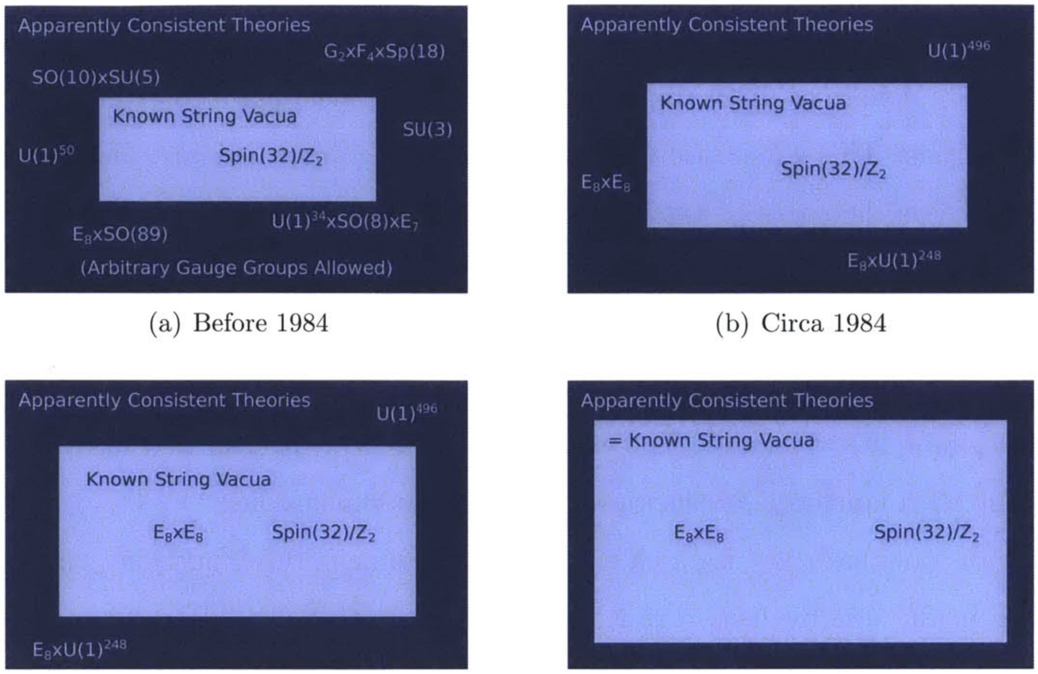
Identifying the intermediate regime lies at the heart of the landscape analysis, as the theories in the intermediate regime are either one of the following three:

1. Previously undiscovered string vacua.
2. Consistent gravity theories that are not string theories.
3. Secretly inconsistent theories that violate consistency conditions that are yet unknown to us.

Hence, the theories in the intermediate regime provide us with a window to previously uncharted territory in the "landscape" of gravity theories. By closely examining theories, one would hopefully be able to achieve one or more of the following:

1. Discover new string vacua.
2. Find new consistent gravity theories that are not embeddable in string theory.
3. Find new constraints on gravity theories.

Notice that the developments of the first superstring revolution can be phrased in terms of the landscape analysis picture. Before [55, 56], the picture of the landscape of ten-dimensional $\mathcal{N} = 1$ supergravity theories was given by figure 1-2(a) — it was understood that type I superstring theory was a consistent theory of gravity, but it was not known whether other theories were consistent or not. After [55, 56], the picture of the landscape changed into figure 1-2(b), *i.e.*, the papers [55, 56] identified the apparently consistent theories, and thereby identified the intermediate regime. The work [57, 58, 59] confirmed that $E_8 \times E_8$ was a consistent theory of gravity, hence changing the picture to 1-2(c). In the landscape analysis language, this work



(a) Before 1984

(b) Circa 1984

(c) Circa 1985

(d) Circa 2010

Figure 1-2: The change of the landscape of ten-dimensional $\mathcal{N} = 1$ supergravity theories throughout the years.

incorporated the $E_8 \times E_8$ supergravity theory into known string vacua. The work [69] showed that the two gauge groups $U(1)^{496}$ and $E_8 \times U(1)^{248}$ were inconsistent, hence obtaining the final picture of the landscape given by figure 1-2(d). Now there is no intermediate regime in the landscape of ten-dimensional supergravity theories with minimal supersymmetry.

The landscape analysis has also been successfully carried out for six-dimensional supergravity theories with $\mathcal{N} = (2, 0)$ supersymmetry[11] in [60]. In this work, gravitational anomalies are used to identify the unique consistent massless spectrum that the theory could have. This spectrum precisely agrees with the massless spectrum of type IIB string theory compactified on a K3 manifold. This result shows that all six-dimensional $\mathcal{N} = (2, 0)$ supergravity theories could be embedded into string theory.

[11]Six-dimensional theories are chiral and hence supersymmetry can be extended in different directions. $\mathcal{N} = (2, 0)$ denotes that the two supercharges of the extended supersymmetry are of the same chirality.

The space of six dimensional supergravity theories with minimal supersymmetry — which we denote as $(1,0)$ supersymmetry — is an ideal place to extend the landscape analysis. This is because while there are strong anomaly constraints that give us a handle on the space of apparently consistent theories [62, 63, 65], the "volume" of the space of apparently consistent theories is quite large. For example, a wide variety of gauge groups and matter content are allowed in the massless spectrum of the theory. At the same time, six-dimensional $(1,0)$ string vacua are well-studied and shown to be quite diverse.[12] Therefore we have a rich, diverse, and also relatively well-controlled landscape of theories to probe in six-dimensions.

It turns out that there is a rich intermediate regime in this landscape. Among the theories in the intermediate regime, some are interesting candidates for new string vacua while some seem to provide circumstantial evidence for undiscovered low-energy constraints. One would be able to gain a better understanding of the string landscape and quantum gravity in general by either assimilating these theories into string vacua or by ruling them out through the discovery of new consistency conditions. Knowledge acquired by this process have practical implications on four-dimensional string model building and phenomenology — the discovery of new string vacua provides new tools for constructing string models, while new consistency conditions provide additional handles on model building.

1.3 Summary of Results and Outline of Thesis

Now that we have set up the context of our investigation, we explain the main results of this thesis — based on the works [115], [116] and [117] — and outline its presentation in this section. Before summarizing the results, let us present some basic facts about six-dimensional $(1,0)$ supergravity theories.

A low-energy six-dimensional $(1,0)$ supergravity theory can be parameterized by

[12]There is extensive literature on six-dimensional $(1,0)$ string vacua as these string models have played an important part in understanding various string dualities in the mid-90's. A small sample of the vast literature is given in the bibliography [61, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112]. Further references can be found in the reviews [113] and [114].

its massless spectrum S, a modulus j, and anomaly coefficients, which we schematically denote by $\{b\}$ [65]. The massless particle spectrum of six-dimensional $(1,0)$ supergravity theories come in four multiplets of the supersymmetry algebra; the gravity multiplet, the tensor multiplet, the vector multiplet, and the hypermultiplet. There is only one gravity multiplet of the theory. The massless spectrum of the theory can be summarized by the number of tensor multiplets T, the gauge group, and the matter content.

The modulus j and the anomaly coefficients $\{b\}$ are $SO(1,T)$ vectors. In particular, j is a unit vector that encodes the vacuum expectation values of the T scalar fields that lie in each tensor multiplet. The low-energy couplings that parametrize the theory can be expressed in terms of j and $\{b\}$. For example, for each simple non-abelian gauge group factor $\mathcal{G}_\kappa$ of the full gauge group, there exists a corresponding anomaly coefficient b_κ. The coefficient for the kinetic term of the $\mathcal{G}_\kappa$ gauge field is given by $j \cdot b_\kappa$ where the inner-product taken by an $SO(1,T)$ metric [86].

The massless spectrum of a six-dimensional $(1,0)$ theory must satisfy anomaly equations that come from a generalized version of the Green-Schwarz factorization condition [80, 86, 97], originally formulated in ten-dimensions [56]. In other words, a set of equations of the form

$$f_i(\{b\}) = F_i(S)\,, \tag{1.1}$$

where f_i and F_i are some functions, must be satisfied. For many spectra, there are not any physically sensible $\{b\}$ satisfying these equations, *i.e.,* only certain massless spectra are allowed by the anomaly equations.

The first step of the landscape analysis we perform in this thesis is to list the apparently consistent theories $(S, j, \{b\})$ that satisfy the anomaly equations and have positive definite kinetic terms for the gauge fields. In the next step, we identify among these theories, those that could be embedded in string theory. In the final step, we make observations on the theories in the intermediate regime and explore their implications.

Even before examining the individual theories in the intermediate regime, it is possible to ask about important qualitative features on the space of apparently consistent theories. One such question is whether the space is bounded, *i.e.,* whether only a finite number of massless spectra are allowed by the anomaly constraints. This question has been answered for theories when the gauge group is non-abelian in [62, 65]. In these references, it was shown that the number of apparently consistent theories with $T < 9$ is bounded while it is not bounded when $T \geq 9$. In [116], whose results we present here, we have extended the analysis to theories with abelian gauge symmetries. It turns out that the of possible combinations of gauge groups — including abelian factors — and non-abelian matter representations is finite when $T < 9$, even when abelian group factors are allowed. There are, however, infinite families of theories with distinct $U(1)$ charge assignments to the matter that cannot be ruled out by using known quantum consistency conditions.

Showing that various bounds exist on the space of apparently consistent theories is one thing, actually drawing the bounds is quite another. In order to compare the space of apparently consistent theories and string vacua, one needs to be able to carry out the latter task of drawing the actual boundary of apparently consistent theories. While this seems to be quite a formidable task for the full space of theories, it turns out to be a much more approachable one when we decide to focus on an interesting subsector of the theories, namely, $T = 0$ theories.

In [115], we have presented a systematic way of constructing the finite set of possible gauge group/matter combinations of non-anomalous six-dimensional $(1,0)$ theories with no tensor multiplets, focusing on the case when the gauge group has only $SU(N)$ factors. Using this method, it is in principle possible to construct and identify all theories in the intermediate regime of the landscape of $T = 0$ supergravity theories. By scanning through the intermediate regime, we were able to identify many theories that are possible candidates for new string vacua. In fact, more recently, some models among these were realized in string theory [112].

Another important lesson we have learned by studying $T = 0$ theories is that there is a systematic obstruction to embedding a large majority of the theories in the intermediate regime. To elaborate, the low-energy parameters of all known string vacua satisfy an inequality known as the "Kodaira constraint" [114, 118, 119]. The majority of the apparently consistent $T = 0$ theories, however, does not satisfy this constraint. An important question that therefore arises is whether the Kodaira constraint is an undiscovered fundamental constraint that applies to all supergravity theories or whether it is an artifact of known string models. Either option would be interesting. The former implies that there is a fundamental constraint on gravity theories that is unknown to us at the present. The latter implies that there is a large class of string vacua that has yet to be discovered. It is worth noting that the Kodaira constraint gets rid of all the infinite classes of apparently consistent non-abelian theories we were able to construct.

While extending the results of the $T = 0$ analysis to $0 < T < 9$ would be technically challenging on the outer-boundary front of the landscape, we have a much better handle on the landscape of known string vacua, given that the gauge group is non-abelian [65]. This is not true, however, if we try to extend the landscape analysis to theories with abelian gauge symmetry. Not only does the outer-boundary of the landscape of six-dimensional theories — as noted previously — qualitatively change, but so does the known string vacua. This is because the non-abelian sector of string vacua is much better understood than the abelian sector. For example, the Kodaira constraint involves only the gravitational anomaly coefficient a and the non-abelian anomaly coefficients. We are not aware of a Kodaira-like constraint that involves abelian anomaly coefficients at the present. Therefore, in order to expand our knowledge of the six-dimensional landscape to theories that have abelian gauge symmetry, it is important that we understand the abelian sector of string vacua better.

A first step in this direction is to identify the string data that correspond to the abelian anomaly coefficients of a six-dimensional string vacuum. An important class of string models in which to investigate this problem is F-theory vacua [118, 119, 120]. F-theory compactifications play a central role in understanding the six-dimensional $(1, 0)$ string landscape in that they accommodate the widest range of known string vacua in six dimensions [64, 65]. In fact, the data of all non-abelian string models known in six dimensions can be in principle embedded in F-theory [114].[13] F-theory string vacua can be thought of as type IIB backgrounds with a non-trivial axio-dilaton profile. A convenient way of thinking about F-theory is to treat it as a fictitious twelve-dimensional theory. Six-dimensional $(1,0)$ vacua can be obtained from F-theory by compactifying it on an elliptically fibered Calabi-Yau threefold. The gravitational anomaly coefficient a and the non-abelian anomaly coefficients of F-theory vacua have a nice interpretation in terms of the geometric data of this elliptically fibered Calabi-Yau manifold [64, 65, 118, 119].

As far as we are aware of, the abelian anomaly coefficients did not have a geometric interpretation before the work [117], whose results we present in this thesis. M-theory/F-theory duality [120] plays a central role in identifying the geometric data that correspond to the abelian anomaly coefficients. Once the geometry of the anomaly coefficients are understood, we can translate the six-dimensional anomaly equations into geometric identities. The resulting identities have a very appealing form, although further work must be done to understand their geometric significance.

The structure of this thesis is as the following. In chapter 2, we review six-dimensional supergravity theories with minimal supersymmetry and investigate the boundary of apparently consistent theories in its landscape. After reviewing previous results on the bounds that anomalies place on the space of non-abelian theories, we systematically construct the space of non-anomalous $T = 0$ theories, and describe its features. We then extend the anomaly analysis to theories with abelian gauge symmetry and investigate how anomalies constrain these theories.

In chapter 3 we investigate the boundary of known string vacua in the six-dimensional landscape. We first explain how the six-dimensional string landscape can be conveniently described in the language of F-theory, and explain how the non-abelian sector of F-theory vacua have a description in terms of the geometry of an elliptically fibered Calabi-Yau threefold. We then investigate the abelian sector of string vacua. We first demonstrate the subtleties of the abelian sector of string vacua

[13] We are not claiming that all known string models in six-dimensions can be embedded in F-theory. It is true, however, that there is no obstruction in embedding the non-abelian sector of any known string model in F-theory vacua. We elaborate on this issue in chapter 3.

through examining heterotic and F-theory backgrounds. We go on to use M-theory/F-theory duality to describe the abelian sector of F-theory backgrounds. In particular, we identify the geometric counterpart of the abelian anomaly coefficients of F-theory vacua.

In chapter 4 we present the lessons learned by comparing the space of apparently consistent theories to the space of known string vacua in the six-dimensional supergravity landscape. We first present the results of examining the intermediate regime of $T = 0$ theories. Next, we describe the intermediate regime of theories with abelian gauge symmetry, and present the challenges of analyzing this space. We also derive the geometric identities that come from the fact that F-theory vacua satisfy six-dimensional anomaly cancellation conditions.

In chapter 5 we summarize the results of this thesis once more and present the interesting questions that arise as a result of our investigations.

As stated at the beginning of this section, the content of this thesis is based on the papers [115], [116] and [117]. The results of [115] were obtained in collaboration with Vijay Kumar and Washington Taylor. The results of [116] were obtained in collaboration with Washington Taylor.

Chapter 2

6D $(1, 0)$ Theories and Anomaly Constraints

In this chapter, we review the low-energy data of six-dimensional $(1, 0)$ supergravity theories and study the bounds placed on these theories by anomaly constraints, *i.e.,* we study the "outer boundary" of apparently consistent theories in the six-dimensional landscape. A major result in the analysis of the "outer boundary" is that there are only a finite number of consistent non-abelian massless spectra when the number of tensor multiplets is less than nine [62, 64]. When $T \geq 9$, there exist infinite classes of theories. We extend upon this result in two directions in this chapter.

While the space of non-abelian theories with $T < 9$ has been shown to be bounded, explicitly constructing these consistent theories is still quite a daunting task. This task is much more approachable for theories with no tensor multiplets. In this chapter, we present a systematic way of building all consistent $T = 0$ theories whose gauge group factors are special unitary. Constructing theories with other gauge groups can be carried out as a straightforward generalization of the methods we have employed here.

We also extend the results [62, 64] to the space of theories with abelian gauge symmetry. It turns out that there are only a finite number of consistent gauge/matter combinations modulo $U(1)$ charge assignments. To elaborate, the number of consistent gauge group/matter representations are finite, but there can be cases where there exists an infinite number of different apparently consistent $U(1)$ charge assignments to a given theory.

This chapter is organized as the following. We first review the data parametrizing the low-energy theory in section 2.1. We then summarize how these theories are constrained by anomaly cancellation conditions when the gauge group is non-abelian in section 2.2 [62, 64]. We then focus on $T = 0$ theories and describe the boundary of apparently consistent theories in this subsector of six-dimensional theories in section 2.3 [115]. In section 2.4 we investigate how the anomaly constraints place bounds on theories with abelian gauge symmetry [116].

2.1 6D $(1, 0)$ Theories and Anomaly Cancellation

In this section we review six-dimensional theories with $\mathcal{N} = (1, 0)$ supersymmetry and anomaly cancellation in these theories. In section 2.1.1 we present an overview of the field content of these theories. We compute the anomaly polynomial in section 2.1.2 and review anomaly cancellation and factorization in section 2.1.3. We give explicit formulae for the anomaly factorization condition in the presence of $U(1)$'s in section 2.1.4 and discuss some salient features of these equations. In section 2.1.5 we discuss aspects of the generalized Green-Schwarz mechanism that come into play when the theory has abelian gauge symmetry, and also explain why this issue can be safely ignored when discussing the massless spectrum. We summarize in section 2.1.6.

2.1.1 The Massless Spectrum

The massless spectrum of the models we consider can contain four different multiplets of the supersymmetry algebra: the gravity and tensor multiplet, vector multiplet, and hypermultiplet. The contents of these multiplets are summarized in table 2.1.

We consider theories with one gravity multiplet. There can in general be multiple tensor multiplets; we denote the number of tensor multiplets by T. When $T = 1$ it is possible to write a Lagrangian for the theory; the self-dual and anti-self-dual tensors can combine into a single antisymmetric tensor. Theories with T tensor multiplets have a moduli space with $SO(1,T)$ symmetry; the T scalars in each multiplet combine into a $SO(1,T)$ vector j that can be taken to have unit norm. We consider theories with arbitrary gauge group and matter content.

Note that a theory with a general number of tensor multiplets can still be defined despite the lack of a covariant Lagrangian. The partition function can be defined by coupling the three-form field strength to a 3-form gauge potential as in [121, 122]. Classical equations of motion can be formulated as in [82, 86]. Supersymmetry and anomaly cancellation may be discussed at the operator level of a theory obtained by quantizing the classical theory defined by these equations.

We write the gauge group for a given theory as[1]

$$\mathcal{G} = \prod_{\kappa=1}^{\nu} \mathcal{G}_\kappa \times \prod_{i=1}^{V_A} U(1)_i \,. \tag{2.1}$$

Lowercase greek letters $\kappa, \lambda, \cdots$ are used to denote the simple non-abelian gauge group factors; lowercase roman letters $i, j, k, \cdots$ are used to denote $U(1)$ factors. ν and V_A denote the numbers of nonabelian and abelian gauge group factors of the theory.

We denote by N the number of irreducible representations of the non-abelian gauge group under which the matter hypermultiplets transform (including trivial representations); we use uppercase roman letters to index these representations. The hypermultiplet representation I transforms in the representation R^I_κ under $\mathcal{G}_\kappa$ and have $U(1)_i$ charge $q_{I,i}$.

We characterize theories by their massless spectrum. There is a slight subtlety we must consider when dealing with $U(1)$ gauge symmetries. It is possible to break $U(1)$ at the linearized level by certain hypermultiplets, called "linear hypermultiplets" in the literature [123]. We will refer to these multiplets simply as "linear multiplets"

[1]The gauge group generally can have a quotient by a discrete subgroup, but this does not affect the gauge algebra, which underlies the anomaly structure analyzed in this thesis.

Multiplet	Field Content
Gravity	$(g_{\mu\nu}, \psi_\mu^+, B_{\mu\nu}^+)$
Tensor	$(\phi, \chi^-, B_{\mu\nu}^-)$
Vector	(A_μ, λ^+)
Hyper	$(4\varphi, \psi^-)$

Table 2.1: Six-dimensional (1,0) supersymmetry multiplets. The signs on the fermions indicate the chirality. The signs on antisymmetric tensors indicate self-duality/anti-self-duality.

throughout this thesis. When a linear multiplet couples to a vector multiplet the two merge into a long (or non-BPS) multiplet and are lifted from the massless spectrum. Once lifted from the massless spectrum, these long multiplets can be safely ignored. This issue is discussed in more detail in section 2.1.5.

2.1.2 The Anomaly Polynomial for Theories with $U(1)$'s

In six-dimensional chiral theories there can be gravitational, gauge and mixed anomalies [55, 66, 67]. The sign with which each chiral field contributes to the anomaly is determined by their chirality.

The 6D anomaly can be described by the method of descent from an 8D anomaly polynomial. The anomaly polynomial is obtained by adding up the contributions of all the chiral fields present in the theory [55]. For the $T = 1$ case this is given in

[87, 108]. In general we obtain

$$
\begin{aligned}
I_8 = &- \frac{1}{5760}(H - V + 29T - 273)[\mathrm{tr}R^4 + \frac{5}{4}(\mathrm{tr}R^2)^2] \\
&- \frac{1}{128}(9 - T)(\mathrm{tr}R^2)^2 \\
&- \frac{1}{96}\mathrm{tr}R^2[\sum_\kappa \mathrm{Tr}F_\kappa^2 - \sum_{I,\kappa} \mathcal{M}_I^\kappa \mathrm{tr}_{R_\kappa^I} F_\kappa^2] \\
&+ \frac{1}{24}[\sum_\kappa \mathrm{Tr}F_\kappa^4 - \sum_{I,\kappa} \mathcal{M}_I^\kappa \mathrm{tr}_{R_\kappa^I} F_\kappa^4 - 6 \sum_{I,\kappa,\lambda} \mathcal{M}_I^{\kappa\lambda} (\mathrm{tr}_{R_\kappa^I} F_\kappa^2)(\mathrm{tr}_{R_\lambda^I} F_\lambda^2)] \\
&+ \frac{1}{96}\mathrm{tr}R^2 \sum_{I,i,j} \mathcal{M}_I q_{I,i} q_{I,j} F_i F_j \\
&- \frac{1}{6}\sum_{I,\kappa,i} \mathcal{M}_I^\kappa q_{I,i} (\mathrm{tr}_{R_\kappa^I} F_\kappa^3) F_i - \frac{1}{4}\sum_{I,\kappa,i,j} \mathcal{M}_I^\kappa q_{I,i} q_{I,j} (\mathrm{tr}_{R_\kappa^I} F_\kappa^2) F_i F_j \\
&- \frac{1}{24}\sum_{I,i,j,k,l} \mathcal{M}_I q_{I,i} q_{I,j} q_{I,k} q_{I,l} F_i F_j F_k F_l \,.
\end{aligned} \tag{2.2}
$$

$\mathcal{M}_I$ is the size of the representation I that is given by

$$\mathcal{M}_I = \prod_\kappa d_{R_\kappa^I}\,, \tag{2.3}$$

where d_{R_κ} is the dimension of the representation R_κ of $\mathcal{G}_\kappa$. Similarly, $\mathcal{M}_I^\kappa$ ($\mathcal{M}_I^{\kappa\lambda}$) is the number of $\mathcal{G}_\kappa$ ($\mathcal{G}_\kappa \times \mathcal{G}_\lambda$) representations in I, which is given by

$$\mathcal{M}_I^\kappa = \prod_{\mu\neq\kappa} d_{R_\mu^I} \qquad (\mathcal{M}_I^{\kappa\lambda} = \prod_{\mu\neq\kappa,\lambda} d_{R_\mu^I}) \tag{2.4}$$

respectively. V and H are the number of massless vector multiplets and hypermultiplets in the theory. They are given by

$$V \equiv V_{NA} + V_A \equiv \sum_\kappa d_{\mathrm{Adj}_\kappa} + V_A, \qquad H \equiv \sum_I \mathcal{M}_I \tag{2.5}$$

where d_{Adj_κ} is the dimension of the adjoint representation of gauge group $\mathcal{G}_\kappa$. V_{NA} is the number of non-abelian vector multiplets in the theory. The integer N, which is the number of irreducible representations of the non-abelian gauge group, plays an important role in bounding the number of $U(1)$'s. We use 'tr' to denote the trace in the fundamental representation, and 'Tr' to denote the trace in the adjoint. Multiplication of forms should be interpreted as wedge products throughout this thesis unless stated otherwise.

2.1.3 Anomaly Cancellation and Factorization

The Green-Schwarz mechanism [56] can be generalized to theories with more than one tensor multiplet when the anomaly polynomial is factorizes in the following form [86, 97]:

$$I_8 = -\frac{1}{32}\Omega_{\alpha\beta}X_4^\alpha X_4^\beta\,, \tag{2.6}$$

where Ω is a symmetric bilinear form (or metric) in $SO(1,T)$ and X_4 is a four form that is an $SO(1,T)$ vector. X_4 can be written as

$$X_4^\alpha = \frac{1}{2}a^\alpha \mathrm{tr}R^2 + \sum_\kappa \left(\frac{2b_\kappa^\alpha}{\lambda_\kappa}\right)\mathrm{tr}F_\kappa^2 + \sum_{ij} 2b_{ij}^\alpha F_i F_j\,, \tag{2.7}$$

where we define b_{ij} to be symmetric in i,j. The a and b's are $SO(1,T)$ vectors and α are $SO(1,T)$ indices. Note that the anomaly coefficients for the $U(1)$'s can be written in this way due to the fact that the field strength is gauge invariant on its own [101, 102]. Under linear redefinitions of the $U(1)$'s b_{ij} transforms as a bilinear, *i.e.*,

$$F_i = M_i^j F_j', \quad b_{ij}' = M_i^k M_j^l b_{kl}, \quad M \in GL(V_A, \mathbb{R})\,. \tag{2.8}$$

The λ_κ's are normalization factors that are fixed by demanding that the smallest topological charge of an embedded $SU(2)$ instanton is 1. λ_κ is actually equal to the Dynkin index of the fundamental representation of the gauge group $\mathcal{G}_\kappa$. The values of λ_κ for given $\mathcal{G}_\kappa$ are listed in table 2.2 for all the simple groups. b_κ turn out to form

	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
λ	1	2	1	2	6	12	60	6	2

Table 2.2: Normalization factors for the simple groups.

an integral $SO(1,T)$ lattice when we include these normalization factors [65].

There is an important fact related to the factors $\lambda(\mathcal{G})$ worth noting for future reference. Let us define the normalized basis $\{T_i\}$ for the Cartan sub-algebra of $\mathcal{G}$ such that

$$\mathrm{tr}_f T_i T_j = \delta_{ij} \,. \tag{2.9}$$

This provides an unambiguous normalization for the root lattice of a given Lie group. Note that two Lie groups with the same Lie algebra can have different normalizations of the root lattice if their fundamental representations differ. We may define the "coroot basis" for the Cartan sub-algebra as

$$\mathcal{T}_I \equiv \frac{2\alpha_I^i T_i}{\langle \alpha_I, \alpha_I \rangle} \tag{2.10}$$

where α_I^i are the coordinates of the I'th simple root. $\mathcal{T}_I$ have the following properties:

1. The charge of the root vector E_α under $\mathcal{T}_I$ is

   $$\mathcal{T}_I |\alpha\rangle = \frac{2\langle \alpha_I, \alpha \rangle}{\langle \alpha_I, \alpha_I \rangle} |\alpha\rangle \,. \tag{2.11}$$

   In particular for the simple roots of the Lie algebra,

   $$\mathcal{T}_I |\alpha_J\rangle = C_{IJ} |\alpha_J\rangle \,. \tag{2.12}$$

   where C_{IJ} are the elements of the Cartan matrix. We note that the Cartan matrix is determined by the gauge algebra, rather than the gauge group. For example, it is the same for $SO(3)$ and $SU(2)$.

2. The charge of any weight vector $|\beta\rangle$ under $\mathcal{T}_I$ is

   $$\frac{2\langle \alpha_I, \beta \rangle}{\langle \alpha_I, \alpha_I \rangle} \,, \tag{2.13}$$

   which is always integral, by definition of weight vectors.

3. For two basis elements among $\{\mathcal{T}_I\}$,

$$\frac{1}{\lambda(\mathcal{G})}\mathrm{tr}\mathcal{T}_I\mathcal{T}_J = \mathcal{C}_{IJ} \tag{2.14}$$

where $\mathcal{C}$ is the normalized inner product matrix of the coroots, *i.e.*,

$$\mathcal{C}_{IJ} = \frac{1}{\lambda(\mathcal{G})}\frac{4\langle\alpha_I,\alpha_J\rangle}{\langle\alpha_I,\alpha_I\rangle\langle\alpha_J,\alpha_J\rangle}\,. \tag{2.15}$$

Just as with the Cartan matrix, the normalized coroot matrix $\mathcal{C}$ is determined uniquely by the gauge algebra.

Proofs of these statements are given in appendix A.1.

The gauge-invariant three-form field strengths are given by

$$H^\alpha = dB^\alpha + \frac{1}{2}a^\alpha\omega_{3L} + 2\sum_\kappa \frac{b^\alpha_\kappa}{\lambda_\kappa}\omega^\kappa_{3Y} + 2\sum_{ij} b^\alpha_{ij}\omega^{ij}_{3Y}, \tag{2.16}$$

where ω_{3L} and ω_{3Y} are Chern-Simons 3-forms of the spin connection and gauge fields respectively. If the factorization condition (2.6) is satisfied, anomaly cancellation can be achieved by adding the local counterterm

$$\delta\mathcal{L}_{GS} \propto -\Omega_{\alpha\beta}B^\alpha \wedge X^\beta_4\,. \tag{2.17}$$

Meanwhile, supersymmetry determines the kinetic term for the gauge fields to be (up to an overall factor) [86, 101, 102]

$$-\sum_\kappa(\frac{j\cdot b_\kappa}{\lambda_\kappa})\mathrm{tr}(F_\kappa \wedge *F_\kappa) - \sum_{ij}(j\cdot b_{ij})(F_i \wedge *F_j)\,, \tag{2.18}$$

where j is the unit $SO(1,T)$ vector that parametrizes the T scalars in the tensor multiplets. The inner product of j and the b vectors are defined with respect to the metric Ω. There must be a value of j such that all the gauge fields have positive definite kinetic terms. This means that there should be some value of j such that all $j\cdot b_\kappa$ are positive and such that $j\cdot b_{ij}$ is a positive definite matrix with respect to i,j.

If we did not have any $U(1)$'s, (2.6) would be the only way in which the anomaly can be cancelled. When we have abelian vector multiplets, however, a generalized version of the Green-Schwarz mechanism is available [96]. In this case, it is possible to cancel terms in the 8-form anomaly polynomial that are proportional to

$$F \wedge X_6 \,, \tag{2.19}$$

where X_6 is a six form, by a counter-term in the action of the form

$$-C \wedge X_6 \,, \tag{2.20}$$

where C is a Stückelberg 0-form that belongs to a linear multiplet. The coupling of C to the vector boson V is given by

$$\frac{1}{2}(\partial_\mu C - V_\mu)^2 \,, \tag{2.21}$$

which is what we mean by C being a Stückelberg 0-form. The anomalous gauge boson V recieves a mass, hence rendering the $U(1)$ broken; the abelian vector multiplet is lifted from the massless spectrum by coupling to the linear multiplet by the Stückelberg mechanism. When all the anomalous $U(1)$'s are lifted and we look at the pure massless spectrum of the theory, all the gravitational anomalies and gauge/mixed anomalies induced by the massless fields are cancelled completely by two forms through the conventional Green-Schwarz mechanism. The lesson is that when we are discussing the massless spectrum, this generalized version of the Green-Schwarz mechanism does not come into play and can be safely ignored.[2] We elaborate further on this issue in section 2.1.5.

[2]The situation is quite the opposite when we are taking the top-down approach, for example when we are constructing theories from string compactifications. When working downward from the high-energy end, it is important to figure out which $U(1)$ vector bosons that naively seem to be massless are lifted by this mechanism.

2.1.4 The Factorization Equations

We are now ready to write down the factorization equations in the presence of $U(1)$'s. The factorization equations come from demanding that the anomaly polynomial (2.2) factorize in the form (2.6). This yields the following anomaly cancellation conditions.

Gravitational Anomaly Equations

$$\begin{aligned} 273 &= H - V + 29T \\ a \cdot a &= 9 - T \end{aligned} \tag{2.22}$$

These equations come from demanding that pure gravitational anomalies are cancelled. Here, H denotes the number of hypermultiplets and V denotes the number of vector multiplets.

The first equation is sometimes referred to as the "gravitational anomaly bound." This is because of the following reason. As will be seen, only the information related to hypermultiplets charged under gauge fields show up in the other anomaly equations, *i.e.,* one has the freedom to add neutral hypermultiplets to the theory and not affect the mixed and gauge anomaly equations. Therefore, as long as the number of charged hypermultiplets of the theory is below $273 + V - 29T$, one can add neutral hypermultiplets to the theory to cancel the gravitational anomaly. In this sense, the first equation of (2.22) provides an upperbound on the number of charged hypermultiplets of the theory for given T and gauge group.

Mixed Anomaly Equations

$$\begin{aligned} a \cdot (\frac{b_\kappa}{\lambda_\kappa}) &= \frac{1}{6}(A_{\text{Adj}_\kappa} - \sum_I \mathcal{M}_I^\kappa A_\kappa^I) \\ a \cdot b_{ij} &= -\frac{1}{6} \sum_I \mathcal{M}_I q_{I,i} q_{I,j} \end{aligned} \tag{2.23}$$

These equations should be satisfied for each gauge group $\mathcal{G}_\kappa$, $U(1)_i$, and $U(1)_j$. The inner products on the left-hand-side of the equations are inner products with respect to the $SO(1,T)$ metric Ω. The group theory factor A_R is defined to be

$$\mathrm{tr}_R F_\kappa^2 = A_R \mathrm{tr} F_\kappa^2 \tag{2.24}$$

for a given representation R of the gauge group $\mathcal{G}_\kappa$.

We can write the mixed anomaly equations in slightly different notation:

$$\begin{aligned} a \cdot (\frac{b_\kappa}{\lambda_\kappa}) &= \frac{1}{6}(A_{\mathrm{Adj}_\kappa} - \sum_R x_R A_R) \\ a \cdot b_{ij} &= -\frac{1}{6}\sum_I x_{q_i,q_j} q_i q_j \end{aligned} \tag{2.25}$$

Here x_R is the number of hypermultiplets of representation R of gauge group $\mathcal{G}_\kappa$ and x_{q_i,q_j} is the number of hypermultiplets with charge (q_i, q_j) under $U(1)_i \times U(1)_j$.

Gauge Anomaly Equations

$$\begin{aligned} 0 &= B_{\mathrm{Adj}_\kappa} - \sum_I \mathcal{M}_I^\kappa B_\kappa^I \\ (\frac{b_\kappa}{\lambda_\kappa})^2 &= \frac{1}{3}(\sum_I \mathcal{M}_I^\kappa C_\kappa^I - C_{\mathrm{Adj}_\kappa}) \\ (\frac{b_\kappa}{\lambda_\kappa}) \cdot (\frac{b_\mu}{\lambda_\mu}) &= \sum_I \mathcal{M}_I^{\kappa\mu} A_\kappa^I A_\mu^I \\ 0 &= \sum_I \mathcal{M}_I^\kappa E_\kappa^I q_{I,i} \\ (\frac{b_\kappa}{\lambda_\kappa}) \cdot b_{ij} &= \sum_I \mathcal{M}_I^\kappa A_\kappa^I q_{I,i} q_{I,j} \\ b_{ij} \cdot b_{kl} + b_{ik} \cdot b_{jl} + b_{il} \cdot b_{jk} &= \sum_I \mathcal{M}_I q_{I,i} q_{I,j} q_{I,k} q_{I,l} \end{aligned} \tag{2.26}$$

These equations should be satisfied for all $\mathcal{G}_\kappa \neq \mathcal{G}_\lambda$, and for all $U(1)_i$, $U(1)_j$, $U(1)_k$ and $U(1)_l$. For each representation R of group $\mathcal{G}_\kappa$ the group theory coefficients B_R and C_R are defined by

$$\mathrm{tr}_R F^4 = B_R \mathrm{tr} F^4 + C_R (\mathrm{tr} F^2)^2 \tag{2.27}$$

In the event that there is only one fourth order invariant for the given gauge group—as is with for example, $SU(2)$—we define $B_R = 0$. Also, E is defined by

$$\mathrm{tr}_R F^3 = E_R \mathrm{tr} F^3 \,. \tag{2.28}$$

As was with the case of mixed anomaly equations, it is convenient to write the gauge anomaly equation using slightly different notation:

$$\begin{aligned}
0 &= B_{\mathrm{Adj}_\kappa} - \sum_R x_R B_R \\
(\frac{b_\kappa}{\lambda_\kappa})^2 &= \frac{1}{3}(\sum_R x_R C_R - C_{\mathrm{Adj}_\kappa}) \\
(\frac{b_\kappa}{\lambda_\kappa}) \cdot (\frac{b_\mu}{\lambda_\mu}) &= \sum_I x_{RS} A_R A_S \\
0 &= \sum_{R,q_i} x_{R,q_i} q_i E_R \\
(\frac{b_\kappa}{\lambda_\kappa}) \cdot b_{ij} &= \sum_{R,q_i,q_j} x_{R,q_i,q_j} q_i q_j A_R \\
b_{ij} \cdot b_{kl} + b_{ik} \cdot b_{jl} + b_{il} \cdot b_{jk} &= \sum_{q_i,q_j,q_k,q_l} x_{q_i,q_j,q_k,q_l} q_i q_j q_k q_l
\end{aligned} \tag{2.29}$$

As before, x_R is the number of hypermultiplets of representation R of gauge group $\mathcal{G}_\kappa$. x_{RS} is the number of hypermultiplets of representation $R \times S$ of gauge group $\mathcal{G}_\kappa \times \mathcal{G}_\mu$, while x_{R,q_i} is the number of hypermultiplets of representation R of gauge group $\mathcal{G}_\kappa$ with charge q_i under $U(1)_i$. x_{R,q_i,q_j} is the number of hypermultiplets of representation R of gauge group $\mathcal{G}_\kappa$ with charge (q_i, q_j) under $U(1)_i \times U(1)_j$, and x_{q_i,q_j,q_k,q_l} is the number of hypermultiplets that have charge (q_i, q_j, q_k, q_l) under $U(1)_i \times U(1)_j \times U(1)_k \times U(1)_l$.

It was shown in [65] using the anomaly equations that the $SO(1,T)$ vector a and the non-abelian anomaly coefficients b_κ can be embedded in an integral lattice Λ. It was subsequently shown that quantum consistency conditions impose that Λ must further be embeddable in a unimodular lattice [124].

2.1.5 Linear Multiplets and Generalized Green-Schwarz Anomaly Cancellation

In this section we discuss linear multiplets and their role in the generalized Green-Schwarz anomaly cancellation mechanism. We first discuss how two different types of hypermultiplets can be distinguished when we consider their representation under $SU(2)_R$. Then we show how each multiplet couples to vector multiplets. In particular, we show how a linear multiplet can couple to an abelian vector multiplet and form a long multiplet. Next we depict the role that linear multiplets play in the generalized Green-Schwarz anomaly cancellation mechanism. Lastly we show that we may ignore long multiplets formed in this way and the generalized Green-Schwarz mechanism when we are discussing the massless spectrum of the theory. Most of the information on linear multiplets given in this section can be found in [125].

There are two different kinds of hypermultiplets in supersymmetric 6D theories with 8 supercharges. The scalar components of the hypermultiplet can transform either as a complex $\mathbf{2}$ or a real $\mathbf{3}+\mathbf{1}$ under the $SU(2)_R$ symmetry of the theory. We refer to the first type of hypermultiplet simply as a "hypermultiplet," and the second kind of hypermultiplet as a "linear multiplet". As far as their contribution to the gravitational anomaly are concerned, the two kinds of hypermultiplets behave identically. The fermions of the linear multiplet are not charged under any gauge group, so the contribution of the linear multiplet to the anomaly is equivalent to that of a neutral hypermultiplet, as shown shortly.

As stated above, under the $SU(2)_R$ symmetry, the scalar components of the hypermultiplet transform as a complex $\mathbf{2}$. The spinors, on the other hand are neutral, *i.e.,* singlets $(\mathbf{1})$. Meanwhile, the scalar components of the linear multiplet transform as a real $\mathbf{3}+\mathbf{1}$. The spinors transform as $\mathbf{2}$'s.

To see how these multiplets couple to other fields, it is useful to reduce to four dimensions on a two-torus and write out the Lagrangian in terms of $\mathcal{N}=1$ superfields. Both multiplets, when dimensionally reduced, are $\mathcal{N}=2$ fields that consist of two chiral superfields. The hypermultiplets can transform in a non-trivial representation of the gauge group and consist of two chiral superfields Q and $\tilde{Q}$. In this case, the representation of Q must be the conjugate of that of $\tilde{Q}$. It is well known that this multiplet couples to the $\mathcal{N}=2$ vector multiplet that consists of a vector multiplet V and a chiral multiplet Φ in the adjoint representation as

$$\int d^4xd^4\theta\,(Q^\dagger e^V Q + \tilde{Q}^\dagger e^{-V}\tilde{Q}) + \int d^4xd^2\theta\,\tilde{Q}^T\Phi Q + (h.c.)\,. \tag{2.30}$$

Meanwhile, the linear multiplets couple to other fields in quite a different manner [125]. They cannot couple to gauge fields in the standard way, as the Lagrangian would not be $SU(2)_R$ invariant in this case. They can couple to $U(1)$ gauge fields, however. The linear multiplet consists of two chiral fields C and B and couples to $U(1)$ gauge fields as

$$\int d^4xd^4\theta\,(\frac{1}{2}(iC - iC^\dagger - V)^2 + B^\dagger B) - \frac{1}{\sqrt{2}}\int d^4xd^2\theta\,B\Phi + (h.c.)\,. \tag{2.31}$$

Writing the scalar of C as $(\pi_3 + i\phi)$ and the scalar of B as $(\pi_1 + i\pi_2)$, the kinetic terms for the scalars become

$$-\int d^4x((\partial_\mu\phi - \frac{1}{2}A_\mu)^2 + (\partial_\mu\pi_i)^2)\,. \tag{2.32}$$

The ϕ can be gauged away using the gauge transformation

$$A_\mu \to A_\mu + \partial_\mu\Lambda, \qquad \phi \to \phi + \frac{1}{2}\Lambda\,, \tag{2.33}$$

and the $U(1)$ gauge field obtains mass $1/2$. The $U(1)$ gauge field has recived a mass by the Stückelberg mechanism.

By integrating out the F-terms of the linear multiplet, we see that the scalar in Φ recieves the same mass$(1/2)$. Meanwhile, the fermions do not couple to the gauge field, and hence only contribute to gravitational anomalies. They only couple to the fermions in V and Φ through Dirac mass terms, *i.e.,* fermions of C and B pair up with fermions of V and Φ into two Dirac fermions of mass $1/2$.

Combining the auxiliary fields of V and Φ, we get three real auxiliary fields (the 'D fields' for the $\mathcal{N}=2$ vector multiplet) that are in the $\mathbf{3}$ of $SU(2)_R$. These couple to the scalars transforming as the $\mathbf{3}$:

$$-\int d^4x(\pi_i D^i) \tag{2.34}$$

Expanding around a vacuum with $\pi_i = 0$, the $U(1)$ vector multiplet and linear multiplet together form a long $\mathcal{N} = 2$ multiplet with 5 scalars, 2 Dirac fermions, and a vector field, all of mass 1/2 in units of the mass parameter. Note that this long massive spin-1 multiplet is not chiral, as the fermions are Dirac.

When we have linear multiplets, they may be used to cancel anomalies. As discussed in section 2.1.3, it is possible to cancel anomalies of the form

$$F_i \wedge X_6\,, \tag{2.35}$$

where X_6 is a six form, by adding the term

$$-\phi \wedge X_6\,. \tag{2.36}$$

ϕ is a Stückelberg 0-form inside a linear multiplet.

In order for the generalized anomaly cancellation to work, we must have a linear multiplet at our disposal. If we do not have such a linear multiplet, we cannot get rid of the term and hence the theory would be anomalous. In case we have such a multiplet, through the Stückelberg mechanism, we expect the linear multiplet to be eaten to form a long massive spin-1 multiplet. Schematically, we may write

$$L_i = V_i + H_i\,, \tag{2.37}$$

where L_i denotes the long multiplet, V_i denotes the $U(1)$ vector multiplet, and H_i the linear multiplet.

So we see that all the vector bosons of $U(1)$ gauge symmetries whose anomalies are cancelled in this fashion must be massive and must form a long multiplet. These long multiplets are non-chiral and hence do not contribute to gravitational anomalies. Furthermore none of the fields inside this multiplet are charged under other gauge groups. Therefore, we see that these multiplets contribute neither to gravitational anomalies nor to unbroken gauge/mixed anomalies.

By this logic we can further state that all long multiplets obtained by $U(1)$ gauge bosons coupling to linear multiplets do not contribute to the anomaly polynomial. Therefore, we may ignore all the long multiplets — or vector/linear multiplet pairs that couple — when we are discussing gravitational anomalies and gauge/mixed anomalies concerning unbroken gauge symmetry, *i.e.,* gauge symmetry of the massless spectrum.

Long multiplets and hence the generalized Green-Schwarz mechanism may thus be ignored when we are discussing the massless spectrum of the theory. In other words, when we are constructing low-energy effective theories, writing down anomalous $U(1)$'s and then lifting them is a redundant procedure. We may safely restrict our attention to the massless spectrum whose anomalies are all cancelled by two-forms; the factorization condition (2.6) should hold for these theories.

2.1.6 Summary

A six-dimensional $\mathcal{N} = (1,0)$ theory is characterized by its massless spectrum S, the vacuum expectation value of the scalars present in the theory, and the anomaly coefficients a, b_κ and b_{ij}. The massless spectrum is specified by the following data:

1. The number of tensor multiplets T.
2. The gauge group
$$\mathcal{G} = \prod_{\kappa=1}^{N} \mathcal{G}_\kappa \times \prod_{i=1}^{V_A} U(1)_i \,. \tag{2.38}$$
3. The hypermultiplet matter content.

The vacuum expectation value of the scalars in the tensor multiplet is given by a $SO(1,T)$ unit vector j. The anomaly coefficients are $SO(1,T)$ vectors, and in particular b_{ij} is also a bilinear form which transforms under the linear redefinitions of the $U(1)$'s. The massless matter content and the anomaly coefficients must satisfy the anomaly equations (2.22), (2.23) and (2.26).

The anomaly coefficients determine the invariant field strength of the tensors (2.16), the Green-Schwarz term of the quantum effective action (2.17), and the corrected kinetic term for the gauge fields (2.18). Quantum consistency conditions demand that a, b_κ must be embeddable into a unimodular lattice.

2.2 Non-abelian Theories

In this section we review some facts about the space of apparently consistent non-abelian theories in the six-dimensional $(1,0)$ supergravity landscape based on [62, 65, 114]. The main result of these works is the fact that there are only finitely many theories — parameterized by the low-energy spectrum S, the anomaly coefficients a, b_κ and the modulus j — that satisfy the following conditions[3]:

1. The gauge symmetry is non-abelian.

2. S, a and b_κ satisfy the anomaly equations.

3. The gauge kinetic terms $j \cdot b_\kappa$ are positive.

4. $T < 9$.

When $T < 9$, a is a time-like $SO(1,T)$ vector as

$$a \cdot a = 9 - T\,. \tag{2.39}$$

Now for time-like vector a, if $a \cdot b = 0$ and $b^2 = 0$ for some $SO(1,T)$ vector b, then $b = 0$, and accordingly $j \cdot b = 0$. This fact was repeatedly used in the proof for boundedness of non-abelian theories. Therefore the boundedness proof breaks down

[3]We do not go in to the details of this proof in this section. Later in this thesis, however, we prove statements on how theories with abelian gauge symmetry are bounded using almost identical methods as [62, 65]. For more details, see section A.4.

when $T \geq 9$. As a consequence, infinite families of apparently consistent theories exist when the condition $T < 9$ is relaxed. We conclude this section by listing three such infinite families.

The first family consists of an infinite family with an arbitrary gauge group $\mathcal{G} = \prod_\kappa \mathcal{G}_\kappa$. If we let the matter content to be such that there is a single adjoint in each of the factors, the anomaly equation boils down to

$$a \cdot a = 9 - T, \quad a \cdot b_\kappa = 0, \quad b_\kappa^2 = 0, \quad b_\kappa \cdot b_\lambda = 0\,, \tag{2.40}$$

for all κ and pairs κ, λ. When $T < 9$, one cannot find $SO(1,T)$ vectors that satisfy these conditions and have positive definite kinetic terms. For $T \geq 9$, however, this is possible. Taking the $SO(1,T)$ metric to be $\mathrm{diag}(+1,-1,-1,\cdots)$, all the anomaly equations can be satisfied by setting the vectors

$$\begin{aligned} a &= (-3, 1 \times T) \\ b_\kappa &= (3, (-1) \times 9, 0 \times (T-9)) \end{aligned} \tag{2.41}$$

for all κ. We used the notation $x \times n$ to mean that n consecutive entries are equal to x. The gauge kinetic terms $j \cdot b$ become positive for the choice

$$j = (1, 0 \times T)\,. \tag{2.42}$$

The second family consists of the infinite family of spectra found by Schwarz in [126]. Anomaly coefficients that satisfy the anomaly equations for these spectra always exists, but it turns out that the kinetic terms for the gauge fields cannot have positive definite kinetic terms for $T < 9$. This is shown using the aforementioned property of $SO(1,T)$ vectors [65]. As in the previous case, however, the anomaly coefficients for these spectra can have give positive kinetic terms when $T \geq 9$. For example, when $\mathcal{G} = SU(M) \times SU(M)$ and the matter consists of two bifundamental matter fields, the anomaly equations boil down to

$$a \cdot a = 9 - T, \quad a \cdot b_1 = a \cdot b_2 = 0, \quad b_1^2 = b_2^2 = -2, \quad b_1 \cdot b_2 = 2\,. \tag{2.43}$$

Setting the $SO(1,T)$ metric to diag$(+1,-1,-1,\cdots)$, the vectors

$$\begin{aligned} a &= (-3, 1 \times T) \\ b_1 &= (1,1,1,1,0 \times (T-3)) \\ b_2 &= (2,0,0,0,(-1) \times 6, 0 \times (T-9)) \end{aligned} \tag{2.44}$$

satisfy the anomaly equations. The gauge kinetic terms $j \cdot b$ become positive for the choice

$$j = (1, 0 \times T)\,. \tag{2.45}$$

There is also an infinite family of theories with an unbounded number of tensor multiplets [65]. For example, there exists a family of theories with $\mathcal{G} = E_8^k$, no matter with $T = 9 + 8k$ with $k \geq 1$. The anomaly equations are given by

$$-a \cdot b_\kappa = -10, \quad b_\kappa^2 = -12, \quad b_\kappa \cdot b_\lambda = 0 \tag{2.46}$$

for $\kappa \neq \lambda$. The anomaly coefficients are given by

$$\begin{aligned} a &= (-3, 1 \times 8(k+1)) \\ b_\kappa &= (-1, 0 \times (4\kappa - 4), -1, -1, -1, -3, 0 \times (8k + 8 - 4\kappa)) \end{aligned} \tag{2.47}$$

for $\kappa = 1, \cdots, k$. The choice of j that makes all the gauge kinetic terms positive is given by

$$j = (-\sqrt{4k+9}, 0 \times 4k, 1 \times (4k+8))\,. \tag{2.48}$$

2.3 Non-abelian $T = 0$ Theories

In this section, we examine the space of apparently consistent $T = 0$ theories. For technical simplicity, we restrict our attention to theories having gauge groups with nonabelian structure

$$\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_k = SU(M_1) \times \cdots \times SU(M_k) . \tag{2.49}$$

The analysis of this section can be carried over with minor modifications to more general nonabelian gauge group structures.

We only sketch the strategy for constructing the space of apparently consistent $T = 0$ theories in this section, and describe details of the constructed theories in section 4.1, where we present the results of the full landscape analysis. We proceed by first reviewing the low-energy constraints of $T = 0$ theories in section 2.3.1, and then explaining how we can construct the full space of apparently consistent theories in section 2.3.2.

2.3.1 Review of Constraints

In this section, we review the low-energy constraints of $T = 0$ theories with gauge group

$$\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_k = SU(M_1) \times \cdots \times SU(M_k) . \tag{2.50}$$

When $T = 0$ the anomaly cancellation conditions (2.22), (2.23) and (2.26) can be written in terms of a set of integers b_κ associated with the simple factors G_κ of the gauge group

$$
\begin{aligned}
H_{\text{total}} - V = H_{\text{neutral}} + H - V = 273 && (2.51)\\
3b_\kappa = \frac{1}{6}\left[\sum_R x_R A_R - A_{\text{adj}}\right] && (2.52)\\
0 = \sum_R x_R B_R - B_{\text{adj}} && (2.53)\\
b_\kappa^2 = \frac{1}{3}\left[\sum_R x_R C_R - C_{\text{adj}}\right] && (2.54)\\
b_\kappa b_\lambda = \sum_{RS} x_{RS} A_R A_S. && (2.55)
\end{aligned}
$$

Recall that x_R denotes the number of matter fields which transform in the irreducible representation R of gauge group factor $\mathcal{G}_\kappa$. Similarly, x_{RS} denotes the number of matter fields transforming under representation $R \times S$ of $\mathcal{G}_\kappa \times \mathcal{G}_\lambda$. The group theory coefficients A_R, B_R, C_R were defined in equations (2.24) and (2.27). The anomaly coefficients should be integers due to the fact that a, b_κ must form a one-dimensional unimodular lattice [124] when $T = 0$.

In obtaining equation (2.52), we have used the fact that

$$a^2 = 9 - T = 9\,. \tag{2.56}$$

There is an ambiguity in fixing the sign of a relative to the scalar "modulus" j. The word "modulus" is in quotes because actually there is no scalar modulus when $T = 0$. There is, however, a discrete choice in choosing the sign of the unit vector j. This choice individually is not physical, but the relative sign of a and j does matter. We note that once a is chosen to be -3, additional considerations yield $j = 1$ [115].

In this section and in section 4.1, we denote by H the number of matter hypermultiplets carrying nonabelian charges for notational simplicity. We have used H_{neutral} to denote the number of neutral hypermultiplets. Note that because we have specialized to models with simple gauge group factors $SU(M_\kappa)$, the normalization factors λ_κ appearing in the anomaly cancellation conditions are all unity ($\lambda_\kappa = 1$) and do not appear in our equations.

In addition to local anomalies, quantum consistency requires the absence of global anomalies [127]. For models with $SU(M)$ gauge group factors, the absence of global anomalies is guaranteed for any model without local anomalies when the anomaly coefficients form a integral lattice. This result is proven in Appendix A.2.

As mentioned above, there is no scalar modulus when $T = 0$, *i.e.,* the vector j can be fixed to $j = 1$. Hence the coefficient of the kinetic term of the $SU(M_\kappa)$ gauge fields is given by b_κ. By imposing positivity of the kinetic terms, we obtain $b_\kappa > 0$. In a general 6D supergravity theory, the tensor multiplet moduli define the coupling constants, or the strength of the gauge interactions relative to gravity. Theories with $T = 0$ are, therefore, intrinsically gravitational with all interaction strengths set by the Planck scale.

2.3.2 Strategy for Construction

In this section, we explain the method we have used to construct the space of apparently consistent non-abelian $T = 0$ theories. We first sketch the method of how to construct this space of theories and then justify important features of $T = 0$ theories that enable our approach to work.

In constructing the full space of apparently consistent theories, the fact that the anomaly equations depend primarily on the integers b_κ associated with each gauge group factor separately, plays an important role. It can be seen from the anomaly equations that only the cross-term component (2.55) of the anomaly factorization condition depends upon more than one distinct b_κ. By using the other anomaly conditions we can constrain the gauge group factors $SU(M_\kappa)$ and matter transforming under each factor independently. We can then treat these factors and associated matter as "blocks" which can be combined to build models with multiple gauge group factors. This general approach is discussed in [65] and used there to construct $T = 1$ models with gauge groups which are products of $SU(M)$ factors with a restricted class of representations. The main reason we are able to use this approach to actually construct the full space of $T = 0$ theories is because the anomaly coefficients are all positive integers. This fact is responsible for the following desirable features of $T = 0$ theories.

First, there are only a finite number of individual blocks that satisfy the anomaly equations (2.52), (2.53), and (2.54) on their own. Furthermore, we are able to construct this full set of individual blocks of the theory. This is a key result for $T = 0$ theories and we devote the latter part of this section on explaining how this is achieved.

Second, since b_κ are non-zero integers, (2.55) implies that any two gauge group factors of a theory must have matter jointly charged under it. If one has the full list of the finite number of individual blocks of the theory, this implies that there are only a finite number of ways to put them together. This is because only a finite number of blocks are joinable to a given block through jointly charged matter.

For example, the number of blocks one can join to an $SU(10)$ block B with 14 fundamentals ($\square$) and 3 antisymmetrics ($\begin{smallmatrix}\square\\ \square\end{smallmatrix}$) is bounded by 8 — the best case is when it is possible to have the 7 pairs of fundamentals of the block to be bifundamentals ($\square\times\square$) that join an $SU(2)$ block to B, and the 3 antisymmetrics to be in the representation $\begin{smallmatrix}\square\\ \square\end{smallmatrix} \times \square$ that joins B to an $SU(3)$ block. This, of course, is an overestimation as we have not used any anomaly conditions in the process of determining this number. In general, if a block has r distinct representations and the number of representations i in the block is given by n_i, the block can be joined to at most

$$\lfloor\frac{n_1}{2}\rfloor + \lfloor\frac{n_2}{2}\rfloor + \cdots + \lfloor\frac{n_r}{2}\rfloor \tag{2.57}$$

other blocks. Since there are only a finite number of individual blocks, only a finite number of blocks are joinable to a given block.

These two features make it possible for one to construct the full set of apparently consistent non-abelian theories when $T = 0$ by the following steps:

1. Construct all the individual blocks satisfying the equations (2.52), (2.53), and (2.54).

2. Find all the combinations of blocks that can be joined together using (2.55).

3. Eliminate theories that violate the gravitational anomaly bound (2.51).

For the rest of this section, we show that there are only a finite number of blocks that satisfy the single-block anomaly equations (2.52), (2.53), and (2.54). We begin by examining the properties of the group theory coefficients $A_R,\ B_R,\ C_R$ in more detail. As discussed for example in [62] (see also [87]), these group theory coefficients can be computed for any particular representation using two diagonal generators $T_{12},\ T_{34}$ which, in the fundamental representation, take the form

$$(T_{12})_{ab} = \delta_{a1}\delta_{b1} - \delta_{a2}\delta_{b2} \tag{2.58}$$

$$(T_{34})_{ab} = \delta_{a3}\delta_{b3} - \delta_{a4}\delta_{b4} \tag{2.59}$$

The group theory factors $A_R,\ B_R,\ C_R$ can be computed in terms of traces of these generators. For $SU(M), M > 3$, we have

$$A_R = \frac{1}{2}\text{tr}_R T_{12}^2 \tag{2.60}$$

$$B_R + 2C_R = \frac{1}{2}\text{tr}_R T_{12}^4 \tag{2.61}$$

$$C_R = \frac{3}{4}\text{tr}_R T_{12}^2 T_{34}^2 \tag{2.62}$$

In these traces, we sum over all basis states in the representation R, which can be represented in terms of the Young tableaux with various labelings of the associated Young diagram. For $SU(2)$ and $SU(3)$ there is no fourth order Casimir, or generator T_{34}, so we can take $B_R = 0$ and use (2.61) to compute C_R. We will find it useful to work with the linear combination

$$g_R := \frac{1}{12}\left(2C_R + B_R - A_R\right) = \frac{1}{24}\left(\text{tr}_R T_{12}^4 - \text{tr}_R T_{12}^2\right) . \tag{2.63}$$

Since in any given state in the representation $T_{12}^2 \leq T_{12}^4$, we see that

$$g_R \geq 0, \qquad \forall R . \tag{2.64}$$

Rep.	Dimension	A_R	B_R	C_R	g_R
$\square$	M	1	1	0	0
Adjoint	M^2-1	$2M$	$2M$	6	1
$\ydiagram{1,1}$	$\frac{M(M-1)}{2}$	$M-2$	$M-8$	3	0
$\ydiagram{2}$	$\frac{M(M+1)}{2}$	$M+2$	$M+8$	3	1
$\ydiagram{1,1,1}$	$\frac{M(M-1)(M-2)}{6}$	$\frac{M^2-5M+6}{2}$	$\frac{M^2-17M+54}{2}$	$3M-12$	0
$\ydiagram{2,1}$	$\frac{M(M^2-1)}{3}$	M^2-3	M^2-27	$6M$	$M-2$
$\ydiagram{3}$	$\frac{M(M+1)(M+2)}{6}$	$\frac{M^2+5M+6}{2}$	$\frac{M^2+17M+54}{2}$	$3M+12$	$M+4$
$\ydiagram{2,2}$	$\frac{M^2(M+1)(M-1)}{12}$	$\frac{M(M-2)(M+2)}{3}$	$\frac{M(M^2-58)}{3}$	$3(M^2+2)$	$\frac{(M-1)(M-2)}{2}$

Table 2.3: Values of the group-theoretic coefficients A_R, B_R, C_R, dimension and genus for some representations of $SU(M)$, $M \geq 4$. For $SU(2)$ and $SU(3)$, A_R is given in table, while $B_R = 0$ and C_R is computed by adding formulae for $C_R + B_R/2$ from table with $M = 2, 3$.

For representations given by Young diagrams with a single column there are no states with $|\langle T_{12}\rangle| > 1$ and therefore $g_R = 0$; all other representations have $g_R > 0$.

For a gauge group factor $SU(M)$ with corresponding anomaly integer b, we can take a linear combination of the anomaly conditions (2.52), (2.53), (2.54) to get

$$\sum_R x_R\, g_R = \frac{1}{2}\left(2g_{\text{adj}} + b^2 - 3b\right) = \frac{(b-1)(b-2)}{2} \tag{2.65}$$

where we have used $g_{\text{adj}} = C_{\text{adj}}/6 = 1$. We denote this number as the "genus" of a block, for reasons that will be clear later on. Some examples of group theory coefficients, dimensions, and genera are shown in table 2.3.

Now let us show that there are only a finite number of blocks satisfying the equations (2.52), (2.53), and (2.54). For any fixed M and b, there are only a finite number of solutions to these equations. An efficient way of finding all solutions is given by the following. (2.65) gives a bound on the sum of the non-negative values g_R associated with the matter representations transforming under $SU(M)$. This gives a finite partition problem, to which all solutions can be found. Each solution of the partition problem corresponds to a set of values for the x_R associated with representations with nonzero genus g_R. As noted above, the representations with $g_R > 0$ are all associated with Young diagrams having more than one column. We can then fix the x_R for all R with $g_R > 0$ and treat (2.54) as a second partition problem. Since all C_R are positive except for the fundamental representation, this gives a set of possible combinations of coefficients x_R for all representations besides the fundamental. We can then use (2.53) to determine the number of fundamental representations, which must be nonnegative. As an example of how this analysis works let us consider the set of blocks with $b = 1$ and $b = 2$.

Let us begin with models with $b = 1$. From (2.65) we have

$$b = 1: \qquad 2\sum_R x_R g_R = (b-1)(b-2) = 0\,. \tag{2.66}$$

Thus, $x_R = 0$ for any representation with $g_R > 0$, and we cannot include any representations other than those with a single column. The anomaly condition (2.54) becomes

$$\sum_R x_R C_R = 9\,. \tag{2.67}$$

For $M > 7$, the coefficients C_R satisfy $C_R > 9$ for all one-column representations other than the two-index antisymmetric (A2) and fundamental (F) representations. So in these cases the only solution is $x_{A2} = 3$. The anomaly condition (2.53) then becomes

$$\sum_R x_R B_R = x_F + 3(N-8) = B_{\rm adj} = 2M\,, \tag{2.68}$$

so

$$x_F = 24 - M\,. \tag{2.69}$$

Thus, for $b = 1$ there are no possible blocks with $M > 24$, and the only possible blocks with $M > 7$ are $SU(M)$ factors with matter content

$$(24-M)\times\square + 3\times\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}, \quad (b = 1, M \leq 24, H - V = (2 + 45M - M^2)/2 \leq 273)\,. \tag{2.70}$$

(Recall that when describing the hypermultiplet matter content of any block or model we denote by H the number of matter hypermultiplets which carry nonabelian charges; as long as this quantity satisfies $H - V \leq 273$, uncharged hypermultiplets can be added to saturate the gravitational anomaly condition (2.51).)

For $M \leq 7$, other $b = 1$ blocks are possible. It is easy to verify that including the 3-antisymmetric (A3) representation at the second step of the above analysis for $SU(7)$ gives a block satisfying the anomaly cancellation conditions with

$$SU(7): \qquad 22 \times \square + 1 \times \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}, \quad (b = 1, H - V = 141)\,. \tag{2.71}$$

A similar block can be constructed for $SU(6)$ with 20 fundamental, one A2, and one A3 representation. Since for $SU(5)$ the A3 and A2 representations are conjugate (and therefore treated as equivalent in this analysis), this exhausts the range of possibilities for $b = 1$. Note that all these blocks automatically satisfy the gravitational anomaly bound $H - V \leq 273$.

A similar analysis for $b = 2$ again allows only single-column representations, which now restrict $M \leq 12$ and includes $SU(M)$ blocks of the form

$$(48 - 4M) \times \square + 6 \times \begin{smallmatrix}\square\\\square\end{smallmatrix}, \quad (b = 2, M \leq 12, H - V = 1 + 45M - 2M^2 \leq 273) \tag{2.72}$$

for all $M \leq 12$. Other $b = 2$ blocks are possible for $6 \leq M \leq 10$: blocks with single 3-antisymmetric (A3) representations are possible at $M = 10, 9$ with $H - V > 273$ and at $M = 8, 7, 6$ with $H - V \leq 273$. For $SU(6)$ there are also blocks with two and three A3 representations, and for $SU(7)$ there is a block with two A3 representations; all these blocks satisfy the gravitational anomaly bound $H - V \leq 273$. There is also a single $b = 2$ block with gauge group $SU(8)$ and a 4-antisymmetric (A4) representation

$$SU(8): \qquad 32 \times \square + 1 \times \begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}, \quad (b = 2, H - V = 263)\,. \tag{2.73}$$

This exhausts the range of possibilities for $b = 2$ blocks.

It can be shown that the equations (2.52), (2.53), and (2.54) impose that b is bounded above for each $SU(M)$. This is proven in appendix A.3 using the Weyl character formula. The bound on b for each M is given in table A.1. In particular, b must be less than or equal to 2 when $M \geq 16$. We have, however, seen above that

M	max b	(total blocks)	# $SU(M)$ models
13-24	1 (1)	(1)	1
12	2 (2)	(2)	2
11	2 (3)	(4)	2
10	2 (4)	(6)	2
9	3 (4)	(8)	3
8	8 (8)	(22)	15
7	4 (7)	(28)	16
6	6 (8)	(147)	48
5	8 (14)	(186)	23
4	16 (34)	(3893)	207
3	597 (597)		10100
2	$24297 \leq b_{\max} < 36647$		$\sim 5 \times 10^7$

Table 2.4: A summary of the possible distinct matter representations for gauge group factors $SU(M)$. The numbers in parentheses refer to possible blocks without constraint on the number of hypermultiplets, while the numbers without parentheses refer to possible anomaly-free models with a single nonabelian factor with total gauge group $SU(M)$. The number of blocks not individually satisfying gravitational anomaly bound becomes very large at $M = 3$, as does the number of blocks for $M = 2$ even with the gravitational anomaly constraint. We have not precisely computed the number of blocks in these categories.

when $b \leq 2$, $M \leq 24$. We therefore have proven that (M, b) can only take a finite number of values, which in turn implies that there are only a finite number of blocks that satisfy the single-block anomaly equations.

We have explicitly constructed all $SU(M)$ blocks with $M \geq 4$. We have, however, constructed only a subset of $SU(3)$ and $SU(2)$ blocks due to the fact that the sheer number of blocks turns out to be too large. We have summarized the number of blocks for each $SU(M)$ in table 2.4.

2.4 Theories with Abelian Gauge Symmetry

In the previous sections, we have reviewed that the number of apparently consistent non-abelian theories with $T < 9$ tensor multiplets is finite, and have presented a way to construct the full set of theories when $T = 0$. In light of these results it is natural to ask whether such nice bounds exist when we allow abelian gauge group factors. It is useful to divide this question into two parts. That is, we ask;

I. Whether the number of different gauge/matter structures is finite when we ignore the charges of the matter under the $U(1)$'s.

II. Whether given the gauge/matter structure, the number of distinct combinations of $U(1)$ charges each matter multiplet can have is finite.

In this section we show that the answer to the first question is "yes" when $T < 9$. As is the case with non-abelian theories, when $T \geq 9$ we can generate an infinite class of theories in which the bounds that hold for $T < 9$ theories are violated. An example of such an infinite class is given in section 2.4.2.

In addressing the second question, it is important to note that theories with multiple $U(1)$ gauge symmetries (say $U(1)^n$) are defined up to arbitrary linear redefinitions of the gauge symmetry. If we assume that all the $U(1)$'s are compact and normalize the unit charge to be 1 for each $U(1)$ factor, the theories are defined up to $SL(n, \mathbb{Z})$.

From this fact, we may deduce that there are an infinite number of distinct $U(1)$ charge assignments possible for certain non-anomalous gauge/matter structures. This is because there are many known examples of theories with two $U(1)$ factors and at least one uncharged scalar, so that the non-anomalous gauge group can be written in the form $U(1)^2 \times \mathcal{G}_0$. Since any linear combination of the two $U(1)$'s is a non-anomalous $U(1)$ gauge symmetry, it is possible to construct an infinite class of apparently consistent 6D supergravity theories with gauge group $U(1) \times \mathcal{G}_0$ by simply removing the other $U(1)$ along with a neutral scalar from the spectrum.

Hence we see that the answer to the second question is negative. We may now ask, however,

III. Whether all infinite families of $U(1)$'s could be generated in the trivial manner presented above.

IV. Whether additional quantum consistency conditions that are unknown to us at the present could be employed to constrain the set of $U(1)$ charges in a given theory.

Regarding question III, we find that there are non-trivially generated infinite families of $U(1)$ charge solutions. The last question is addressed in later chapters.

This section is organized as follows. In section 2.4.1 we develop a useful technical tool in analyzing the abelian anomaly equations. In section 2.4.2 we address the first question. In section 2.4.3 we address the second and third questions. In particular, we present examples of infinite classes of $T = 1$ theories with $U(1)$'s that are trivially/nontrivially generated. We also discuss subtleties arising in the case $T = 0$, where there are no tensor multiplets. We summarize the results of this section in section 2.4.4.

2.4.1 The Abelian Anomaly Equations

In this section we develop a useful formalism in which to manipulate the abelian anomaly equations. After first understanding the anomaly equations in this picture, we apply this formalism to specific examples with $T = 1$ and with $T = 0$.

It is useful to summarize the mixed (2.23) and gauge (2.26) anomaly constraints involving abelian gauge groups by the following polynomial identities:

$$a \cdot P(x_i) = -\frac{1}{6}\sum_I \mathcal{M}_I f_I(x_i)^2 \tag{2.74}$$

$$0 = \sum_I \mathcal{M}_I^\kappa E_\kappa^I f_I(x_i) \tag{2.75}$$

$$b_\kappa \cdot P(x_i) = \lambda_\kappa \sum_I \mathcal{M}_I^\kappa A_\kappa^I f_I(x_i)^2 \tag{2.76}$$

$$P(x_i) \cdot P(x_i) = \frac{1}{3}\sum_I \mathcal{M}_I f_I(x_i)^4 \tag{2.77}$$

Here we have defined the $SO(1,T)$ scalar and vector polynomials

$$f_I(x_i) \equiv \sum_i q_{I,i} x_i \tag{2.78}$$

$$P^\alpha(x_i) \equiv \sum_i b_{ij}^\alpha x_i x_j \,. \tag{2.79}$$

The reason that $U(1)$ factorization conditions can be written as polynomial identities is because the field strengths of the $U(1)$'s behave like numbers rather than matrices in the anomaly polynomial. We note that the x_i are auxiliary variables and do not have any physical significance. Recall that the i indices index the $U(1)$ gauge groups while the I indices index all the matter representations of the theory. $q_{I,i}$ is the charge of representation I under $U(1)_i$.

A theory with charges $q_{I,i}$ assigned to the hypermultiplets is only consistent if there exist b_{ij} satisfying these equations that give a positive-definite kinetic matrix $j \cdot b_{ij}$ for the $U(1)$ gauge fields. It is useful to define the charge vector with respect to $U(1)_i$ whose components are the charges of the N nonabelian representations:

$$\vec{q_i} \equiv (q_{1,i}, q_{2,i}, \cdots, q_{N,i}) \tag{2.80}$$

There is a $GL(V_A, \mathbb{R})$ symmetry of the $U(1)$ anomaly equations that originates from the fact that there is a freedom of redefining $U(1)$'s. Recall that V_A is the number of the abelian gauge group factors. If there are multiple $U(1)$'s one could take some new linear combination of them to define a new set of non-anomalous $U(1)$'s. The equations are invariant under

$$\begin{pmatrix} \vec{q_1} \\ \vec{q_2} \\ \vdots \\ \vec{q_{V_A}} \end{pmatrix} \rightarrow M \begin{pmatrix} \vec{q_1} \\ \vec{q_2} \\ \vdots \\ \vec{q_{V_A}} \end{pmatrix}, \quad (b^\alpha)_{ij} \rightarrow \left(M^t (b^\alpha) M\right)_{ij}, \tag{2.81}$$

for $M \in GL(V_A, \mathbb{R})$, as expected from (2.8). We have denoted (b^α) to be the matrix whose (i,j) element is b^α_{ij}. When we are discussing properly quantized charges of compact $U(1)$'s the linear redefinitions of the $U(1)$'s must be given by elements of $SL(V_A, \mathbb{Z}) \subset GL(V_A, \mathbb{R})$. In this section, however, we merely use the fact that the anomaly equations are invariant under $GL(V_A, \mathbb{R})$ as a tool for obtaining bounds on the number of $U(1)$'s we can add to a given theory. Therefore, we do not need to be concerned with the issue of integrality of charges.

The factorization equations, combined with the positive-definite condition on b_{ij}, impose stronger constraints on the theory when $T < 9$. This is because a is timelike when $T < 9$:

$$a \cdot a = 9 - T > 0 \tag{2.82}$$

When a is timelike,

$$a \cdot y = 0, \quad y \cdot y \geq 0 \qquad \Rightarrow \qquad y = 0 \tag{2.83}$$

for any arbitrary $SO(1,T)$ vector y. This fact is used in [62] to bound the number of theories with nonabelian gauge groups, and is also crucial in bounding the space of theories with abelian factors. In particular, this fact implies that the charge vectors $\vec{q}_i$ must be linearly independent in order to get a positive definite kinetic term for the $U(1)$'s when $T < 9$. If they are not, there exists non-zero (x_i) such that $f_I(x_i) = 0$ for all I since

$$\vec{f}(x_i) \equiv (f_1(x_i), \cdots, f_N(x_i)) = \sum_i x_i \vec{q}_i \,. \tag{2.84}$$

For such x_i, we see that

$$a \cdot P(x_i) = 0, \qquad P(x_i) \cdot P(x_i) = 0 \,. \tag{2.85}$$

This implies that $P(x_i) = 0$, which in turn implies that $j \cdot P(x_i) = 0$, *i.e.,*

$$\sum_{ij} (j \cdot b)_{ij} x_i x_j = 0 \,. \tag{2.86}$$

This would mean that the kinetic term is not positive-definite. Hence we have proven that in order for the kinetic term to be positive-definite, $\vec{q}_i$ must be linearly independent when $T < 9$. This in particular means that we cannot have a massless $U(1)$ vector under which nothing is charged, *i.e.,* that when $T < 9$, the trivial solutions to the $U(1)$ factorization equations where all the charges are set to $q_{I,i} = 0$ are not acceptable. The analogous connection in 10D between $U(1)$ charges and the BF^2 term, which is related by supersymmetry to the gauge kinetic term [86], also played a key role in the analysis in [69] showing that the ten-dimensional supergravity theories with gauge group $U(1)^{496}$ and $E_8 \times U(1)^{248}$ are inconsistent.

The fact that $\vec{q}_i$ are all linearly independent for $T < 9$ also implies that

$$P(x_i) \cdot P(x_i) = \sum_I \mathcal{M}_I f_I(x_i)^4 > 0 \tag{2.87}$$

for all non-zero x_i as $f_I(x_i)$ cannot be made simultaneously zero for all I. We make use of (2.87) in bounding the set of abelian theories in section 2.4.2. Now let us examine some examples using this formalism.

Examples : $T = 1$

Let us examine the abelian anomaly cancellation conditions for examples that have one tensor multiplet. As discussed earlier, $T = 1$ theories have a Lagrangian description, unlike theories with other T values. The $SO(1, T)$ basis most commonly used for $T = 1$ theories in the literature is

$$\Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \quad b = \frac{1}{2}\lambda_\kappa \begin{pmatrix} \alpha_\kappa \\ \tilde{\alpha}_\kappa \end{pmatrix}, \quad j = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\phi} \\ e^{-\phi} \end{pmatrix} \tag{2.88}$$

for which the factorization condition becomes

$$I_8 = -\frac{1}{16}(\mathrm{tr}R^2 - \sum_\kappa \alpha_\kappa \mathrm{tr}F_\kappa^2 - \sum_{ij} \alpha_{ij} F_i F_j) \wedge (\mathrm{tr}R^2 - \sum_\kappa \tilde{\alpha}_\kappa \mathrm{tr}F_\kappa^2 - \sum_{ij} \tilde{\alpha}_{ij} F_i F_j) \,. \tag{2.89}$$

For the abelian factors we have

$$b_{ij} = \frac{1}{2} \begin{pmatrix} \alpha_{ij} \\ \tilde{\alpha}_{ij} \end{pmatrix}, \tag{2.90}$$

and the gravitational anomaly constraint becomes

$$H - V = 244 \,. \tag{2.91}$$

The kinetic term for the antisymmetric tensor is given by

$$\mathcal{L} = -\frac{1}{2}e^{-2\phi}(dB - \omega) \cdot (dB - \omega)\,, \tag{2.92}$$

where ϕ is the dilaton. We define the Chern-Simons forms ω and $\tilde{\omega}$ as

$$d\omega = \frac{1}{16\pi^2}(\mathrm{tr}R^2 - \sum_{\kappa} \alpha_\kappa \mathrm{tr}F_\kappa^2 - \sum_{ij} \alpha_{ij} F_i F_j) \tag{2.93}$$

$$d\tilde{\omega} = \frac{1}{16\pi^2}(\mathrm{tr}R^2 - \sum_{\kappa} \tilde{\alpha}_\kappa \mathrm{tr}F_\kappa^2 - \sum_{ij} \tilde{\alpha}_{ij} F_i F_j) \tag{2.94}$$

The variation of the two form under gauge transformations becomes

$$\delta B = -\frac{1}{16\pi^2}(\sum_{\kappa} \alpha_\kappa \mathrm{tr}\Lambda_\kappa F_\kappa + \sum_{ij} \alpha_{ij} \Lambda_i F_j) \tag{2.95}$$

and the anomaly can be gotten rid of by adding the term

$$-B \wedge d\tilde{\omega} \tag{2.96}$$

to the Lagrangian. Supersymmetry determines the kinetic term for the gauge fields to be

$$-\sum_{\kappa}(\alpha_\kappa e^{\phi} + \tilde{\alpha}_\kappa e^{-\phi})\mathrm{tr}F_\kappa \wedge *F_\kappa - \sum_{ij}(\alpha_{ij} e^{\phi} + \tilde{\alpha}_{ij} e^{-\phi}) F_i \wedge *F_j\,. \tag{2.97}$$

For a consistent theory without instabilities there must be a value of the dilaton such that all the gauge fields have positive kinetic terms. This means that the matrix

$$\gamma_{ij} \equiv \alpha_{ij} e^{\phi} + \tilde{\alpha}_{ij} e^{-\phi} = 2\sqrt{2} j \cdot b_{ij} \tag{2.98}$$

must be positive definite for some value of ϕ. Also, in order for the distinct $U(1)_i$ vector multiplets to be independent degrees of freedom, γ_{ij} must be non-degenerate.

In order to discuss the factorization equations coming from terms with abelian gauge field factors, in addition to $f_I(x_i) = q_{I,i} x_i$ it is convenient to define the quadratic forms

$$F(x_i) = \sum_{ij} \alpha_{ij} x_i x_j \qquad \tilde{F}(x_i) = \sum_{ij} \tilde{\alpha}_{ij} x_i x_j. \tag{2.99}$$

These are the components of $P^\alpha(x)$ defined through (2.79).

The factorization condition can then be summarized by the polynomial identities

$$0 = \sum_I (\mathcal{M}_I^\kappa E_\kappa^I) f_I(x_i) \qquad \text{for all } \kappa \tag{2.100}$$

$$F(x_i) + \tilde{F}(x_i) = \frac{1}{6} \sum_I \mathcal{M}_I f_I(x_i)^2 \tag{2.101}$$

$$\tilde{\alpha}_\kappa F(x_i) + \alpha_\kappa \tilde{F}(x_i) = 4 \sum_I (\mathcal{M}_I^\kappa A_\kappa^I) f_I(x_i)^2 \qquad \text{for all } \kappa \tag{2.102}$$

$$F(x_i)\tilde{F}(x_i) = \frac{2}{3} \sum_I \mathcal{M}_I f_I(x_i)^4 \tag{2.103}$$

The basis chosen for the $U(1)$ factors is defined up to $GL(V_A, \mathbb{R})$:

$$\begin{pmatrix} \vec{q_1} \\ \vec{q_2} \\ \vdots \\ q\vec{V_A} \end{pmatrix} \to M \begin{pmatrix} \vec{q_1} \\ \vec{q_2} \\ \vdots \\ q\vec{V_A} \end{pmatrix}, \quad (\alpha)_{ij} \to \big(M^t(\alpha)M\big)_{ij}, \quad (\tilde{\alpha})_{ij} \to \big(M^t(\tilde{\alpha})M\big)_{ij}. \tag{2.104}$$

(α) and $(\tilde{\alpha})$ denote the matrices whose (i, j) element is α_{ij} and $\tilde{\alpha}_{ij}$, respectively.

As proven in the last section, since $T = 1 < 9$, the charge vectors $\{\vec{q_i}\}$ are linearly independent for solutions of the factorization equations that give a non-degenerate kinetic term for some value of the dilaton. Linear independence of $\vec{q_i}$ imposes positive-definiteness on both α_{ij} and $\tilde{\alpha}_{ij}$. The reason is that the r.h.s.'s of (2.101) and (2.103) are both positive for any real x_i if $\vec{q_i}$ are linearly independent. This is because $\vec{f}(x_i)$ cannot be zero for any real x_i. Hence $F(x_i)$ and $\tilde{F}(x_i)$ are positive for all real x_i. Therefore, α_{ij} and $\tilde{\alpha}_{ij}$ both have to be positive definite.

Let us examine two examples where we can see the abelian anomaly cancellation equations at play. The first example is given by orbifold compactifications of the $E_8 \times E_8$ heterotic string theory [87]. This theory has gauge group $E_7 \times E_8 \times U(1)$

with 10 **56**'s and 66 singlets with respect to E_7. Nothing is charged under the E_8. This matter structure solves the non-abelian factorization equations. The non-abelian part of the anomaly polynomial factorizes to

$$-\frac{1}{16}(\mathrm{tr}R^2 - \frac{1}{6}\mathrm{tr}F^2_{E_7} - \frac{1}{30}\mathrm{tr}F^2_{E_8}) \wedge (\mathrm{tr}R^2 - \mathrm{tr}F^2_{E_7} + \frac{1}{5}\mathrm{tr}F^2_{E_8}) \,. \tag{2.105}$$

We index the hypermultiplet representations **56** by $I = 1, \cdots, 10$ and the singlets by $I = 11, \cdots, 76$. Since there is only one $U(1)$, there is only a single $\alpha = \alpha_{11}$ and a single $\tilde{\alpha} = \tilde{\alpha}_{11}$. Also, $f_I(x) = q_I x$.

Therefore, the anomaly equations can be obtained by plugging in

$$F(x) = \alpha x, \quad \tilde{F}(x) = \tilde{\alpha} x, \quad f_I(x) = q_I x \tag{2.106}$$

to equations (2.100)-(2.103). Since E_7 and E_8 do not have third order invariants, and no matter is charged under E_8, we obtain

$$-\frac{1}{5}\alpha + \frac{1}{30}\tilde{\alpha} = 0 \tag{2.107}$$

$$\alpha + \tilde{\alpha} = \frac{1}{6}(56\sum_{I=1}^{10} q_I^2 + \sum_{I=11}^{76} q_I^2) \tag{2.108}$$

$$\alpha + \frac{1}{6}\tilde{\alpha} = 4\sum_{I=1}^{10} q_I^2 \tag{2.109}$$

$$\alpha\tilde{\alpha} = \frac{2}{3}(56\sum_{I=1}^{10} q_I^4 + \sum_{I=11}^{76} q_I^4) \tag{2.110}$$

This can be re-written as

$$56\sum_{I=1}^{10} q_I^4 + \sum_{I=11}^{76} q_I^4 = 36\left(\sum_{I=1}^{10} q_I^2\right)^2 = \frac{9}{196}\left(\sum_{I=11}^{76} q_I^2\right)^2 . \tag{2.111}$$

Five distinct charge assignments that give solutions to these equations can be obtained by different abelian orbifold — by which we mean an orbifold whose orbifold group is abelian — compactifications. For example, there is a Z_8 orbifold compactification that assigns the charges

$$q_1 = q_2 = -3/8, \quad q_3 = q_4 = q_5 = -1/4, \quad q_8 = q_9 = q_{10} = 0, \quad q_6 = q_7 = -1/8,$$
$$q_{11} = \cdots = q_{30} = 1/8, \quad q_{31} = \cdots = q_{34} = -7/8, \quad q_{35} = q_{44} = 1/4,$$
$$q_{45} = \cdots = q_{50} = -3/4, \quad q_{51} = \cdots = q_{54} = 3/8, \quad q_{55} = \cdots = q_{58} = -5/8,$$
$$q_{59} = \cdots = q_{77} = 1/2, \quad q_{78} = \cdots = q_{66} = -1/2$$

to the hypermultiplets. The anomaly coefficients for these charge assignments are

$$\alpha = 1, \qquad \tilde{\alpha} = 6. \tag{2.112}$$

All five solutions from abelian orbifolds are given in table 1 of [87].

We present one more example that proves to be useful later in this section. Consider the gauge group $SU(13) \times U(1)$ with 4 two-index anti-symmetric, 6 fundamental and 23 singlet representations of $SU(13)$. These solve the anomaly equations that do not concern the $U(1)$ field strengths. The non-abelian part factorizes to

$$-\frac{1}{16}(\mathrm{tr}R^2 - 2\mathrm{tr}F^2_{SU(13)}) \wedge (\mathrm{tr}R^2 - 2\mathrm{tr}F^2_{SU(13)}) \,. \tag{2.113}$$

Denoting the charges of hypermultiplets in the antisymmetric/fundamental/singlet representations as $a_x(x = 1, \cdots, 4)/f_y(y = 1, \cdots, 6)/s_z(z = 1, \cdots, 23)$ the anomaly equations become

$$0 = \sum_x 9a_x + \sum_y f_y \tag{2.114}$$
$$\alpha + \tilde{\alpha} = \frac{1}{6}(\sum_x 78a_x^2 + \sum_y 13f_y^2 + \sum_z s_z^2) \tag{2.115}$$
$$2\alpha + 2\tilde{\alpha} = 4(\sum_x 11a_x^2 + \sum_y f_y^2) \tag{2.116}$$
$$\alpha\tilde{\alpha} = \frac{2}{3}(\sum_x 78a_x^4 + \sum_y 13f_y^4 + \sum_z s_z^4) \tag{2.117}$$

If there exist for given a_x, f_y, s_z a solution $\alpha, \tilde{\alpha}$ to these equations, the anomaly polynomial factorizes into

$$-\frac{1}{16}(\mathrm{tr}R^2 - 2\mathrm{tr}F^2_{SU(13)} - \alpha F^2_{U(1)}) \wedge (\mathrm{tr}R^2 - 2\mathrm{tr}F^2_{SU(13)} - \tilde{\alpha} F^2_{U(1)}) \tag{2.118}$$

We identify infinite classes of charge assignments and $\alpha, \tilde{\alpha}$ values that solve these equations in section 2.4.3.

Examples : $T = 0$

We now examine the abelian anomaly cancellation conditions for examples with $T = 0$. In the case $T = 0$ all the $SO(1,T)$ vectors a, b, j reduce to numbers. As explained in the previous section we may set

$$\Omega = 1, \qquad a = -3, \qquad j = 1\,. \tag{2.119}$$

Positivity of the kinetic term imposes that the b_κ's be positive and that b_{ij} be a positive definite matrix.

In this case the factorization condition becomes

$$I_8 = -\frac{1}{32}(-\frac{3}{2}\mathrm{tr}R^2 + \sum_\kappa \frac{2b_\kappa}{\lambda_\kappa}\mathrm{tr}F_\kappa^2 + \sum_{ij} 2b_{ij}F_iF_j)^2\,. \tag{2.120}$$

The gravitational anomaly constraint becomes

$$H - V = 273\,. \tag{2.121}$$

The factorization equations coming from $U(1)$'s can be written out by using the quadratic form

$$P(x_i) = \sum_{ij} b_{ij}x_ix_j\,, \tag{2.122}$$

as the polynomial identities

$$0 = \sum_I (\mathcal{M}_I^\kappa E_\kappa^I) f_I(x_i) \qquad \text{for all } \kappa \tag{2.123}$$

$$P(x_i) = \frac{1}{18} \sum_I \mathcal{M}_I f_I(x_i)^2 \tag{2.124}$$

$$P(x_i) = \frac{\lambda_\kappa}{b_\kappa} \sum_I (\mathcal{M}_I^\kappa A_\kappa^I) f_I(x_i)^2 \qquad \text{for all } \kappa \tag{2.125}$$

$$P(x_i)^2 = \frac{1}{3} \sum_I \mathcal{M}_I f_I(x_i)^4 \tag{2.126}$$

The basis chosen for the $U(1)$ factors is as usual defined up to $GL(V_A, \mathbb{R})$ through (2.81). Since $T = 0 < 9$, the charge vectors $\{\vec{q}_i\}$ are linearly independent for solutions of the factorization equations that give a non-degenerate kinetic term for some value of the dilaton.

Now let us examine a few examples in which the anomaly equations come into play. We first consider a theory with gauge group $SU(6) \times U(1)$ with 1 adjoint, 9 two-index anti-symmetric, 18 fundamental and 31 singlet representations of $SU(6)$. These solve the non-abelian anomaly factorization equations. The factorized non-abelian anomaly polynomial is

$$-\frac{1}{32}(-\frac{3}{2}\text{tr}R^2 + 6\text{tr}F_{SU(6)}^2)^2 . \tag{2.127}$$

Denoting the charge of hypermultiplets in the adjoint/antisymmetric/fundamental/singlet representation as $d/a_x(x = 1, \cdots, 9)/f_y(y = 1, \cdots, 9)/s_z(z = 1, \cdots, 31)$ the anomaly equations become

$$0 = \sum_x 5a_x + \sum_y f_y \tag{2.128}$$

$$b = \frac{1}{18}(35d^2 + \sum_x 15a_x^2 + \sum_y 6f_y^2 + \sum_z s_z^2) \tag{2.129}$$

$$3b = (12d^2 + \sum_x 4a_x^2 + \sum_y f_y^2) \tag{2.130}$$

$$b^2 = \frac{1}{3}(35d^4 + \sum_x 15a_x^4 + \sum_y 6f_y^4 + \sum_z s_z^4) \tag{2.131}$$

For charges and b satisfying these equations, the anomaly polynomial of the theory factorizes into

$$-\frac{1}{32}(-\frac{3}{2}\mathrm{tr}R^2 + 6\mathrm{tr}F_{SU(6)}^2 + 2bF_{U(1)}^2)^2 \,. \tag{2.132}$$

Finding an apparently consistent supergravity theory with this gauge group amounts to identifying values for b and the charges d, a_x, f_y, s_z so that (2.128) through (2.131) are satisfied. If we assume that the $U(1)$ is compact and the charges are integers then this is a system of Diophantine equations over the integers. In general, classifying solutions to such a system of equations can be a highly nontrivial problem in number theory.

A particularly interesting class of examples are pure abelian theories. In this case the only non-trivial abelian anomaly equations are the last equations of equation (2.23) and equation (2.26) ((2.128) and (2.131) for $T = 0$). For a theory with a given number of abelian vector multiplets, there is a lower bound on the number of charged multiplets it must have. When the number of charged hypermultiplets saturate this bound, the charges that the hypermultiplets carry is severely restricted. Such theories have a particularly simple structure and are interesting to study further.

As an example, consider the case of a purely abelian theory when $T = 0$ and $V_A = 1$. We denote the charges of the X charged hypermultiplets in the theory by

$q_1, \cdots, q_X \neq 0$. Then we must solve

$$\begin{aligned} 18b &= \sum_I q_I^2, \\ 3b^2 &= \sum_I q_I^4. \end{aligned} \tag{2.133}$$

Using the inequality

$$(\sum_I q_I^2)^2 \leq X(\sum_I q_I^4), \tag{2.134}$$

we see that

$$X \geq 108. \tag{2.135}$$

When X is equal to 108, *i.e.,* when the number of charged hypermultiplets saturates the lower bound, the only solutions to the equations (2.133) are

$$q_I = \pm Q \quad \text{for all } I, \quad b = 6Q^2. \tag{2.136}$$

Similarly for any pure abelian $T = 0$ theory, using (2.123) and (2.126) we can show that the following relation between V_A and the number of charged hypermultiplets X holds

$$\frac{324\, V_A}{V_A + 2} \leq X \leq V_A + 273. \tag{2.137}$$

The proof is given in appendix A.5. Hence, as above, when $V_A = 1$ there must be at least 108 charged hypermultiplets; likewise, when $V_A = 2$ there must be at least 162 charged hypermultiplets. As seen in the $V_A = 1$ case, in the marginal cases when X exactly saturates this bound, the solutions to the charge equations are particularly simple. From (2.137) it follows that the maximum possible number of $U(1)$ factors that can be included in any $T = 0$ theory with no nonabelian gauge group is $V_A \leq 17$.[4]

A family of marginal/nearly marginal $T = 0$ theories with gauge group $U(1)^k, k \leq 7$ can be obtained by Higgsing an $SU(8)$ theory with one adjoint hypermultiplet and

[4](2.137) alone implies that $V_A \leq 17$ or $V_A \geq 32$. An additional constraint following from equations (2.123) and (2.126) is needed to obtain the desired bound. We derive this constraint and show that indeed $V_A \leq 17$ in appendix A.5.

Gauge Group	$\cdot$	$U(1)$	$U(1)^2$	$U(1)^3$	$U(1)^4$	$U(1)^5$	$U(1)^6$	$U(1)^7$
X	0	108	162	198	225	243	252	252
$324\,V_A/(V_A+2)$	0	108	162	194.4...	216	231.4...	243	252
X'	273	166	113	78	52	35	27	28

Table 2.5: The number of charged hypermultiplets X for pure abelian theories obtained by Higgsing the adjoint of the $SU(8)$ theory with one adjoint and nine antisymmetrics. We have also tabulated the number of uncharged hypermultiplets in the theory, $X' = (273 + V_A - X)$.

nine antisymmetric hypermultiplets. The number of charged hypermultiplets X for the various pure abelian theories one obtains by Higgsing the adjoint of this theory in different ways is summarized in table 4.3.

2.4.2 Bounds on $T < 9$ Theories With U(1)'s

We now address the first (I) of the four questions raised at the beginning of this section. That is, we prove that the number of different gauge/matter structures — specified by the gauge group and the non-abelian representation of the matter — is finite for theories with $T < 9$, when we ignore the charge of the matter under the $U(1)$'s.

The strategy we pursue is the following. First, we prove that in a non-anomalous theory, the number of $U(1)$'s is bounded by a number determined by the non-abelian gauge/matter content. We prove that the relations

$$V_A \leq (T+2)\sqrt{2N} + 2(T+2) \tag{2.138}$$

$$V_A \leq (T+2)(T+\frac{7}{2}) + (T+2)\sqrt{2V_{NA} + (T^2 - 51T + \frac{2225}{4})} \tag{2.139}$$

hold for non-anomalous theories with $T < 9$, where V_A is the rank of the abelian gauge group, V_{NA} is the number of nonabelian vector multiplets, and N is the number of hypermultiplet representations. These bounds imply that the number of $U(1)$'s one could add to a non-abelian theory is finite. We note that these bounds are in no sense optimal; they could be improved by a more careful analysis. These inequalities, however, will be sufficient for the purpose of proving that there is a finite bound on theories with $T < 9$.

Next, we define the concept of "curable theories" as non-abelian theories with $H - V > 273 - 29T$ that can be made non-anomalous by adding $U(1)$ vector fields and without changing the non-abelian gauge/matter structure. Curable theories are defined so that all non-anomalous theories with abelian gauge symmetry can be obtained by adding $U(1)$'s either to non-anomalous theories, or to curable theories. We then show that the number of curable theories is finite for $T < 9$, which combined with our other results, implies that the number of gauge/matter structures possible for non-anomalous theories with $T < 9$ is finite.

Lastly, we construct an infinite class of non-anomalous theories with an unbounded number of $U(1)$'s and $T \geq 9$.

Bound on Number of $U(1)$ Factors

In this section we prove equations (2.138) and (2.139) for non-anomalous theories with $T < 9$. Given a gauge group

$$\mathcal{G} = \prod_{\kappa=1}^{\nu} \mathcal{G}_\kappa \times \prod_{i=1}^{V_A} U(1)_i \,, \tag{2.140}$$

we show that the bound on V_A can be given as a function of the number of nonabelian vector multiplets

$$V_{NA} = \sum_\kappa d_{\mathrm{Adj}_\kappa} \tag{2.141}$$

and N, the number of nonabelian matter representations. This can be done by making use of equation (2.77), which is equivalent to the last equation of (2.26) :

$$\begin{aligned} P(x_i) \cdot P(x_i) &= \frac{1}{3} \sum_I \mathcal{M}_I f_I(x_i)^4 \\ \Leftrightarrow b_{ij} \cdot b_{kl} + b_{ik} \cdot b_{jl} + b_{il} \cdot b_{jk} &= \sum_I \mathcal{M}_I q_{I,i} q_{I,j} q_{I,k} q_{I,l} \end{aligned} \tag{2.142}$$

We should be looking for integral solutions of this equation for $b_{ij}, q_{I,i}$, but for now we simply determine the conditions for the equations to have real solutions. These conditions impose a bound on V_A, which also is a bound for integral solutions. These equations have a $GL(V_A, \mathbb{R})$ invariance summarized by (2.104) where the matrices M now can be taken to be real.

We first state the following useful

Fact : For $(T+1)$ symmetric $n \times n$ matrices $S_1, \cdots, S_{T+1}$, there exists a matrix $M \in GL(n, \mathbb{R})$ such that for $\tau \equiv \lceil n/(T+2) \rceil$ the matrices $S'_\alpha = M^t S_\alpha M$ satisfy

$$(S'_\alpha)_{kl} = 0 \qquad \text{for distinct } k, l \leq \tau \tag{2.143}$$

for all $\alpha = 1, \cdots, (T+1)$.

Proof : First pick an arbitrary n-dimensional vector e_1. Then generate the set of $(T+2)$ vectors

$$V_1 = \{e_1, S_1 e_1, \cdots, S_{(T+1)} e_1\}\,. \tag{2.144}$$

When $1 < n/(T+2)$ there always exists a non-zero vector that is orthogonal to these $(T+2)$ vectors. Pick one and call it e_2. Then generate the set of $(T+2)$ vectors

$$V_2 = \{e_2, S_1 e_2, \cdots, S_{(T+1)} e_2\}\,. \tag{2.145}$$

When $2 < n/(T+2)$ there always exists a non-zero vector that is orthogonal to the set $V_1 \cup V_2$ of vectors. Pick one and call it e_3. By iterating this process we can obtain τ non-zero mutually orthogonal vectors,

$$e_1, \cdots, e_\tau \tag{2.146}$$

such that

$$e_i^t S_\alpha e_j = 0 \quad \text{for } i \neq j \tag{2.147}$$

for all α. We can then choose vectors $e_{\tau+1}, \cdots, e_n$ that together with $e_1, \cdots e_\tau$ form a basis of $\mathbb{R}^n$. Define

$$M = (e_1 \cdots e_n)\,, \tag{2.148}$$

where e_i are column vectors. It is clear that $\det M \neq 0$ and that for $S'_\alpha = M^t S_\alpha M$

$$(S'_\alpha)_{kl} = 0 \qquad \text{for distinct } k, l \leq \tau \tag{2.149}$$

for all α. □

Due to this fact, there exists a matrix $M \in GL(V_A, \mathbb{R})$ such that for all α

$$M_{ik} M_{jl} b^\alpha_{kl} = 0 \quad \text{for distinct } i, j \leq \tau \tag{2.150}$$

for any solution of (2.142). We have defined

$$\tau = \left\lceil \frac{V_A}{T+2} \right\rceil . \tag{2.151}$$

This means that the existence of a solution of (2.142) implies the existence of a solution of the same equations with

$$\vec{b}_{kl} = 0 \quad \text{for distinct } k, l \leq \tau \,. \tag{2.152}$$

Therefore, we may from now on assume that this condition is true.

For ordered pairs (i, j) with $i < j \leq \tau$, we define the vectors

$$\vec{Q}_{ij} \equiv (\sqrt{\mathcal{M}_1} q_{1,i} q_{1,j}, \sqrt{\mathcal{M}_2} q_{2,i} q_{2,j}, \cdots, \sqrt{\mathcal{M}_N} q_{N,i} q_{N,j}) \,. \tag{2.153}$$

Then we have

$$\vec{Q}_{ij} \cdot \vec{Q}_{kl} = \sum_I \mathcal{M}_I q_{I,i} q_{I,j} q_{I,k} q_{I,l} = b_{ij} \cdot b_{kl} + b_{ik} \cdot b_{jl} + b_{il} \cdot b_{jk} = 0 \tag{2.154}$$

for ordered pairs $(i, j) \neq (k, l)$. Also from the last equation of (2.26), we have

$$\vec{Q}_{ij} \cdot \vec{Q}_{ij} = \sum_I \mathcal{M}_I q^2_{I,i} q^2_{I,j} = b_{ii} \cdot b_{jj} + 2 b_{ij} \cdot b_{ij} = b_{ii} \cdot b_{jj} > 0. \tag{2.155}$$

The last inequality holds due to the fact that $\vec{b}_{ii}$ are timelike vectors since

$$|\vec{b}_{ii}|^2 = \frac{1}{3}\sum_I \mathcal{M}_I q_{I,i}^4 > 0\,, \tag{2.156}$$

as $\vec{q}_i$ cannot be a zero vector. The inner-product of two timelike $SO(1,T)$ vectors cannot be zero, and $\sum_I \mathcal{M}_I q_{I,i}^2 q_{I,j}^2$ cannot be negative and hence the inequality in (2.155).

Therefore, $\vec{Q}_{ij}$ are non-zero mutually orthogonal vectors for $i < j \leq \tau$. Thus, we have $\tau(\tau-1)/2$ non-zero orthogonal vectors in an N-dimensional space. Hence

$$\frac{1}{2}(\frac{V_A}{T+2} - 1)(\frac{V_A}{T+2} - 2) \leq \frac{\tau(\tau-1)}{2} \leq N \leq H \leq V_{NA} + V_A + 273 - 29T. \tag{2.157}$$

Using the two inequalities

$$\frac{1}{2}(\frac{V_A}{T+2} - 1)(\frac{V_A}{T+2} - 2) \leq N \tag{2.158}$$
$$\frac{1}{2}(\frac{V_A}{T+2} - 1)(\frac{V_A}{T+2} - 2) \leq V_{NA} + V_A + 273 - 29T, \tag{2.159}$$

we obtain the bounds

$$V_A \leq (T+2)\sqrt{2N} + 2(T+2) \tag{2.160}$$
$$V_A \leq (T+2)(T+\frac{7}{2}) + (T+2)\sqrt{2V_{NA} + (T^2 - 51T + \frac{2225}{4})}\,, \tag{2.161}$$

as promised. We have used the fact that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for non-negative a, b to simplify the first inequality. The second inequality simply follows from solving (2.159) for V_A when the inequality is saturated. This result implies that given a non-anomalous non-abelian theory, the number of $U(1)$'s one could add to the theory keeping it non-anomalous is bounded.

The equations we have used also apply to pure abelian theories. This is because we have not used any constraint coming from the non-abelian structure of the theory; we have only used the equation (2.142). Hence we can obtain a bound on the number of $U(1)$'s when the theory is purely abelian:

$$V_A \leq (T+2)(T+\frac{7}{2}) + (T+2)\sqrt{T^2 - 51T + \frac{2225}{4}} \tag{2.162}$$

Note that this bound is substantially weaker than the tighter bound $V_A \leq 17$ for $T = 0$. (For $T = 0$ this bound states that $V_A \leq 54$, while we show that $V_A \leq 17$ in appendix A.5.)

Curability and Finiteness of Curable Theories

We define "curable" theories to be non-abelian theories that violate the gravitational anomaly bound $H - V > 273 - 29T$, but whose anomaly polynomial can nonetheless be made factorizable by adding $U(1)$ vector multiplets and some singlet hypermultiplets in such a way that the gravitational bound is satisfied. There should also exist values for the scalars in the tensor multiplets that make the kinetic terms of all gauge fields positive in the resulting non-anomalous theory. We also assume that these theories do not have any hypermultiplets that are singlets under the non-abelian gauge group.

From this definition it is clear that all non-anomalous theories with abelian gauge symmetry can be obtained by the following steps.

1. Begin with a theory without abelian gauge group factors that is either non-anomalous or curable.

2. Add abelian vector multiplets and (possibly) hypermultiplets in the trivial representation of the non-abelian gauge group.

3. Assign $U(1)$ charges to the matter.

We note that it is clear that the number of $U(1)$'s one could add to a given curable theory is finite, since it is bounded by (2.160) and (2.161). From this it is evident that the crucial remaining step in obtaining bounds on theories with abelian gauge symmetry is showing that the number of curable theories is bounded.

As an example of a curable theory, consider the $T = 0$ theory with gauge group and matter content

$$SU(9): \qquad 26\times\Box + 1\times\begin{array}{c}\Box\\[-4pt]\Box\end{array} + 1\times\begin{array}{c}\Box\\[-4pt]\Box\\[-4pt]\Box\end{array}, \quad (H - V = 274)\,. \tag{2.163}$$

Although this theory violates the gravitational anomaly bound, it satisfies the other gauge/mixed anomaly equations with $b = 2$. (Recall that when $T = 0$ the anomaly coefficients a, b are numbers.) The theory (2.172) can be cured by adding a single $U(1)$ vector multiplet and assigning charges to the matter in the following way

$$\begin{aligned} SU(9)\times U(1): \quad & 6\times(\Box,+1) + 6\times(\Box,-1) + 14\times(\Box,\cdot) + \\ & 1\times(\begin{array}{c}\Box\\[-4pt]\Box\end{array},\cdot) + 1\times(\begin{array}{c}\Box\\[-4pt]\Box\\[-4pt]\Box\end{array},\cdot), \qquad (H - V = 273)\,. \end{aligned} \tag{2.164}$$

The anomaly polynomial of the final theory factorizes to

$$I_8 = -\frac{1}{32}(-\frac{3}{2}\mathrm{tr}R^2 + 2\mathrm{tr}F^2_{SU(9)} + 6F^2_{U(1)})^2\,. \tag{2.165}$$

Since abelian vector multiplets and singlet hypermultiplets do not appear in a curable theory and do not contribute to the nonabelian gauge/mixed anomalies, it is clear that the anomaly polynomial of a curable theory takes the form

$$\begin{aligned} I_8 = & -\frac{(H - V - 273 + 29T)}{5760}\left(\mathrm{tr}R^4 + \frac{5}{4}(\mathrm{tr}R^2)^2\right) \\ & - \frac{1}{32}\Omega_{\alpha\beta}\left(\frac{1}{2}a^\alpha\mathrm{tr}R^2 + \sum_\kappa(\frac{2b^\alpha_\kappa}{\lambda_\kappa})\mathrm{tr}F^2_\kappa\right)\left(\frac{1}{2}a^\beta\mathrm{tr}R^2 + \sum_\kappa(\frac{2b^\beta_\kappa}{\lambda_\kappa})\mathrm{tr}F^2_\kappa\right)\,, \end{aligned} \tag{2.166}$$

where $H - V$ is larger than $273 - 29T$. One might think that any theory of this type is naively curable, since we could apparently add an arbitrary number of $U(1)$ vector multiplets under which no matter field is charged, so that $H - V' = 273 - 29T$. The kinetic term for these vector fields, however, would be degenerate — in fact zero — if we do so. In fact, in many cases the bounds on the number of $U(1)$ factors that can be added to a theory make it impossible to cure nonabelian theories with anomalies of the form (2.166).

We have reviewed the fact that the number of distinct nonabelian gauge groups and matter representations possible for theories with $T < 9$ and no $U(1)$ factors is finite in section 2.2 [62, 65]. The bound $H_{\text{charged}} - V \leq 273 - 29T$ from the gravitational anomaly condition played a key role in this proof, limiting the number of charged hypermultiplets that could appear in a theory with any given nonabelian gauge group. To prove that the number of curable theories is also finite for $T < 9$ we need an analogous constraint on the number of hypermultiplets for theories with $U(1)$ factors. We now find such a bound, using the bounds (2.160) and (2.161) on the number of $U(1)$ factors that can be added to a curable theory.

Suppose a theory is curable by adding V_A $U(1)$ vector multiplets and H' hypermultiplets. Then using (2.160) we obtain

$$\begin{aligned} 273 - 29T &\geq (H-V)_{\text{cured theory}} = H - V + H' - V_A \\ &\geq H - V + H' - \sqrt{2}(T+2)\sqrt{H'+N} - 2(T+2) \\ &= (H-V) + (\sqrt{H'+N} - \frac{(T+2)}{\sqrt{2}})^2 - N - (\frac{1}{2}T^2 + 4T + 6)\,, \end{aligned} \tag{2.167}$$

where H, V and N denote the numbers of hypermultiplets, vector multiplets and hypermultiplet representations in the initial non-abelian theory. Since $H' \geq 0$, when $N \leq (T+2)^2/2$ we have

$$(\sqrt{H'+N} - \frac{(T+2)}{\sqrt{2}})^2 - N \geq -\frac{(T+2)^2}{2}\,, \tag{2.168}$$

while when $N \geq (T+2)^2/2$ we have

$$(\sqrt{H'+N} - \frac{(T+2)}{\sqrt{2}})^2 - N \geq (\sqrt{N} - \frac{(T+2)}{\sqrt{2}})^2 - N = -\sqrt{2}(T+2)\sqrt{N} + \frac{(T+2)^2}{2}\,. \tag{2.169}$$

Thus, any curable theory satisfies one of the following two constraints:

$$H - V \leq 273 - 29T + (T^2 + 6T + 8) \tag{2.170}$$

$$H - V - \sqrt{2}(T+2)\sqrt{N} \leq 273 - 29T + (2T + 4) \tag{2.171}$$

Curable theories therefore must satisfy the non-abelian gauge/mixed anomaly equations among (2.23), (2.26) and one of these modified gravitational anomaly constraints.

This result suggests that the proof in [62, 65] can be modified to show that the number of curable theories are in fact finite. There it was shown that the H of theories that obey the non-abelian factorization equations — and can have a positive kinetic term — grew faster than the V of the theory when V became large. This in turn implied that V must be bounded for theories that satisfy the the non-abelian factorization equations and respect the $H - V$ bound. We have shown that curable theories must obey the same non-abelian factorization equations with the $H - V$ constraint modified. Fortunately, this constant is only modified by a term subleading in $N < H$. This suggests that the boundedness of curable theories can be shown along the same lines as the proof of boundedness of non-abelian theories. This is indeed the case, though the added term proportional to $\sqrt{N}$ complicates some parts of the analysis. The details of the full proof of this statement are presented in appendix A.4.

We note that the equations (2.170) and (2.171) enable us to identify many uncurable theories with ease. For example, it can be shown that the $T = 0$ theory with gauge group and matter content

$$SU(7): \qquad 27 \times \square + 1 \times \ydiagram{2,1}, \quad (H - V = 351) \tag{2.172}$$

is uncurable, since

$$H - V > 273 - 29T + (T^2 + 6T + 8) = 281 \tag{2.173}$$

$$H - V > 273 - 29T + (2T + 4) + \sqrt{2}(T + 2)\sqrt{N} = 277 + 2\sqrt{56} = 291.9... \tag{2.174}$$

To summarize, we have defined 'curable theories' to be supergravity theories that satisfy the following conditions:

1. The gauge group is non-abelian.

2. The theory has no singlet hypermultiplets.

3. $H - V > 273 - 29T$

4. The theory can be made non-anomalous by adding $U(1)$ vector fields that are independent degrees of freedom, as well as possibly adding singlet hypermultiplets.

5. In the resulting non-anomalous theory, there exists a choice for the scalars in the tensor multiplets that makes the kinetic terms of all gauge fields positive.

We have proven the latter three of the following facts:

1. The number of non-anomalous non-abelian theories is finite [62, 65].

2. The number of $U(1)$'s one can add to non-anomalous theories is finite.

3. The number of curable theories is finite.

4. The number of $U(1)$'s one can add to curable theories is finite.

As pointed out in the beginning of this section, any non-anomalous theory with $U(1)$'s can be constructed by adding abelian vector multiplets and neutral hypermultiplets to a non-anomalous or curable theory with no abelian gauge symmetry. Hence it follows that there is only a finite number of distinct gauge/matter structures a 6D $(1,0)$ theory could have even when we allow abelian components to the gauge group. In particular, this implies that the total rank of the gauge group is bounded, even when we admit abelian factors in the gauge group.

Infinite Classes of Theories with $T \geq 9$

In this section, we show that for $T \geq 9$ a bound cannot be imposed on the number of $U(1)$'s as we have done in the case $T < 9$. We first show that there are certain classes of theories to which one could add an arbitrary number of $U(1)$'s, and discuss why this is not possible when $T < 9$ and show an example of an infinite class of non-anomalous theories with an unbounded number of $U(1)$'s. We end by presenting a class of theories with an unbounded number of $U(1)$'s and tensor multiplets.

Suppose we have a theory $\mathcal{T}_0$ with gauge group $\mathcal{G}_0$ that satisfies all the anomaly equations and has an $SO(1,T)$ unit vector j_0 that satisfies $j_0 \cdot b_\kappa > 0$ for all gauge groups κ. Denote the number of vector and hypermultiplets of this theory as V_0 and H_0.

Suppose an $SO(1,T)$ vector b that satisfies the following conditions exists:

1. b is light-like, *i.e.*, $b^2 = 0$.

2. $a \cdot b = 0$.

3. $b_\kappa \cdot b = 0$ for all κ.

4. $b \cdot j_0 > 0$.

Recall that in the case $T < 9$ it is impossible for a vector b to satisfy conditions 1, 2 and 4 at the same time. In that case a is a time-like vector and if 1 and 2 are satisfied, b must be a zero vector. This is what prevented us from having a $U(1)$ with nothing charged under it.

The situation is quite different when $T \geq 9$; in this case a vector b satisfying the four conditions above is not ruled out in general. Once such a b is available one could construct theory $\mathcal{T}_k$ from $\mathcal{T}_0$ with the following properties.

1. The gauge group is $\mathcal{G}_k = \mathcal{G}_0 \times U(1)^k$.

2. The matter content is that of $\mathcal{T}_0$ with k neutral hypermultiplets added.

3. Nothing is charged under the $U(1)$'s, *i.e.*, $q_{I,j} = 0$ for all I, j.

4. The non-abelian anomaly coefficients are given by b_κ.

5. The abelian anomaly coefficients are given by $b_{ij} = \delta_{ij} b$.

6. The tensor multiplet scalar vacuum expectation value is given by j_0.

By adding the k neutral hypermultiplets, the gravitational anomaly condition,

$$H_k - V_k = (H_0 + k) - (V_0 + k) = H_0 - V_0 = 273 - 29T \tag{2.175}$$

is satisfied. The non-abelian anomaly factorization conditions are all satisfied by definition. We find that all the $U(1)$ anomaly equations among (2.23) and (2.26) are also satisfied as both sides of the equation turn out to be 0. Also,

$$j_0 \cdot b_{ij} = (j_0 \cdot b)\delta_{ij} \tag{2.176}$$

is a positive definite matrix by the assumption that $b \cdot j_0 > 0$. Therefore, this theory satisfies all the anomaly equations and has a sensible kinetic term. Since this is true for any k we find that we could add an infinite number of $U(1)$'s to $\mathcal{T}_0$.

The simplest of these classes of theories is given when there are no non-abelian factors. A $U(1)^k$ theory with $273 - 29T + k$ neutral hypermultiplets and a, b_{ij} given by

$$a = (-3, 1 \times T, 0, \cdots, 0), \quad b_{ij} = b\delta_{ij} \quad \text{for } b = (3, (-1) \times 9, 0, \cdots, 0) \tag{2.177}$$

satisfies all the factorization equations. $x \times n$ denotes that n consecutive components have the same value x. Defining

$$j = (1, 0, 0, 0, 0, \cdots, 0)\,, \tag{2.178}$$

we find that the matrix for the kinetic term of the vector multiplets

$$j \cdot b_{ij} = 3\,\delta_{ij} \tag{2.179}$$

is positive definite. k is bounded below by $29T - 273$ but has no upper-bound.

Let us end by giving an example of a class of theories with an indefinitely increasing number of $U(1)$ factors and tensor multiplets. A simple example can be constructed when the gauge group is $U(1)^{29k}$, $T = k$ and the matter is given by 273 neutral hypermultiplets. a, b_{ij} given by

$$a = (-3, 1 \times T, 0, \cdots, 0), \quad b_{ij} = b\delta_{ij} \quad \text{for } b = (3, (-1) \times 9, 0, \cdots, 0) \tag{2.180}$$

These anomaly coefficients trivially satisfy the anomaly equations for the given spectrum. When

$$j = (1, 0, 0, 0, 0, \cdots, 0), \tag{2.181}$$

we find that the matrix for the kinetic term of the vector multiplets

$$j \cdot b_{ij} = 3\,\delta_{ij} \tag{2.182}$$

is positive definite. k is bounded below by 9 but has no upper-bound.

2.4.3 Infinite Classes of Non-anomalous Theories with $U(1)$'s

In this section, we investigate the second and third questions (II and III) posed at the beginning of section 2.4, beginning with II: Given the gauge/matter content of the theory — by which we mean that we fix the gauge group and the representations of the hypermultiplets with respect to the non-abelian part of the gauge group — are there an infinite number of solutions to the $U(1)$ charge equations? We denote these $U(1)$ charge equations "hypercharge" equations.

As pointed out previously, there are infinite families of solutions that may be "trivially generated" in the following sense. There certainly exist solutions of the anomaly equations with gauge group $\mathcal{G} = \mathcal{G}_0 \times U(1)^2$. In such a case, denoting the charge vectors with respect to the two $U(1)$'s $\vec{q}_1$ and $\vec{q}_2$, any linear combination $\vec{Q} = r\vec{q}_1 + s\vec{q}_2$ solves the anomaly equation for $\mathcal{G}' = \mathcal{G}_0 \times U(1)$ with the same matter structure. On top of the anomaly cancellation conditions, we may demand that additional consistency conditions be obeyed [124, 128, 129, 130]. Three such conditions are applicable to six-dimensional supergravity theories with compact $U(1)$ abelian factors:

1. **Charge Integrality Constraint :** All charges of particles should be integral with respect to the minimal charge of the $U(1)$'s.

2. **Minimal Charge Constraint :** The greatest common divisor of the charges of all particles under each $U(1)$ should coincide with the minimal charge — or inverse of the periodicity — of the $U(1)$.

3. **Unimodularity Constraint :** The string charge lattice spanned by the anomaly coefficients should be embeddable in a unimodular lattice.

The first and second constraints do not stop us from generating an infinite familiy because if the initial theory with $\mathcal{G} = \mathcal{G}_0 \times U(1)^2$ satisfied the charge integrality constraint and the minimal charge constraint, the new theory would also satisfy this constraint when r, s are taken to be mutually prime integers. The unimodularity constraint does not help either, as we see shortly.

Let us depict the situation with the simplest example. For $\mathcal{G} = U(1)^2$, $T = 1$ the following charges on the 246 hypermultiplets of the theory solve the anomaly equations. Assume that there are 48 hypermultiplets with charge $(0,1)$, 48 hypermultiplets with charge $(1,0)$, 48 hypermultiplets with charge $(1,1)$ and 102 neutral hypermultiplets. Written in terms of charge vectors

$$\vec{q}_1 = (1 \times 96, 0 \times 48, 0 \times 102), \qquad \vec{q}_2 = (0 \times 48, 1 \times 96, 0 \times 102)\,, \tag{2.183}$$

where $q \times n$ denotes that n consecutive components have the same value q. The only non-trivial anomaly equations concerned are

$$\frac{1}{6}(48x_1^2 + 48(x_1+x_2)^2 + 48x_2^2) = (\alpha_{11} + \tilde{\alpha}_{11})x_1^2 + 2(\alpha_{12} + \tilde{\alpha}_{12})x_1x_2 + (\alpha_{22} + \tilde{\alpha}_{22})x_2^2 \tag{2.184}$$

$$\frac{2}{3}(48x_1^4 + 48(x_1+x_2)^4 + 48x_2^4) = (\alpha_{11}x_1^2 + 2\alpha_{12}x_1x_2 + \alpha_{22}x_2^2)(\tilde{\alpha}_{11}x_1^2 + 2\tilde{\alpha}_{12}x_1x_2 + \tilde{\alpha}_{22}x_2^2) \tag{2.185}$$

Both equations are satisfied by the choice

$$\alpha_{11} = \alpha_{22} = 2\alpha_{12} = \tilde{\alpha}_{11} = \tilde{\alpha}_{22} = 2\tilde{\alpha}_{12} = 8\,. \tag{2.186}$$

Therefore

$$\vec{Q} = (r \times 48, (r+s) \times 48, s \times 48, 0 \times 102) \tag{2.187}$$

satisfy the equations

$$\frac{1}{6}(48r^2 + 48(r+s)^2 + 48s^2) = 16r^2 + 16rs + 16s^2 \tag{2.188}$$

$$\frac{2}{3}(48r^4 + 48(r+s)^4 + 48s^4) = (8r^2 + 8rs + 8s^2)^2 . \tag{2.189}$$

It is easy to see that this choice of charges solves the anomaly equation for $\mathcal{G} = U(1)$ with $\alpha = \tilde{\alpha} = (8r^2 + 8rs + 8s^2)$. Therefore, we obtain an infinite class of solutions to the anomaly equations for $\mathcal{G} = U(1)$.

It is clear that imposing the charge integrality constraint and the minimal charge constraint does not stop us from generating this infinite family as we may take r and s to be mutually prime integers. Now we show that the unimodularity constraint is also satisfied when r and s are integers.

It is useful to notice that when $T = 1$, a sufficient condition for the unimodularity constraint is that all the anomaly coefficients α and $\tilde{\alpha}$ defined in section 2.4.1 are even integers. This is because if all α and $\tilde{\alpha}$ are even integers, all string charge vectors

$$a = \begin{pmatrix} -2 \\ -2 \end{pmatrix}, \quad b = \frac{1}{2}\begin{pmatrix} \alpha \\ \tilde{\alpha} \end{pmatrix} \tag{2.190}$$

are embeddable in the unimodular lattice spanned by

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.191}$$

with inner product structure Ω as defined in (2.88). When r, s are integers, α and $\tilde{\alpha}$ of the $U(1)$ are both equal to $(8r^2 + 8rs + 8s^2)$, which is an even integer. Therefore, the unimodularity constraint does not rule out this infinite class of theories.

The natural follow-up question to ask is whether there is some gauge/matter structure that permits an infinite number of distinct solutions to the hypercharge equations that cannot be lifted to a theory with more $U(1)$'s. It turns out that there are infinite classes of solutions to anomaly equations of a theory with gauge group $\mathcal{G}_0 \times U(1)$ that cannot be lifted to $\mathcal{G}_0 \times U(1)^2$. The example we examine is the theory with gauge group $SU(13) \times U(1)$ that we presented at the end of section 2.4.1. There we found a solution to the non-abelian factorization condition with 4 antisymmetrics, 6 fundamentals and 23 singlets in the $SU(13)$. The non-abelian part of the factorized polynomial is

$$-\frac{1}{16}(\mathrm{tr}R^2 - 2\mathrm{tr}F^2_{SU(13)}) \wedge (\mathrm{tr}R^2 - 2\mathrm{tr}F^2_{SU(13)}) . \tag{2.192}$$

Denoting the charge of hypermultiplets in the antisymmetric/fundamental/singlet representation as $a_x(x = 1, \cdots, 4)/f_y(y = 1, \cdots, 6)/s_z(z = 1, \cdots, 23)$ the anomaly equations become

$$9\sum_x a_x + \sum_y f_y = 0 \tag{2.193}$$

$$78\sum_x a_x^4 + 13\sum_y f_y^4 + \sum_z s_z^4 = \frac{3}{2}\alpha\tilde{\alpha} \tag{2.194}$$

$$78\sum_x a_x^2 + 13\sum_y f_y^2 + \sum_z s_z^2 = 6\alpha + 6\tilde{\alpha} \tag{2.195}$$

$$44\sum_x a_x^2 + 4\sum_y f_y^2 = 2\alpha + 2\tilde{\alpha} \tag{2.196}$$

There is an ansatz that solves this equation given by

$$(a_x) = (-3a - \frac{2}{3}f, a, a, a) \tag{2.197}$$

$$(f_y) = (f, f, f, f, f, f) \tag{2.198}$$

$$(s_z) = ((6a + f) \times 18, 0, 0, 0, 0, 0) \tag{2.199}$$

where in the last line we mean that 18 of the s_z take the value $(6a + f)$ while five take 0. This ansatz satisfies the first equation and renders the third and fourth equations equivalent. Then the second and third equation can be solved with respect to $\alpha, \tilde{\alpha}$ to yield

$$\frac{\alpha}{f^2}, \frac{\tilde{\alpha}}{f^2} = \frac{2}{9}\left[(49 + 198t + 594t^2) \pm \sqrt{39}(1 + 6t)\sqrt{23 - 24t - 36t^2}\right] \tag{2.200}$$

where we have defined $t = a/f$. It is easy to see that $\alpha, \tilde{\alpha}$ are real as long as

$$\frac{-2 - 3\sqrt{3}}{6} \leq t \leq \frac{-2 + 3\sqrt{3}}{6}. \tag{2.201}$$

Both α, $\tilde{\alpha}$ are positive when t is in this range. Hence we see that there are an infinite number of integral hypercharge solutions to the equations (2.193)-(2.196) that give allowed values of $\alpha, \tilde{\alpha}$.

It is clear that this theory cannot be lifted to a theory with gauge group $SU(13) \times U(1)^2$. Although the ansatz for the given solution seems to imply that this theory can be lifted, for example by choosing the charges for one $U(1)$ to be proportional to a and the charges for the other $U(1)$ to be proportional to f, the fact that a/f must lie in a certain range implies that there must be an obstruction to doing this. The obstruction is that if one tries to lift the theory to a theory with gauge group $SU(13) \times U(1)^2$, the matrices α_{ij} and $\tilde{\alpha}_{ij}$ of this theory cannot be made into positive definite real matrices as is required for the $U(1)$ gauge fields to have positive-definite kinetic terms.

The next question to ask is whether there is an infinite subclass of these theories that satisfy all three quantum consistency conditions introduced at the beginning of this section. Generating a subclass of theories that satisfy the integrality constraint and the minimum charge constraint is not difficult. For example, by taking f and a to be mutually prime integers and f to be a multiple of 3, one can generate an infinite class of solutions that satisfy these two constraints. These conditions, however, do not lead to the unimodularity constraint.

In order to construct a subclass of theories that satisfy all three constraints, let us examine whether there exists an infinite number of rational values of t that make the right hand side of (2.200) rational. This problem boils down to the question of whether the equation

$$23 - 24t - 36t^2 = 39q^2 \tag{2.202}$$

admits an infinite number of solutions with rational t and q. We find that there indeed are an infinite number of rational solutions to this equation using methods outlined in chapter 7 of [131]. When

$$\frac{a}{f} = t = \frac{13k^2 - 234k - 51}{24(13k^2 + 3)} \tag{2.203}$$

for k rational we find that

$$\frac{\alpha}{f^2} = \frac{13}{144(3 + 13k^2)^2}\left[6687 + 54756k + 94458k^2 - 124956k^3 + 39455k^4\right] \tag{2.204}$$

$$\frac{\tilde{\alpha}}{f^2} = \frac{13}{144(3 + 13k^2)^2}\left[2475 + 37908k + 170274k^2 - 29484k^3 + 9035k^4\right] . \tag{2.205}$$

Hence we find that the number of non-anomalous theories with $SU(13) \times U(1)$ with this particular type of matter content is infinite.

To be clear, we now spell out the explicit subclass of theories that satisfy all three quantum consistency conditions. Setting $k = r/s$ for integers r and s in the above equations, we find that when

$$a = 13r^2 - 234rs - 51s^2 \tag{2.206}$$

$$f = 24(13r^2 + 3s^2) , \tag{2.207}$$

α and $\tilde{\alpha}$ take on the values

$$\alpha = 52\left[6687s^4 + 54756s^3r + 94458s^2r^2 - 124956sr^3 + 39455r^4\right] \tag{2.208}$$

$$\tilde{\alpha} = 52\left[2475s^4 + 37908s^3r + 170274s^2r^2 - 29484sr^3 + 9035r^4\right] , \tag{2.209}$$

which are even integers. As discussed early on in this section, this implies that the string charge lattice can be embedded in a unimodular lattice. It is clear that this ansatz assigns integer charges to all the fields and hence the charge integrality constraint is also satisfied. If a and $(-3a - 2f/3)$ are mutually prime, the minimal charge constraint is also satisfied. There are an infinite number of integer pairs (r, s) that render a and $(-3a - 2f/3)$ mutually prime. In fact, we can show that when

$$r = 84n + 43 \tag{2.210}$$

$$s = 182n + 92 \tag{2.211}$$

for integer n, a and $(-3a-2f/3)$ are mutually prime. This fact is proven in appendix A.6.

We have found a particular gauge/matter structure with one $U(1)$ that has an infinite number of distinct solutions to the hypercharge equations for $T = 1$. Furthermore the theory cannot be lifted to a theory with two $U(1)$'s for these hypercharge assignments.

The situation is rather subtle for the case of $T = 0$. The equations (2.123)-(2.126) make it clear that any infinite class of solutions to the anomaly equation with charge vectors of the form

$$\vec{Q} = r\vec{q}_1 + s\vec{q}_2 \tag{2.212}$$

for one $U(1)$ can be lifted to $U(1)^2$. As in the $T = 1$ case there are a plethora of examples of gauge/matter structure that admit an infinite family of hypercharge solutions in this way. If, however, we want to identify an infinite class of theories that satisfy anomaly equations for a single $U(1)$ factor that cannot be extended to $U(1)^2$, we cannot have a simple linear ansatz as in the $T = 1$ case. Examining some specific examples of $T = 0$ theories gives interesting number theory problems that in some cases seem to have infinite $U(1)$ families that cannot be extended to $U(1)^2$ models, but we do not go into the details of these constructions here.

2.4.4 Summary

In this section, we have considered 6D supergravity theories with $(1, 0)$ supersymmetry with abelian as well as nonabelian gauge group factors. The following statements have been proven for such theories when the number of tensor multiplets T satisfies $T < 9$:

1. The number of abelian vector multiplets is bounded above by (2.160) and (2.161). The upper bound is determined by the nonabelian gauge/matter content.

2. The number of possible gauge groups and nonabelian matter content is finite, though there are families with infinite numbers of possible distinct $U(1)$ charges.

From (2), it immediately follows that

3. There is a global bound on the rank of the gauge group of any non-anomalous 6D $\mathcal{N} = (1, 0)$ theory with $T < 9$.

When $T \geq 9$, infinite classes of theories can be constructed, just as it was with non-abelian theories.

Chapter 3

6D $(1,0)$ String Vacua

In the previous chapter, we have investigated the boundary of apparently consistent theories of the six-dimensional $(1,0)$ supergravity landscape. In the present chapter, we carry out the second step of the landscape analysis by investigating the boundary of known string vacua.

An important observation that can be made about string models in six-dimensions is that their non-abelian sectors have a natural embedding into F-theory vacua [64, 65, 114]. In fact, F-theory provides a unifying framework in which to include string vacua with all values of T with a wide variety of gauge groups. Six-dimensional F-theory vacua can be obtained by compactifying the twelve-dimensional theory on an elliptically fibered Calabi-Yau threefold X [118, 119]. The low-energy data of an F-theory vacuum can be extracted from the geometric data of manifold X. We review how the geometric language of F-theory is ideal for describing low-energy data of six-dimensional supergravity theories and how non-abelian sectors of string models can be embedded into this framework.

The abelian sector of a string vacuum is more difficult to deal with. The difficultly arises from an aspect of abelian gauge symmetry discussed in section 2.1.5. Abelian vector fields — vector fields under which no other vector field is charged — can be coupled at the linear level to Stückelberg fields and be lifted from the massless spectrum. Therefore, even simple information such as the number of abelian gauge fields is not trivial to obtain.

There are, however, tools to understand the abelian sector of F-theory backgrounds. In particular, M-theory/F-theory duality is useful in understanding the abelian spectrum and the anomaly coefficients of F-theory vacua. In this chapter, we show in detail how to extract information of the abelian sector of six-dimensional F-theory vacua from the geometric data of the Calabi-Yau threefold it is compactified on.

This chapter is organized as the following. In section 3.1 we review how the geometric data of F-theory compactifications translate into low-energy data of the six-dimensional supergravity theory, and explain how non-abelian sectors of string vacua can be naturally embedded in this framework. In particular we introduce type IIB intersecting brane models and magnetized brane backgrounds and show how the non-abelian sector of these vacua can be embedded into F-theory data. In section 3.2 we investigate the subtleties in determining the abelian sector of string vacua. We utilize the example of magnetized brane backgrounds. Finally in section 3.3 we use M-theory/F-theory duality to extract low-energy data of the abelian sector of F-theory vacua. In particular, we determine the spectrum, the charges of matter under abelian gauge groups and the abelian anomaly coefficients from the geometry of the compactification manifold.

3.1 F-theory and the Non-abelian Sector

In this section, we explain why F-theory is an ideal framework to describe six-dimensional $(1,0)$ supergravity theories and string vacua. In particular, we explain why it is possible to embed the non-abelian sector of all known string vacua into F-theory.

We begin by explaining how low-energy parameters of six-dimensional supergravity theories can naturally be embedded into F-theory in section 3.1.1. In particular, there is a beautiful map between anomaly coefficients and geometric data of F-theory vacua [64, 65, 118, 119]. This map is crucial in embedding other string vacua into F-theory — we explain how this is done in section 3.1.1 as well. Next we go on to examine two examples of string vacua — type IIB intersecting brane models in section 3.1.2 and magnetized brane models in section 3.1.3— and see how the non-abelian sector of these backgrounds can naturally be embedded in the framework of F-theory.

3.1.1 6D $(1,0)$ F-theory Vacua and Embeddability

In this section, we review six-dimensional F-theory vacua and how to embed the non-abelian sector of supergravity theories and other string vacua in them. We first give a general review F-theory backgrounds following the presentation of [65]. We then explain an important feature of F-theory vacua, *i.e.,* that they all satisfy the "Kodaira constraint," and how one can embed the non-abelian sector of a supergravity theory satisfying this constraint to F-theory. We conclude the section by explaining how the Kodaira constraint is obeyed in various six-dimensional string vacua and noting that the non-abelian sector of all known string vacua therefore can be in principle embedded in F-theory.

F-theory backgrounds can be thought of as non-perturbative type-IIB backgrounds with seven-branes which are not necessarily mutually local. When we are compactifying F-theory on some elliptically fibered manifold, the base of the manifold $\mathcal{B}$ can be thought of as the space we are compactifying type IIB string theory on. The value of the axio-dilaton varies over the base; this value is identified with the complex structure of the elliptic fiber of the fibration. In order to get $\mathcal{N}=(1,0)$ supersymmetry, the total space of the fibration must be a Calabi-Yau threefold. This fact places restrictions on $\mathcal{B}$ [65]. Two relevant conditions that $\mathcal{B}$ must satisfy are that $h^{2,0}(\mathcal{B})=h^{1,0}(\mathcal{B})=0$ and that $K\cdot K=10-h^{1,1}(\mathcal{B})$ where K is the canonical class of $\mathcal{B}$. We note that the first condition implies that $H_2(\mathcal{B})\cong H^{1,1}(\mathcal{B})$.

The fiber degenerates on complex codimension-one submanifolds of the base. These submanifolds can be thought of as the submanifolds the seven-branes wrap. The type of degeneration determines the nature of the seven-brane and tells us the non-abelian gauge group we get in the six-dimensional theory [118, 119, 132, 133, 134, 135]. The codimension-two singularities can be thought of as intersecting points of the seven-branes. These contain information on the local matter we obtain

[97, 112, 133, 134, 135, 136, 137, 138, 139, 140]. There are various ways of deriving the gauge/matter content coming from a given elliptically fibered Calabi-Yau manifold. One way is to blow up these codimension-one and codimension-two singularities and to compute intersection numbers. We explain this procedure in more detail in section 3.3.

A beautiful fact is that the geometrical data of an elliptically fibered Calabi-Yau threefold can be encoded as vectors in an integral lattice. More precisely, geometric data such as the canonical divisor class, the Kähler class of the base manifold, and the algebraic two-cycles (or divisors) the seven-branes wrap can be expressed as a vector in the $H_2 \cong H^{1,1}$ lattice of the base manifold. It turns out that this integral lattice is precisely the lattice Λ that parametrizes the low energy theory [64, 65].

To provide the complete map between geometrical data and the low-energy data, we begin by noting that the number of tensor multiplets T satisfies $T = h^{1,1}(\mathcal{B}) - 1$. The H_2 lattice of the base manifold is an $SO(1,T)$ lattice [65]. The anomaly coefficients and the modulus j — which parameterizes the vacuum expectation values of the scalars in the tensor multiplets — are vectors in this lattice, and hence correspond to two-cycles in the base. The a vector corresponds to the canonical divisor class of the base, the j vector corresponds to the Kähler class of the base, and the b_κ vectors corresponds to the locus of brane κ. The "type of degeneration" along the locus b_κ — more precisely the singularity type and the monodromy of the fiber — determines the gauge group $\mathcal{G}_\kappa$.[1] For example, we have listed the gauge algebra obtained from singularity types when there is no monodromy in table 3.1. Note that since b_κ must be effective, $j \cdot b_\kappa > 0$.

There is an important constraint on the vectors a and b_κ that comes from the Calabi-Yau condition. The discriminant locus of an elliptic fibration is given by a divisor class Δ in the base $\mathcal{B}$. This can be decomposed into a sum of components

$$\Delta = \sum_\kappa c_\kappa b_\kappa + Y \tag{3.1}$$

[1]More precisely, the singularity determines the gauge algebra rather than the gauge group. This distinction can be ignored for our purposes.

Singular Fiber	Gauge Group	c_κ
I_n	$SU(n)$	n
II	none	2
III	$SU(2)$	3
IV	$SU(3)$	4
I_n^*	$SO(2n+8)$	$n+6$
II^*	E_8	10
III^*	E_7	9
IV^*	E_6	8

Table 3.1: Singular fibers and their associated gauge group in the absence of monodromy. c denotes the coefficients appearing in the Kodaira formula.

where b_κ, recall, are irreducible effective divisors giving rise to non-abelian gauge factors. Y is a residual effective divisor. The coefficients c_κ are determined by the codimension-one singularity along b_κ. We have listed c_κ for singularities with no monodromy in table 3.1. Although the locus Y does not induce any enhanced gauge group, it produces codimension-two singularities by colliding with the brane loci b_κ. The Kodaira condition stating that the total space of the elliptic fibration is a Calabi-Yau manifold is

$$-12a = \Delta = \sum_\kappa c_\kappa b_\kappa + Y\,. \tag{3.2}$$

This relation implies that $(-a)$ lies inside the Mori cone. Meanwhile, since Y is effective, it must have non-negative volume. This condition implies that

$$-12j \cdot a \geq \sum_\kappa c_\kappa j \cdot b_\kappa \tag{3.3}$$

as j is the normalized Kähler form of the base. We call this inequality, the "Kodaira constraint."

Therefore there is a clear route to mapping the non-abelian sector of a given low-energy theory to an F-theory model using this map given that the parameters of the low-energy theory satisfy the Kodaira constraint. We begin by choosing an appropriate base $\mathcal{B}$ with $h^{1,1}(\mathcal{B}) = T + 1$.[2] Then we tune the complex structure

[2]This choice is not quite simple as T increases and the number bases to choose from increases. It is known, however, that the number of bases to choose from is finite [141, 142]. A systematic study of the bases that can be used has been undertaken in [143].

of the elliptically fibered Calabi-Yau manifold so that it has the codimension-one singularity structure encoded in the non-abelian anomaly coefficients b_κ. Once the codimension-one singularity structure is established, we tune the complex structure further to obtain the required matter structure.

There is no proof that this procedure would work in general. It is true, however, that when the gap between $-12a$ and $\sum_\kappa c_\kappa b_\kappa$ is quite large, and when the matter representations are simple, this procedure is expected to produce the desired F-theory model. This expectation is based on the fact that the procedure sketched above is bound to be executable when the numbers of complex parameters one needs to tune to obtain the desired singularity structure is small.

Some explanation on the term "simple matter" is due. By saying that a given matter representation is "simple" we are implying that they come from codimension-two singularities that arise without much tuning. What we mean is the following. After one tunes the complex structure of a given elliptically fibered Calabi-Yau manifold to reproduce some desired codimension-one singularity structure, there are still some degrees of freedom in the complex structure left over. Setting those complex variables to generic values, one in general obtains codimension-two singularities that give these "simple" matter representations. Tuning the complex structure further, one can obtain more "complicated" matter. For example, the codimension-two singularity structure that yields fundamental($\square$), antisymmetric($\begin{smallmatrix}\square\\\square\end{smallmatrix}$) and adjoint matter of an $SU(M)$ gauge group is generic for many single block models [64, 115]. In particular, when there is an A_M singularity over a smooth curve of genus g, there are $2g - 2$ adjoint hypermultiplets in the massless spectrum [119, 144, 145]. One can, however, tune the complex structure so that the curve has a double point. For each double point, it can be shown that an adjoint hypermultiplet can be traded for an antisymmetric and a symmetric hypermultiplet [97].

As noted above, in the case of $SU(M)$ groups, codimension-two singularities that generate fundamental($\square$), antisymmetric($\begin{smallmatrix}\square\\\square\end{smallmatrix}$), symmetric($\square\square$) matter is well known [97, 119, 136]. Adjoint matter is generated when there is a codimension-one singularity over a curve of genus greater than one [119, 144, 145]. Jointly charged matter arises when two enhanced singularities collide at a point. The singularity structure that give rise to the bifundamental($\square \times \square$) is well understood. More exotic matter have been constructed in F-theory from more complicated codimension-two singularities, but we postpone their discussion to section 4.1.

To our knowledge, all perturbative string vacua under-saturate the Kodaira bound, and only have simple types of matter.[3] In particular, we find that

$$-12a > -na \geq \sum_{\kappa} c_{\kappa} b_{\kappa} \tag{3.4}$$

for some $n < 12$. The inequality implies that the difference between the left-hand-side and the right-hand-side lies inside the Mori cone. Note that this inequality makes sense due to the fact that $-a \cdot j > 0$. The Kodaira bound in many of these string constructions is manifested as tadpole constraints. These facts are the basis for claiming that the non-abelian sectors of all known string vacua are embeddable into F-theory. In the following two subsections, we examine the examples of type IIB intersecting brane models and magnetized brane backgrounds from this point of view.[4]

3.1.2 Type IIB Intersecting Brane Models

In this section, we introduce type IIB intersecting brane models. We find the massless spectrum and anomaly coefficients of the non-abelian gauge groups. We check that the matter structure is simple and that the Kodaira bound is under-saturated for this set of models.

Type IIB intersecting brane models are models constructed by compactifying type IIB string theory on a compact manifold with D-branes and O-planes wrapping compact cycles. A nice general review on this subject can be found in [148]. In this section, we focus on the case where type IIB is compactified on a $K3$ surface to six

[3] A more extensive discussion of this fact can be found in [114].

[4] We note that in the case of these models, explicit F-theory duals exist [99, 146, 147]. We, however, use these examples to sketch the general strategy to embed the non-abelian sector of string vacua into F-theory.

Representation	Multiplicity
$\square_\kappa \times \square_\lambda$	$\pi_\kappa \cdot \pi'_\lambda$
$\square_\kappa \times \bar{\square}_\lambda$	$\pi_\kappa \cdot \pi_\lambda$
$\begin{array}{c}\square\\[-6pt]\square\end{array}_\kappa$	$\frac{1}{2}\pi_\kappa \cdot (\pi'_\kappa + \pi_{O7})$
$\square\!\square_\kappa$	$\frac{1}{2}\pi_\kappa \cdot (\pi'_\kappa - \pi_{O7})$
Adjoint in $SU(N_\kappa)$	$\frac{1}{2}\pi_\kappa \cdot \pi_\kappa + 1$

Table 3.2: The number of hypermultiplets of each representation in an intersecting brane model.

dimensions following [106, 114].

Type IIB compactified on a K3 surface yields a six-dimensional supergravity theory with $(2,0)$ supersymmetry. To obtain $(1,0)$ supersymmetry, one can wrap D7-branes and O7-planes on cycles in the $K3$ surface when the $K3$ surface admits an anti-holomorphic involution $\bar{\sigma}$. The O-plane lies on a cycle π_{O7} that is invariant under this involution. Meanwhile, groups of M_κ coincident D7-branes wrapping a cycle π_κ can be thought of as wrapping its image under the involution $\pi'_\kappa \equiv \bar{\sigma}(\pi_\kappa)$ simultaneously in the presence of the O7-plane. The compactness of $K3$ imposes that the brane tensions cancel:

$$\sum_\kappa M_\kappa(\pi_\kappa + \pi'_\kappa) = 8\pi_{O7}\,. \tag{3.5}$$

Given a intersecting brane configuration such that the O7-plane wraps the cycle π_{O7} and that M_a coincident D7-branes wrap the cycle π_κ for $\kappa = 1, \cdots, k$, the non-abelian gauge group is given by

$$SU(M_1) \times \cdots \times SU(M_k) \tag{3.6}$$

and the hypermultiplets for each gauge group is given by table 3.2 [106, 114]. The dots between cycles are intersection numbers of the two cycles involved.

Using the the tadpole constraint and properties of the involution, one finds that the anomaly equations become

$$
\begin{aligned}
a \cdot a &= \frac{1}{2}\pi_{O7} \cdot \pi_{O7} = 9 - T && (3.7)\\
a \cdot b_\kappa &= -\frac{1}{2}\pi_{O7} \cdot (\pi_\kappa + \pi'_\kappa) && (3.8)\\
b_\kappa \cdot b_\kappa &= \frac{1}{2}(\pi_\kappa + \pi'_\kappa) \cdot (\pi_\kappa + \pi'_\kappa) && (3.9)\\
b_\kappa \cdot b_\lambda &= \frac{1}{2}(\pi_\kappa + \pi'_\kappa) \cdot (\pi_\lambda + \pi'_\lambda) \quad (\kappa \neq \lambda) && (3.10)
\end{aligned}
$$

The inner-product on the left-hand-side of the equations is, as usual, an $SO(1,T)$ inner-product while the inner-product on the right-hand-side is the intersection product in the $K3$ manifold.

We find that we can identify the integral cycles invariant under the antiholomorphic involution

$$
-\frac{1}{\sqrt{2}}\pi_{O7}, \quad \frac{1}{\sqrt{2}}(\pi_\kappa + \pi'_\kappa) \tag{3.11}
$$

with the anomaly coefficients of the low-energy theory

$$
a = -\frac{1}{\sqrt{2}}\pi_{O7}, \quad b_\kappa = \frac{1}{\sqrt{2}}(\pi_\kappa + \pi'_\kappa)\,. \tag{3.12}
$$

Now the tadpole constraint becomes

$$
\sum M_\kappa b_\kappa = -8a\,. \tag{3.13}
$$

Hence we see that the Kodaira bound is under-saturated for these models. All the matter that appear are also the type that can be generated from F-theory by known methods. We note that there are additional consistency conditions that must be imposed for supersymmetry, namely the fact that the D-branes and O-planes should be calibrated under the real part of the complex structure two-form. We, however, did not need them in deriving the anomaly equations of the low-energy theory and also in finding the tadpole condition. The additional constraint may be required to restrict the cycles $(\pi_\kappa + \pi'_\kappa)$ and π_{O7} to a $T+1$ dimensional plane with signature $(1,T)$ within the homology lattice so that the identifications (3.12) could indeed be made.

3.1.3 Magnetized Brane Backgrounds

In this section, we introduce magnetized brane models and compute the massless spectrum and anomaly coefficients of the non-abelian gauge groups. We confirm that the matter structure is simple and that the Kodaira bound is under-saturated.

Magnetized brane backgrounds are $SO(32)$ heterotic string compactifications on a $K3$ manifold with $U(1)$ gauge fluxes. These models were widely studied in the context of the dual type I theory [103, 104, 105]. Here we use the formalism of [108, 109, 110, 111].

The equations of motion, which come from imposing that there is a covariantly constant spinor in the internal manifold, imposes that we have a gauge bundle over $K3$. The $U(1)$ background gauge bundle can be written as,

$$\bar{F} = 4\pi \sum_i f_i T^i \tag{3.14}$$

where T^i are generators of the Cartan subgroup of $SO(32)$ with the normalization,

$$\text{tr} T_i^2 = 2 \tag{3.15}$$

By the assumption that F is in the Cartan subalgebra, the equations of motion impose that f_i are harmonic two-forms. Harmonic two-forms have a natural inner product

$$f_i \cdot f_j = 4 \int_{K3} f_i \wedge f_j \,. \tag{3.16}$$

We restrict ourselves to "multiple stack models" where we define the background field to be of the form,

$$F = i \begin{pmatrix} 0 & -f_1 & 0 & 0 & & & & & & & & & & & & \\ f_1 & 0 & 0 & 0 & & & & & & & & & & & & \\ 0 & 0 & 0 & -f_1 & & & & & & & & & & & & \\ 0 & 0 & f_1 & 0 & & & & & & & & & & & & \\ & & & & 0 & -f_2 & & & & & & & & & & \\ & & & & f_2 & 0 & & & & & & & & & & \\ & & & & & & \ddots & & & & & & & & & \\ & & & & & & & 0 & -f_n & & & & & & & \\ & & & & & & & f_n & 0 & & & & & & & \\ & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & & & & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.17}$$

where there are $M_1, \cdots, M_n$ blocks of $-if_1\sigma_y, \cdots, -if_n\sigma_y$ matrices along the diagonal with

$$2M_1 + \cdots + 2M_n = 32 - 2M\,. \tag{3.18}$$

Our background could be written more concisely as,

$$\bar{F} = 4\pi \sum_{a=1}^{n} f_a \mathcal{T}_a \tag{3.19}$$

for

$$\mathcal{T}_a \equiv \left[\sum_{i=(M_1+\cdots+M_{a-1}+1)}^{(M_1+\cdots+M_a)} T^i \right] \tag{3.20}$$

Note that the $\mathcal{T}_a$ do not satisfy $\mathrm{tr}\mathcal{T}_a^2 = 2$ in general. It is clear that the gauge group

Representation	Number	$U(1)$ charge
$(N_a, \bar{N}_b) + (\bar{N}_a, N_b)$	$-2-(f_a-f_b)^2$	$(1,-1)$, $(-1,1)$
$(N_a, N_b)+(\bar{N}_a, \bar{N}_b)$	$-2-(f_a+f_b)^2$	$(1,1)$, $(-1,-1)$
Antisym. in $U(N_a)$ + c.c.	$-2-4f_a^2$	2, -2
$(N_a, 2M) + (\bar{N}_a, 2M)$	$-2-f_a^2$	1, -1
Neutral	20	-

Table 3.3: The number of hypermultiplets of each representation and their $U(1)$ charge.

is broken to

$$\prod_{a=1}^{n} U(M_a) \times SO(2M) \tag{3.21}$$

in this background.

The tadpole cancellation condition imposed by the equations of motion becomes in this language

$$\sum_a M_a f_a^2 = -24\,. \tag{3.22}$$

There are additional constraints imposed on f_a by supersymmetry. Denoting the holomorphic(antiholomorphic) and Kähler 2-forms as $\Omega(\bar{\Omega})$ and J, the f_a must satisfy

$$f_a \cdot \Omega = 0, \qquad f_a \cdot \bar{\Omega} = 0, \qquad f_a \cdot J = 0\,. \tag{3.23}$$

The magentic brane models are parametrized precisely by f_a obeying such constraints which can be embedded into the integral lattice, $H^2(K3, \mathbb{Z})$. It turns out to be enough to demand that f_a spans a negative definite plane in the $H^2(K3, \mathbb{Z})$ lattice [111]. Since this implies that $f_a^2 < 0$, and since the $H^2(K3, \mathbb{Z})$ lattice is even, this automatically implies that the number of $U(M_a)$ gauge factors $n \leq 12$.

The matter content of the dimensionally reduced theory has been worked out many times in the literature — for example, in [83] — and we simply state the results. The ten-dimensional gravity multiplet decomposes in to one gravity multiplet, one tensor multiplet and twenty hypermultiplets. The ten-dimensional $SO(32)$ vector multiplet decomposes in to six-dimensional multiplets of the gauge group $\prod_{a=1}^{n} U(M_a) \times SO(2M)$ and charged hypermultiplets. The number of hypermultiplets can be obtained by using index theorems. It is summarized in table 3.3. We can normalize the $U(1)_a$ charges so that the fundamental of $SU(M_a)$ has charge 1 under this $U(1)$. The correpsonding $U(1)$ charges are written out in the table for future reference.

The anomaly polynomial of the nonabelian sector can be computed explicitly from the matter content. It is given by

$$I_8 \propto [\mathrm{tr}R^2 + \sum_a 4(f_a^2+1)\mathrm{tr}F_a^2 + 2\mathrm{tr}F_{2M}^2] \wedge [\mathrm{tr}R^2 - \sum_a 2\mathrm{tr}F_a^2 - \mathrm{tr}F_{2M}^2] \tag{3.24}$$

F_a denote the field strengths for the $SU(M_a)$ gauge fields and F_{2m} denotes the field strength for the $SO(2M)$ gauge field. The anomaly coefficient for $SU(M_a)$ and $SO(2M)$ are given by

$$b_a = \frac{1}{2}\begin{pmatrix} -4(f_a^2+1) \\ 2 \end{pmatrix}, \quad b_{2M} = \frac{1}{2}\begin{pmatrix} -4 \\ 2 \end{pmatrix}. \tag{3.25}$$

We have used the fact that for $SO(2M)$, $\lambda = 2$. Note that we are using the conventional metric and a vector

$$\Omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} -2 \\ -2 \end{pmatrix}. \tag{3.26}$$

It is clear that by the two constraints (3.18) and (3.22),

$$\sum c_\kappa b_\kappa = \sum_a M_a b_a + (M+2)b_{2M} = \begin{pmatrix} 12 \\ 18 \end{pmatrix} < -9a < -12a\,. \tag{3.27}$$

This inequality holds since for any value of the modulus

$$j = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\phi} \\ e^{-\phi} \end{pmatrix}, \tag{3.28}$$

the inner product

$$j \cdot (-9a - \sum_a M_a b_a - (M+2) b_{2M}) = 3\sqrt{2} e^{-\phi} > 0\,. \tag{3.29}$$

We therefore find that these models have simple matter structure, and also undersaturate the Kodaira bound, as desired.

We must expose a caveat in our discussion here. All of our discussions have been perturbative — we have assumed that we are working in a background in which ten-dimensional $SO(32)$ supergravity provides a valid description of the physics. It must be noted, however, that $U(1)$ backgrounds have the subtlety that the K3 manifold on which they are compactified over are often singular. This point is discussed in detail in [111].

This is due to the fact that the K3 manifold can have vanishing cycles due to (3.23) and can have orbifold singularities for some f_a. It is known that worldsheet instantons smooth these singularities when there is no gauge bundle [149] but in the cases of interest, we have non-trivial gauge bundles precisely at the orbifold point. Furthermore, this corresponds to a small instanton limit, so we expect an enhanced gauge symmetry. So to analyze the the low-energy physics properly, we must factor in the non-perturbative effects that come from the small instanton limit. Therefore, the validity of our analysis only holds in the perturbative sector of this background.

There is a way to move out of this small instanton limit. We have nineteen moduli that blow up the orbifold singularity of the theory. Turning on the nineteen moduli renders the manifold non-singular, but can potentially break the gauge symmetries. This is because that we generically have to turn on vacuum expectation values for hypermultiplets charged under some of the gauge groups. A careful analysis of this phenomenon along the lines of [110, 125, 150, 151, 152, 153] must be carried out to understand these points of enhanced gauge symmetry properly. We note that these points have also been studied in the context of F-theory in [154, 155].

3.2 Non-trivialities of the Abelian Sector : An Example

As stressed many times throughout this thesis, the abelian sector of string models involve subtleties that do not arise for the non-abelian sector. For example, even determining the abelian gauge symmetry itself is an involved process in a wide variety of string constructions, including heterotic, orbifold, intersecting brane, fractional brane and F-theory models [80, 81, 87, 96, 98, 100, 107, 108, 109, 110, 118, 119, 156, 157, 158, 159, 160, 161, 162, 163, 164, 165, 166, 167]. The subtlety comes, as pointed out in the beginning of this chapter, from the fact that abelian vector fields in the spectrum that are naively massless can be lifted at the linear level by coupling to Stückelberg fields. In this section, we demonstrate the process of determining the abelian sector of magnetized brane backgrounds introduced in section 3.1.3, and find that additional constraints on the vacuum construction is needed to find massless abelian gauge fields.

From the discussion in section 3.1.3, we might naively expect that there are n $U(1)$'s, where n is the number of $U(M_a)$ gauge group factors. Many of these $U(1)$'s, however, should be lifted by the generalized Green-Schwarz mechanism we have described in section 2.1.5. To see how the generalized Green-Schwarz mechanism comes into play, let us examine the full anomaly polynomial of the magnetized brane background given in section 3.1.3. As before, we use F_a (A_a) to denote the $SU(M_a)$ field strength(vector field,) G_a ($\mathcal{A}_a$) to denote the $U(1)_a$ field strength(vector field,) and $F_{2M}(A_{2M})$ to denote the $SO(2M)$ field strength(vector field.) With these conventions, the anomaly polynomial becomes

$$
\begin{aligned}
I_8 =& [\mathrm{tr}R^2 + \sum_a 4(f_a^2+1)\mathrm{tr}F_a^2 + \sum_a 4M_a(f_a^2+1)G_a^2 + 2\mathrm{tr}F_{2M}^2] \\
&\wedge[\mathrm{tr}R^2 - \sum_a 2\mathrm{tr}F_a^2 - \sum_a 2M_aG_a^2 - \mathrm{tr}F_{2M}^2] \\
&-\frac{8}{3}\sum_a M_aG_a \wedge \sum_b (f_a\cdot f_b)[-\frac{1}{4}M_b(\mathrm{tr}R^2)G_b + 4\mathrm{tr}F_b^3 + 12(\mathrm{tr}F_b^2)G_b + 4M_aG_b^3]
\end{aligned}
\tag{3.30}
$$

Let us first find the six-dimensional terms relevant to the Green-Schwarz mechanism. These are terms that involve the six-dimensional B field. The kinetic term for the B field is found to be,

$$
-\frac{1}{2}e^{-2\phi}(dB-\omega)\cdot(dB-\omega) \tag{3.31}
$$

with

$$
d\omega \propto \mathrm{tr}R^2 - \sum_a 2\mathrm{tr}F_a^2 - \sum_a 2M_aG_a^2 - \mathrm{tr}F_{2M}^2 \tag{3.32}
$$

Part of the ten-dimensional Green-Schwarz term descends to,

$$
-B\wedge d\tilde{\omega} \tag{3.33}
$$

with

$$
d\tilde{\omega} \propto \mathrm{tr}R^2 + \sum_a 4(f_a^2+1)\mathrm{tr}F_a^2 + \sum_a 4M_a(f_a^2+1)G_a^2 + 2\mathrm{tr}F_{2M}^2\,. \tag{3.34}
$$

Therefore we see that the Green-Schwarz mechanism cancels the first term in (3.30), as it should.

Before finding the terms relevant to the generalized Green-Schwarz mechanism, let us examine the neutral hypermultiplets which will participate in the anomaly cancellation. The four scalars of a neutral hypermultiplet transform under $SU(2)_R$ as a $\mathbf{1}+\mathbf{3}$. For nineteen of the neutral hypermultiplets, the $\mathbf{1}$ comes from the ten dimensional tensor reduced over nineteen harmonic two-forms ω_k, *i.e.*,

$$B_{10} \ni \sum_k b^k \omega_k \,. \tag{3.35}$$

These two-forms ω_k are orthogonal to the Kähler form J and the holomorphic and anti-holomorphic 2-forms, Ω and $\bar{\Omega}$. The **3** come from geometric moduli of the K3 manifold. For one neutral hypermultiplet, the **1** is the volume modulus of the internal manifold. The **3** come from reducing the ten-dimensional tensor over the Kähler, holomorphic and anti-holomorphic two-forms [108].

Now let us dimensionally reduce the Lagrangian of the ten-dimensional heterotic string theory to observe the generalized Green-Schwarz mechanism at play. Defining

$$\int_{K3} \omega_k \wedge f_a = f_{a,k} \tag{3.36}$$

we obtain the kinetic term

$$-\frac{1}{2} e^{-2\phi} (db^k - \frac{1}{2} \sum_a M_a f_{a,k} A_a)^2 \tag{3.37}$$

for the Stückelberg fields by dimensionally reducing the ten-dimensional kinetic term of the tensor field. Part of the ten-dimensional Green-Schwarz term descends to

$$\sum_k \left(b^k \wedge \sum_a f_{a,k} [-\frac{1}{4} M_a (\mathrm{tr} R^2) G_a + 4 \mathrm{tr} F_a^3 + 12 (\mathrm{tr} F_a^2) G_a + 4 M_a G_a^3] \right) . \tag{3.38}$$

Therefore we see that the generalized Green-Schwarz mechanism gets rid of the second term in (3.30). We note that the reason all this is possible by using only nineteen harmonic forms is because of the relations (3.23).

The $U(1)$'s being lifted can precisely be read off of the kinetic term (3.37) of the linear multiplets. A useful way to write out the nineteen components of the f_a that parametrize the $U(1)$ instanton background is to organize them into column vectors:

$$f \equiv \begin{pmatrix} M_1 f_1 & M_2 f_2 & \cdots & M_n f_n \end{pmatrix} \tag{3.39}$$

Now (3.37) implies that the rank of this matrix is precisely the number of $U(1)$'s being lifted. In order for some $U(1)$ to survive, the rank of this matrix should be less than the number of $U(1)$'s, n. As seen in the last section $n \leq 12 < 19$ which is the number of rows. Therefore, in order for there to be an unlifted $U(1)$, $f_1, \cdots, f_n$ should not be linearly independent. In this case, there exist n component vectors $v_1, \cdots, v_r$ with $r = n - \text{rank} f$ such that

$$\sum_{a=1}^{n} M_a v_{i,a} f_{a,k} = 0 \tag{3.40}$$

for all $k = 1, \cdots, 19$. $f_{a,k}$ are the components of the two-forms f_a given in equation (3.36). Then the r abelian vector fields

$$\mathrm{A}_i = \sum_a v_{i,a} \mathcal{A}_a \tag{3.41}$$

survive the Stückelberging and remain in the low-energy spectrum.

Therefore, we have demonstrated that in order to properly identify the abelian gauge sector, a careful treatment of the Stückelberg mechanism is necessary for magnetized brane backgrounds. In this example, we found that additional constraints on the gauge fluxes f_a must be imposed in order for there to be massless abelian gauge fields.

3.3 M-theory/F-theory Duality and the Abelian Sector of F-theory

We have demonstrated non-trivial aspects of abelian gauge symmetry of string backgrounds through the example of magnetized brane backgrounds in the previous section. Such non-trivialities show up in F-theory vacua as well. Unlike the non-abelian sector, there is a global flavor to the abelian sector of F-theory backgrounds. For example, even simple information such as the number of massless abelian vector fields is encoded in global data of the full manifold [119, 168, 169]. This should be contrasted to the case of the non-abelian sector of the theory as its dynamics could be determined locally — only the near-brane geometry mattered in understanding it.

It is not clear how to probe the abelian sector of F-theory backgrounds directly. This is because the degrees of freedom of the underlying twelve-dimensional theory and their interactions — if they exist at all — are unclear at the moment. In order to understand the abelian sector of the theory, it turns out to be more convenient to take an intersection theory-based[5] approach in the M-theory dual of the F-theory background [132, 144, 145, 172]. By recovering the low-energy data of the six-dimensional F-theory background using this duality, we can identify the geometric meaning of the abelian anomaly coefficients by comparing the coefficients of topological terms obtained from both sides [172, 173, 174].

We proceed in three steps. In section 3.3.1, we first review M-theory/F-theory duality and obtain basic information about the massless spectrum of F-theory from the M-theory side. In the process we obtain equation (3.49), which is an identity relating various topological data of the manifold. In section 3.3.2, we demonstrate how M-theory/F-theory duality can be used to recover the low-energy data of the non-abelian sector. Finally in section 3.3.3, we identify the geometric counterparts of the low-energy data of the abelian sector in an analogous way.

3.3.1 M/F-theory Duality

The duality between M and F-theory [120] provides the clearest way to see how the low-energy dynamics of gauge bosons and matter content arise in F-theory backgrounds.[6] F-theory compactified on $X \times S^1$—where X is an elliptically fibered Calabi-Yau threefold with a section—is dual to M-theory compactified on X. In the five-dimensional M-theory background, all the Kähler deformations of X become available, unlike in the six-dimensional theory. These moduli on the F-theory side are given by the size of the S^1 and the Wilson lines of the gauge fields along the S^1. By turning these moduli on to generic values, we may resolve the singular manifold

[5] Some standard texts on intersection theory are [170] and [171].

[6] A great review on F-theory and M-theory/F-theory duality can be found in [175].

6D	5D
Gravity	1×(Gravity)+1×(Vector)
Tensor	1×(Vector)
Vector	1×(Vector)
Hyper	1×(Hyper)

Table 3.4: Six-dimensional (1,0) supersymmetry multiplets and their descendants in five dimensions when compactified on a circle.

X to $\hat{X}$. This is equivalent to going to the Coulomb branch of the non-abelian gauge theory, as the five-dimensional vector multiplet has a real adjoint scalar. We can recover the fibration limit $\hat{X} \to X$ as we turn off all the Wilson lines and take the radius of the S^1 to infinity. In this sense, the six-dimensional theory can be thought of as a "decompactification limit" of the M-theory background. We use the terms "decompactification limit," "F-theory limit," and "fibration limit" interchangeably.

Now let us recover the massless spectrum of the six-dimensional theory from the geometrical data of $\hat{X}$. When we compactify the six-dimensional theory with $\mathcal{N} = (1,0)$ supersymmetry on S^1, we get a five-dimensional $\mathcal{N} = 2$ theory with 8 supercharges. The short multiplets of the six-dimensional theory descend to short multiplets of the five-dimensional theory as shown in table 3.4. By resolving X to $\hat{X}$ we have turned on Wilson lines, and hence all multiplets charged under the Cartan sub-algebra of the full gauge group become massive. We denote these multiplets "charged multiplets." Charged multiplets descend from either vector multiplets or hypermultiplets. Therefore the six-dimensional massless spectrum can be recovered from the five-dimensional theory by identifying the massless multiplets and the charged multiplets that become massless in the decompactification limit.

There is nothing special about the Cartan basis. The Wilson lines turned on are generic and mutually commuting, and hence we can always find a Cartan subgroup of which the Wilson lines are elements of. We note that the Cartan sub-algebra of the full gauge algebra consists of the direct sum of the Cartan sub-algebra of the individual gauge groups. For abelian groups, the Cartan sub-algebra is equal to the full gauge algebra.

Let us first identify the massless fields of the five-dimensional theory [176, 177, 178]. M-theory compactified on the fully resolved manifold $\hat{X}$ has $h^2(\hat{X}) = h^{1,1}(\hat{X})$ massless vector fields coming from descending the three-form on the harmonic two-forms of $\hat{X}$. Among these, one vector field is inside the five-dimensional gravity multiplet and the others belong to vector multiplets. The two-forms are Poincaré dual to four-cycles in $\hat{X}$, that is, for any harmonic two-form ω there exists a four-cycle Σ in $\hat{X}$ such that for any two-cycle c in $\hat{X}$,

$$\int_c \omega = \Sigma \cdot c \tag{3.42}$$

where the right-hand side denotes the intersection number between the two cycles. Therefore, for each massless vector field, there is a corresponding four-cycle. On the F-theory side, one of these vector fields come from KK-reducing the graviton along the S^1, while $(T+1) = h^{1,1}(\mathcal{B})$ come from KK-reducing the one self-dual and T anti-self dual tensor fields. The rest come from vectors in the six-dimensional vector multiplets that are either abelian or in the Cartan of a non-abelian gauge group.

Also, there are $h^3(\hat{X}) = h^{2,1}(\hat{X})+1$ massless hypermultiplets in the five-dimensional spectrum. In the decompactification limit, all of these hypermultiplets become six-dimensional neutral hypermultiplets—hypermultiplets that are not charged under any vector field in the Cartan.

Now let us identify the charged multiplets. These come from M2 branes wrapping complex curves of $\hat{X}$. Since the charged multiplets should become massless in the decompactification limit, they should come from M2 branes wrapping curves that shrink in the fibration limit. As we move along the Coulomb branch to recover the full non-abelian gauge symmetry of X, two types of curves shrink to zero size.

1. **Type I :** Isolated rational curves that shrink to zero size in the limit $\hat{X} \to X$.

2. **Type F :** Rational curves fibered over a curve that shrink to zero size in the limit $\hat{X} \to X$.

These curves are all rational curves; they are topologically $\mathbb{P}^1$'s as only these types of curves can shrink in $\hat{X}$ [179]. We index the curves of type I by r and denote them

c_r, and index the curves of type F by ρ and denote them χ_ρ. We use g_ρ to denote the genus of the curve χ_ρ is fibered over. The curve a type F curve is fibered over is either a curve in the base or its branched cover [132]. In the fibration limit $\hat{X} \to X$, the type F curves shrink into points on the singular fibers along codimension-one loci in the base while the type I curves shrink into points on singular fibers at codimension-two loci.

A curve of type I contributes one hypermultiplet, while a curve of type F fibered over a curve of genus g contributes $2g$ hypermultiplets and 2 vector multiplets to the BPS spectrum [144, 145]. By quantizing the zero mode of an M2 brane wrapping a curve of type I, one obtains a half-hypermultiplet. Together with another half-hypermultiplet that comes from quantizing the zero modes of an anti-brane wrapping the same curve, a curve of type I contributes one hypermultiplet. Meanwhile, $2g$ half-hypermultiplets and one vector multiplet come from quantizing the zero-modes of an M2 brane wrapping a curve of type F fibered over a curve of genus g. Also the same number of multiplets come from quantizing the zero-modes of an anti-M2 brane wrapping the same curve. By definition, these multiplets become massless in the decompactification limit, and are in the massless spectrum of the six-dimensional theory.

The charge of a charged BPS particle coming from a brane wrapping a rational curve c under a vector field A_Σ coming from reducing the eleven-dimensional three-form field on the harmonic three-form ω is given by

$$\pm \int_c \omega = \pm \Sigma \cdot c\,, \tag{3.43}$$

where Σ is the four-cycle that is Poincaré dual to ω. The sign depends on whether the brane is an M2 brane or and anti-M2 brane. While the charge of a vector multiplet is unambiguous, there is an overall sign ambiguity in defining charges of the hypermultiplet. A hypermultiplet consists of two half-hypermultiplets with opposite charges under any abelian or non-abelian Cartan gauge field; one coming from M2 branes wrapped on a curve and the other coming from an anti-brane wrapped on the same curve. We fix the sign of the charge of a hypermultiplet coming from a curve c under a gauge field A_Σ as

$$\int_c \omega = \Sigma \cdot c\,. \tag{3.44}$$

Meanwhile, the charge of vector or hypermultiplets under the vector multiplets coming from shrinking rational curves of type F can be obtained by considering the algebra of BPS states in the Calabi-Yau manifold as described in [180, 181]. Some of the multiplets that are uncharged under the vector fields in the Cartan sub-algebra can be charged under vector fields that come from shrinking type F rational curves.

Let us summarize what we have learned. There are $h^{1,1}(\hat{X})$ massless vector fields in the five-dimensional M-theory background. In the decompactification limit, two of them belong to the gravity multiplet, $T = (h^{1,1}(\mathcal{B}) - 1)$ of them belong to the tensor multiplets and the rest of them belong to the vector multiplets that are either abelian or in the Cartan of the non-abelian gauge groups. There are $h^{2,1}(\hat{X}) + 1$ massless hypermultiplets in the five-dimensional theory. In the decompactification limit, they are hypermultiplets uncharged under the Cartan/abelian vector multiplets. There are $(\sum_r 1 + \sum_\rho 2g_\rho)$ massive hypermultiplets and $(\sum_\rho 2)$ massive vector multiplets, which in the decompactification limit, are hyper/vector multiplets charged under the abelian or non-abelian Cartan vector multiplets.

Since we have accounted for all the vector and hypermultiplets of the six-dimensional theory from the geometric data of $\hat{X}$, the gravitational anomaly constraint

$$H - V + 29T = 273\,, \tag{3.45}$$

can be written in terms of this data. The number of six-dimensional vector multiplets and hypermultiplets are given by

$$V = (h^{1,1}(\hat{X}) - 2 - T) + \sum_\rho 2 \tag{3.46}$$

$$H = (h^{2,1}(\hat{X}) + 1) + \sum_r 1 + \sum_\rho 2g_\rho\,. \tag{3.47}$$

Thus, the gravitational anomaly constraint can be re-written as

$$270 - 30T + (h^{1,1}(\hat{X}) - h^{2,1}(\hat{X})) = \sum_r 1 + \sum_\rho (2g_\rho - 2)\,. \tag{3.48}$$

Using the fact that $K \cdot K = 9 - T$ for the canonical divisor K of $\mathcal{B}$, and that $\chi_{\hat{X}} = 2(h^{1,1}(\hat{X}) - h^{2,1}(\hat{X}))$ for the Euler characteristic $\chi_{\hat{X}}$ of $\hat{X}$, we find

$$30K \cdot K + \frac{1}{2}\chi_{\hat{X}} = \sum_r 1 + \sum_\rho (2g_\rho - 2)\,. \tag{3.49}$$

3.3.2 The Non-Abelian Sector

In this section, we continue the analysis of the M-theory/F-theory duality. We first classify the vector fields of the five-dimensional theory in a useful way. Then we recover the low-energy data of the non-abelian sector from the geometric data of $\hat{X}$. The results turn out to be consistent with that of section 3.1. Most of the discussion in this section can be found in [132, 133, 134, 135, 136] but we have rephrased them in a way more convenient for our purposes.

Let us first classify the vector fields of the five-dimensional theory in a useful way. Recall there is a one-to-one correspondence between the massless five-dimensional vector fields and four-cycles of $\hat{X}$. There are four types of four-cycles in $\hat{X}$.

1. **Type $\hat{\mathbf{Z}}$:** The zero section; $\hat{Z} \leftrightarrow \hat{\zeta}$.

2. **Type B :** Four-cycles obtained by fibering the elliptic fiber f over two-cycles $H_0, \cdots, H_T$ in the base $\mathcal{B}$; $B_0, \cdots, B_T \leftrightarrow \beta_0, \cdots, \beta_T$.

3. **Type C :** Monodromy invariant four-cycles that are locally type F rational curves fibered over a curve in the base $\mathcal{B}$; $T_1, \cdots, T_r \leftrightarrow \tau_1, \cdots, \tau_R$.

4. **Type $\hat{\mathbf{S}}$:** Non-zero sections of the fibration; $\hat{S}_1, \cdots, \hat{S}_{V_A} \leftrightarrow \hat{\sigma}_1, \cdots, \hat{\sigma}_{V_A}$.

The lowercase greek letters denote the Poincaré dual two-forms in the resolved manifold. The type $\hat{\mathrm{S}}$ four-cycles are generators of the non-torsion part of the Mordell-Weil group of the elliptic fibration. The number V_A is the Mordell-Weil rank of the elliptic fibration [119, 168, 169].

We note that the intersection of type B cycles satisfy

$$B_\alpha \cdot B_\beta = (H_\alpha \cdot H_\beta)_{\mathcal{B}} f \equiv \Omega_{\alpha\beta} f \,, \tag{3.50}$$

where the subscript $\mathcal{B}$ means that we are taking the intersection of curves on the base and f is the fiber class of the elliptic fibration. We emphasize once more that H_α are the basis elements of $H_2(\mathcal{B})$. From this relation we also see that the triple intersection products among the type B cycles are zero, as the type B cycles do not intersect a generic fiber. The Ω is a symmetric invertible $SO(1,T)$ bilinear form, or an $SO(1,T)$ "metric." We denote

$$\Omega^{\alpha\beta} \equiv (\Omega^{-1})_{\alpha\beta} \tag{3.51}$$

and raise and lower $SO(1,T)$ indices by Ω.

We postpone the discussion of the four cycles of type $\hat{\mathrm{S}}$ to section 3.3.3 and focus on the first three types of cycles. We make the following

(Claim 1)

1. The vector field Z obtained by the three-form KK-reduced along $\zeta = \hat{\zeta} - \frac{1}{2}(\hat{Z} \cdot \hat{Z} \cdot B^\alpha)\beta_\alpha$ can be identified with the vector field coming from KK-reducing the six-dimensional metric along S^1 in the decompactification limit. It is inside the five dimensional gravity multiplet.

2. The vector fields $B_0, \cdots, B_T$ obtained by the three form KK-reduced along $\beta_0, \cdots, \beta_T$ can be identified with the vector fields obtained by KK-reducing the $(T+1)$ six-dimensional tensor fields along S^1 in the decompactification limit.

3. The vector fields $\mathcal{A}_1, \cdots, \mathcal{A}_R$ obtained by the three-form KK-reduced along τ_i can be identified with the vector fields obtained by KK-reducing the six-dimensional non-abelian vector fields in the coroot basis of the Cartan of each gauge group along S^1 in the decompactification limit.

For convenience, we abuse the term "duality" throughout the rest of this thesis in the following way; we say that a vector field is "dual to" a four-cycle S when it is obtained by KK-reducing the eleven-dimensional three-form on a two-form that is Poincaré dual to S.

We denote the Poincaré dual four-cycle of ζ as

$$Z = \hat{Z} - \frac{1}{2}(\hat{Z} \cdot \hat{Z} \cdot B^\alpha) B_\alpha \tag{3.52}$$

Z has been defined so that

$$B_\alpha \cdot Z \cdot Z = 0 \tag{3.53}$$

for all α, as can be checked explicitly. Also, as the B_α do not intersect the fiber, $Z \cdot f = \hat{Z} \cdot f = 1$. We denote this four-cycle a **type Z** four-cycle.

Let us verify the third entry of (Claim 1) first. We can always organize the basis of type C cycles in a convenient way. For each non-abelian gauge group $\mathcal{G}_\kappa$, there exists a curve b_κ in the base over which the fiber takes the Kodaira fiber-type corresponding to $\mathcal{G}_\kappa$. We use b_κ to denote both the actual curve and its class in the base. The fiber at b_κ consists of groups of type F rational curves. There is a canonical way of choosing linearly independent monodromy invariant groups of these rational curves, which we discuss in length in appendix B.1. If we denote this group as $\gamma_{I,\kappa}$ for each κ, the four cycles obtained by fibering $\gamma_{I,\kappa}$ over b_κ are dual to the vector field corresponding to $\mathcal{T}_{I,\kappa}$, the I'th element of the coroot basis of the Cartan of $\mathcal{G}_\kappa$. This is because, as we have checked in appendix B.1, the intersection numbers reproduce the charges of the charged adjoint multiplets under $\mathcal{T}_{I,\kappa}$ correctly.

To be more precise, let us denote $T_{I,\kappa}$ to be the four cycle obtained by fibering $\gamma_{I,\kappa}$ over the curve b_κ. As checked case by case for each Lie group in B.1, for each κ we can find type F rational curves that correspond to the simple roots $\alpha_{I,\kappa}$ of the Lie group $\mathcal{G}_\kappa$. Let us denote those curves $\chi_{I,\kappa}$. Then,

$$T_{I,\kappa} \cdot \chi_{J,\lambda} = -\delta_{\kappa\lambda} \frac{2\langle \alpha_{I,\kappa}, \alpha_{J,\kappa} \rangle}{\langle \alpha_{I,\kappa}, \alpha_{I,\kappa} \rangle} = -\delta_{\kappa\lambda} C_{IJ,\kappa}\,, \tag{3.54}$$

where C_κ is the Cartan matrix of $\mathcal{G}_\kappa$. All type F rational curves χ_ρ can be written as linear combinations of $\chi_{I,\kappa}$. The intersection numbers between these curves and $T_{I,\kappa}$ precisely reproduce the charges all the negative roots of each gauge group. By wrapping branes and anti-branes along these type F curves, one recovers all the charged adjoint vector fields of the theory. In the F-theory limit, these charged vector fields become massless, and along with the vector fields dual to $T_{I,\kappa}$ form the vector multiplet of gauge group $\mathcal{G}_\kappa$.

Therefore we can group the type C cycles according to their gauge groups *i.e.,*

$$T_{I,\kappa} \; : \; (T_{1,1}, \cdots, T_{R_1,1}), \cdots, (T_{1,N}, \cdots, T_{R_N,N})\,. \tag{3.55}$$

These are dual to non-abelian gauge field components

$$\mathcal{A}_{I,\kappa} \; : \; (\mathcal{A}_{1,1}, \cdots, \mathcal{A}_{R_1,1}), \cdots, (\mathcal{A}_{1,N}, \cdots, \mathcal{A}_{R_N,N}) \tag{3.56}$$

of the coroot basis elements of the Cartans of the non-abelian gauge groups;

$$\mathcal{T}_{I,\kappa} \; : \; (\mathcal{T}_{1,1}, \cdots, \mathcal{T}_{R_1,1}), \cdots, (\mathcal{T}_{1,N}, \cdots, \mathcal{T}_{R_N,N})\,. \tag{3.57}$$

Meanwhile, hypermultiplets obtained by wrapping M2 branes around clusters of type I curves in the resolved manifold also form representations under the non-abelian gauge groups. The representations can be determined by computing the intersection number of all the type I curves in each "cluster" with each $T_{I,\kappa}$. Note that the intersection number between $T_{I,\kappa}$ and any rational curve is integral. This is consistent with the fact that the charge of any weight vector is integral under elements of the coroot basis.

There is one question we raise before we go further. The type B cycles and type Z cycles do not intersect any of the shrinking curves, *i.e.,*

$$B_\alpha \cdot c_\tau = B_\alpha \cdot \chi_\rho = Z \cdot c_\tau = Z \cdot \chi_\rho = 0\,. \tag{3.58}$$

A given shrinking rational curve sits at a point above the base, and hence B_α, which is a fibration over a curve in the base, can always be smoothly deformed to avoid intersecting it. Z does not intersect any shrinking curves since the zero section does not touch any singularities in the fibration limit. Therefore, a vector field obtained by reducing the 11D three-form over some type C cycle T, and a vector field dual to $T + xZ + t^\alpha B_\alpha$ should reproduce the same charges. We claim that x and t^α can be fixed to zero by comparing the coefficient for the Chern-Simons five-form for the vector fields.

We now justify the two claims about vector fields Z and B^α by examining the Chern-Simons term in the five-dimensional theory in a generic point in the Coulomb branch. It is given by [178]

$$(\mathcal{S}_x \cdot \mathcal{S}_y \cdot \mathcal{S}_z) \int A^x \wedge F^y \wedge F^z \,, \tag{3.59}$$

where $\mathcal{S}_x$ are the dual four-cycles of the two-forms each gauge field is KK-reduced upon. The coefficient is the triple intersection of the four-cycles involved.

The intersection numbers are given by

$$\begin{aligned}
&\frac{1}{4}(9-T)(ZZZ) + \frac{3}{2}\delta_{\kappa\lambda}\mathcal{C}_{IJ,\kappa}(K\cdot b)_{\mathcal{B}}(ZT_{I,\kappa}T_{J,\lambda}) \\
&+ 3\Omega_{\alpha\beta}(ZB_\alpha B_\beta) - 3\delta_{\kappa\lambda}\mathcal{C}_{IJ,\kappa}b_{\alpha,\kappa}(B_\alpha T_{I,\kappa}T_{J,\lambda}) \\
&+ (\text{triple intersections among } T\text{'s})
\end{aligned} \tag{3.60}$$

in standard polynomial notation[7]; the coefficient of the term $(\mathcal{S}_x\mathcal{S}_y\mathcal{S}_z)$ is the intersection number $(\mathcal{S}_x \cdot \mathcal{S}_y \cdot \mathcal{S}_z)$ with multiplicities, *i.e.*, the polynomial is defined as

$$\sum_{x,y,z} (\mathcal{S}_x \cdot \mathcal{S}_y \cdot \mathcal{S}_z)(\mathcal{S}_x\mathcal{S}_y\mathcal{S}_z)\,. \tag{3.61}$$

[7]We have not computed the triple intersections among the type T cycles, as we do not need them for the purposes of this thesis. We note that these terms have been computed and matched with the F-theory side in [182].

b_κ^α are the $SO(1,T)$ coordinates of b_κ, *i.e.*,

$$b_\kappa = b_\kappa^\alpha H_\alpha \,. \tag{3.62}$$

$\mathcal{C}_\kappa$ is the normalized coroot inner-product matrix for gauge group $\mathcal{G}_\kappa$ defined in section 2.1.4 and discussed extensively in appendix B.1. K is the canonical divisor of the base.

Let us explain this result first. It is more convenient to obtain the intersection numbers using $\hat{Z}$, so we use the $\hat{Z}$ rather than Z. Intersection numbers involving Z can be obtained straightforwardly from those involving $\hat{Z}$.

We first note that

$$\hat{Z} \cdot X \cdot Y = (X|_\mathcal{B} \cdot Y|_\mathcal{B})_\mathcal{B} \tag{3.63}$$

since $\hat{Z}$ is the normal bundle of the base $\mathcal{B}$. $X|_\mathcal{B}$ is the two-cycle on $\mathcal{B}$—or more precisely, the zero section $\hat{Z}$—obtained by restricting the four-cycle X to $\hat{Z}$. The manifold $\hat{X}$ is Calabi-Yau and hence by the adjunction formula

$$\hat{Z}|_\mathcal{B} = K \,. \tag{3.64}$$

Also,

$$B_\alpha|_\mathcal{B} = H_\alpha \,. \tag{3.65}$$

Hence

$$\begin{gathered} \hat{Z} \cdot \hat{Z} \cdot \hat{Z} = (K \cdot K)_\mathcal{B} = (9-T), \quad \hat{Z} \cdot \hat{Z} \cdot B_\alpha = (K \cdot H_\alpha)_\mathcal{B} = K_\alpha, \\ \hat{Z} \cdot B_\alpha \cdot B_\beta = (H_\alpha \cdot H_\beta)_\mathcal{B} = \Omega_{\alpha\beta} \,. \end{gathered} \tag{3.66}$$

By construction, $\hat{Z}$ and $T_{I,\kappa}$ are disjoint; the zero section does not touch any of the singularities. Hence for any four cycle X,

$$\hat{Z} \cdot T_{I,\kappa} \cdot X = 0 \,. \tag{3.67}$$

Also, by (3.50)

$$B_\alpha \cdot B_\beta \cdot T_{I,\kappa} = \Omega_{\alpha\beta} f \cdot T_{I,\kappa} \,. \tag{3.68}$$

Since $T_{I,\kappa}$ are rational curves fibered along a curve in the base, it does not intersect with a generic fiber. Hence the intersection number is 0.

It is shown in appendix B.1 that

$$B_\alpha \cdot T_{I,\kappa} \cdot T_{J,\lambda} = -b_{\alpha,\kappa}(\gamma_{I,\kappa} \cdot T_{J,\lambda}) = -\delta_{\kappa\lambda} \mathcal{C}_{IJ} b_{\alpha,\kappa} \,. \tag{3.69}$$

κ and λ must be the same in order to get a non-zero result since $T_{I,\kappa}$ does not intersect type F rational curves fibered over a different locus.

It is convenient to express this relation using the projection π to the base manifold. More precisely, we define $\pi(C)$ of some two-cycle C in $\hat{X}$ to be the projection of C to the $H_2(\mathcal{B}) \cong H^{1,1}(\mathcal{B})$ lattice of the base manifold. As pointed out in the introduction, the projection of C to the base can in general be a linear combination of two, one and zero-cycles in the base. We treat one-cycles and zero-cycles to be null vectors in $H_2(\mathcal{B})$. Then π is defined so that for any two-cycle C in $\hat{X}$,

$$\pi(C) = (C \cdot B^\alpha) H_\alpha \quad \Leftrightarrow \quad B_\alpha \cdot C = (H_\alpha \cdot \pi(C))_{\mathcal{B}} = \pi(C)_\alpha \,. \tag{3.70}$$

Therefore (3.69) can be rewritten as

$$\pi(T_{I,\kappa} \cdot T_{J,\lambda}) = -\delta_{\kappa\lambda} \mathcal{C}_{IJ} b_{\alpha,\kappa} H^\alpha = -\delta_{\kappa\lambda} \mathcal{C}_{IJ} b_\kappa \,. \tag{3.71}$$

Now let us investigate the six-dimensional F-theory background compactified on the singular manifold X and then further compactified on S^1. Let us denote the vector fields obtained by KK-reduction on S^1 in the following way:

1. Z' is the vector field obtained by KK-reducing the six-dimensional metric. It is inside the gravity multiplet.

2. B' are the vector fields obtained by KK-reducing the $(T+1)$ tensors.

3. $\mathcal{A}'$ are the vector fields obtained by KK-reducing the non-abelian vector fields

in the coroot basis of the Cartan of the gauge group.

Let us denote the anomaly coefficients for the non-abelian gauge fields as b'_κ. In section 3 of [173], the coefficients of the Chern-Simons term of the KK-reduced five-dimensional theory on a generic point in the Coulomb branch is worked out explicitly. The intersection polynomial is given by

$$\begin{aligned}&\Omega_{\alpha\beta}(Z'B'_\alpha B'_\beta) - 2\delta_{\kappa\lambda}\mathcal{C}_{IJ,\kappa}b'_{\alpha,\kappa}(B'_\alpha \mathcal{A}'_{I,\kappa}\mathcal{A}'_{J,\lambda})\\ &+ (\text{triple intersections among } \mathcal{A}\text{'s})\end{aligned} \tag{3.72}$$

up to an overall constant—that we denote K_{int}—in the "decompactification limit," *i.e.,* when the vacuum expectation value of the scalars in the vector multiplets and the inverse radius of the S^1 go to zero [173]. Note that we have used the non-trivial fact that λ_κ is chosen so that the Cartan generators $\{\mathcal{T}_{I,\kappa}\}$ of $\mathcal{G}_\kappa$ in the coroot basis satisfy

$$\frac{1}{\lambda_\kappa}\mathrm{tr}\mathcal{T}_{I,\kappa}\mathcal{T}_{J,\kappa} = \mathcal{C}_{IJ,\kappa}\,. \tag{3.73}$$

This intersection form agrees with (3.60) up to terms that do not involve B when we identify $b_\kappa = b'_\kappa$ — which is indeed true for non-abelian gauge fields [64, 65] — and take Z and B_α to be proportional to Z' and B'_α. The terms that involve B cannot receive corrections for the following reason. The corrections to these Chern-Simons terms come from one-loop integrals of five-dimensional fermions [144]. The only way that terms involving B could receive corrections on the F-theory side is if some six-dimensional fermion in a short multiplet couples to the tensor field B in a way that reduces to

$$\bar{\psi}B_\mu\Gamma^\mu\psi \tag{3.74}$$

in five dimensions. There are no such couplings so these terms are not modified [182]. Meanwhile, the vector field $Z_\mu \sim g_{\mu 5}$ can couple in this manner to charged fermions in short multiplets. One-loop contributions of these fermions generate the first two terms of (3.60) [182].

We note that we have a well defined normalization prescription for Z and B given in the following way. There is an unambiguous prescription for the normalization of non-abelian gauge fields on both sides; they were normalized to reproduce the charges of the coroot lattice. This implies that $\mathcal{A}$ and $\mathcal{A}'$ are indeed identical. Then we can fix the proportionality constant of the B with respect to B' by using the fact that

$$b_\kappa = b'_\kappa \,. \tag{3.75}$$

This in turn fixes the proportionality constant of Z with respect to Z'. Fixing the normalization of B and Z is important in determining what the abelian anomaly coefficients are.

We have verified (Claim 1) by comparing the Chern-Simons five form on the M-theory and F-theory side. Note that if we add type B or type Z cycles to type C cycles, the intersection polynomial becomes modified. In particular, terms of form (ZBT) would appear, which do not and cannot appear on the F-theory side in the decompactification limit.

3.3.3 The Abelian Sector

In this section we find the four-cycles dual to KK-reduced abelian gauge fields and identify the abelian anomaly coefficients. We make the following

(Claim 2) The vector fields $A_1, \cdots, A_{V_A}$ dual to type S four-cycles S_i — which we shortly define — can be identified with the vector fields obtained by KK-reducing the six-dimensional abelian vector fields along S^1 in the decompactification limit.

We construct the **type S** four-cycles in the following way. For each type Ŝ four-cycle $\hat{S}_i$, define the corresponding type S four-cycle S_i as

$$S_i = \hat{S}_i - (\hat{S}_i \cdot f)\hat{Z} - ((\hat{S}_i - (\hat{S}_i \cdot f)\hat{Z}) \cdot \hat{Z} \cdot B^\alpha)B_\alpha + \sum_{I,J,\kappa} (\hat{S}_i \cdot \chi_{I,\kappa})(C_\kappa^{-1})_{IJ} T_{J,\kappa} \tag{3.76}$$

where κ labels the non-abelian gauge groups of the six-dimensional theory and I labels their simple roots. $(C_\kappa^{-1})_{IJ}$ is the (I, J) component of the inverse of the Cartan matrix C_κ. Recall that $\chi_{I,\kappa}$ is the type F cycle corresponding to the simple roots $\alpha_{I,\kappa}$

of $\mathcal{G}_\kappa$. Type S cycles are defined so that

1. $S_i \cdot f = 0$.

2. $S_i \cdot \hat{Z} \cdot B_\alpha = 0$.

3. $S_i \cdot \chi_\rho = 0$.

The first and second identities can be checked easily by using intersection identities given in the previous section. We note that the first condition implies that

$$S_i \cdot B_\alpha \cdot B_\beta = \Omega_{\alpha\beta} S_i \cdot f = 0\,. \tag{3.77}$$

We also note that the second condition implies that

$$S_i \cdot \hat{Z} \cdot X = (S_i|_\mathcal{B} \cdot X|_\mathcal{B})_\mathcal{B} = (X|_\mathcal{B})^\alpha S_i \cdot \hat{Z} \cdot B_\alpha = 0 \tag{3.78}$$

for any four-cycle X.

Since all χ_ρ are homologically equivalent to a sum of $\chi_{I,\kappa}$, the third identity needs to be checked only for all $\chi_{I,\kappa}$. This can be done:

$$\begin{aligned} S_i \cdot \chi_{I,\kappa} &= \hat{S}_i \cdot \chi_{I,\kappa} + \sum_{J,K,\lambda} (\hat{S}_i \cdot \chi_{J,\lambda})(C_\lambda^{-1})_{JK}(T_{K,\lambda} \cdot \chi_{I,\kappa}) \\ &= \hat{S}_i \cdot \chi_{I,\kappa} + \sum_{J,K,\lambda} (\hat{S}_i \cdot \chi_{J,\lambda})(C_\lambda^{-1})_{JK}(-\delta_{\lambda\kappa} C_{KI,\kappa}) \\ &= \hat{S}_i \cdot \chi_{I,\kappa} - \hat{S}_i \cdot \chi_{I,\kappa} = 0\,. \end{aligned} \tag{3.79}$$

Meanwhile,

$$S_i \cdot T_{I,\kappa} \cdot B_\alpha = b_{\alpha,\kappa}(S_i \cdot \gamma_{I,\kappa})\,. \tag{3.80}$$

Recall that $\gamma_{I,\kappa}$ is the monodromy invariant fiber of $T_{I,\kappa}$ over b_κ. As can be seen in appendix B.1, the $\gamma_{I,\kappa}$ are linear combinations of $\chi_{J,\kappa}$. Therefore, it follows that

$$S_i \cdot T_{I,\kappa} \cdot B_\alpha = 0 \tag{3.81}$$

for any $T_{I,\kappa}$ and B_α.

Equation (3.76) is the threefold analog of the map Shioda used to map rational sections of an elliptically fibered surface to points in the Néron-Severi lattice of that surface [183].[8] Once the rational sections were mapped to a lattice, a number-valued inner-product on the sections could be defined. In our case there is an $H_2(\mathcal{B})$ vector-valued inner-product on type S cycles. It is $-\pi(S_i \cdot S_j)$. We claim that these are the anomaly coefficients of the abelian gauge groups.

Now we verify that vector fields dual to type S cycles can indeed be identified with the KK-reduced six dimensional abelian vector fields in the decompactification limit. We verify that none of the vector multiplets are charged under A_i and that the coefficients of the Chern-Simons five-forms have the proper form. The first point is easily checked since $S_i \cdot \chi_\rho = 0$ implies that none of the charged vector multiplets are charged under A_i. S_i, however, can intersect with curves of type I, *i.e.*, hypermultiplets can be charged under abelian gauge fields.

The triple intersection polynomial, when incorporating S_i becomes

$$\begin{aligned}&\frac{1}{4}(9-T)(ZZZ) + \frac{3}{2}\delta_{\kappa\lambda}\mathcal{C}_{IJ,\kappa}(K\cdot b)_{\mathcal{B}}(ZT_{I,\kappa}T_{J,\lambda}) - \frac{3}{2}(K\cdot\pi(S_i\cdot S_j))_{\mathcal{B}}(ZS_iS_j)\\&+3\Omega_{\alpha\beta}(ZB_\alpha B_\beta) - 3\delta_{\kappa\lambda}\mathcal{C}_{IJ,\kappa}b_{\alpha,\kappa}(B_\alpha T_{I,\kappa}T_{J,\lambda}) + 3\pi(S_i\cdot S_j)_\alpha(B_\alpha S_iS_j)\\&+(\text{triple intersections among } T,\, S)\end{aligned} \tag{3.82}$$

We have explained the absence of the terms (SBB), (SBZ), (STZ) and (STB). Coefficients of the (SSB) terms follow from the definition of the projection π.

Adding the contributions of A'—the vector fields obtained by KK-reducing the six-dimensional abelian vector fields—to equation (3.72), the tree-level intersection polynomial on the F-theory side is given by

$$\begin{aligned}&\Omega_{\alpha\beta}(Z'B'_\alpha B'_\beta) - 2\delta_{\kappa\lambda}\mathcal{C}_{IJ,\kappa}b'_{\alpha,\kappa}(B'_\alpha \mathcal{A}'_{I,\kappa}\mathcal{A}'_{J,\lambda}) - 2b_{\alpha,ij}(B'_\alpha A'_{I,\kappa}A'_{J,\lambda})\\&+(\text{triple intersections among } \mathcal{A}\text{'s and } A\text{'s})\end{aligned} \tag{3.83}$$

[8]The image of the rational sections of an elliptically fibered K3 manifold $\mathcal{M}$ under the Shioda map—which are two-cycles—are also dual to the abelian vector fields of the eight-dimensional supergravity theory obtained by compacifying F-theory on $\mathcal{M}$ [120, 179, 184, 185].

up to the same overall constant K_{int} defined below (3.72). Recall that $b_{\alpha,ij}$ are the $SO(1,T)$ vector components of the abelian anomaly coefficients. We see that the intersection polynomial (3.82) matches with (3.83) up to terms not involving B, when we normalize Z and B with respect to Z' and B' according to the prescription given at the end of the previous section. The matching of the intersection polynomial concludes the justification of (Claim 2).

Furthermore, if we normalize the gauge fields A'_i so that the charge of the hypermultiplet coming from branes wrapping c_r is $c_r \cdot S_i$, we can equate A'_i and A_i. Then due to the normalization prescription of B_α we have given in the previous section, we can unambiguously equate

$$b_{ij} = -\pi(S_i \cdot S_j)\,. \tag{3.84}$$

This is the main result of this section.

3.3.4 Summary

Let us summarize our findings. F-theory compactified on $X \times S^1$—where X is an elliptically fibered Calabi-Yau threefold with a section—is dual to M-theory compactified on X. We have identified the massless field content of the six-dimensional theory from the M-theory dual. In the process, we have proven equation (3.49).

The vector fields of the six-dimensional theory KK-reduce along S^1 to vector fields in five dimensions. In the M-theory dual, the KK-reduced vector fields have the following origins:

1. **Abelian Vector Fields :** KK-reduction of the 11D three-form on two-forms dual to four-cycles of type S.

2. **Non-abelian Vector Fields in the Cartan of the Gauge Group:** KK-reduction of the 11D three-form on two-forms dual to four-cycles of type C.

3. **Non-abelian Vector Fields Not in the Cartan of the Gauge Group:** M2 branes and anti-branes wrapping curves of type F.

The definitions of the various types of cycles are given in section 3.3.2. We note once again that we abuse the term "duality" in the following way; we say that a vector field is "dual to" a four-cycle $\mathcal{S}$ when it is obtained by KK-reducing the eleven-dimensional three-form on a two-form Poincaré dual to $\mathcal{S}$. The abelian and non-abelian Cartan vector multiplets are dual to four-cycles that do not intersect the fiber.

Let us elaborate on the construction of cycles of type S. Type S cycles S_i are constructed from four-cycles that are the generators of the rational sections through the Shioda map (3.76). The anomaly coefficient of the abelian vector fields can be identified with the opposite vector of the projection of the the intersection of two type S four-cycles to the $H^{1,1}$ lattice of the base:

$$b_{ij} = -\pi(S_i \cdot S_j)\,. \tag{3.85}$$

All the fields charged under abelian or non-abelian Cartan vector fields come from M2 branes wrapping shrinking rational curves. Rational curves of type I—or isolated rational curves—contribute one hypermultiplet each to the massless spectrum in the decompactification limit: a brane and an anti-brane wrapping a given curve contribute a half-hypermultiplet each, which together form one hypermultiplet. Rational curves of type F—or fibered rational curves—contribute $2g$ hypermultiplets where g is the genus of the curve over which the rational curve is fibered. As mentioned above, a type F rational curve also contributes two vector multiplets to the massless spectrum of the six-dimensional theory, each obtained by either wrapping a brane or an anti-brane.

A charged hypermultiplet consists of two half-hypermultiplets each coming from wrapping an M2 brane or anti-brane on a curve. There is an overall sign ambiguity in defining charges of hypermultiplets. We use the convention that a hypermultiplet coming from wrapping branes and anti-branes on a rational curve C has charge $C\cdot\mathcal{S}$ under the vector field dual to a four-cycle $\mathcal{S}$. Meanwhile, each vector field coming from wrapping M2 branes(anti-branes) on the type F curve χ_ρ has charge $\chi_\rho\cdot\mathcal{S}(-\chi_\rho\cdot\mathcal{S})$ under the vector multiplet dual to a four-cycle $\mathcal{S}$, respectively.

Chapter 4

Lessons Learned

We have seen in section 2 that the anomaly constraints of six dimensional supergravity theories with minimal supersymmetry render the space of potentially consistent theories quite manageable under certain assumptions. For example, we have reviewed that the number of possible non-anomalous massless spectra is bounded when the number of tensor multiplets T is smaller than nine and there are no abelian gauge group factors [62, 65].

In particular, we have seen that when $T = 0$, all the anomaly equations simplify so that there is a systematic way of constructing all non-anomalous models given the gauge group. Therefore, we are now in a position to compare the non-anomalous models with all known string vacua for $T = 0$. Comparing the set of non-anomalous theories and the set of string vacua, we identify many theories that can readily be embedded into string theory, as well as theories that are candidates for new string vacua.

We also find many of theories that we do not know how to incorporate into any known string vacua. These are theories whose anomaly coefficients violate the Kodaira bound

$$\sum_{\kappa} \nu_\kappa (j \cdot b_\kappa) \leq -12 j \cdot a \tag{4.1}$$

that we have introduced in the previous chapter. In fact, as can be seen from table 4.1, these theories turn out to populate the bulk of the $T = 0$ landscape. If these

M	# $SU(M)$ models	# satisfy Kodaira
13-24	1	1
12	2	2
11	2	2
10	2	2
9	3	3
8	15	14
7	16	16
6	48	47
5	23	16
4	207	154
3	10100	262
2	$\sim 5 \times 10^7$	176
$U(1)$	∞	?

Table 4.1: The table of numbers of non-anomalous $T = 0$ theories with gauge group $SU(M)$ for various M. There are no non-anomalous theories when $M > 24$. The number of all theories that are non-anomalous are given in the second column. The number of theories that satisfy the Kodaira condition in addition are given in the third column. The number of non-anomalous theories with $M = 2$ are very large and have not been computed precisely. Note when the gauge group is abelian there exist an infinite number of non-anomalous theories. There is no known analogue of the Kodaira constraint for abelian theories.

theories indeed are undiscovered string vacua, there must be a large part of the string landscape that is currently undiscovered. If these are secretly inconsistent, there must be some fundamental reason that the Kodaira bound should hold for these supergravity theories. The Kodaira bound has other intriguing features that make it attractive as a candidate for a fundamental constraint of supergravity. We show in section 4.2 that in fact all the infinite families of $T \geq 9$ theories constructed in section 2.2 violate the Kodaira bound. We therefore have identified an important question regarding the consistency of quantum supergravity theories through the landscape analysis:

(Q1) Is the Kodaira constraint a fundamental constraint of quantum supergravity?

Meanwhile, we have shown in section 2.4 that the situation becomes less tractable when the gauge group has abelian factors. For example, it has been shown that while the number of allowed gauge groups and non-abelian matter representations are bounded when $T < 9$, there exist infinite classes of theories generated by assigning different $U(1)$ charges to the matter.

The immediate question that arises in this context is whether such infinite classes of theories are consistent. This is a hard question to answer. To our knowledge, there are no consistency conditions that could rule out the simple examples of infinite classes of theories given in section 2.4.3, but at the same time, there is no guarantee that these examples are consistent. We may be less ambitious and ask whether there is an obstruction to embedding all of these theories in string theory. This question is still a difficult one to answer, as there is no known abelian generalization of the Kodaira constraint that would restrict the charge structure of $U(1)$ theories. By examining the table 4.1, however, it is easy to observe that the $U(1)$ theories fit the general trend of anomaly equations being less restrictive for groups with less structure. This table strongly suggests that a generalized version of the Kodaira constraint might be attainable in string theory. Hence, another important question arising from our landscape analysis is:

(Q2) Is there a generalized version of the Kodaira constraint for theories with abelian gauge symmetry (in string theory) ?

A practical strategy to pursue in this direction is to ask whether there is an obstruction in incorporating the infinite class of theories to known string vacua, and hope to gain insight from it. In section 3.3 we have made some progress in this direction by identifying a geometrical object that corresponds to the abelian anomaly coefficients for F-theory vacua. Although we have not been able to go further to identify a generalized version of the Kodaira constraint, some interesting results follow from carrying the analysis further. The dictionary of section 3.3 enables us to translate the anomaly equations that hold for six-dimensional $(1, 0)$ theories to intersection theory equalities of Calabi-Yau threefolds — we present these equations in section

4.3.

The structure of this chapter is as the following. We present the results from the analysis of $T = 0$ theories in section 4.1. Then we discuss the status of the landscape of theories with abelian gauge symmetry in 4.2. In particular, we review the infinite classes of theories that could be generated for these theories and pose the question of how to think about them. Finally, in section 4.3 we decribe the intersection theory equalities that could be obtained by translating the anomaly equations to geometric terms.

4.1 Non-abelian $T = 0$ Theories

In this section, we compare the space of apparently consistent $T = 0$ theories with gauge group

$$\mathcal{G} = SU(M_1) \times \cdots \times SU(M_k) \tag{4.2}$$

with the space of string vacua, in particular, F-theory vacua. According to the dictionary reviewed in section 3.1, F-theory models with $T = 0$ are obtained by compactifying on an elliptically fibered Calabi-Yau threefold with base $\mathbb{P}^2$ [118, 119]. The set of toric Calabi-Yau threefolds with base $\mathbb{P}^2$ have recently been systematically studied in [186]. Also, a detailed study of the matter sector of $T = 0$ F-theory compactifications is given in [112].

We can classify the set of apparently consistent $T = 0$ theories constructed by the strategy sketched in section 2.3 into three groups:

1. Theories readily embeddable in F-theory
2. Candidate theories for new string vacua
3. Theories we do not know how to embed in string theory

As reviewed in section 3.1.1, there is no clear obstruction to embedding any apparently consistent theory that satisfies the Kodaira bound (3.3) into F-theory. In other words, any theory that satisfies the Kodaira constraint has a chance of being an F-theory vacuum. As reviewed in section 2.3, the anomaly coefficients of $T = 0$ vacua are integers b_κ, and the gravitational anomaly coefficient is given by $a = -3$. Using the group theory coefficients given in section 3.1.1, we find that the Kodaira bound (3.3) for theories with gauge group $SU(M_1) \times \cdots \times SU(M_k)$ is given by

$$M_1 b_1 + M_2 b_2 + \cdots + M_k b_k \leq 36 \,. \tag{4.3}$$

If a theory satisfies the Kodaira constraint and only has conventional, or simple, matter — matter that is known to be generated in F-theory vacua — it is expected to be obtainable from F-theory by tuning the complex structure in a generic way. We classify these theories to be readily embeddable.

Meanwhile, we find theories that satisfy the Kodaira bound but that have exotic matter structure that has not been constructed in F-theory previously. Although there is exotic matter, there is no clear reason why these theories cannot be F-theory vacua with complicated singularities unstudied before. We therefore expect these theories to be embeddable into F-theory, with non-trivial tuning of the complex structure. These theories are candidate theories for new string vacua. A systematic study of such exotic vacua was initiated in the recent work [112].

Finally there are — in fact, quite many — theories that violate the Kodaira bound. As discussed in the introduction of this chapter, we do not have a framework in which to understand these theories. These theories might be exotic string vacua that have not been constructed yet, or inconsistent theories violating an undiscovered consistency condition.

In this section, we systematically scan the space of apparently consistent $T = 0$ theories and see how these theories fall into each category described above. As explained in section 2.3, we construct theories by putting together blocks of single gauge group factors. Therefore, it is convenient to classify theories in the landscape by the number of blocks it is made of. We examine the structure of the single building "blocks" as well as single block models in section 4.1.1. We then proceed to examine the structure of two-block and multi-block models in sections 4.1.2 and 4.1.3. We summarize our findings in section 4.1.4.

4.1.1 Singe Blocks in $T = 0$ Theories

In this section, we examine the single $SU(M)$ blocks that satisfy the gauge anomaly equations (2.52), (2.53) and (2.54):

$$3b_\kappa = \frac{1}{6}\left[\sum_R x_R A_R - A_{\text{adj}}\right] \tag{4.4}$$

$$0 = \sum_R x_R B_R - B_{\text{adj}} \tag{4.5}$$

$$b_\kappa^2 = \frac{1}{3}\left[\sum_R x_R C_R - C_{\text{adj}}\right] \tag{4.6}$$

and also the single block models that consist of one of these blocks. Recall from section 2.3 that $M \leq 24$ for all blocks that satisfy these equations.

Let review facts about blocks with $b = 1$ or 2 shown in section 2.3. For $M > 7$, the only possible blocks with $M > 7$ are $SU(M)$ factors with matter content

$$(24 - M) \times \square + 3 \times \begin{smallmatrix}\square\\\square\end{smallmatrix}, \quad (b = 1, M \leq 24, H - V = (2 + 45M - M^2)/2 \leq 273)\,. \tag{4.7}$$

(In this section, we use H to denote the number of charged hypermultiplets of a given block.)

For $M \leq 7$, other $b = 1$ blocks are possible. For $SU(7)$ there exists the block

$$SU(7): \qquad 22 \times \square + 1 \times \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}, \quad (b = 1, H - V = 141)\,. \tag{4.8}$$

A similar block can be constructed for $SU(6)$ with 20 fundamental, one A2, and one A3 representation. For $SU(5)$ the A3 and A2 representations are conjugate (and therefore treated as equivalent in this analysis), this exhausts the range of possibilities for $b = 1$. All these blocks automatically satisfy the gravitational anomaly bound $H - V \leq 273$, and hence can give single block models.

For $b = 2$ there are again only single-column representations. Now $M \leq 12$. The generic blocks have the form

$$(48-4M)\times\Box+6\times\begin{smallmatrix}\Box\\\Box\end{smallmatrix}, \qquad (b=2, M\leq 12, H-V=1+45M-2M^2\leq 273) \quad (4.9)$$

for all $M\leq 12$. There are other $b=2$ blocks when $6\leq M\leq 10$. Blocks with single 3-antisymmetric (A3) representations are possible for $M=10,9$ with $H-V>273$ and at $M=8,7,6$ with $H-V\leq 273$. For $SU(6)$ there are also blocks with two and three A3 representations, and for $SU(7)$ there is a block with two A3 representations; all these blocks satisfy the gravitational anomaly bound $H-V\leq 273$. There is also a single $b=2$ block with gauge group $SU(8)$ and a 4-antisymmetric (A4) representation

$$SU(8): \qquad 32\times\Box+1\times\begin{smallmatrix}\Box\\\Box\\\Box\\\Box\end{smallmatrix}, \quad (b=2, H-V=263). \qquad (4.10)$$

This completes the list of all possible $b=2$ blocks.

Continuing to $b=3$, there is now a nonzero contribution to the genus,[1]

$$b=3: \qquad 2\sum_R x_R\, g_R=(b-1)(b-2)=2. \qquad (4.13)$$

There is, therefore, necessarily a matter representation with more than one column, which has $g_R=1$. The only possibilities are the adjoint and two-index symmetric representations for general N (note that the representation $\begin{smallmatrix}\Box&\Box\\\Box&\end{smallmatrix}$ in Table 2.3 for $SU(3)$

[1]Some comments are due on the genus g of a block given by $(b-1)(b-2)/2$ and the genus g_R of each matter representation defined by

$$g_R=\frac{1}{12}\left(2C_R+B_R-A_R\right). \qquad (4.11)$$

For models with an F-theory construction, the anomaly integer b is the degree of the curve realizing the corresponding gauge group. The quantity $g:=\sum_R x_R g_R=(b-1)(b-2)/2$ is then the (arithmetic) genus of this curve. In F-theory, the number of adjoint hypermultiplets in the low-energy theory is given by the geometric genus g_g of the curve. The genus-degree formula for a general, possibly singular, curve relates the arithmetic and geometric genera

$$g=(b-1)(b-2)/2=g_g+\sum_P\frac{m_P(m_P-1)}{2}, \qquad (4.12)$$

where the sum is over all singular points P of the curve, and m_P is the multiplicity at point P [187]. This relationship provides a clue towards realizing general matter representations in F-theory through codimension-2 singularities. This point has been explored further in [112].

has $g_R = 1$, but is also the adjoint of $SU(3)$). For each choice of representation saturating the genus $g = 1$, there are various possible combinations of n-antisymmetric single-column representations which can solve the partition problem for the C's. The largest N for which a one-block model appears with $b = 3$ which satisfies the gravitational anomaly bound on the number of hypermultiplets is $N = 9$; the matter content of this model is

$$SU(9): \qquad 5 \times \Box + 4 \times \begin{smallmatrix}\Box\\\Box\end{smallmatrix} + 1 \times \begin{smallmatrix}\Box\\\Box\\\Box\end{smallmatrix} + 1 \times \text{Adj}, \qquad (b = 3, H - V = 273)\,. \tag{4.14}$$

Note that the blocks listed explicitly above (4.7, 4.8, 4.9, 4.10, 4.14) have $H - V \leq 273$, and therefore, by adding neutral hypermultiplets, can be completed to anomaly-free low-energy supergravity theories with single factor gauge groups $G = SU(M)$. The model in (4.14) precisely saturates the gravitational anomaly bound with $H - V = 273$. This model therefore has no neutral hypermultiplets and is "rigid" in the sense that deformation along any scalar modulus will break the symmetry of the model. As we will see, many of the most exotic matter representations arise in such rigid models.

All the models described above furthermore satisfy the Kodaira bound from F-theory $\sum_\kappa b_\kappa M_\kappa = bM \leq 36$. We might therefore expect that these models have F-theory realizations. While the fundamental and antisymmetric matter representations have standard F-theory realizations, however, the 3-index and 4-index representations are more exotic. These representations were also encountered in $T = 1$ models in [65]. In the case of the 3-index representations, a codimension two singularity structure has been identified in F-theory which realizes this matter representation for $N = 6, 7, 8$ [134] through local enhancement of the singularity type to E_6, E_7 and E_8 respectively. We are not aware, however, of any known F-theory realization of the 4-index antisymmetric representation, or of the 3-index antisymmetric representation for $M = 9$. Progress in this direction has recently been made by [112].

We have systematically analyzed the set of all possible $SU(M)$ blocks with arbitrary matter representations for $T = 0$ and any M. A summary of the results of

M	max b	(total blocks)	$SU(M)$ models	Kodaira models
13-24	1 (1)	(1)	1	1
12	2 (2)	(2)	2	2
11	2 (3)	(4)	2	2
10	2 (4)	(6)	2	2
9	3 (4)	(8)	3	3
8	8 (8)	(22)	15	14
7	4 (7)	(28)	16	16
6	6 (8)	(147)	48	47
5	8 (14)	(186)	23	16
4	16 (34)	(3893)	207	154
3	597 (597)		10100	262
2	$24297 \leq b_{\max} < 36647$		$\sim 5 \times 10^7$	176

Table 4.2: A summary of possible distinct $SU(M)$ blocks. The numbers in parentheses refer to possible blocks without the gravitational anomaly constraint imposed, while the numbers without parentheses refer to possible single block $SU(M)$ models. The last column gives the number of single factor models which satisfy the Kodaira constraint $bM \leq 36$ needed for an F-theory realization. The number of blocks not individually satisfying the gravitational anomaly bound becomes very large at $M = 3$, as does number of blocks for $M = 2$ even with the gravitational anomaly constraint. We have not precisely computed the number of blocks in these categories.

this analysis — part of which was already shown in table 2.4 — appears in table 4.2. We carried out this analysis by finding all of the finite number of solutions for the partition problem for each combination of M and b, within the bounded range of b's for which a solution can be found for each M. As noted before, we have explicitly computed all blocks for $M \geq 4$, dividing the set into those which do or do not individually satisfy the gravitational anomaly bound $H - V \leq 273$. For $SU(2)$ and $SU(3)$, the total number of blocks becomes quite large. For $SU(3)$ we have only explicitly computed the number of blocks which individually satisfy the gravitational anomaly bound, and for $SU(2)$ we have only estimated the number of blocks and their range and computed some specific examples, as described below. The detailed analysis of upper bounds on b for each fixed M is given in appendix A.3.

We now describe briefly a few interesting aspects of the results summarized in Table 4.2 and highlight a few specific blocks of interest.

$M > 8$:

For $M > 9$, there are known F-theory realizations of all matter representations appearing in all single-block models. Furthermore, the Kodaira constraint is satisfied for all single blocks with $M \geq 8$. Thus, it seems likely that all the single-block $SU(M)$ models with $M > 9$ which are anomaly-free can be realized in F-theory. The only unusual representation which arises at $M = 9$ is the 3-index antisymmetric representation mentioned above in the model (4.14).

$M = 8$:

At $M = 8$ we find several novel features. As mentioned above, there is an $SU(8)$ model with a 4-index antisymmetric representation. There is also a somewhat exotic model with

$$SU(8): \qquad 1 \times \ydiagram{2,2} \quad (b = 8, H - V = 273)\,. \tag{4.15}$$

This is the only $SU(8)$ model containing a block with $b > 4$ and is another example of a model with rigid symmetry. There is no known F-theory realization of the "box" matter representation appearing in this model, although a singularity structure that could possibly give this representation for a low-rank gauge group was studied in [112]. Furthermore, this model violates the Kodaira condition ($bM = 64 > 32$). Nonetheless, the numerology seems to work out rather nicely for this model, suggesting that there may possibly be some new class of string compactification which could realize this model.

$M \leq 6$

At $M = 6$ and below, the range of possible representations expands significantly, and models which violate the Kodaira condition begin to proliferate. There is one model at $M = 6$ which has another exotic representation

$$SU(6): \qquad 2 \times \ydiagram{1,1} + 2 \times \ydiagram{1,1,1} + 2 \times \ydiagram{2,1,1,1} + 2 \times \text{Adj}, \quad (b = 6, H - V = 273)\,. \tag{4.16}$$

This is another example of a model with rigid symmetry, although this model is (just barely) within the Kodaira bound.

At $M = 5$ and below an increasing range of exotic representations becomes possible. At the end of this section we summarize the set of representations which can be realized in models satisfying the Kodaira condition for any M. One particularly simple and interesting block with $M = 4$ is

$$SU(4): \qquad 1 \times \ydiagram{2,2} + 64 \times \ydiagram{1}, \qquad (b = 4, H - V = 261)\,. \tag{4.17}$$

For models not satisfying the Kodaira bound, an even wider range of representations can be realized; for example, for $N = 4$ there are single block models violating the Kodaira bound which have the representations $\ydiagram{4}$ and $\ydiagram{3,2,1}$. Most of these exotic representations appear in models which precisely saturate or almost saturate the gravitational anomaly bound. For example, one $SU(4)$ model at $b = 16$ has

$$SU(4): \qquad 3 \times \ydiagram{3,2,1} + 3 \times \ydiagram{2,2} + 1 \times \ydiagram{4}, \qquad (b = 16, H - V = 272)\,. \tag{4.18}$$

At $M = 3$ the range of possibilities increases still further. The distribution of blocks across values of b is rather non-uniform. There are an enormous range of blocks not satisfying the gravitational anomaly bound and having $b < 500$ which we have not attempted to completely enumerate. Among those blocks individually satisfying the gravitational anomaly bound, most are distributed across values of $b < 70$, with more blocks at values of b divisible by 3. The most blocks satisfying the gravitational anomaly bound occur at $b = 24$ (910 blocks). There are only a few values of $b > 70$ with allowed such blocks, including 44 blocks at $b = 93$, followed by 3 blocks at $b = 105$ and single blocks each at $b = 153, 168, 408$ and 597. The matter content for $b = 597$ is given by

$$SU(3): \qquad 1 \times \ydiagram{6}\,(\mathrm{S6}) + 1 \times \quad (\mathrm{S21}) \qquad (b = 597, H - V = 273)\,. \tag{4.19}$$

Even without imposing the gravitational anomaly bound, there are only blocks possible for three distinct values of $b > 500$. At $b = 521$ there are 79,151 different blocks possible with $H - V \leq 1000$; at $b = 522$ there are 40 such blocks. The only block possible with $b > 522$ is (4.19). It is striking that the largest possible $SU(3)$ block precisely saturates the gravitational anomaly bound.

For $SU(2)$ we have not computed all blocks explicitly, even restricting to blocks satisfying the gravitational anomaly bound, as the number of possibilities is very large. The best upper bound we have found for b for $SU(2)$ is 36,647 (see Appendix B). We have sampled the distribution by computing the number of blocks satisfying the gravitational anomaly bound for multiples of 20, $b = 20k$, up to $b = 1000$, and for multiples of 250 up to $b = 20,000$. The number of blocks at fixed b seems to peak around $b = 420$, where there are 65,459 distinct $SU(2)$ blocks. The number of blocks starts to drop significantly after $b = 1000$, with for example 11,121 blocks at $b = 1000$, 835 blocks at $b = 2000$, and 12 blocks at $b = 4000$. As for $N = 3$, however, there are individual blocks out to much larger values of b. We have found blocks satisfying $H - V \leq 273$ for b up to 24,297, though there are probably sporadic blocks appearing for larger b up to close to the bound of 36,647 (though these must be rare; for example 24,297 is the only value of b between 24,000 and 25,500 which admits a block). Based on the partial data we have computed, we estimate the number of blocks satisfying the gravitational anomaly bound to be on the order of 5×10^7. The total number of blocks without imposing the gravitational anomaly constraint is much larger, but still finite. An example of an $SU(2)$ block with a very large value of b satisfying the gravitational anomaly bound is the following block with $b = 10,750$

$$
\begin{aligned}
SU(2): \qquad & 1 \times \square\!\square (\mathrm{S2}) + 1 \times (\mathrm{S3}) + 1 \times (\mathrm{S4}) + 1 \times (\mathrm{S5}) + 1 \times (\mathrm{S6}) \\
& +1 \times (\mathrm{S17}) + 1 \times (\mathrm{S55}) + 1 \times (\mathrm{S69}) + 1 \times (\mathrm{S85}), \\
& (b = 10750, H - V = 252)\,.
\end{aligned} \tag{4.20}
$$

An example of a block with larger b which violates the gravitational anomaly bound is

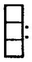
$$
\begin{aligned}
SU(2): \qquad & 24530 \times \square + 8380 \times \square\!\square + 1 \times (\mathrm{S12}) + 1 \times (\mathrm{S29}) + 1 \times (\mathrm{S43}) \\
& +1 \times (\mathrm{S113}), \qquad (b = 18000, H - V = 74398)\,.
\end{aligned} \tag{4.21}
$$

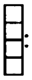
This block, in fact, wildly violates the gravitational anomaly bound, and it can be shown fairly easily that no model satisfying the gravitational anomaly bound can contain this block. For $SU(2)$ there are many such single blocks at large b that satisfy the single block anomaly equations but violate the gravitational anomaly bound. Thus, as the rank decreases the gravitational anomaly bound becomes a more important constraint in restraining the class of allowed models, even though the gravitational anomaly bound alone is sufficient to prove that the number of blocks is finite.

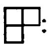
We conclude this description of single $SU(M)$ factor matter blocks in $T = 0$ models with a brief summary of all novel representations which can appear in single block models satisfying the F-theory Kodaira constraint, but for which no F-theory realization is known. There is no argument we are aware of which rules out these representations in F-theory; indeed it seems likely that some of these representations can be realized by new codimension-two singular structures, some which are suggested in [112]. Note that further representations can appear when multiple blocks are considered, so this list is not a complete list of all possible matter types for $T = 0$ models.

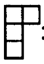
As review in section 3.1.1, matter representations with standard F-theory constructions are the fundamental ($\square$), 2-antisymmetric ($A2 = \begin{smallmatrix}\square\\\square\end{smallmatrix}$), and adjoint representations [119]. The 2-symmetric ($S2 = \square\square$) was identified in terms of a double point singularity in F-theory in [97] and the local singularity structure associated with 3-antisymmetric representations ($\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$) have also been identified in F-theory for $SU(6)$ [135, 136, 134], $SU(7)$ [136, 134], and $SU(8)$ [134].

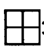
The novel matter representations which can appear in a model satisfying the Kodaira constraint, where the gauge group has a single nonabelian factor $SU(M)$ are as follows

: Appears for $SU(M)$, $M = 9, 8, 7, 6$.

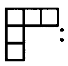
: Appears for $SU(8)$ as in the single block model (4.10)

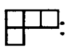
: Appears for $SU(M)$, $M = 5, 4$ (Adjoint for $SU(3)$).

: Appears for $SU(5)$ (Adjoint for $SU(4)$).

: Appears for $SU(4)$.

: Appears for $SU(M)$, $M = 4, 3, 2$.

: Appears for $SU(4)$.

: Appears for $SU(3)$.

: Appears for $SU(2)$.

: Appears for $SU(2)$.

4.1.2 Two-factor combinations

In principle, given the complete list of all possible single blocks one can construct all multi-block models satisfying the gravitational anomaly bound by simply considering all possible ways in which matter can be multiply charged between blocks in a fashion compatible with equation (2.55):

$$b_\kappa b_\lambda = \sum_{R,S} x_{RS} A_R A_S \,. \tag{4.22}$$

Since the number of jointly charged hypermultiplets grows quickly as the number of blocks increases, the ways of combining multiple blocks are actually quite constrained. We have used the complete analysis of single blocks to construct in this fashion all possible two-block models with gauge group $SU(M) \times SU(P)$ for $4 \leq M \leq P$. We present here some examples of the features which can appear in such two-block models.

From the cross-term anomaly constraint (2.55), it follows that any pair of blocks must share matter which transforms under each gauge group factor, satisfying the summation relation

$$b_\kappa b_\lambda = \sum_{RS} x_{RS} A_R A_S \,, \tag{4.23}$$

where R and S are representations in $SU(M_\kappa)$ and $SU(M_\lambda)$ respectively. The simplest type of matter charged under two gauge group factors is bifundamental matter, familiar from various string constructions. In this case $A_R = A_S = 1$. There is a simple family of two-block models with matter content of the form

$$\begin{aligned} G &= SU(M) \times SU(24-M) \\ b_1 = b_2 &= 1 \\ \text{matter} &= 3\left(\begin{smallmatrix}\square\\\square\end{smallmatrix} \times \cdot\right) + 3\left(\cdot \times \begin{smallmatrix}\square\\\square\end{smallmatrix}\right) + 1(\square \times \square)\,. \end{aligned} \tag{4.24}$$

Another family of models takes the form

$$\begin{aligned} G &= SU(M) \times SU(12-M) \\ b_1 = b_2 &= 2 \\ \text{matter} &= 6\left(\begin{smallmatrix}\square\\\square\end{smallmatrix} \times \cdot\right) + 6\left(\cdot \times \begin{smallmatrix}\square\\\square\end{smallmatrix}\right) + 4(\square \times \square) \end{aligned} \tag{4.25}$$

for $N \leq 12$. The family of models (4.25), including the single block model with $b = 2, M = 12$ were previously constructed by Schellekens using Gepner models [188].

There are a variety of other two-block combinations possible with bifundamental matter and higher values of b's. When we consider larger values of b_κ, b_λ, more interesting combinations can also arise. There are some models which contain representations of the form $\begin{smallmatrix}\square\\\square\end{smallmatrix} \times \square$. For example, the two-block model with largest $M \leq P$ with such a representation has gauge group and matter content

$$\begin{aligned} G &= SU(5) \times SU(7) \\ (b_1, b_2) &= (4, 2) \\ \text{matter} &= 2(\square \times \cdot) + 1\left(\begin{smallmatrix}\square\\\square\end{smallmatrix} \times \cdot\right) + 3(\square\!\square \times \cdot) + 2\left(\cdot \times \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}\right) + 2\left(\begin{smallmatrix}\square\\\square\end{smallmatrix} \times \square\right) + 2(\square \times \square) \\ H - V &= 273\,. \end{aligned} \tag{4.26}$$

There is a similar model with gauge group $SU(5) \times SU(6)$, but with $SU(5)$ adjoints instead of symmetric representations.

$$
\begin{aligned}
G &= SU(5) \times SU(6) \\
(b_1, b_2) &= (4,2) \\
\text{matter} &= 4(\ydiagram{1} \times \cdot) + 3(\text{Adj} \times \cdot) + 3(\cdot \times \ydiagram{1,1,1}) + 2(\ydiagram{1,1} \times \ydiagram{1}) + 2(\ydiagram{1} \times \ydiagram{1}) \\
H - V &= 273\,.
\end{aligned}
\tag{4.27}
$$

These models both saturate the gravitational anomaly, have similar representation content, and satisfy the Kodaira constraint.

As the rank of the gauge group factors drops, more exotic matter multiplets charged under two factors appear. For example, for $SU(4) \times SU(4)$ there are models containing matter which transforms in a non-trivial non-fundamental representation of two gauge groups. One example is given by the model

$$
\begin{aligned}
G &= SU(4) \times SU(4) \\
(b_1, b_2) &= (2,2) \\
\text{matter} &= 32(\ydiagram{1} \times \cdot) + 32(\cdot \times \ydiagram{1}) + 1(\ydiagram{1,1} \times \ydiagram{1,1}) \\
H - V &= 262\,.
\end{aligned}
\tag{4.28}
$$

Another interesting class of models are those which contain two blocks $SU(M) \times SU(P)$ for large P and small M. For example we find the following three models

$$
\begin{aligned}
G &= SU(2) \times SU(24) \\
(b_1, b_2) &= (88,1) \\
\text{matter} &= 1(\ydiagram{8} \times \cdot) + 1(\ydiagram{12} \times \cdot) + 1(\ydiagram{2} \times \ydiagram{1,1}) \\
H - V &= 272\,.
\end{aligned}
\tag{4.29}
$$

$$
\begin{aligned}
G &= SU(3) \times SU(24) \\
(b_1, b_2) &= (22, 1) \\
\text{matter} &= 1(\Box\!\Box\!\Box\!\Box\!\Box\!\Box \times \cdot) + 1(\Box \times \begin{smallmatrix}\Box\\\Box\end{smallmatrix}) \\
H - V &= 273\,.
\end{aligned}
\tag{4.30}
$$

$$
\begin{aligned}
G &= SU(2) \times SU(19) \\
(b_1, b_2) &= (27, 1) \\
\text{matter} &= 1(\Box \times \cdot) + 2(\Box\!\Box \times \cdot) + 1(\Box\!\Box\!\Box \times \cdot) + 1(\Box\!\Box\!\Box\!\Box \times \cdot) \\
&\quad +1(\Box\!\Box\!\Box\!\Box\!\Box \times \cdot) + 1(\Box\!\Box\!\Box\!\Box\!\Box\!\Box \times \cdot) \\
&\quad +1(\Box \times \begin{smallmatrix}\Box\\\Box\end{smallmatrix}) + 1(\Box\!\Box\!\Box \times \Box) \\
H - V &= 273\,.
\end{aligned}
\tag{4.31}
$$

These are the only multiblock models with a gauge group larger than $SU(18)$ that has non-bifundamental jointly charged matter. These models all severely violate the Kodaira bound. It is perhaps interesting to note that models containing $SU(M)$ factors with $M = 20, 21, 22, 23$ cannot have jointly charged matter other than bifundamental matter as in the family of models (4.24)

4.1.3 Matter transforming under more than two factors

We have also considered models containing more than two blocks which when taken together satisfy the gravitational anomaly bound, and which contain matter charged under more than two gauge group factors. A limited class of such multiply-charged matter representations are known to appear in F-theory constructions. In particular tri-fundamental representations of $SU(2) \times SU(2) \times SU(M)$ can arise at a point where the singularity structure is enhanced to D_{M+2} [136], and tri-fundamentals of $SU(2) \times SU(3) \times SU(M)$ can be realized from E_{M+3} singularities for $M \leq 5$. In [65] apparently consistent low-energy models with $T = 1$ containing tri-fundamental matter charged under the three gauge group factors $SU(2) \times SU(3) \times SU(6)$ were identified. While we have not done a completely systematic search, we have identified a number of the interesting matter structures of this type which can arise in $T = 0$ models. We list here some of the possibilities. While this list is not necessarily comprehensive, it should serve to demonstrate the kinds of multiply-charged matter representations which may be possible.

3-charged matter

As for $T = 1$, at $T = 0$ we find tri-fundamental matter charged under $SU(2) \times SU(3) \times SU(6)$. Such matter appears in the following 3-block model

$$
\begin{aligned}
G &= SU(2) \times SU(3) \times SU(6) \\
(b_1, b_2, b_3) &= (3, 2, 1) \\
\text{matter} &= 1(\square \times \square \times \square) + 36(\square \times \cdot \times \cdot) + 30(\cdot \times \square \times \cdot) + 12(\cdot \times \cdot \times \square) \\
&\quad + 1(\square\!\square \times \cdot \times \cdot) + 3(\cdot \times \cdot \times \begin{smallmatrix}\square\\ \square\end{smallmatrix}) \\
H - V &= 272\,.
\end{aligned} \tag{4.32}
$$

Matter charged under $SU(2) \times SU(4) \times SU(4)$ appears in the model

$$
\begin{aligned}
G &= SU(2) \times SU(4) \times SU(4) \\
(b_1, b_2, b_3) &= (2, 4, 4) \\
\text{matter} &= 2(\square \times \square \times \square) + 4(\cdot \times \square \times \square) + 2(\cdot \times \begin{smallmatrix}\square\\ \square\end{smallmatrix} \times \begin{smallmatrix}\square\\ \square\end{smallmatrix}) \\
&\quad + 8(\square \times \cdot \times \cdot) + 3(\cdot \times \text{Adj} \times \cdot) + 3(\cdot \times \cdot \times \text{Adj}) \\
H - V &= 273\,.
\end{aligned} \tag{4.33}
$$

There is also matter charged under $SU(3) \times SU(3) \times SU(3)$, appearing in the model

$$
\begin{aligned}
G &= SU(3) \times SU(3) \times SU(3) \qquad (4.34) \\
(b_1, b_2, b_3) &= (2, 2, 2) \\
\text{matter} &= 1(\square \times \square \times \square) + 1\,[(\square \times \square \times \cdot) + \text{cyclic}] + 27\,[(\square \times \cdot \times \cdot) + \text{cyclic}] \\
H - V &= 273\,.
\end{aligned}
$$

Both these models containing tri-fundamental matter satisfy the Kodaira constraint. There is also an interesting combination of 3 blocks of the form (4.17) which contains matter charged under $SU(4) \times SU(4) \times SU(4)$.

$$
\begin{aligned}
G &= SU(4) \times SU(4) \times SU(4) \qquad (4.35) \\
(b_1, b_2, b_3) &= (4, 4, 4) \\
\text{matter} &= 4(\square \times \square \times \square) + 1\,\left[(\ydiagram{2,2} \times \cdot \times \cdot) + \text{cyclic}\right] \\
H - V &= 271\,.
\end{aligned}
$$

It is possible to combine four SU(3) blocks to have multiple tri-fundamentals between groups of 3 of the SU(3)'s

$$
\begin{aligned}
G &= SU(3) \times SU(3) \times SU(3) \times SU(3) \qquad (4.36) \\
(b_1, b_2, b_3, b_4) &= (3, 3, 3, 3) \\
\text{matter} &= 1\,[(\square \times \square \times \square \times \cdot) + \text{cyclic}] + 3\,[(\square \times \square \times \cdot \times \cdot) + 5 \text{ permutations}] \\
&\quad +1\,\left[(\ydiagram{2,1} \times \cdot \times \cdot \times \cdot) + \text{cyclic}\right] \\
H - V &= 270\,.
\end{aligned}
$$

Matter charged under more than three factors

We have found a few exotic models in which matter can be charged under more than 3 gauge group factors.

There is a combination of 4 SU(2) factors carrying a 4-fundamental in a model which satisfies the Kodaira constraint

$$\begin{aligned} G &= SU(2)^4 \\ b_i &= 4 \\ \text{matter} &= 2(\square\times\square\times\square\times\square) + 8\,[(\square\times\square\times\cdot\times\cdot) + 5 \text{ permutations}] \\ &\quad +3\,[(\square\!\square\times\cdot\times\cdot\times\cdot) + \text{cyclic}] \\ H-V &= 248\,. \end{aligned} \tag{4.37}$$

And there is a more exotic combination of 8 SU(2) factors at $b = 8$ where each block has 128 fundamental representations and one S4 (5-dimensional) representation

$$\begin{aligned} G &= SU(2)^8 \\ b_i &= 8 \\ \text{matter} &= 1(\square\times\square\times\square\times\square\times\square\times\square\times\square\times\square) \\ &\quad +1\,[(\square\!\square\!\square\!\square\times\cdot\times\cdot\times\cdot\times\cdot\times\cdot\times\cdot\times\cdot) + \text{cyclic}] \\ H-V &= 272\,. \end{aligned} \tag{4.38}$$

4.1.4 Summary

Let us summarize the results from the landscape analysis on the space of $T = 0$ theories:

Candidates for new string vacua:

We have identified a number of $SU(M)$ matter representations which are not ruled out by low-energy consistency conditions, but whose realization in string theory is not yet known. Some of the novel matter representations we have found are compatible with the Kodaira constraint, and may be realized by new codimension-two singularities in F-theory.

Theories Violating the Kodaira bound:

All F-theory — and to our knowledge, string theory — realizations of $T = 0$ 6D theories satisfy the Kodaira constraint, as elaborated in section 3.1. There are many — but a finite number of — models which satisfy the anomaly cancellation condition on the low-energy theory but which violate the Kodaira constraint. We do not have a framework to think about these theories at the present. Whether the Kodaira constraint can be derived from a yet unknown fundamental principle of quantum gravity, or whether it is an artifact of the constraints on the landscape of known string models is an important question that this investigation presents. We investigate this question further for the rest of the thesis.

Models with rigid symmetry:

Many of the most unusual matter representations we have found live in models which either completely or almost completely saturate the gravitational anomaly bound $H - V \leq 273$. When this bound is saturated, there are no uncharged hypermultiplets, and any deformation of the model will break the symmetry and reduce the matter content. Thus, these models are delicately balanced configurations which exist only at specific points in the moduli space of 6D supergravity theories. Many of the explicit models we have found which go outside the domain of established F-theory constructions turn out to precisely saturate the gravitational anomaly bound and exhibit remarkable numerological/group theory structure, suggesting that perhaps some novel stringy mechanism may enable the existence of these theories as quantum-complete theories of supergravity.

Diversity at low rank:

When the rank of a gauge group factor is large, in general we find that the associated models contain matter associated with well-known F-theory singularity types and clearly satisfy the Kodaira constraint. As the rank of the factors decreases, however, more exotic types of matter appear and more models arise which violate the Kodaira constraint. Models containing only $SU(2)$ factors become difficult to classify, and admit a wide range of representations. This observation matches with the results of section 2.4, where it is shown that an infinite number of charge solutions exist for $U(1)$ models.

4.2 Theories with Abelian Gauge Symmetry

As we have seen in sections 2.4, 3.2 and pointed out in numerous places in the thesis, the space of six-dimensional theories with abelian gauge symmetry is not as well controlled as the space of non-abelian theories, to say the least. There are infinite classes of apparently consistent theories for any T, and the abelian sector of the string landscape is not well understood. For example, how many different "kinds" of Mordell-Weil sections — which would lead to different $U(1)$ theories — an elliptically fibered Calabi-Yau manifold could have is not well known.

The tables 4.1 and 4.2, however, gives us reason to hope that there may be, at least on the string side, a generalized version of the Kodaira constraint that truncates the infinite families of theories that exist in abelian theories. These tables indicate that there being an infinite number of apparently consistent charge assignments to an abelian theory fits the general trend of theories with less structure having a wider variety of allowed matter. If there indeed exists a generalized version of the Kodaira constraint, it would truncate the infinite classes of theories along with the charge integrality, minimality and unimodularity constraints [124, 128, 129, 130]. For example, let us consider the first infinite class of theories considered in section 2.4.3. In these $T = 1$ theories with gauge group $U(1)$, there are 48 hypermultiplets with charge r, 48 hypermultiplets with charge s, 48 hypermultiplets with charge $(r + s)$, and 102 neutral hypermultiplets. The anomaly coefficient is given by

$$(\alpha, \tilde{\alpha}) = (8r^2 + 8rs + 8s^2, 8r^2 + 8rs + 8s^2)\,, \tag{4.39}$$

while the gravitational anomaly coefficient is given by

$$a = (-2, -2)\,. \tag{4.40}$$

It is clear that if the charge is quantized so that r and s are integral and mutually prime, then α and $\tilde{\alpha}$ would be unbounded in magnitude. If there exists some condition that imposes a bound on α and $\tilde{\alpha}$, it would truncate this infinite family.

A bound on the magnitude of abelian anomaly coefficients have strong implications for $T < 9$, since for such values of T, the infiniteness of the space of apparently consistent abelian theories lie entirely in the infinite choice charge assignments. Any infinite choice of charge assignments for a given gauge/matter representation structure would lead to one of the abelian anomaly coefficients to diverge, given that charges and anomaly coefficients are quantized. Therefore if a generalized version of the Kodaira constraint — which bounds the magnitude of abelian anomaly coefficients — exists for string vacua, it will pick out a finite subset of the infinite set of apparently consistent abelian models with $T < 9$.

A generalized version of the Kodaira constraint is expected to truncate the infinite classes of $T \geq 9$ theories also. In the case of non-abelian theories, the Kodaira constraint actually truncates all the infinite classes of $T \geq 9$ theories constructed in section 2.2. For example, let us consider the first family with arbitrary gauge group $\mathcal{G} = \prod_\kappa \mathcal{G}_\kappa$ with anomaly coefficients

$$\begin{aligned} a &= (-3, 1 \times T) \\ b_\kappa &= (3, (-1) \times 9, 0 \times (T-9)) \end{aligned} \tag{4.41}$$

for all κ and modulus

$$j = (1, 0 \times T)\,. \tag{4.42}$$

As before, we use the notation $x \times n$ to mean that n consecutive entries are equal to

x. Then the Kodaira constraint becomes

$$\sum c_\kappa (b_\kappa \cdot j) = \sum 3c_\kappa \leq 36 \tag{4.43}$$

which imposes that

$$\sum c_\kappa \leq 12 \,. \tag{4.44}$$

This constrains the gauge group and truncates the infinite family to a finite subset. It can be checked that the Kodaira constraint truncates the other two infinite families constructed in section 2.2.

These observations suggest that a generalized version of the Kodaira constraint should exist at least for F-theory vacua, since F-theory can only allow a finite number of possible $U(1)$ charges for any class of theories. The result of section 3.3, in which the geometric counterpart of abelian anomaly coefficients are identified for F-theory vacua, is a first step in the direction of identifying such a generalized constraint for the abelian sector. We have so far not been able to go further towards achieving this goal. We, however, can use the geometric characterization of the abelian sector given in section 3.3 to translate the gauge/graviational and mixed anomaly equations into non-trivial geometric identities. We present the results in the following section.

Let us end this section by examining an interesting class of abelian F-theory backgrounds that might give us further insight into charge constraints of F-theory vacua upon further investigation. These are the pure abelian $T = 0$ theories we have introduced in section 2.4. Recall that a family of $T = 0$ theories with gauge group $U(1)^k, k \leq 7$ can be obtained by Higgsing an $SU(8)$ theory with one adjoint and nine antisymmetric matter. The number of charged hypermultiplets X for the various pure abelian theories one obtains by Higgsing the adjoint of this theory in different ways is summarized in table 4.3. There exists an F-theory construction of this $SU(8)$ model, which has $b = 3$, through an explicit Weierstrass model in [112], and hence all the seven theories of this family should be embeddable in F-theory, although we have not worked out the details of this Higgsing.

Explicit F-theory compactifications are known for the first four theories on table

Gauge Group	$\cdot$	$U(1)$	$U(1)^2$	$U(1)^3$	$U(1)^4$	$U(1)^5$	$U(1)^6$	$U(1)^7$
X	0	108	162	198	225	243	252	252
$324\, V_A/(V_A+2)$	0	108	162	194.4...	216	231.4...	243	252
X'	273	166	113	78	52	35	27	28

Table 4.3: The number of charged hypermultiplets X for pure abelian theories obtained by Higgsing the adjoint of the $SU(8)$ theory with one adjoint and nine antisymmetrics. We have also tabulated the number of uncharged hypermultiplets in the theory, $X' = (273 + V_A - X)$.

4.3. Since these theories have $T = 0$, they can be obtained by F-theory compactifications on Calabi-Yau threefolds that are elliptic fibrations of $\mathbb{P}^2$ [118, 119]. Such a Calabi-Yau threefold that is non-singular can be expressed as a degree 18 hypersurface in the projective space $\mathbb{P}[1,1,1,6,9]$ [189], which is denoted by

$$X_{18}[1,1,1,6,9]^{2,272}. \tag{4.45}$$

The subscript denotes the degree of the hypersurface, the number in the brackets parametrize the projective space, and the two superscripts denote the $h_{1,1}$ and $h_{2,1}$ values of the manifold. Using M-theory/F-theory duality explained in detail in section 3.3, one can see that when $T = 0$ the total rank of the gauge group is given by $(h_{1,1}-2)$ and the number of uncharged hypermultiplet is given by $(h_{2,1}+1)$. It is easy to check that the data of this manifold reproduces the first theory in table 4.3.

There is a general process by which one can replace the fiber-type of an elliptically fibered manifold to generate a different manifold [189]. From the point of view of stringy geometry, one can understand this as a conifold transition between topologically distinct manifolds [190]. Three manifolds can be generated from $X_{18}[1,1,1,6,9]^{2,272}$ by successive conifold transitions. They are given by

$$X_{12}[1,1,1,3,6]^{3,165}, \qquad X_9[1,1,1,3,3]^{4,112}, \qquad X_{6,6}[1,1,1,3,3,3]^{5,77}. \tag{4.46}$$

At a generic point in the complex moduli space, theories obtained by compactifying on these manifolds do not have nonabelian gauge symmetry. Comparing the numbers with table 4.3, we find that the massless spectrum of the six-dimensional theories obtained by F-theory compactifications on the three manifolds of (4.46) coincides with the massless spectrum of the second, third and fourth theories of table 4.3 with gauge groups $U(1)^k, k = 1,2,3$. We do not know how to continue this process to construct an explicit geometry realizing a theory with gauge group $U(1)^4$.

This simple set of theories seems to be an ideal place to start examining aspects of abelian gauge symmetry for F-theory vacua. Hopefully insight gained from studying these simple examples will lead to a better understanding of how charges of abelian gauge symmetry are constrained in F-theory.

4.3 Intersection Theory

Due to the identifications made in the section 3.3, the mixed/gauge anomaly equations can be reformulated into equalities between intersection numbers obtained in the resolved Calabi-Yau threefold. Remarkably, they can be summarized in two equalities. They are given by the following:

$$\begin{aligned}&\pi(\mathcal{S}_1\cdot\mathcal{S}_2)\cdot\pi(\mathcal{S}_3\cdot\mathcal{S}_4)+\pi(\mathcal{S}_1\cdot\mathcal{S}_3)\cdot\pi(\mathcal{S}_2\cdot\mathcal{S}_4)+\pi(\mathcal{S}_1\cdot\mathcal{S}_4)\cdot\pi(\mathcal{S}_2\cdot\mathcal{S}_3)\\&=\sum_r(c_r\cdot\mathcal{S}_1)(c_r\cdot\mathcal{S}_2)(c_r\cdot\mathcal{S}_3)(c_r\cdot\mathcal{S}_4)+\sum_\rho(2g_\rho-2)(\chi_\rho\cdot\mathcal{S}_1)(\chi_\rho\cdot\mathcal{S}_2)(\chi_\rho\cdot\mathcal{S}_3)(\chi_\rho\cdot\mathcal{S}_4)\end{aligned} \tag{4.47}$$

and

$$6K\cdot\pi(\mathcal{S}_1\cdot\mathcal{S}_2)=\sum_r(c_r\cdot\mathcal{S}_1)(c_r\cdot\mathcal{S}_2)+\sum_\rho(2g_\rho-2)(\chi_\rho\cdot\mathcal{S}_1)(\chi_\rho\cdot\mathcal{S}_2) \tag{4.48}$$

when

$$f\cdot\mathcal{S}_n=0\,. \tag{4.49}$$

As in the previous chapter, c_r denote the isolated (type I) rational curves, while χ_ρ denote the fibered (type F) rational curves. Recall that by definition c_r and χ_ρ are curves that shrink to zero area in the fibration limit. g_ρ denotes the genus of the curve over which rational curve χ_ρ is fibered. We have used K to denote the canonical class divisor of the base.

π is the projection to the base manifold. More precisely, $\pi(C)$ of some two-cycle C in $\hat{X}$ is the projection of C to the $H_2(\mathcal{B})$ lattice of the base manifold. The intersection between projected curves are taken in the base, while all the other intersections are taken inside the full manifold. Recall that for any two-cycle C in $\hat{X}$

$$\pi(C) = (C \cdot B^\alpha) H_\alpha \quad \Leftrightarrow \quad B_\alpha \cdot C = (H_\alpha \cdot \pi(C))_{\mathcal{B}} = \pi(C)_\alpha \,, \tag{4.50}$$

for the basis elements H_α of $H_2(\mathcal{B})$. Recall that B_α are type B four-cycles obtained by fibering the elliptic fiber over H_α.

As seen in section 3.3, any four-cycle that does not intersect the fiber is a linear combination of four-cycles of type B, S, or C. One can easily check that to prove equations (4.47) and (4.48) for any four-cycle with zero intersection with the fiber, it is enough to prove them in the case when all $\mathcal{S}_n$ are among the basis elements $\{B_\alpha, T_{I,\kappa}, S_i\}$. We can carry out this procedure in the following steps.

1. We first show that these equations trivially hold when one of $\mathcal{S}_n$ is of type B.

2. We then show that these equations hold when all four four-cycles $\mathcal{S}_n$ are of type S or of type C.

3. Finally we show the validity of the equations when there are both four-cycles of type S and C among $\mathcal{S}_n$ thereby concluding the proof of these equations.

The details of these steps are unilluminating, but the basic idea is simple. For the rest of the section, we carry out step 1 explicitly and sketch the idea behind showing steps 2 and 3. We have carried out steps 2 and 3 explicitly in appendix B.2.

Let us prove equations (4.47) and (4.48) in the case that one of the four cycles is of type B. Without loss of generality, let $\mathcal{S}_1 = B_\alpha$. All shrinking two-cycles do not intersect $\mathcal{S}_1$. Therefore the right-hand sides of both equations are 0. Meanwhile, for any S such that $S \cdot f = 0$

$$B_\alpha \cdot B_\beta \cdot S = \Omega_{\alpha\beta} f \cdot S = 0 \tag{4.51}$$

for all B_β and therefore

$$\pi(B_\alpha \cdot S) = (B_\beta \cdot B_\alpha \cdot S) H^\beta = 0 \,. \tag{4.52}$$

Therefore $\pi(\mathcal{S}_n \cdot \mathcal{S}_1) = 0$ for $n = 2, 3, 4$ and hence the left-hand sides of the two equations are also zero.

When all $\mathcal{S}_n$ are either of type C or S, the equations (4.47) and (4.48) become more interesting. In this case, the gauge anomaly equations (2.26) lead to (4.47) and the mixed anomaly equations (2.23) lead to (4.48). As can be seen in section 3.3, each gauge field A_x in the Cartan subalgebra of the full gauge group is dual to a four-cycle $C_x \in \{T_{I,\kappa}, S_i\}$ in the resolved Calabi-Yau manifold $\hat{X}$. If we restrict our attention to only these gauge fields, the anomaly polynomial takes the structure of an abelian theory. In particular, the anomaly coefficients of $F_x F_y$ are given by $-\pi(C_x \cdot C_y)$ — they are bilinear forms in the x index and are vectors in the H_2 lattice of the base. Therefore, by plugging in elements of the Cartan to the gauge/mixed anomaly equations, the inner-products between anomaly coefficients on left-hand sides reproduce the intersection numbers between between various $\pi(\mathcal{S}_i \cdot \mathcal{S}_j)$ of (4.47) and (4.48).

The right-hand sides of the gauge/mixed anomaly equations (2.26)/(2.23), are given by the sum of products of the charges of "charged multiplets" under vector fields dual to $C_x \in \{T_{I,\kappa}, S_i\}$. The charged multiplets come from quantizing zero-modes of the M2 branes and anti-branes wrapping type I or type F curves. A type I curve c_r contributes one hypermultiplet with charge $c_r \cdot C_x$, while a type F curve χ_ρ contributes $2g_\rho$ hypermultiplets of charge $\chi_\rho \cdot C_x$ and two vector multiplets each with charge $\pm\chi_\rho \cdot C_x$ under the vector field dual to C_x [144, 145]. Therefore the right-hand sides of equations (4.47)/(4.48) are reproduced by plugging in elements of the Cartan to the right-hand sides of the gauge/mixed anomaly equations. This concludes the proof of the two equations.

Together with the gravitational anomaly equation (3.49), the equations (4.47) and (4.48) can be rewritten as

$$3\pi(\mathcal{S}\cdot\mathcal{S})\cdot\pi(\mathcal{S}\cdot\mathcal{S}) = \sum_r (c_r\cdot\mathcal{S})^4 + \sum_\rho (2g_\rho - 2)(\chi_\rho\cdot\mathcal{S})^4 \tag{4.53}$$

$$6K\cdot\pi(\mathcal{S}\cdot\mathcal{S}) = \sum_r (c_r\cdot\mathcal{S})^2 + \sum_\rho (2g_\rho - 2)(\chi_\rho\cdot\mathcal{S})^2 \tag{4.54}$$

$$30K\cdot K + \frac{1}{2}\chi_{\hat{X}} = \sum_r 1 + \sum_\rho (2g_\rho - 2) \tag{4.55}$$

for any $\mathcal{S}\cdot f = 0$. It is easy to check that these set of equations are equivalent to equations (4.47) and (4.48) and (3.49) by setting

$$\mathcal{S} = \sum_{i=1}^{4} t_i\mathcal{S}_i \quad \text{or} \quad \mathcal{S} = \sum_{i=1}^{2} t_i\mathcal{S}_i \tag{4.56}$$

and treating both sides of the equations as polynomials of t_i and comparing coefficients. Noting that the Euler characteristic of a point is 1 and that the Euler characteristic of a genus g curve is $(2-2g)$, we can finally write the equations in the appealing form:

$$3\pi(\mathcal{S}\cdot\mathcal{S})\cdot\pi(\mathcal{S}\cdot\mathcal{S}) = \sum_c \chi_M(c)(c\cdot\mathcal{S})^4 \tag{4.57}$$

$$6K\cdot\pi(\mathcal{S}\cdot\mathcal{S}) = \sum_c \chi_M(c)(c\cdot\mathcal{S})^2 \tag{4.58}$$

$$30K\cdot K + \frac{1}{2}\chi_{\hat{X}} = \sum_c \chi_M(c) \tag{4.59}$$

for any $\mathcal{S}\cdot f = 0$.

The sum of the right-hand-sides of the equations are over all — both type I and F — curves c that shrink in the fibration limit. $\chi_M(c)$ is the Euler characteristic of the moduli space of the curve in the manifold $\hat{X}$ graded by the complex dimension of the moduli space, *i.e.*,

$$\chi_M(c) = \begin{cases} 1 & \text{(Moduli space of } c \text{ is a point)} \\ 2g-2 & \text{(Moduli space of } c \text{ is a curve of genus } g) \end{cases} \tag{4.60}$$

We note that the geometric implications of the mixed and gauged anomaly equations have been studied in [133, 134] for non-abelian gauge groups, but have not been put into the form we have presented here. The implications of the third equation — coming from the gravitational anomaly constraint — has also been studied previously [132, 133, 134], although not quite in the language that we have used. An interesting fact is that the equation (4.59) can be translated into a threefold analogue [189] of the Sethi-Vafa-Witten formula [191] for elliptically fibered Calabi-Yau threefolds with various fiber types. We note that the Sethi-Vafa-Witten formula was originally derived for elliptically fibered Calabi-Yau fourfolds in Weierstrass form, and has been extended to more general elliptically fibered fourfolds [189, 192, 193]. While these equations are aesthetically pleasing, we do not yet have much insight into how much they add to what we already know about the geometry of Calabi-Yau threefolds. Understanding the origin and implications of these equations geometrically and possibly generalizing them in a meaningful way would be an interesting direction of inquiry.

Chapter 5

Conclusions and Outlook

Let us conclude this thesis by summarizing the major lessons that we have learned from examining the landscape of six-dimensional $(1,0)$ supergravity theories and exploring directions for future developments.

Candidates for new string vacua

We have identified theories with exotic matter that have a high chance of being new string — in particular, F-theory — vacua. If these are indeed verified to be new F-theory vacua, it would also have implications on four-dimensional F-theory vacua, since the codimension-two singularities that generate the exotic matter would also generate the same kind of matter for four-dimensional backgrounds. A thorough investigation of $T = 0$ global models with exotic matter has been initiated in [112].

The Kodaira bound

We have found that many — in fact, most — apparently consistent non-abelian theories violate the Kodaira bound (3.3), which all string models known to us satisfy. The Kodaira bound essentially states that the weighted sum of the inverse gauge couplings of the non-abelian gauge groups is bounded above by the coupling of a higher curvature term. The question whether this bound comes from a fundamental principle of quantum gravity, or whether it comes from artificial constraints particular to the known string vacua must be answered to gain a full understanding of the six-dimensional supergravity landscape.

One hope is that the Kodaira bound is indeed a fundamental constraint of quantum supergravity theory that follows from basic properties such as unitarity or causality [194]. Indeed, constraints on higher curvature terms in gravity have been imposed by demanding such basic properties, especially in the context of AdS/CFT [195, 196, 197, 198]. There also is an intuition that probe solitons contain non-trivial information of the theory [89, 199, 200, 201] and can be used to detect hidden pathologies of supergravity theories [202]. Putting these results together, one may speculate that conformal field theories dual to the near horizon geometry of BPS solitons of six-dimensional supergravity theories might exhibit pathologies when the Kodaira bound is violated. Six-dimensional supergravity theories with a non-abelian gauge group have extremal dyonic strings [203, 204]. One might be able to find additional constraints on the gravity theory by imposing the consistency of the dual CFT living on these strings.

Existence of a generalized Kodaira bound

We have seen that the space of apparently consistent abelian theories is not well controlled due to the infinite number of apparently consistent charge assignments possible to a given theory. The space of known string vacua is also less well-characterized due to the lack of a generalized version of the Kodaira bound that involves the abelian anomaly coefficients. A major result we have presented is that the infiniteness of the space of apparently consistent theories with abelian gauge symmetry lies solely in the infinite number of charge assignments a given gauge/matter structure could have. Therefore a subspace satisfying a generalized version of the Kodaira constraint would be finite given that we demand that the abelian charges are quantized.

This strongly motivates the search for a generalized version of the Kodaira constraint that applies to theories with abelian gauge symmetry. If the Kodaira constraint could be derived from low-energy methods mentioned above, one would guess that the generalized Kodaira constraint could be obtained by a slight generalization of those methods. From the string theory/F-theory point of view, however, deriving a generalized Kodaira constraint seems to require qualitatively different methods. The hope is that the intersection theory based techniques we have used in this thesis to analyze the abelian sector could be used to derive a generalized Kodaira constraint for F-theory vacua. Whether this could be achieved remains to be seen. One thing for certain is that a thorough study of the abelian sector of F-theory itself should be carried out in order to understand the charge constraints of abelian gauge fields.

The intersection equations

We were able to derive non-trivial geometric equalities that hold for elliptically fibered Calabi-Yau threefolds by using gravitational/gauge and mixed anomaly equations of the six-dimensional theory obtained by compactifying F-theory on it. The mathematical origin of these equations is unclear. An interesting question is whether such equations could be derived by geometric means. Another is whether these equations could be generalized in a meaningful way. Answering these questions could hopefully lead to a deeper understanding of the geometry of Calabi-Yau threefolds.

An intriguing observation that we can make is that the formulae strongly resemble the instanton-corrected triple intersections introduced in [205]. In mirror symmetry, these instanton sums are taken in the large Kähler structure limit. In our case, however, the mixed anomaly equations seem to be related to an instanton-corrected triple intersection form in a singular limit. We do not understand this connection fully at this moment.

Understanding the interaction between anomaly constraints and consistency conditions that the geometry and various fluxes of four-dimensional F-theory constructions must satisfy is expected to be more involved. This is because four dimensional F-theory backgrounds have much richer structure than six-dimensional backgrounds.[1] There has, however, been beautiful work [165] in which constraints on "hypercharge fluxes" on F-theory $SU(5)$ GUT models with $U(1)$ symmetries — referred to as

[1]An incomplete list of references on the structure of four-dimensional F-theory backgrounds is given in the bibliography [163, 164, 167, 174, 175, 206, 207, 208, 209, 210, 211, 212, 213, 214]. A nice review of this subject and further references can be found in [215].

"generalized Dudas-Palti relations" [216] — are derived by four-dimensional anomaly cancellation conditions. The generalized Dudas-Palti relations provide a good handle on F-theory GUT models with $U(1)$ symmetries [165, 166]. It would be interesting to expand the anomaly analysis to more general F-theory constructions and see if one could understand the constraints that anomaly cancellation imposes upon the various building-blocks of four-dimensional F-theory models in the language of intersection theory.

Some final words

The space of six-dimensional supergravity theories with minimal supersymmetry has provided us with surprising insights into quantum gravity and string theory. There is a long way to go, however, to achieve an understanding of this space of theories at the level of, say, ten-dimensional theories. If the Kodaira constraint and its abelian generalization could indeed be shown to be fundamental constraints of quantum gravity, it would be a major breakthrough in proving the string universality conjecture in six-dimensions [63]. According to the current picture of the six-dimensional landscape we have, this seems to be the best-case scenario among the many imaginable options. Attaining string universality would be, to say the least, quite an important result — it would provide strong evidence toward an affirmative answer to the vague question we have posed at the beginning of this thesis:

(Q') Are all consistent quantum gravity theories, string theories?

History shows, however, that one could never be sure as nature has many surprises hidden up her sleeves. One could only hope that she or he is ready enough to discover those surprises which are sure to be pleasant and beautiful in their unexpected way.

Appendix A

Appendices for Chapter 2

A.1 Some Lie Algebra

In this section, we review some relevant Lie algebra to understand the group factors λ_κ introduced in table 2.2 of section 2.1.3, and their relation to the normalized coroot matrix $\mathcal{C}_{IJ}$. Almost all of what is discussed in this section can be found in standard texts such as [217, 218].

For a given Lie group $\mathcal{G}$ and its Lie algebra $\mathfrak{g}$, let us define the generators of Cartan sub-algebra $\{T_i\}$. Let us normalize the Cartan generators so that

$$\mathrm{tr} T_i T_j = \delta_{ij} \tag{A.1}$$

where the trace is taken in the fundamental representation. We can diagonalize all the other generators of the Lie group with respect to $\{T_i\}$. Each such generator is uniquely labelled by its eigenvalue under $\{T_i\}$, *i.e.*,

$$[T_i, E_\alpha] = \alpha_i E_\alpha \,. \tag{A.2}$$

In other words, there is a one-to-one correspondence between the vectors α and the generators of the Lie group. These vectors α are called the roots of the Lie algebra.

Notice that α will scale with a change of normalization in T_i. Since we have normalized the Cartan generators in an unambiguous way with respect to the definition of $\mathcal{G}$, the normalization of α are also fixed. This is because the weights β_s of the fundamental representation of $\mathcal{G}$ must satisfy

$$\sum_s \beta_{s,i}^2 = 1 \tag{A.3}$$

for each i, where $\beta_{s,i}$ is the i coordinate value of β_s. This condition fixes the normalization of the weight lattice. In this sense, we can say that α are the roots of the Lie group $\mathcal{G}$, with a slight abuse of terminology.

Now let us determine $\lambda(\mathcal{G})$ with respect to these vectors. Recall that $\lambda(\mathcal{G})$ is a normalization factor fixed by demanding that the smallest topological charge of an embedded $SU(2)$ instanton is 1. This definition can be rephrased in the following way.

For any given Lie group $\mathcal{G}$, we may find an $SU(2)$ subgroup. Hence we may always find an $SU(2)$ sub-algebra $\mathfrak{s}$ generated by a subgroup of the generators of the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$, *i.e.,* there exist elements S_i, $i = 1,2,3$ of the Lie algebra that satisfy

$$[S_i, S_j] = i\epsilon_{ijk} S_k \,. \tag{A.4}$$

From this relation, one can deduce that

$$2\mathrm{tr} S_1^2 = 2\mathrm{tr} S_2^2 = 2\mathrm{tr} S_3^2 \tag{A.5}$$

in any representation. Let us call this value $l(\mathfrak{s})$ where the trace is taken in the fundamental representation. The normalization of the S_i are fixed; if we multiply them by a factor, the defining commutation relation does not hold anymore. Therefore, for all the $SU(2)$ sub-algebras $\mathfrak{s}$ of Lie algebra $\mathfrak{g}$, the $l(\mathfrak{s})$ is a well-defined number. We define $\lambda(\mathcal{G})$ to be,

$$\lambda(\mathcal{G}) = \min_{\{\mathfrak{s}\}} l(\mathfrak{s}) \tag{A.6}$$

where $\{\mathfrak{s}\}$ are all the $SU(2)$ sub-algebras of $\mathfrak{g}$. For example, in $SU(2)$ the generators that satisfy the $SU(2)$ sub-algebra — in the fundamental representation — are given by $S_i = \frac{1}{2}\sigma_i$ where σ_i are the Pauli matrices. It is clear that $2\mathrm{tr}S_1^2 = 1$. For $SO(4)$, the generators that satisfy the $SU(2)$ sub-algebra with minimum $l(\mathfrak{s})$ are,

$$S_1 = \begin{pmatrix} 0 & i/2 & 0 & 0 \\ -i/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i/2 \\ 0 & 0 & i/2 & 0 \end{pmatrix}, \; S_2 = \begin{pmatrix} 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \end{pmatrix}, \tag{A.7}$$

and

$$S_3 = \begin{pmatrix} 0 & 0 & 0 & i/2 \\ 0 & 0 & -i/2 & 0 \\ 0 & -i/2 & 0 & 0 \\ i/2 & 0 & 0 & 0 \end{pmatrix}. \tag{A.8}$$

In this case, $l(\mathfrak{s}) = 2\mathrm{tr}S_1^2 = 2$.

Now it can be shown that for any root α

$$[\frac{\alpha \cdot T}{\langle\alpha, \alpha\rangle}, E_\alpha] = E_\alpha, \quad [\frac{\alpha \cdot T}{\langle\alpha, \alpha\rangle}, E_{-\alpha}] = -E_\alpha, \quad [E_\alpha, E_{-\alpha}] \propto \alpha \cdot T, \tag{A.9}$$

where we have defined

$$\alpha \cdot T \equiv \alpha_i T_i \,. \tag{A.10}$$

We may use the freedom to rescale E_α so that the proportionality constant in (A.9) is $\langle\alpha, \alpha\rangle^{-1}$. Then

$$\frac{E_\alpha + E_{-\alpha}}{2}, \quad \frac{E_\alpha - E_{-\alpha}}{2i}, \quad \frac{\alpha \cdot T}{\langle\alpha, \alpha\rangle} \tag{A.11}$$

generate a $\mathfrak{su}(2)$ subalgebra $\mathfrak{s}(\alpha)$ of $\mathfrak{g}$. Then

$$l(\mathfrak{s}(\alpha)) = 2\mathrm{tr}(\frac{\alpha \cdot T}{\langle\alpha, \alpha\rangle})^2 = \frac{2}{\langle\alpha, \alpha\rangle}\,. \tag{A.12}$$

Every $\mathfrak{su}(2)$ sub-algebra can be embedded into the Lie algebra in this way by a change of basis, so we find that

$$\lambda(\mathcal{G}) = \frac{2}{\langle\alpha, \alpha\rangle_{\max}} \tag{A.13}$$

where $\langle \alpha, \alpha \rangle_{\max}$ is the length squared of the longest root of the Lie algebra.

Now let us examine properties of $\{\mathcal{T}_I\}$, which are the coroot basis for the Cartan generators. They are defined to be

$$\mathcal{T}_I \equiv \frac{2\alpha_I \cdot T}{\langle \alpha_I, \alpha_I \rangle} \tag{A.14}$$

where α_I are the simple roots of the Lie group. The charges of the root vectors E_β under $\mathcal{T}_I$ are given as

$$[\mathcal{T}_I, E_\beta] = \frac{2\alpha_{I,i}}{\langle \alpha_I, \alpha_I \rangle}[T_i, E_\beta] = \frac{2\langle \alpha_I, \beta \rangle}{\langle \alpha_I, \alpha_I \rangle}. \tag{A.15}$$

In particular,

$$[\mathcal{T}_I, E_{\alpha_J}] = \frac{2\langle \alpha_I, \alpha_J \rangle}{\langle \alpha_I, \alpha_I \rangle} = C_{IJ}. \tag{A.16}$$

Now let us examine

$$\frac{1}{\lambda(\mathcal{G})}\mathrm{tr}\mathcal{T}_I\mathcal{T}_J. \tag{A.17}$$

Using (A.13) we find that

$$\frac{1}{\lambda(\mathcal{G})}\mathrm{tr}\mathcal{T}_I\mathcal{T}_J = \frac{2\langle \alpha, \alpha \rangle_{\max}\langle \alpha_I, \alpha_J \rangle}{\langle \alpha_I, \alpha_I \rangle\langle \alpha_J, \alpha_J \rangle} = \mathcal{C}_{IJ}. \tag{A.18}$$

This is exactly the inner-product matrix for the coroot lattice normalized such that the shortest coroot has length 2. Note that although we had to refer to the group $\mathcal{G}$ in defining $\mathcal{T}_I$, the matrix $\mathcal{C}_{IJ}$ only depends on the Lie algebra due to the dividing out by $\lambda(\mathcal{G})$. For example, $\mathcal{C} = (2)$ for both $SU(2)$ and $SO(3)$.

For simply laced groups, $\mathcal{C}_{IJ}$ coincides with the Cartan matrix C_{IJ}. For non-simply laced groups $\mathcal{C}$ and C are different. $\mathcal{C}$ for B_n and C_n are given by

$$\mathcal{C}(B_n) = \begin{pmatrix} 2 & -1 & \cdots & 0 & 0 & 0 & 0 \\ -1 & 2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & -1 & 0 & 0 \\ 0 & 0 & \cdots & -1 & 2 & -1 & 0 \\ 0 & 0 & \cdots & 0 & -1 & 2 & -2 \\ 0 & 0 & \cdots & 0 & 0 & -2 & 4 \end{pmatrix} \tag{A.19}$$

and

$$\mathcal{C}(C_n) = \begin{pmatrix} 2 & -2 & 0 & 0 & \cdots & 0 & 0 \\ -2 & 4 & -2 & 0 & \cdots & 0 & 0 \\ 0 & -2 & 4 & -2 & \cdots & 0 & 0 \\ 0 & 0 & -2 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 4 & -2 \\ 0 & 0 & 0 & 0 & \cdots & -2 & 4 \end{pmatrix} . \tag{A.20}$$

For B_n we have defined α_n to be the simple root with the different(short) norm, *i.e.*,

$$\langle \alpha_1, \alpha_1 \rangle = \langle \alpha_2, \alpha_2 \rangle = \cdots = \langle \alpha_{n-1}, \alpha_{n-1} \rangle = 2\langle \alpha_n, \alpha_n \rangle . \tag{A.21}$$

For C_n we have defined α_1 to be the simple root with the different(long) norm, *i.e.*,

$$\langle \alpha_1, \alpha_1 \rangle = 2\langle \alpha_2, \alpha_2 \rangle = \cdots = 2\langle \alpha_{n-1}, \alpha_{n-1} \rangle = 2\langle \alpha_n, \alpha_n \rangle . \tag{A.22}$$

Note that the coroot corresponding to a long/short root becomes a short/long coroot, respectively.

$\mathcal{C}$ for F_4 and G_2 are given by

$$\mathcal{C}(F_4) = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 4 \end{pmatrix}, \quad \mathcal{C}(G_2) = \begin{pmatrix} 2 & -3 \\ -3 & 6 \end{pmatrix} \tag{A.23}$$

respectively. For F_4 we have taken α_1 and α_2 to be the long roots, *i.e.*,

$$\langle \alpha_1, \alpha_1 \rangle = \langle \alpha_2, \alpha_2 \rangle = 2\langle \alpha_3, \alpha_3 \rangle = 2\langle \alpha_4, \alpha_4 \rangle \,. \tag{A.24}$$

For G_2 we have taken α_1 to be the long root, *i.e.*,

$$\langle \alpha_1, \alpha_1 \rangle = 3\langle \alpha_2, \alpha_2 \rangle \,. \tag{A.25}$$

For each non-simply laced group, we have aligned the roots so that they are decreasing in norm.

A.2 Global anomalies

In this section we prove that locally non-anomalous blocks with anomaly coefficient b and gauge group $SU(2)$ or $SU(3)$ are free of global anomalies of the kind addressed in [127] if and only if $b \cdot b$ is integral. The problem with $SU(2)$ and $SU(3)$ charged chiral fermions is that the fermion measure might obtain a phase under global gauge transformations, which are gauge transformations that are not homotopic to the identity. This happens for only for the gauge groups $SU(2)$ and $SU(3)$ among the SU groups in six dimensions because $\pi_6(SU(2)) = \mathbb{Z}_{12}$, $\pi_6(SU(3)) = \mathbb{Z}_6$ while π_6 is trivial for the other $SU(N)$ gauge groups.

Let us first consider $SU(3)$ in six dimensions. Defining the global gauge transformation that generates $\pi_6(SU(3)) = \mathbb{Z}_6$ as g, we need to determine the phase $2\pi\alpha_r$ a chiral fermion measure in representation r acquires when acted on by g. Note that

α_r is defined up to integers.

This problem was essentially solved in [219], but let us phrase it in a language convenient for our purposes. The result of [219, 220] is that if an $SU(4)$ representation R is broken into $\sum_i r_i$ of $SU(3)$ representations r_i in a canonical embedding then

$$\sum \alpha_{r_i} = \frac{B_R}{3!} \tag{A.26}$$

Let us define two generators of $SU(4)$, T_{12}, T_{34} which, in the fundamental representation, take the form

$$(T_{12})_{ab} = \delta_{a1}\delta_{b1} - \delta_{a2}\delta_{b2} \tag{A.27}$$

$$(T_{34})_{ab} = \delta_{a3}\delta_{b3} - \delta_{a4}\delta_{b4} \tag{A.28}$$

The group theory factors B_R, C_R can be computed in terms of traces of these generators, by their definitions:

$$B_R + 2C_R = \frac{1}{2}\mathrm{tr}_R T_{12}^4 \tag{A.29}$$

$$C_R = \frac{3}{4}\mathrm{tr}_R T_{12}^4 T_{34}^4 \tag{A.30}$$

Note that C_R is a multiple of 3 as $\mathrm{tr}_R T_{12}^4 T_{34}^4$ is integral, and is a multiple of 4. This can be seen from looking at which Young tableaux contribute to the trace for a given representation (for an explicit proof, see [65]). If R is broken into $\sum_i r_i$, it is clear that

$$B_R + 2C_R = \frac{1}{2}\mathrm{tr}_R T_{12}^4 = \sum_i \frac{1}{2}\mathrm{tr}_{r_i} T_{12}^4 = 2\sum_i C_{r_i} \tag{A.31}$$

and therefore

$$2\sum_i C_{r_i} \equiv B_R \mod 6 \equiv \sum_i 6\alpha_{r_i} \mod 6 \tag{A.32}$$

Since the measure of a chiral fermion in the trivial representation does not acquire a phase under global transformations, and by taking $R = 4$ we find that $\alpha_3 = C_3/3$. It is straightforward to carry this through by going through the representations of

$SU(4)$, and showing by induction that actually

$$\alpha_r = \frac{C_r}{3} \tag{A.33}$$

For $SU(2)$ the situation is similar. Denoting the generator of $\pi_6(SU(2)) = \mathbb{Z}_{12}$ by g' we need to find the phase $2\pi\alpha_{r'}$ the chiral fermion measure in representation r' of $SU(2)$ acquires when acted on by g'.

Let us embed $SU(2)$ into $SU(3)$. It is known(for example as stated in [219]), that g' maps to g when we do the embedding. Therefore we see that if an $SU(3)$ representation r is broken into $\sum_i r'_i$ of $SU(2)$ representations r'_i in a canonical embedding

$$\sum_i \alpha_{r'_i} = \alpha_r = \frac{C_r}{3} = \sum_i \frac{C_{r'_i}}{3} \tag{A.34}$$

By an almost word-by-word duplication of the argument for $SU(3)$, we obtain

$$\alpha_{r'} = \frac{C_{r'}}{3} \tag{A.35}$$

This is a satisfying result, because we see from the anomaly equations on the C factors

$$\sum_i x_r \alpha_r - \alpha_{\text{adj}} = \sum_i x_r \frac{C_r}{3} - \frac{C_{\text{adj}}}{3} = b \cdot b\,. \tag{A.36}$$

The far left hand side of this equation is the phase (divided by 2π) the fermion measure obtains under the global transformation given by the generator of π_6 of the gauge group. Therefore the condition that the given gauge group does not have any global gauge anomalies is equivalent to the condition that the far right hand side of the above equation is integral. Therefore an $SU(2)$ or $SU(3)$ block with anomaly coefficient b does not have global gauge anomalies if and only if $b \cdot b$ is integral.

A.3 Proof of bounds on b

Using group theory identities, one can find constraints on the matter content of individual blocks of a given theory. In this Appendix, we show that even without the $H - V$ constraint we can bound the degree b of an $SU(M)$ block just from group theory constraints when $T = 0$. The only equations we use are the anomaly cancellation conditions for a given gauge group:

$$18b + A_{\text{adj}} = \sum_R x_R A_R \tag{A.37}$$

$$B_{\text{adj}} = \sum_R x_R B_R \tag{A.38}$$

$$3b^2 + C_{\text{adj}} = \sum_R x_R C_R \tag{A.39}$$

In section A.3.1 we make some useful statements based on the Weyl character formula. In section A.3.2 we see how this bounds b for gauge groups larger than $SU(3)$. In section A.3.3 we discuss the process of bounding b's for the gauge groups $SU(2)$ and $SU(3)$. In section A.3.4 we summarize the results. A useful reference for this section is [218], chapters 12 and 13.

A.3.1 The Weyl Character Formula

We use the Weyl character formula (equation (XIII.37) of [218])

$$\text{tr}_\lambda e^\rho = \frac{\sum_{w\in W} \text{sign}(w) e^{\langle \lambda+\delta, w\rho\rangle}}{\sum_{w\in W} \text{sign}(w) e^{\langle \delta, w\rho\rangle}} \tag{A.40}$$

In this formula tr_λ denotes the trace of the representation with highest weight vector λ. ρ is an element of the Lie algebra $\rho = \rho_a T^a$ where T^a is the Cartan basis, and $(\rho_1, \cdots, \rho_r)$ are coordinates on the weight space. Brackets denote inner products in the weight space. R^+ is the set of positive roots of the Lie algebra, and W is the

Weyl group. The vector δ is defined to be half the sum of the positive roots

$$\delta = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \tag{A.41}$$

For $\rho = s\delta$ the equation simplifies to

$$\begin{aligned} &\mathrm{tr}_\lambda e^{s\delta} \\ &= \prod_{\alpha \in R^+} \left(\frac{e^{\frac{s}{2}\langle \alpha, \lambda+\delta \rangle} - e^{-\frac{s}{2}\langle \alpha, \lambda+\delta \rangle}}{e^{\frac{s}{2}\langle \alpha, \delta \rangle} - e^{-\frac{s}{2}\langle \alpha, \delta \rangle}} \right) \\ &= \prod_{\alpha \in R^+} \frac{\langle \alpha, \lambda+\delta \rangle}{\langle \alpha, \delta \rangle} \prod_{\alpha \in R^+} \left(\frac{1 + \frac{1}{6}\langle \alpha, \lambda+\delta \rangle^2 (\frac{s}{2})^2 + \frac{1}{120}\langle \alpha, \lambda+\delta \rangle^4 (\frac{s}{2})^4 + \cdots}{1 + \frac{1}{6}\langle \alpha, \delta \rangle^2 (\frac{s}{2})^2 + \frac{1}{120}\langle \alpha, \delta \rangle^4 (\frac{s}{2})^4 + \cdots} \right) \end{aligned} \tag{A.42}$$

This is due to the relation (equation (XIII.13) of [218])

$$\sum_{w \in W} \mathrm{sign}(w) e^{\langle \delta, w\rho \rangle} = \prod_{\alpha \in R^+} (e^{\frac{1}{2}\langle \alpha, \rho \rangle} - e^{-\frac{1}{2}\langle \alpha, \rho \rangle}) \tag{A.43}$$

which holds for an arbitrary vector ρ.

Expanding in s, and looking at the terms of order 0, 2, and 4, we find that

$$\begin{aligned} D_\lambda = \mathrm{tr}_\lambda 1 &= \prod_{\alpha \in R^+} \frac{\langle \alpha, \lambda+\delta \rangle}{\langle \alpha, \delta \rangle} \\ A_\lambda(\mathrm{tr}_f \delta^2) = \mathrm{tr}_\lambda \delta^2 &= \frac{D_\lambda}{12} \sum_{\alpha \in R^+} (\langle \alpha, \lambda+\delta \rangle^2 - \langle \alpha, \delta \rangle^2) \\ B_\lambda(\mathrm{tr}_f \delta^4) + C_\lambda(\mathrm{tr}_f \delta^2)^2 = \mathrm{tr}_\lambda \delta^4 &= \frac{D_\lambda}{120} \sum_{\alpha \in R^+} (-\langle \alpha, \lambda+\delta \rangle^4 + \langle \alpha, \delta \rangle^4) \\ &+ \frac{D_\lambda}{48} (\sum_{\alpha \in R^+} (\langle \alpha, \lambda+\delta \rangle^2 - \langle \alpha, \delta \rangle^2))^2 \end{aligned} \tag{A.44}$$

Now $\mathrm{tr}_f \delta^2$ and $\mathrm{tr}_f \delta^4$ are explicitly calculable. We consider $SU(M)$ groups with the normalization $\mathrm{tr}_f T^a T^b = 2\delta_{ab}$. We use the fact that for $SU(M)$ the positive roots are given by

$$\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} + \alpha_j \tag{A.45}$$

for $i \le j$ where α_i for $i = 1, 2, \cdots, N-1$ are the simple roots of $SU(M)$ whose Cartan matrix is given by

$$(\alpha_i \cdot \alpha_j) = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix} \tag{A.46}$$

Then,

$$\delta = \frac{1}{2}\sum_{i\leq j} \alpha_{ij} = \sum_{i=1}^{M-1} \frac{i(M-i)}{2}\alpha_i \tag{A.47}$$

By explicit calculation one may show that

$$\langle \alpha_i, \delta \rangle = 1 \tag{A.48}$$

and therefore taking the dual basis of $\{\alpha_i\}$ to be $\{\beta_i\}$,

$$\delta = \sum_i \beta_i \tag{A.49}$$

Now we use the fact that the highest weight vector for the fundamental representation is β_1. Then using (A.44) we find that

$$\begin{aligned} \mathrm{tr}_f \delta^2 &= \frac{M}{12}\sum_{i\leq j}(\langle \alpha_{ij}, \beta_1 + \delta\rangle^2 - \langle \alpha_{ij}, \delta\rangle^2) \\ &= \frac{M}{12}\sum_{i\leq j}((\delta_{i1} + (j-i+1))^2 - (j-i+1)^2) \\ &= \frac{M}{12}\sum_{j=1}^{M-1}((j+1)^2 - j^2) = \frac{M(M-1)(M+1)}{12} \end{aligned} \tag{A.50}$$

and likewise

$$
\begin{aligned}
\mathrm{tr}_f\delta^4 &= \frac{M}{120}\sum_{i\le j}(-\langle\alpha_{ij},\beta_1+\delta\rangle^4+\langle\alpha_{ij},\delta\rangle^4)+\frac{M}{48}\sum_{i\le j}(\langle\alpha_{ij},\beta_1+\delta\rangle^2-\langle\alpha_{ij},\delta\rangle^2))^2\\
&= \frac{M}{120}\sum_{i\le j}(-(\delta_{i1}+(j-i+1))^4+(j-i+1)^4)+\frac{M}{48}(N^2-1)^2\\
&= \frac{M}{120}\sum_{j=1}^{M-1}(-(j+1)^4+j^4)+\frac{M}{48}(M^2-1)^2\\
&= -\frac{M}{120}(M^4-1)+\frac{M}{48}(M^2-1)^2\\
&= \frac{1}{240}M(M-1)(M+1)(3M^2-7)
\end{aligned}
\tag{A.51}
$$

Furthermore, plugging in the equation for A_λ to the equation for the fourth order invariants and dividing both sides by $(\mathrm{tr}_f\delta^2)^2$ we find that

$$
y_M B_\lambda + C_\lambda \le \frac{3A_\lambda^2}{D_\lambda} \tag{A.52}
$$

where we have defined

$$
y_M \equiv \frac{\mathrm{tr}_f\delta^4}{(\mathrm{tr}_f\delta^2)^2} = \frac{3(3M^2-7)}{5M(M^2-1)} \tag{A.53}
$$

All the results of the current section hold for $SU(2)$ and $SU(3)$ also if we set $B_\lambda = 0$ by hand, which we can certainly do for these groups.

A.3.2 Restriction on b

For a single $SU(M)$ block with $M \ge 4$ define

$$
\frac{\sum_R x_R(y_M B_R + C_R)}{\sum_R x_R A_R} = \frac{3b^2+6+2My_M}{18b+2M} \equiv \eta \tag{A.54}
$$

This means that there must exist a representation R_0 with

$$
\frac{y_M B_{R_0} + C_{R_0}}{A_{R_0}} \ge \eta \tag{A.55}
$$

since by definition, $y_M B_{R_0} + C_{R_0}$ and A_{R_0} are positive. Then by inequality (A.52),

$$(\frac{D_{R_0}}{3})\eta \leq A_{R_0} \leq \sum_R x_R A_R = 18b + 2M \tag{A.56}$$

Plugging in the definition of η, we find that

$$D_{R_0} \leq \frac{324(b + M/9)^2}{b^2 + (2 + 2y_M M/3)} \tag{A.57}$$

The maximum value for the right hand side of the above equation is obtained for $b = (18/M + 6y_M)$ and plugging in this value of b we obtain the inequality

$$D_{R_0} \leq 324 + \frac{4M^2}{2 + 2y_M M/3} \equiv D_M \tag{A.58}$$

Hence we obtain

$$\frac{b^2 + 2 + 2My_M/3}{18b + 2M} = \frac{\eta}{3} \leq \max_{R:D_R<D_M}(\frac{A_R}{D_R}) \tag{A.59}$$

This procedure gives an upper bound b_u on b, for all $M \geq 4$. For $M = 2, 3$ we are able to obtain slightly improved bounds as we have explicit expressions for A_R, C_R, D_R for these groups. This is helpful since the enumeration of $SU(2)$ and $SU(3)$ blocks takes much more time numerically compared to blocks with $M \geq 4$. We will elaborate on this in section A.3.3.

Implementing (A.59) to obtain an upper bound of b is a rather tedious, but straightforward task. In practice we must find all representations with $D_R < D_M$ and find the maximum value of A_R/D_R among those representations. This can be done by using the following useful

Fact : Given two representations R_1 and R_2 represented by young-diagrams Y_1, Y_2, if Y_2 can be obtained by attaching columns of blocks to the right of Y_1, then necessarily $D_{R_1} < D_{R_2}$.

This follows simply from the fact that the dimension of a representation is associated with the number of distinct labelings of the boxes which are horizontally non-decreasing and vertically increasing. Adding columns to the right, there is always at least one labeling of Y_2 for each labeling of Y_1 by simply repeating entries on each row in the added columns. For example, if we define

$$Y_1 = \ydiagram{4,4,4,3,2}, \qquad Y_2 = \ydiagram{5,5,4,3,2}, \qquad Y_3 = \ydiagram{5,5,4,4,2} \tag{A.60}$$

the dimension of representation Y_1 is smaller than that of Y_2. Meanwhile, it is not guaranteed that the dimension of Y_1 is smaller than the dimension of Y_3.

Starting from single column representations one may span a tree of young diagrams by attaching columns of varying length to the right until one runs into a diagram with $D_R > D_M$. Since the dimension strictly increases at each step, all the branches of the tree will eventually terminate and one can obtain all the representations with bounded dimension.

Although we have thus found an upper bound on b for each group $SU(M)$, which in principle makes the problem of enumerating blocks into a finite algorithm, in practice it is helpful to reduce the bound somewhat to make the enumeration of blocks more tractable. It turns out that we can further restrict b by utilizing the condition

$$A_{R_0} \leq 18b_u + 2M \tag{A.61}$$

That is, it can be the case that

$$\frac{b^2 + 2 + 2My_M/3}{18b + 2M} = \frac{\eta}{3} \leq \max_{\substack{R:D_R<D_M \\ \text{and } A_R \leq 18b_u+2M}} \left(\frac{A_R}{D_R}\right) \tag{A.62}$$

further restricts b below the bound coming from (A.59).

For example, in the case of $SU(7)$ one finds that

$$\max_{R:D_R<386} \left(\frac{A_R}{D_R}\right) = \frac{495}{492} = 1.07\cdots \tag{A.63}$$

and hence

$$b \leq 19\,. \tag{A.64}$$

M	3	4	5	6	7	8	9	10	11	12	13	14	15	≥ 16
b bound	617	126	40	24	14	12	11	6	6	5	3	3	3	2

Table A.1: Upper bound on b for individual block of group $SU(M)$.

But one finds that for $b \leq 19$, $18b + 2M \leq 356$, so

$$\max_{\substack{R:D_R<386 \\ \text{and } A_R\leq 356}} \left(\frac{A_R}{D_R}\right) = \frac{165}{210} = 0.78\cdots \tag{A.65}$$

and hence b is further restricted to the value

$$b \leq 14\,. \tag{A.66}$$

The bound on b obtained this way for $4 \leq M \leq 17$ is given in table A.1. For $M \geq 18$,

$$324 + \frac{4M^2}{2 + 2y_M M/3} < \frac{M(M-1)(M-2)}{6} \tag{A.67}$$

This means that the representation with maximum A/D in an $SU(M)$, $M \geq 18$ block is either the adjoint, symmetric, antisymmetric or fundamental. The maximum value of A/D of these is given by the symmetric and hence it must be the case that

$$\frac{b^2 + 2 + 2My_M/3}{18b + 2M} \leq \frac{2(M+2)}{M(M+1)} \tag{A.68}$$

A little bit of algebra shows that this implies $b \leq 2$ for $M \geq 18$. This completes the data needed for Table A.1. Given the upper bound on b for each M it is therefore a finite problem to enumerate all possible gauge factor + matter "blocks". As noted in section 4.1, in most cases the actual maximum b for each M is smaller than that given in Table A.1. In particular, above $M = 12$ only blocks with $b = 1$ are possible.

A.3.3 Comments on $SU(2)$ and $SU(3)$ Blocks

The b values of the $SU(2)$ blocks can be restricted in an equivalent fashion. The reason we are addressing them separately is because the bound on b obtained for

$SU(2)$ using the equations in the previous section is very high. In particular, the most naive bounds on b for $SU(2)$ is of order 10^5.

We first provide the most naive bounds on b we can get for $SU(2)$ and $SU(3)$. As mentioned in the previous section we can get slightly better bounds because the equations for A, C, D are simple enough to manipulate directly.

For $SU(2)$ all representations are m-symmetric representations. The dimension and group theory coefficients of the representation are

$$A_m = m(m+1)(m+2)/6 \tag{A.69}$$
$$C_m = A_m(3m^2+6m-4)/10 \tag{A.70}$$
$$D_m = m+1\,. \tag{A.71}$$

Also recall that $B_m = 0$ for all representations. The anomaly equations (A.37), (A.39) can be written as

$$\sum_m m(m+1)(m+2)x_m = 108b + 24 \tag{A.72}$$
$$\sum_m m^2(m+2)^2(m+1)x_m = 60b^2 + 36b + 192\,. \tag{A.73}$$

Taking the largest m with $x_m \neq 0$ to be m_M, we find that,

$$\frac{10b^2+6b+32}{18b+4} \leq m_M(m_M+2) \tag{A.74}$$

and hence

$$\begin{aligned}\left(\frac{10b^2+6b+32}{18b+4}\right)^{3/2} &\leq m_M(m_M+2)\sqrt{m_M(m_M+2)} \\ &< m_M(m_M+2)(m_M+1) \leq 108b+24\,.\end{aligned} \tag{A.75}$$

This gives the bound $b \leq 68018$.

The representations of $SU(3)$ can be represented by a pair of numbers x and y which denote the number of two block/one block columns of its young diagram. Then the dimension and group theory coefficients of representation (x, y) are given by

$$\begin{aligned} A_{x,y} &= \frac{1}{24} XY(X+Y)(X^2+Y^2+XY-3) && \text{(A.76)} \\ C_{x,y} &= \frac{1}{120} XY(X+Y)(X^2+Y^2+XY-3)(X^2+Y^2+XY-\frac{9}{2}) && \text{(A.77)} \\ D_{x,y} &= \frac{1}{2} XY(X+Y), && \text{(A.78)} \end{aligned}$$

where we have defined $X = (x+1)$, $Y = (y+1)$. Writing out the anomaly equations and going through a similar process as in the $SU(2)$ case one obtains the bound $b \leq 617$.

Up to now we have been using that fact that in order for a block to satisfy the anomaly equations, we must have some representation R with a large

$$\frac{C_R}{A_R} \geq \frac{3b^2 + C_{\text{adj}}}{18b + A_{\text{adj}}} \sim \mathcal{O}(b)\,. \qquad \text{(A.79)}$$

We have been ruling out b values for which all such large representations R have $A_R > 18b + A_{\text{adj}}$. Finding all solutions to the anomaly equations for $SU(2)$ and $SU(3)$ for the range of b values constrained only by this condition turns out to be a demanding task numerically, and it is useful to further restrict the allowed values of b. To do this, we generalize the strategy employed up to now.

Suppose b_0 is a value not ruled out by the previous arguments. This means that there exists a representation R satisfying

$$\frac{C_R}{A_R} \geq \frac{3b_0^2 + C_{\text{adj}}}{18b_0 + A_{\text{adj}}} \qquad \text{(A.80)}$$

and

$$A_R \leq 18b_0 + A_{\text{adj}} \qquad \text{(A.81)}$$

within the block. Let $S(b_0) = \{R_1, \cdots, R_k\}$ be all the representations that satisfy these two equations. The fact that A, C, D, A/D and C/A are all strictly increasing functions with respect to m in the case of $SU(2)$ and of x and y in the case of $SU(3)$ is helpful in constructing this list.

Now assume that we have the representation R_1 in a block with given b_0. If R_1 satisfied,

$$3b_0^2 + C_{\rm adj} = C_{R_1}, \qquad \text{and} \qquad 18b_0 + A_{\rm adj} = A_{R_1}, \tag{A.82}$$

we would have a solution for a single block whose matter content is given by just one R_1. Suppose this were not the case. [1] By the same line of argument as before we must have a representation R satisfying

$$\frac{C_R}{A_R} > \frac{3b_0^2 + C_{\rm adj} - C_{R_1}}{18b_0 + A_{\rm adj} - A_{R_1}} \tag{A.83}$$

and

$$A_R < 18b_0 + A_{\rm adj} - A_{R_1}. \tag{A.84}$$

If no such representation R exists, the representation R_1 cannot show up in a block with $b = b_0$. If the situation were same for all the representations in $S(b_0) = \{R_1, R_2, \cdots, R_k\}$ we could rule out $b = b_0$. We can iterate this process to further rule out b values.

We have employed this procedure for $SU(2)$ and the initial bound 68,018 has been pulled down to 36,647. For $SU(3)$ we have iterated this process 5 times and were able to rule out 288 values of b in the range $b \leq 617$.

We can describe the problem of constructing single block models in a very concise way as a partition problem for $SU(2)$ and $SU(3)$ as the coefficients of the anomaly equations are all positive for these cases. While the situation is similar for $SU(3)$ only depict the process for $SU(2)$ for simplicity. The problem is to find a combination of representations where the A_R, C_R, D_R values add up to $(4 + 18b, 8 + 3b^2, D)$ with $D \leq 276$. This is a straightforward partition problem, whose algorithmic solution is simplified by the fact that A_m/D_m and C_m/A_m are monotonically increasing functions of m. We have implemented an algorithm which computes all such partitions for fixed b and checked a representative sample of b's in the allowed range as described in the main text.

[1] In fact, we can show that for $SU(2)$ and $SU(3)$ there cannot be a block with a single matter representation, i.e. a block with $\sum_R x_R = 1$.

One last note is that the maximum b value possible for single block models turns out to be within an order of magnitude of the upper bounds for small gauge groups. For $SU(3)$ the maximum b possible for an $SU(3)$ is $b = 597$ while the bound is 617, and for $SU(2)$ we were able to find a block with $b = 24,297$ while the bound is given by $36,647$.

A.3.4 Summary

To summarize, just from the group theory we find that each individual block cannot have a gauge group larger than 24. The b values are bounded as in table A.1 $SU(N)$ with $N \geq 3$. The best bound we have for $SU(2)$ is $36,647$. As discussed in the main text of this thesis, given an upper bound on b, for each fixed $N \leq 24$ we can solve the finite partition problem for each b to enumerate all possible blocks.

A.4 Proof of Bound on Curable Theories

We prove that the number of curable theories as defined in section 2.4.2 is finite for $T < 9$. The crucial fact we use is that for curable theories one of the two following conditions must hold:

$$H - V \leq 273 - 29T + (T^2 + 6T + 8) \tag{A.85}$$

$$H - V - \sqrt{2}(T+2)\sqrt{N} \leq 273 - 29T + (2T + 4) \tag{A.86}$$

where N is the number of hypermultiplet representations of the theory. It is clear from [62, 65] that there could not be an infinite family of theories for which the first condition holds as it requires $H - V$ to be bounded. Therefore, it is sufficient to show that there does not exist an infinite family of curable theories for which the second condition (A.86) holds. It proves convenient to define

$$c(T) \equiv \frac{T+2}{\sqrt{2}}. \tag{A.87}$$

Before presenting the proof of the desired result, we point out that the proof is very similar to that given for non-abelian theories in [62, 65]. Proofs of the existence of bounds on non-anomalous theories are carried out by two steps in these references. First, the authors identify infinite classes of theories that satisfy all the anomaly equations other than the gravitational anomaly bound, and that have positive kinetic terms for the gauge fields. Then they show that it is impossible for all the theories in that infinite class to satisfy the gravitational anomaly bound. This is proven by showing that as the total rank of the gauge group increases, the increase of H is much faster than V. This proves that constructing an infinite class of theories that satisfy all the anomaly cancellation conditions and that have positive kinetic terms for the gauge fields does not exist.

We also take the same approach in proving our bounds. In our case, however, we must prove that the increase of $H - c(T)\sqrt{N}$ is much faster than V for the infinite classes of theories one could construct. Most of our effort will be put in to showing that $\sqrt{N}$ does not increase so fast as to affect the growth of H.

There are infinite classes of theories that this is easy to show. For example, for the class of theories whose H and V exhibit a scaling behavior with respect to the rank of the total gauge group when it becomes large, the arguments presented in [62, 65] can be virtually repeated. This is because equation (A.86) implies that

$$(\sqrt{H} - c(T))^2 - V \leq 277 - 27T + c(T)^2 \,. \tag{A.88}$$

This is because as we have assumed there exist no singlets in curable theories, and hence

$$N \leq \frac{H}{2} < H \tag{A.89}$$

holds. Therefore, the scaling behavior of $(\sqrt{H} - c(T))^2 - V$ and $H - V$ with respect to the rank is equivalent and the boundedness argument for these classes of theories are essentially the same. In particular, for an infinite class of theories whose simple group factors have bounded rank, the proof of boundedness given in [62, 65] can be used with very little adjustments. This is presented in section A.4.1.

It is, however, worth pointing out that for some infinite class of theories, the situation is rather subtle. When there exist simple group factors with unbounded rank in the infinite class of theories, the bound (A.88) becomes too delicate to use. In that case the stronger bound (A.86) turns out to be more useful in proving the existence of bounds of curable theories. We will carry this out in section A.4.2.

We now turn to presenting the complete proof of the bound on curable theories. We proceed by *reductio ad absurdum.* Let us assume there is an infinite family of curable non-abelian theories with gauge group $\prod_\kappa \mathcal{G}_\kappa$. Due to the bound on $H - V - 2c\sqrt{N} > (\sqrt{H} - c)^2 - V - c^2$, we see that theories in an infinite family of curable theories should be unbounded in the dimension of the gauge group. If not, H and V are both bounded, and hence only a finite number of theories can be constructed. There are two ways an unbounded family can occur. These are given as the following:

1. The dimension of each $\mathcal{G}_\kappa$, or equivalently, d_{Adj_κ} is bounded, but the number of simple factors is unbounded in this family.

2. The dimension of a single simple group factor $\mathcal{G}_\kappa$ is unbounded.

We show that both kinds of families cannot exist in the following subsections.

An important fact we use throughout the proof is the fact that,

$$\begin{aligned} H &\geq \begin{pmatrix} \text{Number of pairs of } \mathcal{G}_\iota \neq \mathcal{G}_\kappa \\ \text{for which there exists a jointly charged hypermultiplet.} \end{pmatrix} \\ &= (\text{Number of pairs of } \mathcal{G}_\iota \neq \mathcal{G}_\kappa \text{ with } b_\iota \cdot b_\kappa \neq 0.) \end{aligned} \tag{A.90}$$

This can be shown in the following way.

Suppose a hypermultiplet representation I is charged under $\lambda \geq 2$ gauge groups. Then

$$\mathcal{M}_I \geq 2^\lambda \geq \lambda(\lambda - 1) = \begin{pmatrix} \text{Number of pairs of } \mathcal{G}_\iota \neq \mathcal{G}_\kappa \\ \text{that } I \text{ is charged jointly under.} \end{pmatrix} \tag{A.91}$$

and therefore

$$
\begin{aligned}
H = \sum_I \mathcal{M}_I \geq \sum_I & \begin{pmatrix} \text{Number of pairs of } \mathcal{G}_\iota \neq \mathcal{G}_\kappa \\ \text{that } I \text{ is charged jointly under.} \end{pmatrix} \\
\geq & \begin{pmatrix} \text{Number of pairs of } \mathcal{G}_\iota \neq \mathcal{G}_\kappa \\ \text{for which there exist a jointly charged hypermultiplet.} \end{pmatrix}
\end{aligned} \tag{A.92}
$$

This means that any ordered pair of gauge groups that has matter jointly charged under it contributes at least 1 to H. This proves the inequality in the first line of (A.90).

Since A_κ^R for any representation R of any simple Lie group $\mathcal{G}_\kappa$ is positive and since,

$$
b_\iota \cdot b_\kappa = \sum \lambda_\kappa \lambda_\iota \sum_I \mathcal{M}_I^{\iota\kappa} A_\iota^I A_\kappa^I \geq 0 \tag{A.93}
$$

the necessary sufficient condition for two gauge groups $\mathcal{G}_\iota, \mathcal{G}_\kappa$ to have jointly charged matter is $b_\iota \cdot b_\kappa \neq 0$. Therefore

$$
H \geq \sum_{\iota \neq \kappa,\ b_\iota \cdot b_\kappa \neq 0} 1 = (\text{Number of pairs of } \iota \neq \kappa \text{ with } b_\iota \cdot b_\kappa \neq 0.) \tag{A.94}
$$

This proves the equality in the second line of (A.90).

A.4.1 Case 1 : Bounded Simple Group Factors

Let us assume that there exists an infinite number of curable theories with bounded simple group factors but with unbounded total dimension.

Let's denote the gauge group of this infinite family of theories $\{\mathcal{T}_\Phi\}$ as $\mathcal{G}_\Phi = \prod_{\kappa=1}^{\nu} \mathcal{G}_\kappa$ with $d_{\mathrm{Adj}_\kappa} < D$. Notice that we are denoting the number of gauge group factors, ν. So as $\Phi \to \infty$, $\nu \to \infty$. It is useful to classify the gauge group factors into three types according to their b^2 value:

1. **Type Z :** $b_\kappa^2 = 0$

2. **Type N :** $b_\kappa^2 < 0$

3. **Type P** : $b_\kappa^2 > 0$

Since the dimension of each factor is bounded,

$$(\sqrt{H} - c)^2 \leq H - 2c\sqrt{N} + c^2 \leq 273 - 29T + \nu D + c^2 \equiv B \sim \mathcal{O}(\nu)\,. \tag{A.95}$$

Therefore, the dimension of any representation is bounded also by

$$B' \equiv (\sqrt{B} + c)^2 \sim \mathcal{O}(\nu)\,. \tag{A.96}$$

Let's denote the number of N, Z, P type factors as ν_N, ν_Z, ν_P. Then

$$\nu = \nu_N + \nu_Z + \nu_P\,. \tag{A.97}$$

It is shown in [65] that b_ι, b_κ of any two P type factors $\mathcal{G}_\iota, \mathcal{G}_\kappa$ satisfy $b_\iota \cdot b_\kappa > 0$. Also it is shown that there exists $\nu_N^2/T - \nu_N$ distinct ordered pairs of type N gauge group factors that have matter jointly charged under them. Therefore, using (A.90) we can show that

$$\nu_P(\nu_P - 1) + (\frac{\nu_N^2}{T} - \nu_N) \leq H \leq B' \sim \mathcal{O}(\nu)\,. \tag{A.98}$$

Thus when ν is large,

$$\nu_P, \nu_N \leq \mathcal{O}(\sqrt{\nu}) \ll \nu \tag{A.99}$$

and therefore the majority of gauge group factors are of type Z:

$$\nu_Z \sim \mathcal{O}(\nu) \tag{A.100}$$

From the fact that two lightlike vectors cannot have zero inner product unless they are parallel, it is clear that in order for two Z type gauge groups to have no jointly charged matter their b vectors must be parallel. When we denote the size of the largest collection of parallel type Z vectors as μ there are at least $\nu_Z(\nu_Z - \mu)$

ordered pairs of type Z gauge groups with $b_\iota \cdot b_\kappa \neq 0$. This means that

$$\nu_Z(\nu_Z - \mu) = (\nu_Z - \mu)(\nu - \nu_P - \nu_N) \leq B' , \tag{A.101}$$

so from (A.100) we see that $\nu_Z - \mu$ is of order at most $\mathcal{O}(1)$, *i.e.*, $\nu_Z - \mu$ is bounded as a function of D as we take $\nu \to \infty$. Therfore

$$\mu \sim \nu_Z \sim \mathcal{O}(\nu) . \tag{A.102}$$

Meanwhile, it was shown in [62] that all Z factors satisfy $H - V > 0$ on their own. Since the μ Z factors have no jointly charged matter among themselves,

$$H - V > \mu - D \times [(\nu_Z - \mu) + \nu_p + \nu_N] \sim \mathcal{O}(\nu) , \tag{A.103}$$

i.e., the right hand side of the inequality is unbounded as a function of ν. Then it is clear that,

$$H - V - 2c\sqrt{N} > H - 2c\sqrt{H} - V = (\sqrt{H} - c)^2 - V - c^2 \tag{A.104}$$

is also unbounded as a function of ν. Therefore, $H - V - 2c\sqrt{N}$ cannot be bounded when each simple group factor of the infinite family has bounded rank. This rules out case 1.

A.4.2 Case 2 : Unbounded Simple Group Factors

Let us assume that there exists an infinite number of curable theories with a simple group factor that is unbounded. This is possible if the gauge group contains a classical group $H(\mathcal{N})$ (which is either SU, SO or Sp) with unbounded rank. In this case, there would be an infinite subfamily whose gauge group is given by $H(\mathcal{N}) \times \mathcal{G}_\mathcal{N}$ with fixed classical group type H, and $\mathcal{G}_\mathcal{N} = \prod_{\kappa=1}^{\nu(\mathcal{N})} \mathcal{G}_\kappa$ an arbitrary product of simple gauge groups with $\mathcal{N}$ unbounded. It is shown in [62] that when $\mathcal{N}$ is large, the $\mathcal{H}(\mathcal{N})$ block must be among those given in table A.2.

Group	Matter content	$H' - V'$	$2c\sqrt{N'}$	$a \cdot b$	b^2
$SU(\mathcal{N})$	$2\mathcal{N}\,\square$	$\mathcal{N}^2+1$	$\leq 2c\sqrt{2\mathcal{N}}$	0	-2
	$(\mathcal{N}+8)\,\square + 1\,\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\frac{1}{2}\mathcal{N}^2+\frac{15}{2}\mathcal{N}+1$	$\leq 2c\sqrt{\mathcal{N}+9}$	1	-1
	$(\mathcal{N}-8)\,\square + 1\,\square\!\square$	$\frac{1}{2}\mathcal{N}^2-\frac{15}{2}\mathcal{N}+1$	$\leq 2c\sqrt{\mathcal{N}-7}$	-1	-1
	$16\,\square + 2\,\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$15\mathcal{N}+1$	$\leq 2c\sqrt{18}$	2	0
$SO(\mathcal{N})$	$(\mathcal{N}-8)\,\square$	$\frac{1}{2}\mathcal{N}^2-\frac{15}{2}\mathcal{N}$	$\leq 2c\sqrt{\mathcal{N}-8}$	-1	-1
$Sp(\mathcal{N}/2)$	$(\mathcal{N}+8)\,\square$	$\frac{1}{2}\mathcal{N}^2+\frac{15}{2}\mathcal{N}$	$\leq 2c\sqrt{\mathcal{N}+8}$	1	-1
	$16\,\square + 1\,\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$15\mathcal{N}$	$\leq 2c\sqrt{17}$	2	0

Table A.2: Allowed charged matter for an infinite family of models with gauge group $H(\mathcal{N})$. The last column gives the values of $\alpha, \tilde{\alpha}$ in the factorized anomaly polynomial. H', V' and N' are as defined in the text.

Let us enumerate the hypermultiplet representations charged under $H(\mathcal{N})$ with indices, $I' = 1, \cdots, N'$ and the ones uncharged(and hence charged only under other gauge group factors) as $I'' = N'+1, \cdots, (N'+N'')$. Note that $(N'+N'') = N$. We call the former hypermultiplets I' hypermultiplets and the latter I'' hypermultiplets. We also define

$$H' \equiv \sum_{I'} \mathcal{M}_{I'} \qquad\qquad V' \equiv d_{\mathrm{Adj}_{H(\mathcal{N})}} \tag{A.105}$$

$$H'' \equiv \sum_{I''} \mathcal{M}_{I''} \qquad\qquad V'' \equiv \sum_{\kappa} d_{\mathrm{Adj}_{\mathcal{G}_\kappa}} \tag{A.106}$$

Then

$$H - V = (H'-V') + (H''-V'') \tag{A.107}$$

$$H-V-2c\sqrt{N} \geq (H'-V'-2c\sqrt{N'}) + (H''-V''-2c\sqrt{2N''}), \tag{A.108}$$

where we have used the fact that for positive x and y, $\sqrt{x+y} < \sqrt{x}+\sqrt{y}$.

From the $H-V-2c\sqrt{N}$ constraint, we see that additional gauge groups have to be added to make the theory curable since for large $\mathcal{N}$, $(H'-V'-2c\sqrt{N'}) \sim \mathcal{O}(\mathcal{N}^2)$ or $\sim \mathcal{O}(\mathcal{N})$. We shortly see that this is not possible for arbitrary large $\mathcal{N}$. We explicitly work out the proof for the cases when $(H'-V'-2c\sqrt{N'}) \sim \mathcal{O}(\mathcal{N}^2)$, but the proof generalizes to the other case straightforwardly.

Let us see whether we can find an infinite class of $\mathcal{G}_{\mathcal{N}}$ such that $(H'' - V'' - 2c\sqrt{N''}) \sim -\mathcal{O}(\mathcal{N}^2)$ for large $\mathcal{N}$. There are again two kinds of behavior of $\mathcal{G}_{\mathcal{N}}$ under $\mathcal{N} \to \infty$. It can consist of simple gauge factors of bounded rank, or it can have a simple gauge factor whose rank is unbounded. We consider these cases separately.

Case 2-1 : Rank of Simple Gauge Group Factors of $\mathcal{G}_{\mathcal{N}}$ Bounded

Assume that the dimensions of of the simple gauge group factors are bounded by D. Denoting $\nu(\mathcal{N})$ as the number of gauge group factors, as previously mentioned, we see that

$$H'' - V'' - 2c\sqrt{N} > (\sqrt{H''} - c)^2 - c^2 - D\nu(\mathcal{N}) \tag{A.109}$$

must behave as $-\mathcal{O}(\mathcal{N}^2)$ for large $\mathcal{N}$. Therefore,

$$\nu(\mathcal{N}) \geq \mathcal{O}(\mathcal{N}^2) \,. \tag{A.110}$$

Hence we must have an infinite family of theories where the number of simple gauge group factors of $\mathcal{G}_{\mathcal{N}}$ increase at least as $\mathcal{O}(\mathcal{N}^2)$. Also it is clear that

$$H'' \leq \mathcal{O}(\nu) \tag{A.111}$$

for large $\mathcal{N}$ and therefore,

$$H = H' + H'' \leq \mathcal{O}(\mathcal{N}^2) + \mathcal{O}(\nu) \leq \mathcal{O}(\nu) \,. \tag{A.112}$$

Meanwhile, we know from table A.2 that the size of the representation of I' hypermultiplets with respect to the κ gauge groups can be at most of order $\mathcal{O}(\mathcal{N})$ in our case.[2] Therefore, the maximum number of gauge groups an I' hypermultiplet could be charged under is given by $\mathcal{O}(\log \mathcal{N}) \leq \mathcal{O}(\log \nu)$. Since there are at most $\mathcal{O}(\mathcal{N} \log \mathcal{N}) \leq \mathcal{O}(\sqrt{\nu} \log \nu)$ such factors, there exists an order $\mathcal{O}(\nu) \geq \mathcal{O}(\mathcal{N}^2)$ number of gauge groups among the ν gauge groups $\mathcal{G}_\kappa$ under which only the I'' hypermultiplets

[2] When $(H' - V' - 2c\sqrt{N'}) \sim \mathcal{O}(\mathcal{N})$, the number of I' hypermultiplets can be at most of order $\mathcal{O}(1)$.

are charged. We denote theses gauge groups as $\{\mathcal{G}_{\kappa''}\}$.

Let us denote the size of the set ν'' and the total number of vector multiplets in $\{\mathcal{G}_{\kappa''}\}$ as V'''. Note that $\nu'' \sim \mathcal{O}(\nu)$. Then

$$H'' - V'' - 2c\sqrt{N''} \geq H'' - V''' - 2c\sqrt{N''} - D\mathcal{O}(\sqrt{\nu}\log\nu)\,. \tag{A.113}$$

Defining the number of P, N, and Z type factors in $\{\mathcal{G}_{\kappa''}\}$ as ν''_P, ν''_N and ν''_Z as before repeating the steps of case 1 we can show that

$$\nu''_P, \nu''_N \leq \mathcal{O}(\sqrt{\nu''}) \ll \nu''\,, \tag{A.114}$$

and therefore that

$$\nu''_Z \sim \mathcal{O}(\nu'')\,. \tag{A.115}$$

Also, denoting the size of the largest collection of parallel type Z vectors as μ'' we may show that

$$\mu'' \sim \mathcal{O}(\nu'') \tag{A.116}$$

as in case 1. We may finally show as in case 1 that

$$\begin{aligned} H'' - V''' &> \mu'' - D \times [(\nu_Z - \mu'') + \nu''_P + \nu''_N] \\ &\sim \mathcal{O}(\nu'') \sim \mathcal{O}(\nu)\,, \end{aligned} \tag{A.117}$$

and hence that

$$H'' - V''' - 2c\sqrt{N''} > (\sqrt{H''} - c)^2 - V''' - c^2 \geq \mathcal{O}(\nu)\,. \tag{A.118}$$

Putting this result together with (A.113) we find that

$$H'' - V'' - 2c\sqrt{N''} > \mathcal{O}(\nu) - D\mathcal{O}(\sqrt{\nu}\log\nu) \sim \mathcal{O}(\nu)\,. \tag{A.119}$$

Hence $(H'' - V'' - 2c\sqrt{N''})$ cannot behave as $-\mathcal{O}(\mathcal{N}^2)$ for large $\mathcal{N}$. We have come a long way to show that there exists a simple gauge group factor in $\mathcal{G}_{\mathcal{N}}$ that is unbounded in rank.

Case 2-2 : A Simple Gauge Group Factor of Unbounded Rank in $\mathcal{G}_{\mathcal{N}}$

In this case, there must be an infinite family of theories with

$$\hat{H}(\mathcal{N}) \times H(\mathcal{P}) \times \mathcal{G}_{\mathcal{N},\mathcal{P}} \tag{A.120}$$

with unbounded $\mathcal{N}$ and $\mathcal{P}$ where $\hat{H}$ and H are given classical groups. It is clear that both gauge groups have to come from table A.2.

Unless $H' - V' - 2c\sqrt{N'}$ for $\hat{H}(\mathcal{N}) \times H(\mathcal{P})$ is bounded, by the same arguments as case 2-1 we can show that $\mathcal{G}_{\mathcal{N},\mathcal{P}}$ contains a gauge group factor of unbounded rank. By the same investigation as in [65] we find that all combinations that have bounded $H' - V' - 2c\sqrt{N'}$ cannot have positive definite kinetic terms.

Hence we are led to the conclusion that there must be an infinite family of theories with

$$\tilde{H}(\mathcal{N}) \times \hat{H}(\mathcal{P}) \times H(\mathcal{Q}) \times \mathcal{G}_{\mathcal{N},\mathcal{P},\mathcal{Q}}\,. \tag{A.121}$$

$\mathcal{N}$, $\mathcal{P}$ and $\mathcal{Q}$ are unbounded and $\tilde{H}$, $\hat{H}$ and H are given classical groups. All three unbounded gauge groups must be from table A.2.

It also is the case that $H' - V' - 2c\sqrt{N'}$ for $\tilde{H}(\mathcal{N}) \times \hat{H}(\mathcal{P}) \times H(\mathcal{Q})$ must be bounded in order for $\mathcal{G}_{\mathcal{N},\mathcal{P},\mathcal{Q}}$ to have no gauge group factor of unbounded rank. It turns out to be impossible to find such a family with bounded $H' - V' - 2c\sqrt{N'}$ for which there exists a j vector that gives a positive kinetic term.

Our proof is concluded by the fact that one cannot construct an infinite family of theories that consists of four blocks from table A.2 for which all the ranks of the individual gauge group factors go to infinity. □

A.5 A Bound on the Number of Vector Multiplets for Pure Abelian Theories with $T = 0$

In the case that $T = 0$ and the gauge group is purely abelian, we can obtain a lower bound on the number of charged hypermultiplets as a function of the the total rank of the gauge group. Similar bounds may be obtained for other values of $T < 9$ though they are less stringent.

We label the $U(1)$ gauge groups by $i = 1, \cdots, V_A$. The gravitational anomaly condition imposes that the number of hypermultiplets is equal to $V_A + 273$. We denote the number of charged hypermultiplets to be $X \leq (V_A + 273)$, and label them by $I = 1, \cdots, X$.

For $T = 0$, the vectors

$$\vec{q_i} \equiv (q_{1,i}, q_{2,i}, \cdots, q_{X,i}) , \tag{A.122}$$

whose components are the charges of the X charged hypermultiplets under $U(1)_i$, must satisfy

$$108 \sum_I f_I(x_i)^4 = (\sum_I f_I(x_i)^2)^2 . \tag{A.123}$$

This follows from (2.124) and (2.126) where, as before, we have defined

$$f_I(x_i) = q_{I,i} x_i . \tag{A.124}$$

In order for the kinetic term matrix proportional to

$$b_{ij} = \sum_I q_{I,i} q_{I,j} \tag{A.125}$$

to be positive definite, $\vec{q_i}$ must be linearly independent. This was explained at the end of section 2.1.4. Therefore, using the $GL(V_A)$ invariance of the equation, we can redefine $\vec{q_i}$ so these vectors become orthogonal. It is convenient to normalize them to have norm $\sqrt{108}$, *i.e.*,

$$\sum_I q_{I,i} q_{I,j} = \sqrt{108} \delta_{ij} . \tag{A.126}$$

(Note that the $q's$ are not necessarily integers in this basis.)

Plugging this into (A.123) and expanding, we find that $q_{I,i}$ must also satisfy

$$\sum_I q_{I,i}^4 = 1 \tag{A.127}$$

$$\sum_I q_{I,i}^2 q_{I,j}^2 = \frac{1}{3} \qquad \text{for } i, j \text{ distinct} \tag{A.128}$$

$$\sum_I q_{I,i}^3 q_{I,j} = 0 \qquad \text{for } i, j \text{ distinct} \tag{A.129}$$

$$\sum_I q_{I,i}^2 q_{I,j} q_{I,k} = 0 \qquad \text{for } i, j, k \text{ distinct} \tag{A.130}$$

$$\sum_I q_{I,i} q_{I,j} q_{I,k} q_{I,l} = 0 \qquad \text{for } i, j, k, l \text{ distinct}\,. \tag{A.131}$$

Defining the vectors

$$\vec{Q}_i \equiv (q_{1,i}^2, q_{2,i}^2, \cdots, q_{X,i}^2) \tag{A.132}$$

$$\vec{A} \equiv \frac{1}{\sqrt{X}}(1, 1, \cdots, 1)\,, \tag{A.133}$$

and

$$\vec{Q}_{ij} \equiv (q_{1,i}q_{1,j}, q_{2,i}q_{2,j}, \cdots, q_{X,i}q_{X,j}) \tag{A.134}$$

for $i < j$, the equations obtained from (A.123) and (A.126) can be re-written as,

$$\vec{A} \cdot \vec{Q}_i = \sqrt{\frac{108}{X}} \tag{A.135}$$

$$\vec{Q}_i^2 = 1 \tag{A.136}$$

$$\vec{Q}_i \cdot \vec{Q}_j = \vec{Q}_{ij}^2 = \frac{1}{3} \qquad \text{for } i, j \text{ distinct} \tag{A.137}$$

$$\vec{A} \cdot \vec{Q}_{ij} = \vec{q}_i \cdot \vec{q}_j = 0 \qquad \text{for } i, j \text{ distinct} \tag{A.138}$$

$$\vec{Q}_i \cdot \vec{Q}_{ij} = 0 \qquad \text{for } i, j \text{ distinct} \tag{A.139}$$

$$\vec{Q}_i \cdot \vec{Q}_{jk} = \vec{Q}_{ij} \cdot \vec{Q}_{ik} = 0 \qquad \text{for } i, j, k \text{ distinct} \tag{A.140}$$

$$\vec{Q}_{ij} \cdot \vec{Q}_{kl} = 0 \qquad \text{for } i, j, k, l \text{ distinct} \tag{A.141}$$

It is easy to see that $\vec{Q}_i$ and $\vec{Q}_{ij}$ are all non-zero, since all $\vec{q}_i \neq \vec{0}$. Also $\vec{Q}_i$ and $\vec{Q}_{ij}$ are X-dimensional vectors by definition.

Using the given inner products we can show that

$$|\vec{Q}_1 + \cdots + \vec{Q}_{V_A}|^2 = V_A + \frac{1}{3}V_A(V_A - 1) = \frac{1}{3}V_A(V_A + 2)\,. \tag{A.142}$$

Since $\vec{A}$ is a unit vector by definition,

$$V_A\sqrt{\frac{108}{X}} = \vec{A}\cdot(\sum_i \vec{Q}_i) \leq |\sum_i \vec{Q}_i| = \sqrt{\frac{1}{3}V_A(V_A+2)}\,. \tag{A.143}$$

Hence

$$\frac{324V_A}{V_A+2} \leq X \leq V_A + 273\,, \tag{A.144}$$

as promised. This equation implies that

$$V_A \leq 17 \quad \text{or} \quad V_A \geq 32\,. \tag{A.145}$$

An additional constraint is needed to obtain an upper bound on V_A.

The additional constraint can be obtained by utilizing the full set of vectors $\vec{Q}_{ij}$ and $\vec{Q}_i$ we have defined. Note that by (A.140) and (A.141), $\vec{Q}_{ij}$ are mutually orthogonal. They are also orthogonal to $\vec{Q}_i$ and $\vec{A}$ as can be seen in (A.138), (A.139) and (A.140).

Also, all $\vec{Q}_i$ must be linearly independent. This is because if we assume

$$k_1\vec{Q}_1 + \cdots + k_{V_A}\vec{Q}_{V_A} = 0\,, \tag{A.146}$$

then for non-zero k_i

$$k_1^2 + \cdots k_{V_A}^2 + \frac{2}{3}(k_1k_2 + \cdots + k_{V_A-1}k_{V_A}) = 0\,. \tag{A.147}$$

But the l.h.s. can be rewritten as

$$\frac{2}{3}(k_1^2 + \cdots k_{V_A}^2) + \frac{1}{3}(k_1 + \cdots k_{V_A})^2 = 0 \tag{A.148}$$

and hence the equality cannot hold for non-zero k_i.

Therefore, we find that $\vec{Q}_i$ together with $\vec{Q}_{ij}$ form a set of linearly independent vectors. This means that we must have $V_A(V_A+1)/2$ linearly independent vectors in $X \leq V_A + 273$ dimensional space. Hence,

$$\frac{V_A(V_A+1)}{2} \leq X \leq V_A + 273. \tag{A.149}$$

From this we obtain the bound $V_A \leq 24$.

Put together with the bound (A.145) we obtain

$$V_A \leq 17\,, \tag{A.150}$$

as desired. □

A.6 Proof of Minimal Charge Condition for $SU(13)\times U(1)$ Models

In this section we prove that when

$$r = 84n + 43 = 2 \times 3 \times 7 \times (2n+1) + 1 \tag{A.151}$$

$$s = 182n + 92 = 7 \times 13 \times (2n+1) + 1 \tag{A.152}$$

for integer n, then the integers a and $(-3a - 2f/3)$ for

$$a = 13r^2 - 234rs - 51s^2 \tag{A.153}$$

$$f = 24(13r^2 + 3s^2) \tag{A.154}$$

are mutually prime. Let us define

$$g \equiv gcd(a, -3a - \frac{2}{3}f) = gcd(a, \frac{2}{3}f)\,. \tag{A.155}$$

Our goal is to show that $g = 1$.

We first acknowledge that r and s are mutually prime. This is because

$$gcd(r, s) | (13r - 6s) \tag{A.156}$$

and

$$13r - 6s = 7\,. \tag{A.157}$$

It is clear, however, that 7 and r are mutually prime. Therefore, $gcd(r, s)$ must be 1, and hence r and s must be mutually prime. Meanwhile, a is odd since r is even and s is odd. Therefore g must also be odd, i.e. $2 \nmid g$. Also, g is not divisible by 3. We can show $3 \nmid g$ by noting that $g \mid (13r^2 + 3s^2)$ and that

$$13r^2 + 3s^2 \equiv 1 \pmod 3\,. \tag{A.158}$$

Let us show that $g = 1$. By definition

$$\begin{aligned} g &= gcd(a, \frac{2f}{3}) = gcd(13r^2 - 234rs - 51s^2, 16(13r^2 + 3s^2)) \\ &= gcd(13r^2 - 234rs - 51s^2, 13r^2 + 3s^2)\,, \end{aligned} \tag{A.159}$$

where we have used the fact that $2 \nmid g$. Using standard properties of the greatest common divisor, we further find that

$$\begin{aligned} g &= gcd(13r^2 - 234rs - 51s^2, 13r^2 + 3s^2) \\ &= gcd(-234rs - 54s^2, 13r^2 + 3s^2) \\ &= gcd(-234r - 54s, 13r^2 + 3s^2) = gcd(-18(13r + 3s), 13r^2 + 3s^2) \\ &= gcd(13r + 3s, 13r^2 + 3s^2) \end{aligned} \tag{A.160}$$

In the penultimate line we have used the fact that

$$gcd(s, 13r^2 + 3s^2) = gcd(s, 13r^2) = 1 \tag{A.161}$$

since $s \equiv 1 \pmod{13}$ and s and r are mutually prime. In the last line we have used $2 \nmid g$ and $3 \nmid g$.

Therefore, g must be a divisor of

$$-(13r + 3s)(13 - 3s) + 13(13r^2 + 3s^2) = 48s^2\,. \tag{A.162}$$

We have seen in (A.161) that s is mutually prime with $13r^2 + 3s^2$. Therefore, g is mutually prime with s and hence

$$g|2^3 \times 3\,. \tag{A.163}$$

We, however, know that $2 \nmid g$ and $3 \nmid g$. This proves that $g = 1$, as desired. □

Appendix B

Appendices for Chapter 4

B.1 Lie Algebra and Intersection Theory

In this appendix, we show that the normalized coroot inner-product matrix, and the Cartan matrix defined as

$$\mathcal{C}_{IJ} = \frac{1}{\lambda(\mathcal{G})} \frac{4\langle \alpha_I, \alpha_J \rangle}{\langle \alpha_I, \alpha_I \rangle \langle \alpha_J, \alpha_J \rangle} \tag{B.1}$$

$$C_{IJ} = \frac{2\langle \alpha_I, \alpha_J \rangle}{\langle \alpha_I, \alpha_I \rangle} \tag{B.2}$$

are related to the intersection matrix of cycles obtained by blowing up singular fibers. α_I are simple roots of the Lie algebra. The simple roots are normalized by fixing the normalization of the matrices $\{T_i\}$ that generate the Cartan sub-algebra such that

$$\mathrm{tr} T_i T_j = \delta_{ij} \, , \tag{B.3}$$

where the trace is taken in the fundamental representation. Therefore we see that the normalization of the roots depend on the Lie group, rather than the Lie algebra. The Cartan matrix, however, clearly only depends on the Lie algebra rather than the Lie group from its definition; it is independent of the normalization of the roots. We have shown that the same holds for the normalized coroot matrix $\mathcal{C}_{IJ}$ in appendix A.1.

To make a precise statement relating these matrices to the intersection theory of a resolved codimension-one singularity on the base, let us set up the context. Suppose there is a singular fiber of Calabi-Yau threefold X fibered over a curve b in the base that gives an enhanced gauge symmetry with Lie algebra $\mathfrak{g}$. One can resolve this singularity by blowing up r independent $\mathbb{P}^1$'s, where r is the rank of $\mathfrak{g}$. Denote the r $\mathbb{P}^1$'s as $\chi_1, \cdots, \chi_r$. Also denote the r four-cycles obtained by fibering the $\mathbb{P}^1$'s along b as $C_1, \cdots, C_r$. In the case of non-simply laced gauge groups, a single fiber of C_I might contain multiple copies $\mathbb{P}^1$'s because monodromy of the fibers will map rational curves in the fiber into each other. We denote the monodromy invariant fibers γ_I, so that C_I is obtained by fibering γ_I over b.

The statement is that

$$\begin{aligned} B_\alpha \cdot C_I \cdot C_J &= -\mathcal{C}_{IJ} b_\alpha \\ C_I \cdot c_J &= -C_{IJ} \end{aligned} \tag{B.4}$$

where B_α are the four-cycles that are obtained by fibering the elliptic fiber over elements of $H_2(\mathcal{B})$. We check this statement by comparing the Cartan matrix and the normalized coroot matrix $\mathcal{C}$ computed in section A.1 and the explicit intersection numbers to obtain (B.4).

We verify the equations pictorially. For each gauge algebra, we draw the corresponding tree of resolved rational curves and label the linearly independent curves as α_I and label the monodromy invariant combinations of rational curves γ_J according to [133]. The curves χ_I that M2 branes wrap to give root vectors are identified with α_I. The four cycles C_J dual to non-abelian gauge field components of the coroot basis elements $\mathcal{T}_J$ of the Cartan are obtained by fibering γ_J over b.

We verify that

$$\begin{aligned} \gamma_I \cdot \gamma_J &= -\mathcal{C}_{IJ} \\ \gamma_I \cdot \alpha_J &= -C_{IJ} \end{aligned} \tag{B.5}$$

where the intersections are taken within a local complex dimension two slice of the manifold transverse to b at a generic point in b. These two equations imply (B.4) since

$$B_\alpha \cdot C_I \cdot C_J = b_\alpha(\gamma_I \cdot C_J) = -b_\alpha \mathcal{C}_{IJ}\,, \tag{B.6}$$

$$C_I \cdot \chi_J = C_I \cdot \alpha_J = -C_{IJ}\,. \tag{B.7}$$

The latter equalities of the two equations follow from (B.5) since C_I is a γ_I fibration over b. We note that all the data are defined for the Lie algebra, and not sensitive to the Lie group.

We verify these equations first for simply laced Lie algebras, and then for non-simply laced Lie algebras. All of the facts stated in this appendix either can be found in, or are implicit in [118, 119, 132, 133, 134, 135, 179], but we have stated them in a way that is convenient for our purposes.

B.1.1 Simply Laced Lie Algebras

For simply laced Lie algebras, the monodromy group of the blown-up singular fibers are trivial and the blown-up rational curves form the Dynkin diagram of the corresponding Lie algebra, except possibly for the case of $\mathfrak{su}(2)$. It turns out that $\alpha_I = \gamma_I$ for all the simply laced Lie algebras.

The self-intersection number of a rational curve is (-2) and the intersection number between adjacent rational curves is 1. The intersection number between non-adjacent curves are 0. The intersection number satisfies linearity conditions, *i.e.*,

$$c \cdot (\lambda_1 c_1 + \lambda_2 c_2) = \lambda_1 (c \cdot c_1) + \lambda_2 (c \cdot c_2)\,. \tag{B.8}$$

Based on these rules, we can verify that the resolved fibers that give the A, D, E algebra satisfy (B.5).

I_{n+1} fibers, or possibly the III/IV fiber for A_1/A_2 respectively, give the A_n Lie algebra. The tree of blown-up rational curves of the resolved I_{n+1} fiber—or the

III/IV fiber for A_1/A_2—is depicted in figure B-1. It is clear that

$$(\gamma_I \cdot \gamma_J) = (\gamma_I \cdot \alpha_J) = \begin{pmatrix} -2 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -2 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 1 & -2 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 1 & -2 \end{pmatrix} = -C_{IJ} = -\mathcal{C}_{IJ}\,. \quad \text{(B.9)}$$

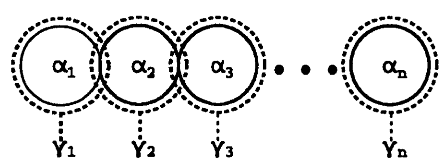


Figure B-1: Resolved fiber for A_n. The curves α corresponding to root vectors are in solid lines while the monodromy invariant fibers γ corresponding to coroots are in dotted lines.

I^*_{n-4} fibers give the D_n Lie algebra. The tree of blown-up rational curves of the resolved I^*_{n-4} fiber is depicted in figure B-2. The intersection matrices are given by

$$(\gamma_I \cdot \gamma_J) = (\gamma_I \cdot \alpha_J) = \begin{pmatrix} -2 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 \end{pmatrix} = -C_{IJ} = -\mathcal{C}_{IJ}\,. \quad \text{(B.10)}$$

The fibers IV^*, III^* and II^* give the E_6, E_7 and E_8 Lie algebra respectively. The tree of blown-up rational curves of the resolved E_n fiber is depicted in figure B-3.

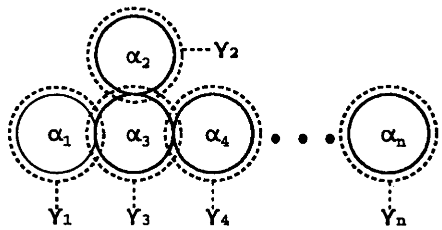


Figure B-2: Resolved fiber for D_n.

The intersection matrices are given by

$$(\gamma_I \cdot \gamma_J) = (\gamma_I \cdot \alpha_J) = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -2 \end{pmatrix} = -C_{IJ} = -\mathcal{C}_{IJ}\,. \tag{B.11}$$

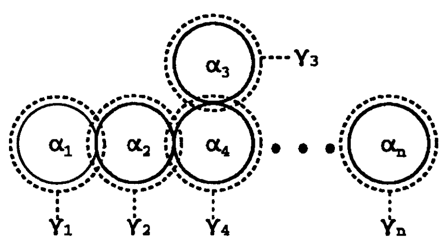


Figure B-3: Resolved fiber for E_n.

$\mathfrak{su}(2)$ can come from an I_3 or IV fiber with a $\mathbb{Z}_2$ monodromy. The resolved fiber that gives the A_1 Lie algebra in this way is given by figure B-4. The $\mathbb{Z}_2$ interchanges the two $\mathbb{P}^1$'s drawn in the figure, and hence the point where the two rational curves touch is singular. γ_1 is the monodromy invariant fiber. It is shown in [132] that the BPS states come from branes wrapping α_1 rather than the the individual components drawn as spheres in the figure. The intersection matrices are given by

$$(\gamma_I \cdot \gamma_J) = (\gamma_I \cdot \alpha_J) = \begin{pmatrix} -2 \end{pmatrix} = -C_{IJ} = -\mathcal{C}_{IJ}\,. \tag{B.12}$$

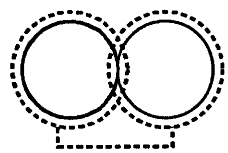


Figure B-4: Resolved fiber for A_1.

B.1.2 Non-simply Laced Lie Algebras

For non-simply laced Lie algebras, the blown-up singular fibers have non-trivial monodromy. The blown-up rational curves form the Dynkin diagram of a larger Lie algebra. Under monodromy, the rational curves are exchanged among themselves. For each fiber, we denote the independent rational curves α_I, and the monodromy invariant components of the fiber γ_I . Let us verify that the resolved fibers that give the A, D, E algebra satisfy (B.5).

The fibers $I^*_{(n-3)}$ with $\mathbb{Z}_2$ monodromy give the B_n Lie algebra. The tree of blown-up rational curves of the resolved $I^*_{(n-3)}$ fiber is depicted in figure B-5. The $\mathbb{Z}_2$ monodromy exchanges the two rational curves in γ_n. The intersection matrices are given by

$$
\begin{aligned}
(\gamma_I \cdot \gamma_J) &= \begin{pmatrix} -2 & 1 & \cdots & 0 & 0 & 0 \\ 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & \cdots & 1 & -2 & 2 \\ 0 & 0 & \cdots & 0 & 2 & -4 \end{pmatrix} = -\mathcal{C}_{IJ}\,, \\
(\gamma_I \cdot \alpha_J) &= \begin{pmatrix} -2 & 1 & \cdots & 0 & 0 & 0 \\ 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & \cdots & 0 & 2 & -2 \end{pmatrix} = -C_{IJ}\,.
\end{aligned}
\tag{B.13}
$$

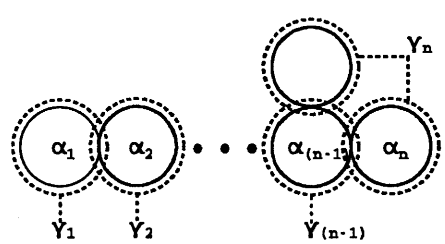


Figure B-5: Resolved fiber for B_n. The curves α corresponding to root vectors are in solid lines while the monodromy invariant fibers γ corresponding to coroots are in dotted lines.

The fibers I_{2n} or $I_{(2n+1)}$ with $\mathbb{Z}_2$ monodromy give the C_n Lie algebra. The trees of blown-up rational curves of the resolved I_{2n} and $I_{(2n+1)}$ fibers are depicted in figure B-6. The $\mathbb{Z}_2$ monodromy exchanges the two rational curves in each γ_I. Just as with the case of $\mathfrak{su}(2)$, α_1 should be taken to be equal to γ_1 [132]. The intersection matrices are given by

$$
\begin{aligned}
(\gamma_I \cdot \gamma_J) &= \begin{pmatrix} -2 & 2 & 0 & \cdots & 0 & 0 \\ 2 & -4 & 2 & \cdots & 0 & 0 \\ 0 & 2 & -4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -4 & 2 \\ 0 & 0 & 0 & \cdots & 2 & -4 \end{pmatrix} = -\mathcal{C}_{IJ}\,, \\
(\gamma_I \cdot \alpha_J) &= \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 & 0 \\ 2 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & -2 \end{pmatrix} = -C_{IJ}\,.
\end{aligned}
\tag{B.14}
$$

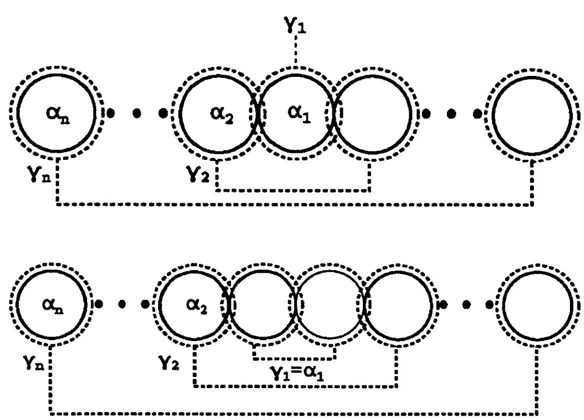


Figure B-6: Resolved fiber for C_n.

The fiber IV^* with $\mathbb{Z}_2$ monodromy gives the F_4 Lie algebra. The tree of blown-up rational curves of the resolved IV^* fiber is depicted in figure B-7. The $\mathbb{Z}_2$ monodromy exchanges the two rational curves in γ_3 and γ_4. The intersection matrices are given by

$$
(\gamma_I\!\cdot\!\gamma_J)=\begin{pmatrix}-2&1&0&0\\1&-2&2&0\\0&2&-4&2\\0&0&2&-4\end{pmatrix}=-\mathcal{C}_{IJ}\,,\ \ (\gamma_I\!\cdot\!\alpha_J)=\begin{pmatrix}-2&1&0&0\\1&-2&1&0\\0&2&-2&1\\0&0&1&-2\end{pmatrix}=-C_{IJ}\,. \tag{B.15}
$$

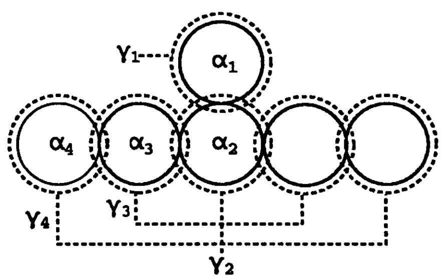


Figure B-7: Resolved fiber for F_4.

The fiber I_0^* with $\mathbb{Z}_3$ or $\mathfrak{S}_3$ monodromy gives the G_2 Lie algebra. The tree of blown-up rational curves of the resolved I_0^* fiber is depicted in figure B-8. The $\mathbb{Z}_3$ or $\mathfrak{S}_3$ monodromy exchanges the three rational curves in γ_2. The intersection matrices are given by

$$
(\gamma_I\cdot\gamma_J)=\begin{pmatrix}-2&3\\3&-6\end{pmatrix}=-\mathcal{C}_{IJ}\,,\quad(\gamma_I\cdot\alpha_J)=\begin{pmatrix}-2&1\\3&-2\end{pmatrix}=-C_{IJ}\,. \tag{B.16}
$$

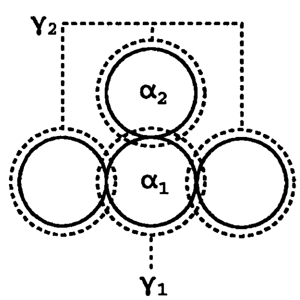


Figure B-8: Resolved fiber for G_2.

B.2 Proof of Intersection Equations for $\mathcal{S}_n$ of Type S or C

We prove the intersection equations

$$\pi(\mathcal{S}_1 \cdot \mathcal{S}_2) \cdot \pi(\mathcal{S}_3 \cdot \mathcal{S}_4) + \pi(\mathcal{S}_1 \cdot \mathcal{S}_3) \cdot \pi(\mathcal{S}_2 \cdot \mathcal{S}_4) + \pi(\mathcal{S}_1 \cdot \mathcal{S}_4) \cdot \pi(\mathcal{S}_2 \cdot \mathcal{S}_3)$$
$$= \sum_r (c_r \cdot \mathcal{S}_1)(c_r \cdot \mathcal{S}_2)(c_r \cdot \mathcal{S}_3)(c_r \cdot \mathcal{S}_4) + \sum_\rho (2g_\rho - 2)(\chi_\rho \cdot \mathcal{S}_1)(\chi_\rho \cdot \mathcal{S}_2)(\chi_\rho \cdot \mathcal{S}_3)(\chi_\rho \cdot \mathcal{S}_4) \tag{B.17}$$

and

$$6K \cdot \pi(\mathcal{S}_1 \cdot \mathcal{S}_2) = \sum_r (c_r \cdot \mathcal{S}_1)(c_r \cdot \mathcal{S}_2) + \sum_\rho (2g_\rho - 2)(\chi_\rho \cdot \mathcal{S}_1)(\chi_\rho \cdot \mathcal{S}_2) \tag{B.18}$$

when

$$\mathcal{S}_n \in \{T_{I,\kappa}, S_i\}\,. \tag{B.19}$$

We first check the case when $\mathcal{S}_n$ are all of type S, then check the case when $\mathcal{S}_n$ are all of type C. Finally we check the case when there is a mixture of type S and C cycles among $\mathcal{S}_n$. We refer to the first equation (B.17) the quartic equation and the second equation (B.18) the quadratic equation throughout this appendix.

B.2.1 Type S Cycles Only

Take $\mathcal{S}_n$ to be type S cycles $\{S_i, S_j, S_k, S_l\}$. $\mathcal{S}_n$ are dual to abelian vector multiplets. Then using the result (3.84)

$$b_{ij} = -\pi(S_i \cdot S_j)\,, \tag{B.20}$$

the last equation of (2.26) can be rewritten in the form

$$\pi(S_i \cdot S_j) \cdot \pi(S_k \cdot S_l) + \pi(S_i \cdot S_k) \cdot \pi(S_j \cdot S_l) + \pi(S_i \cdot S_l) \cdot \pi(S_j \cdot S_k) = \sum_x q_i^x q_j^x q_k^x q_l^x\,. \tag{B.21}$$

We have used x to index all the hypermultiplets in the theory and q_n^x denotes the charge of hypermultiplet x under the $U(1)$ vector field dual to S_n.

We note that all hypermultiplets charged under these vector multiplets come from M2 branes wrapping type I curves, which are precisely c_r. Recall that

$$S_j \cdot \chi_\rho = 0 \tag{B.22}$$

for all ρ by the construction of S-type cycles.

Then since the charge of the hypermultiplet coming from wrapping branes on c_r under vector field n is $c_r \cdot S_i$, the last equation of (2.26) is indeed equivalent to the equation

$$\begin{aligned}&\pi(S_i \cdot S_j) \cdot \pi(S_k \cdot S_l) + \pi(S_i \cdot S_k) \cdot \pi(S_j \cdot S_l) + \pi(S_i \cdot S_l) \cdot \pi(S_j \cdot S_k) \\ &= \sum_r (c_r \cdot S_i)(c_r \cdot S_j)(c_r \cdot S_k)(c_r \cdot S_l) + \sum_\rho (2g_\rho - 2)(\chi_\rho \cdot S_i)(\chi_\rho \cdot S_j)(\chi_\rho \cdot S_k)(\chi_\rho \cdot S_l)\end{aligned} \tag{B.23}$$

since the latter term on the right hand side is zero.

Similarly, since the vector a is identified with the canonical class of the base K, the second equation in (2.23) implies that

$$\begin{aligned}6K \cdot \pi(S_i \cdot S_j) &= -6a \cdot b_{ij} = \sum_x q_i^x q_j^x \\ &= \sum_r (c_r \cdot S_i)(c_r \cdot S_j) + \sum_\rho (2g_\rho - 2)(\chi_\rho \cdot S_i)(\chi_\rho \cdot S_j)\end{aligned} \tag{B.24}$$

for two type S four-cycles S_i, S_j. Hence we have shown that (B.17) and (B.18) hold when S_n are all of type S.

B.2.2 Type C Cycles Only

We prove

$$\begin{aligned}\pi(T_{I,\kappa}\cdot T_{J,\nu})\cdot\pi(T_{K,\lambda}\cdot T_{L,\mu}) &+ (2 \text{ other groupings}) \\ &= \sum_r (c_r\cdot T_{I,\kappa})(c_r\cdot T_{J,\nu})(c_r\cdot T_{K,\lambda})(c_r\cdot T_{L,\mu}) \\ &+ \sum_\rho (2g_\rho - 2)(\chi_\rho\cdot T_{I,\kappa})(\chi_\rho\cdot T_{J,\nu})(\chi_\rho\cdot T_{K,\lambda})(\chi_\rho\cdot T_{L,\mu})\end{aligned} \tag{B.25}$$

and

$$6K\cdot\pi(T_{I,\kappa}\cdot T_{J,\nu}) = \sum_r (c_r\cdot T_{I,\kappa})(c_r\cdot T_{J,\nu}) + \sum_\rho (2g_\rho - 2)(\chi_\rho\cdot T_{I,\kappa})(\chi_\rho\cdot T_{J,\nu}) \tag{B.26}$$

for $T_{I,\kappa}$ of type C. The quartic equation is only non-trivial when the $\kappa, \nu, \lambda, \mu$ are equal in pairs—this includes the case when they are all equal. The quadratic equation is only non-trivial when κ and ν are equal.

The two statements above follow from three facts.

1. $\pi(T_{I,\kappa}\cdot T_{J,\nu})$ satisfies

$$\pi(T_{I,\kappa}\cdot T_{J,\nu}) = -\delta_{\kappa\nu} b_\kappa \mathcal{C}_{IJ,\kappa}\,, \tag{B.27}$$

so the left-hand side of the quartic equation is zero unless $\kappa, \nu, \lambda, \mu$ are equal in pairs. Similarly, the left hand side of the quadratic equation is zero unless κ and ν are equal.

2. Unless $\kappa, \nu, \lambda, \mu$ are equal in pairs,

$$\sum_r (c_r\cdot T_{I,\kappa})(c_r\cdot T_{J,\nu})(c_r\cdot T_{K,\lambda})(c_r\cdot T_{L,\mu}) = \sum_R k_R \mathrm{tr}_R \mathcal{T}_{M,\eta} = 0 \tag{B.28}$$

for some constants k_R and M, η. Similarly, unless κ and ν are equal,

$$\sum_r (c_r\cdot T_{I,\kappa})(c_r\cdot T_{J,\nu}) = \sum_R k_R \mathrm{tr}_R \mathcal{T}_{M,\eta} = 0 \tag{B.29}$$

for some constants k_R and M, η. This is because the hypermultiplets coming from type I cycles always can be organized into representations of the Lie algebra.

3. χ_ρ can be organized into positive(or negative, depending on convention) roots of the simple Lie algebra factors as $\{\chi_\rho\} = \{\chi_{s,\kappa}\}$. Any $\chi_{s,\kappa}$ for a given κ is a linear combination of curves $\chi_{I,\kappa}$ corresponding to the simple roots of $\mathcal{G}_\kappa$. Since

$$\chi_{I,\kappa} \cdot T_{J,\nu} = -\delta_{\kappa\nu} C_{IJ,\kappa}\,, \tag{B.30}$$

the equation

$$(\chi_\rho \cdot T_{I,\kappa})(\chi_\rho \cdot T_{I,\nu}) = 0 \tag{B.31}$$

holds for $\kappa \neq \nu$.

Therefore (B.25) is non-trivial only when $\kappa, \nu, \lambda, \mu$ are equal in pairs, and (B.26) is non-trivial only when $\kappa = \nu$.

Now let us write the anomaly equations in a form more convenient to our purposes. The anomaly equations concerning only non-abelian gauge group factors implies that the following holds for all elements T_κ, T_ν of the Cartan of the gauge groups $\mathcal{G}_\kappa, \mathcal{G}_\nu$:

$$a \cdot b_\kappa \frac{\mathrm{tr} {T_\kappa}^2}{\lambda_\kappa} = \frac{1}{6}(\mathrm{tr}_{\mathrm{Adj}_\kappa} {T_\kappa}^2 - \sum_I x_R \mathrm{tr}_R {T_\kappa}^2) \tag{B.32}$$

$$b_\kappa \cdot b_\kappa (\frac{\mathrm{tr} {T_\kappa}^2}{\lambda_\kappa})^2 = \frac{1}{3}(\sum_I x_R \mathrm{tr}_R {T_\kappa}^4 - \mathrm{tr}_{\mathrm{Adj}_\kappa} {T_\kappa}^4) \tag{B.33}$$

$$b_\kappa \cdot b_\nu (\frac{\mathrm{tr} {T_\kappa}^2}{\lambda_\kappa})(\frac{\mathrm{tr} {T_\nu}^2}{\lambda_\nu}) = \sum_{R,S} x_{RS} \mathrm{tr}_R {T_\kappa}^2 \mathrm{tr}_S {T_\nu}^2 \quad (\kappa \neq \nu) \tag{B.34}$$

Let us take $T_\kappa = t_I \mathcal{T}_{I,\kappa}$ and $T_\nu = s_I \mathcal{T}_{I,\kappa}$ where I runs over the indices of the coroot basis of the Cartan sub-algebra of each gauge group. Expanding the equalities above, we obtain polynomials with respect to t_I and s_I on both sides of the equations. The anomaly equations must hold for any value of t_I and s_I. Hence all the coefficients of the polynomials must be identical. By identifying the coefficients, we obtain

$$
\begin{aligned}
a \cdot b_\kappa \mathcal{C}_{IJ,\kappa} &= \frac{1}{6}(\mathrm{tr}_{\mathrm{Adj}_\kappa} \mathcal{T}_{I,\kappa}\mathcal{T}_{J,\kappa} - \sum_I x_R \mathrm{tr}_R \mathcal{T}_{I,\kappa}\mathcal{T}_{J,\kappa}) \\
b_\kappa \cdot b_\kappa(\mathcal{C}_{IJ,\kappa}\mathcal{C}_{KL,\kappa} + \mathcal{C}_{IK,\kappa}\mathcal{C}_{JL,\kappa} + \mathcal{C}_{IL,\kappa}\mathcal{C}_{JK,\kappa}) &= \\
&\sum_I x_R \mathrm{tr}_R \mathcal{T}_{I,\kappa}\mathcal{T}_{J,\kappa}\mathcal{T}_{K,\kappa}\mathcal{T}_{L,\kappa} - \mathrm{tr}_{\mathrm{Adj}_\kappa} \mathcal{T}_{I,\kappa}\mathcal{T}_{J,\kappa}\mathcal{T}_{K,\kappa}\mathcal{T}_{L,\kappa} \\
b_\kappa \cdot b_\nu \mathcal{C}_{IJ,\kappa}\mathcal{C}_{KL,\nu} &= \sum_{R,S} x_{RS} \mathrm{tr}_R \mathcal{T}_{I,\kappa}\mathcal{T}_{J,\kappa} \mathrm{tr}_S \mathcal{T}_{K,\nu}\mathcal{T}_{L,\nu}
\end{aligned}
\tag{B.35}
$$

for $\kappa \neq \nu$.

We can write all the elements of the right-hand sides as a sum of products of the charge of each vector or hypermultiplet under each Cartan element. Each charged multiplet corresponds to a type I or a type F rational curve, and its charges are given by the intersection numbers of the curve with the four-cycles of type C. Rewriting the right-hand sides of the equations we obtain

$$
\begin{aligned}
-6a \cdot b_\kappa \mathcal{C}_{IJ,\kappa} &= \\
&\sum_r (c_r \cdot T_{I,\kappa})(c_r \cdot T_{J,\kappa}) + \sum_\rho (2g_\rho - 2)(\chi_\rho \cdot T_{I,\kappa})(\chi_\rho \cdot T_{J,\kappa}) \\
b_\kappa \cdot b_\kappa(\mathcal{C}_{IJ,\kappa}\mathcal{C}_{KL,\kappa} + \mathcal{C}_{IK,\kappa}\mathcal{C}_{JL,\kappa} + \mathcal{C}_{IL,\kappa}\mathcal{C}_{JK,\kappa}) &= \\
&\sum_r (c_r \cdot T_{I,\kappa})(c_r \cdot T_{J,\kappa})(c_r \cdot T_{K,\kappa})(c_r \cdot T_{L,\kappa}) \\
&+ \sum_\rho (2g_\rho - 2)(\chi_\rho \cdot T_{I,\kappa})(\chi_\rho \cdot T_{J,\kappa})(\chi_\rho \cdot T_{K,\kappa})(\chi_\rho \cdot T_{L,\kappa}) \\
b_\kappa \cdot b_\nu \mathcal{C}_{IJ,\kappa}\mathcal{C}_{KL,\nu} &= \\
&\sum_r (c_r \cdot T_{I,\kappa})(c_r \cdot T_{J,\kappa})(c_r \cdot T_{K,\nu})(c_r \cdot T_{L,\nu}) \\
&+ \sum_\rho (2g_\rho - 2)(\chi_\rho \cdot T_{I,\kappa})(\chi_\rho \cdot T_{J,\kappa})(\chi_\rho \cdot T_{K,\nu})(\chi_\rho \cdot T_{L,\nu})
\end{aligned}
\tag{B.36}
$$

Note that the curve χ_ρ contributes $2g_\rho$ hypermultiplets and two vector multiplets [144, 145], as explained in section 3.3.1. The vector multiplet always contributes with a negative sign with respect to the contribution of the hypermultiplet to the right hand sides of the equations. The last term of the last equation is zero.

Finally using

$$\pi(T_{I,\kappa} \cdot T_{J,\nu}) = -\delta_{\kappa\nu} b_\kappa \mathcal{C}_{IJ,\kappa}\,, \tag{B.37}$$

and the fact that a is equal to the canonical class K of the base, the three equations translate into

$$\begin{aligned}
&6K \cdot \pi(T_{I,\kappa} \cdot T_{J,\kappa}) = \\
&\qquad \sum_r (c_r \cdot T_{I,\kappa})(c_r \cdot T_{J,\kappa}) + \sum_\rho (2g_\rho - 2)(\chi_\rho \cdot T_{I,\kappa})(\chi_\rho \cdot T_{J,\kappa}) \\
&\pi(T_{I,\kappa} \cdot T_{J,\kappa}) \cdot \pi(T_{K,\kappa} \cdot T_{L,\kappa}) + (\text{2 other groupings}) = \\
&\qquad \sum_r (c_r \cdot T_{I,\kappa})(c_r \cdot T_{J,\kappa})(c_r \cdot T_{K,\kappa})(c_r \cdot T_{L,\kappa}) \\
&\qquad + \sum_\rho (2g_\rho - 2)(\chi_\rho \cdot T_{I,\kappa})(\chi_\rho \cdot T_{J,\kappa})(\chi_\rho \cdot T_{K,\kappa})(\chi_\rho \cdot T_{L,\kappa}) \\
&\pi(T_{I,\kappa} \cdot T_{J,\kappa}) \cdot \pi(T_{K,\nu} \cdot T_{L,\nu}) + (\text{2 other groupings}) = \\
&\qquad \sum_r (c_r \cdot T_{I,\kappa})(c_r \cdot T_{J,\kappa})(c_r \cdot T_{K,\nu})(c_r \cdot T_{L,\nu}) \\
&\qquad + \sum_\rho (2g_\rho - 2)(\chi_\rho \cdot T_{I,\kappa})(\chi_\rho \cdot T_{J,\kappa})(\chi_\rho \cdot T_{K,\nu})(\chi_\rho \cdot T_{L,\nu})
\end{aligned} \tag{B.38}$$

We note that in the last equation, the two other groupings of cycles that are not written down are zero.

B.2.3 Both Type S and C Cycles

The quadratic equation is always trivial when $\mathcal{S}_1$ and $\mathcal{S}_2$ are each of type S and C, for the following reasons.

1. The left hand side of the quadratic equation is trivially zero since

$$\pi(\mathcal{S}_i \cdot T_{I,\kappa}) = 0\,, \tag{B.39}$$

due to the construction of type S cycles.

2. The hypermultiplets come in representations of the non-abelian Lie algebra. Hence,

$$\sum_r (c_r \cdot S_i)(c_r \cdot T_{I,\kappa}) = \sum_R k_R \mathrm{tr}_R \mathcal{T}_{I,\kappa} = 0\,. \tag{B.40}$$

3. S_i do not intersect type χ_ρ curves by construction. Hence,

$$\sum_\rho (2g_\rho - 2)(\chi_\rho \cdot S_i)(\chi_\rho \cdot T_{I,\kappa}) = 0\,. \tag{B.41}$$

Similarly we can show that the non-trivial cases to check for the quartic equation are, without loss of generality, when

1. $\mathcal{S}_1 = \mathcal{T}_{I,\kappa},\ \mathcal{S}_2 = \mathcal{T}_{J,\kappa},\ \mathcal{S}_3 = \mathcal{T}_{K,\kappa},\ \mathcal{S}_4 = S_i$.
2. $\mathcal{S}_1 = \mathcal{T}_{I,\kappa},\ \mathcal{S}_2 = \mathcal{T}_{J,\kappa},\ \mathcal{S}_3 = S_i,\ \mathcal{S}_4 = S_j$.

From the anomaly equations, we can show that for any element of the Cartan T_κ of $\mathcal{G}_\kappa$

$$\begin{aligned} 0 &= \sum_{R,q_i} x_{R,q_i} q_i \mathrm{tr}_R T_\kappa^3 \\ b_\kappa \cdot b_{ij} \frac{\mathrm{tr} T_\kappa^2}{\lambda_\kappa} &= \sum_{R,q_i,q_j} x_{R,q_i,q_j} q_i q_j \mathrm{tr} T_\kappa^2 \end{aligned} \tag{B.42}$$

Setting $T_\kappa = \mathcal{T}_{I,\kappa} t_I$, we can write both sides of the two equations as polynomials with respect to t_I. Since the equality must hold for all values of t_I, the coefficients of the polynomials must match on both sides, and hence

$$\begin{aligned} 0 &= \sum_{R,q_i} x_{R,q_i} \mathrm{tr}_R (\mathcal{T}_{I,\kappa} \mathcal{T}_{J,\kappa} \mathcal{T}_{L,\kappa}) q_i \\ b_\kappa \cdot b_{ij} \mathcal{C}_{IJ} &= \sum_{R,q_i,q_j} x_{R,q_i,q_j} \mathrm{tr}_R (\mathcal{T}_{I,\kappa} \mathcal{T}_{J,\kappa}) q_i q_j \end{aligned} \tag{B.43}$$

As before, we can write all the elements of the right-hand sides as a sum of products of the charge of each vector or hypermultiplet under $\mathcal{T}_{I,\kappa}$, $U(1)_i$ or $U(1)_j$. Each charged hypermultiplet has a corresponding type I rational curve, and its charge is given by the intersection numbers of the curve with the four-cycles of type C or S. Rewriting the right-hand sides of the equations we obtain

$$\begin{aligned} 0 &= \sum_r (c_r \cdot T_{I,\kappa})(c_r \cdot T_{J,\kappa})(c_r \cdot T_{K,\kappa})(c_r \cdot S_i) \\ b_\kappa \cdot b_{ij} \mathcal{C}_{IJ,\kappa} &= \sum_r (c_r \cdot T_{I,\kappa})(c_r \cdot T_{J,\kappa})(c_r \cdot S_i)(c_r \cdot S_j) \end{aligned} \tag{B.44}$$

Using

$$b_\kappa \mathcal{C}_{IJ,\kappa} = -\pi(\mathcal{T}_{I,\kappa} \cdot \mathcal{T}_{J,\kappa}), \quad \pi(\mathcal{T}_{I,\kappa} \cdot S_i) = 0, \quad b_{ij} = -\pi(S_i \cdot S_j)\,, \tag{B.45}$$

we obtain the final expressions by rewriting the equations:

$$\begin{aligned} &\pi(\mathcal{T}_{I,\kappa} \cdot S_i) \cdot \pi(\mathcal{T}_{J,\kappa} \cdot \mathcal{T}_{L,\kappa}) + (\text{2 other groupings}) \\ &\qquad = \sum_r (c_r \cdot T_{I,\kappa})(c_r \cdot T_{J,\kappa})(c_r \cdot T_{K,\kappa})(c_r \cdot S_i) \\ &\qquad + \sum_\rho (2g_\rho - 2)(\chi_\rho \cdot T_{I,\kappa})(\chi_\rho \cdot T_{J,\kappa})(\chi_\rho \cdot T_{K,\kappa})(\chi_\rho \cdot S_i) \\ &\pi(\mathcal{T}_{I,\kappa} \cdot \mathcal{T}_{J,\kappa}) \cdot \pi(S_i \cdot S_j) + (\text{2 other groupings}) \\ &\qquad = \sum_r (c_r \cdot T_{I,\kappa})(c_r \cdot T_{J,\kappa})(c_r \cdot S_i)(c_r \cdot S_j) \\ &\qquad + \sum_\rho (2g_\rho - 2)(\chi_\rho \cdot T_{I,\kappa})(\chi_\rho \cdot T_{J,\kappa})(\chi_\rho \cdot S_i)(\chi_\rho \cdot S_j) \end{aligned} \tag{B.46}$$

Note that the left hand side of the first equation is zero, and that the two other groupings in the second equation are zero. The second term on the right hand side of both equations are zero since $S_i \cdot \chi_\rho = 0$ by construction of type S cycles. □

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