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On the Riemann Tensor in Double Field Theory

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Abstract

Double field theory provides T-duality covariant generalized tensors that are natural extensions of the scalar and Ricci curvatures of Riemannian geometry. We search for a similar extension of the Riemann curvature tensor by developing a geometry based on the generalized metric and the dilaton. We find a duality covariant Riemann tensor whose contractions give the Ricci and scalar curvatures, but that is not fully determined in terms of the physical fields. This suggests that α' corrections to the effective action require α' corrections to T-duality transformations and/or generalized diffeomorphisms. Further evidence to this effect is found by an additional computation that shows that there is no T-duality invariant four-derivative object built from the generalized metric and the dilaton that reduces to the square of the Riemann tensor.

Contents

1 Introduction

Among the celebrated dualities of string theory T-duality is arguably the most intriguing, for it directly hints at novel geometrical structures, transcending the usual framework of differential geometry. Recently, a so-called double field theory has been constructed that manifests some of these features at the level of space-time theories for the massless sector of string theory. Specifically, here the space-time coordinates are doubled in order to realize the 'T-duality group' $O(D,D)$ geometrically, while introducing an $O(D,D)$ covariant constraint that locally removes the dependence on half of the coordinates [1–3]. (See [4–26] for previous work and further developments.)

The formulation of double field theory that is perhaps the most intuitive and which will be used throughout this paper is the generalized metric formulation. The generalized metric $\mathcal{H}_{MN}$ is the $O(D,D)$-valued symmetric tensor

$$\mathcal{H}_{MN} \;=\; \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij}-b_{ik}g^{kl}b_{lj} \end{pmatrix}\,, \tag{1.1}$$

which combines the space-time metric g_{ij} and the Kalb-Ramond two-form b_{ij}. Here, $M,N,\ldots = 1,\ldots,2D$ are fundamental $O(D,D)$ indices, where D denotes the total number of space-time dimensions. Being an element of $O(D,D)$, the generalized metric satisfies

$$\mathcal{H}^{MK}\mathcal{H}_{KN} \;=\; \delta^{M}{}_{N}\;, \tag{1.2}$$

where

$$\mathcal{H}^{MN} \;\equiv\; \eta^{MK}\,\eta^{NL}\,\mathcal{H}_{KL}\;, \qquad \eta^{MN} \;=\; \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}\,, \tag{1.3}$$

and η^{MN} is the $O(D,D)$ invariant metric that will be used to raise and lower $O(D,D)$ indices. The theory also includes the duality invariant dilaton field d related to the standard dilaton ϕ via the field redefinition $e^{-2d} = \sqrt{g}e^{-2\phi}$.

Double field theory features a gauge symmetry parameterized by an $O(D,D)$ vector parameter $\xi^M = (\tilde{\xi}_i\,, \xi^i)$ that combines the diffeomorphism parameter ξ^i and the b-field gauge parameter $\tilde{\xi}_i$. We will refer to this gauge symmetry as 'generalized diffeomorphisms'. It acts on the fundamental variables as:

$$\begin{aligned} \delta_\xi\mathcal{H}_{MN} \;&=\; \xi^P\partial_P\mathcal{H}_{MN} + \big(\partial_M\xi^P - \partial^P\xi_M\big)\mathcal{H}_{PN} + \big(\partial_N\xi^P - \partial^P\xi_N\big)\mathcal{H}_{MP}\;,\\ \delta_\xi\big(e^{-2d}\big) \;&=\; \partial_M\big(\xi^M e^{-2d}\big)\;, \end{aligned} \tag{1.4}$$

where $\partial_M = (\tilde{\partial}^i, \partial_i)$ is the partial derivative with respect to the doubled coordinates $X^M = (\tilde{x}_i, x^i)$. We see that e^{-2d} transforms as a scalar density. The transformation rule in the top line of (1.4) defines a generalized Lie derivative $\delta_\xi\mathcal{H}_{MN} = \widehat{\mathcal{L}}_\xi\mathcal{H}_{MN}$, that can be defined similarly for arbitrary $O(D,D)$ tensors. An $O(D,D)$ tensor transforming under generalized diffeomorphisms with a generalized Lie derivative is called a generalized tensor. The double field theory action can be written as

$$S \;=\; \int dx d\tilde{x}\, e^{-2d}\,\mathcal{R}(\mathcal{H},d)\;, \tag{1.5}$$

where $\mathcal{R}$ is an $O(D,D)$ invariant function of $\mathcal{H}$ and d that is a generalized scalar,

$$\delta_\xi\mathcal{R}(\mathcal{H},d) \;=\; \xi^P\partial_P\mathcal{R}(\mathcal{H},d)\;, \tag{1.6}$$

making the gauge invariance of (1.5) manifest. In order to verify the gauge variation (1.6) the following 'strong constraint' is required:

$$\eta^{MN}\partial_M\partial_N \;=\; \partial^M\partial_M \;=\; 0\;, \tag{1.7}$$

when acting on arbitrary fields and parameters and all their products. This constraint implies that locally all fields depend only on half of the coordinates, e.g., only the x^i.

The scalar $\mathcal{R}$ can be viewed as a generalized scalar curvature: it reduces to the scalar curvature when we set $b = \phi = 0$ and choose the duality frame $\tilde{\partial} = 0$. Moreover, the variation of (1.5) with respect to $\mathcal{H}_{MN}$ gives rise to an $O(D, D)$ tensor $\mathcal{R}_{MN}(\mathcal{H}, d)$ that is in fact a generalized tensor and can be viewed as a generalized Ricci tensor; its non-vanishing components reduce to the Ricci tensor when we set $b = \phi = 0$ and choose the duality frame $\tilde{\partial} = 0$. Given this similarity with the corresponding tensors of Riemannian geometry it is natural to look for a systematic way to construct these curvatures starting with Christoffel-like connections and a generalized version of the Riemann tensor. Indeed, it would be useful to have a T-duality covariant generalization of the full Riemann tensor in order to write general higher-derivative or α' corrections to the effective action.

In searching for a generalized four-index Riemann tensor $\mathcal{R}_{MNPQ}$ it is useful to make a list of properties that we may want this tensor to satisfy:

1. It is a tensor under $O(D, D)$.
2. It is a tensor under generalized diffeomorphisms.
3. It gives the generalized tensors $\mathcal{R}_{MN}$ and $\mathcal{R}$ upon suitable contractions.
4. It is expressed in terms of the physical fields $\mathcal{H}_{MN}$ and d.

Property (1) ensures proper behavior under T-duality and property (2) ensures proper behavior under gauge transformations. Property (3) implies that, as in Riemannian geometry, the Riemann tensor contains the information in Ricci and the information in the scalar curvature.[1] Property (4) means that the tensor is 'physical', or fully determined. We could also demand some additional properties that would establish a close relation of $\mathcal{R}_{MNPQ}$ to the familiar Riemann tensor. In analogy to the situation with $\mathcal{R}_{MN}$ and $\mathcal{R}$ we could demand that

(A) For $b = \phi = 0$ and $\tilde{\partial} = 0$ some components of $\mathcal{R}_{MNPQ}$ reduce to the Riemann tensor.

If property (4) holds, property (A) has a clear meaning. If property (4) does not hold some components of $\mathcal{R}_{MNPQ}$ may be determined and some may not; we need only study the former to test (A).

Some time ago Siegel developed a vielbein formalism with a local $GL(D) \times GL(D)$ tangent space symmetry [4]. Introducing connections for this tangent space symmetry he defined invariant curvatures, but not all connections can be expressed in terms of the physical fields by imposing covariant constraints. The scalar curvature and Ricci tensor can be defined in a way that is independent of the undetermined connections, but there does not appear to be an uncontracted Riemann tensor that depends only on physical fields. Interestingly, in Batalin-Vilkovisky quantization, a formalism based on antisymplectic geometry, a similar phenomenon occurs: connections exist for which their undetermined components drop out of the curvature scalar [28].

In this paper we will revisit these issues in a purely metric-like formalism. We work solely with the generalized metric $\mathcal{H}_{MN}$ and the dilaton and there are no additional gauge redundancies. This is equivalent to Siegel's formulation and may be derived from it by imposing a

[1] The analogy with Riemannian geometry is not complete: there is no contraction of $\mathcal{R}_{MN}$ that gives $\mathcal{R}$.

vielbein postulate that relates the Christoffel-like connections to the spin-connection [7]. This will be briefly explained in the appendix. We find it simpler and more illuminating, however, to present the metric-like formalism in a self-contained fashion. A closely related formulation has been developed before in useful papers by Jeon, Lee, and Park [18,19]. Many of our results have a direct analogue in the frame formalism of Siegel and some have appeared in [18,19]. Finally, generalized geometry [27] also features closely related connections and curvatures; see [26] for a recent concise exposition.

We investigate systematically within the formalism if there is a $\mathcal{R}_{MNPQ}$ that satisfies the properties listed above ((1) through (4), and (A)). Our investigation confirms the existence of a duality covariant generalized Riemann tensor that determines $\mathcal{R}_{MN}$ and $\mathcal{R}$. Thus properties (1), (2), and (3) hold. We find, however, that $\mathcal{R}_{MNPQ}$ is not fully determined in terms of the physical fields: property (4) does not hold. We show that this is a simple consequence of an algebraic Bianchi identity of the Riemann tensor. In fact, property (A) does not hold either: the components of $\mathcal{R}_{MNPQ}$ that do not contain undetermined connections are zero.

The generalized metric formulation differs from Riemannian geometry in that the metric is a constrained object; it satisfies (1.2)–(1.3). As a consequence, there are projectors

$$P_M{}^N \;=\; \frac{1}{2}\big(\delta_M{}^N - \mathcal{H}_M{}^N\big)\;, \qquad \bar{P}_M{}^N \;=\; \frac{1}{2}\big(\delta_M{}^N + \mathcal{H}_M{}^N\big)\;, \tag{1.8}$$

satisfying $P+\bar{P}=1$, $P\bar{P}=0$, $P^2=P$ and $\bar{P}^2=\bar{P}$. They allow us to project onto a 'left-handed' or 'right-handed' subspace. This is the analogue of the factorized tangent space group $GL(D)\times GL(D)$ in the frame formulation, and equivalence of the two formalisms then requires the projectors to be covariantly constant. Jeon, Lee and Park [18, 19] postulate an expression for the Christoffel symbols in terms of the physical fields that satisfies this condition. The resulting 'covariant derivatives', however, do not transform covariantly in general, but only for certain projections and contractions. The reason is that the imposition of covariant constraints only determines part of the connections, and their ansatz effectively sets the undetermined connections to zero, thereby violating covariance. Here we follow a somewhat different route. As in the frame formalism, we work with proper connections and fully covariant expressions by keeping those connection components that are not determined by the physical fields. For the final results on Ricci and scalar curvature tensors for which the undetermined connections drop out, our results are in full agreement with the most recent work [19]. We also establish differential Bianchi identities that have not appeared before in such a metric-like formalism.

An important motivation for this work was the construction of higher-derivative or α' corrections involving the full Riemann tensor. Thus, in the second part of this paper we ask if there is a manifestly $O(D,D)$ invariant function of the generalized metric (1.1), quartic in derivatives, that reduces to the square of the Riemann tensor in some T-duality frame. In fact, even if there is no physical $\mathcal{R}_{MNPQ}$, one can imagine an expression that reproduces the square of the Riemann tensor, but is not the square of an $O(D,D)$ tensor. We find, however, that for general D such a construction is impossible, showing that generic α' corrections cannot be written in terms of the generalized metric defined in terms of g and b as in (1.1).

In hindsight, this result is not too surprising in view of similar results obtained for dimensionally reduced theories. It has been shown by Meissner that α'-corrected supergravity, reduced to one dimension, can be written in a T-duality invariant way if the formula for the generalized metric in terms of the physical fields receives α' corrections [29]. We discuss in the conclusions the possible implications of this fact for our analysis.

2 Christoffel connections and invariant curvatures

In this section we introduce Christoffel-type connections and determine their transformation behavior by requiring that they give rise to derivatives that are covariant under generalized diffeomorphisms. In terms of these connections we define an $O(D,D)$ covariant Riemann tensor that is also a generalized tensor. Next, we impose covariant constraints on the connections in order to express them in terms of the physical fields. It turns out that this leaves undetermined components, which we analyze systematically.

2.1 Connections and curvatures

$O(D,D)$ tensors are said to be generalized tensors if they transform with generalized Lie derivatives under generalized diffeomorphisms parametrized by ξ^M. The generalized Lie derivative is defined on generalized vectors as

$$\begin{aligned} \delta_\xi A^M &= \widehat{\mathcal{L}}_\xi A^M \equiv \xi^N \partial_N A^M + \left(\partial^M \xi_N - \partial_N \xi^M\right) A^N \ , \\ \delta_\xi A_M &= \widehat{\mathcal{L}}_\xi A_M \equiv \xi^N \partial_N A_M + \left(\partial_M \xi^N - \partial^N \xi_M\right) A_N \ , \end{aligned} \tag{2.1}$$

and is defined similarly on tensors with an arbitrary number of upper and lower $O(D,D)$ indices. For a generalized scalar S the generalized Lie derivative is just given by the transport term. The partial derivative of a scalar is a generalized vector since

$$\delta_\xi \left(\partial_M S\right) = \partial_M \left(\xi^P \partial_P S\right) = \xi^P \partial_P (\partial_M S) + \partial_M \xi^P \, \partial_P S - \partial^P \xi_M \, \partial_P S \ , \tag{2.2}$$

where we are allowed to add the last term because it vanishes by the constraint (1.7). Next, we define a covariant derivative of a vector by introducing a connection Γ:

$$\begin{aligned} \nabla_M A_N &\equiv \partial_M A_N - \Gamma_{MN}{}^K A_K \ , \\ \nabla_M A^N &\equiv \partial_M A^N + \Gamma_{MK}{}^N A^K \ . \end{aligned} \tag{2.3}$$

The transformation property of the connection is determined by the condition that the above derivatives be generalized tensors. A short calculation shows that one must have

$$\delta_\xi \Gamma_{MN}{}^P = \widehat{\mathcal{L}}_\xi \Gamma_{MN}{}^P + \partial_M \partial_N \xi^P - \partial_M \partial^P \xi_N \ . \tag{2.4}$$

The first two terms on the right-hand side are familiar and the last one is due to the extra terms in the generalized Lie derivative. That last term implies that the connection cannot be chosen to be symmetric in its first two indices M and N. We will let Δ_ξ denote all non-covariant terms in a trasformation law: $\delta_\xi W = \widehat{\mathcal{L}}_\xi W + \Delta_\xi W$, for any $O(D,D)$ tensor W. We then have

$$\Delta_\xi \Gamma_{MNK} = 2\partial_M \partial_{[N} \xi_{K]} \ , \tag{2.5}$$

where, as usual, we raise and lower all indices with η.

Given these connections we can define curvature and torsion through the commutator of covariant derivatives,

$$\big[\nabla_M, \nabla_N\big] A_K \;=\; -R_{MNK}{}^L A_L - T_{MN}{}^L \nabla_L A_K \,. \tag{2.6}$$

One finds

$$\begin{aligned} R_{MNK}{}^L \;&=\; \partial_M \Gamma_{NK}{}^L - \partial_N \Gamma_{MK}{}^L + \Gamma_{MQ}{}^L \Gamma_{NK}{}^Q - \Gamma_{NQ}{}^L \Gamma_{MK}{}^Q \,, \\ T_{MN}{}^K \;&=\; 2\,\Gamma_{[MN]}{}^K \,. \end{aligned} \tag{2.7}$$

By definition R is antisymmetric on the first two indices,

$$R_{MNK}{}^L \;=\; -R_{NMK}{}^L \,. \tag{2.8}$$

There will also be an antisymmetry in the last two indices after the imposition of constraints. Lowering the L index in $R_{MNK}{}^L$ we have

$$R_{MNKL} \;=\; \partial_M \Gamma_{NKL} - \partial_N \Gamma_{MKL} + \Gamma_{MQL} \Gamma_{NK}{}^Q - \Gamma_{NQL} \Gamma_{MK}{}^Q \,. \tag{2.9}$$

It turns out that neither R nor T is a generalized tensor. The non-covariant transformation of the torsion tensor follows directly by applying (2.5) to the definition in (2.7). The non-covariant transformation of R follows by a slightly longer but straightforward computation. In total, one finds

$$\begin{aligned} \Delta_\xi R_{MNK}{}^L \;&=\; -2\,\partial^P \partial_{[M}\xi_{N]}\,\Gamma_{PK}{}^L \,, \\ \Delta_\xi T_{MN}{}^L \;&=\; -2\,\partial^L \partial_{[M}\xi_{N]} \,. \end{aligned} \tag{2.10}$$

While each of the two terms on the right-hand side of (2.6) fails to transform covariantly the sum must since the left-hand side is manifestly covariant. This can be readily checked; acting with Δ_ξ on the right-hand side of (2.6) gives

$$-\Delta_\xi R_{MNK}{}^L A_L - \Delta_\xi T_{MN}{}^L \nabla_L A_K \;=\; 2\,\partial_{[M}\partial^P \xi_{N]}\,\Gamma_{PK}{}^L A_L + 2\partial_{[M}\partial^P \xi_{N]}\,(\partial_P A_K - \Gamma_{PK}{}^L A_L)\,,$$

where use was made of (2.10). The term with a $\partial_P A_K$ vanishes by the strong constraint and the other two terms cancel each other so that, as expected,

$$\Delta_\xi\big(-R_{MNK}{}^L A_L - T_{MN}{}^L \nabla_L A_K\big) \;=\; 0\,. \tag{2.11}$$

Although R_{MNKL} is not a generalized tensor it can be made so by the simple following modification. Note that the first equation in (2.10) can be written as

$$\Delta_\xi R_{MNKL} \;=\; -2\,\partial_Q \partial_{[M}\xi_{N]}\,\Gamma^Q{}_{KL} \;=\; -\,(\Delta_\xi \Gamma_{QMN})\,\Gamma^Q{}_{KL}\,, \tag{2.12}$$

by use of (2.5). This equation makes it easy to see that $\mathcal{R}_{MNKL}$, defined by

$$\mathcal{R}_{MNKL} \;\equiv\; R_{MNKL} + R_{KLMN} + \Gamma_{QMN}\Gamma^Q{}_{KL} \,, \tag{2.13}$$

is a generalized tensor. By definition $\mathcal{R}$ is symmetric under the interchange of the first and second pair of indices:

$$\mathcal{R}_{MNKL} \;=\; \mathcal{R}_{KLMN}\,. \tag{2.14}$$

The antisymmetry $R_{MNKL} = -R_{NMKL}$ in the first pair of indices does not immediately carry over to $\mathcal{R}_{MNKL}$ but it will after the imposition of constraints on the connection.

2.2 Constraints on the connection

We now impose four constraints in order to determine part of the connections in terms of the physical fields $\mathcal{H}$ and d. These constraints follow from the constraints of Siegel's frame formalism given in [4], as will be reviewed in the appendix, and are also satisfied by the connection-like objects postulated in [19]. The first two set some components of the connection equal to zero and do not involve $\mathcal{H}$ or d. The third constraint involves $\mathcal{H}$ and the fourth involves the dilaton d. As we will see in the following section, the connection is not fully determined by these four constraints.

(1) Covariant constancy of η_{MN}:

$$\nabla_M\eta_{NP} \;=\; \partial_M\eta_{NP}-\Gamma_{MN}{}^{Q}\eta_{QP}-\Gamma_{MP}{}^{Q}\eta_{NQ} \;=\; 0 \quad\Rightarrow\quad \Gamma_{MNP}+\Gamma_{MPN} \;=\; 0\,, \tag{2.15}$$

where we recall that η is a constant matrix and that indices are lowered with η. This equation means that the connection is antisymmetric in the last two indices,

$$\Gamma_{MNP} \;=\; -\Gamma_{MPN}\,. \tag{2.16}$$

(2) Generalized torsion constraint: We demand that the generalized Lie derivative of a vector can be evaluated with an identically looking formula where partial derivatives are replaced by covariant derivatives,

$$\widehat{\mathcal{L}}_\xi V_M \;\equiv\; \xi^N\partial_N V_M + 2\partial_{[M}\xi_{N]}\,V^N \;=\; \xi^N\nabla_N V_M + 2\nabla_{[M}\xi_{N]}\,V^N \;=\; \widehat{\mathcal{L}}_\xi^{\nabla} V_M\,. \tag{2.17}$$

Here $\widehat{\mathcal{L}}_\xi^{\nabla}$ denote the generalized Lie derivative with ∂ replaced by ∇. Put differently, we are setting to zero a generalized torsion tensor $\mathcal{T}$ defined by [26]

$$(\widehat{\mathcal{L}}_\xi^{\nabla} - \widehat{\mathcal{L}}_\xi)V_M \;=\; \mathcal{T}_{MNK}\xi^N V^K\,. \tag{2.18}$$

A short calculation gives [18]

$$\mathcal{T}_{MNK} \;=\; \Gamma_{MNK} - \Gamma_{NMK} + \Gamma_{KMN} \;=\; T_{MNK} + \Gamma_{KMN}\,. \tag{2.19}$$

As defined in (2.18) $\mathcal{T}_{MNK}$ is manifestly a generalized tensor, and this can also be checked directly with (2.5). Our constraint sets this generalized torsion to zero:

$$\mathcal{T}_{MNK} \;=\; \Gamma_{MNK} - \Gamma_{NMK} + \Gamma_{KMN} \;=\; 0\,. \tag{2.20}$$

Using constraint (1) we find that the sum of cyclic index permutations vanishes:

$$\Gamma_{MNK} + \Gamma_{NKM} + \Gamma_{KMN} \;=\; 0\,. \tag{2.21}$$

This property, given constraint (1), is equivalent to the condition that the totally antisymmetric part of the connection vanishes:

$$\Gamma_{[MNK]} \;=\; 0\,. \tag{2.22}$$

(3) Covariant constancy of $\mathcal{H}_{MN}$:

$$\nabla_M\mathcal{H}_{NK} \;=\; \partial_M\mathcal{H}_{NK} - \Gamma_{MN}{}^{P}\mathcal{H}_{PK} - \Gamma_{MK}{}^{P}\mathcal{H}_{NP} \;=\; 0\,. \tag{2.23}$$

(4) Partial integration in presence of dilaton density:

$$\int e^{-2d}\, V\nabla_M V^M \;=\; -\int e^{-2d}\, V^M\nabla_M V\,. \tag{2.24}$$

This condition results in

$$\Gamma_M \;\equiv\; \Gamma_{NM}{}^N \;=\; -2\partial_M d\,. \tag{2.25}$$

Equivalently, this condition means that the covariant divergence of a vector is computed using the density e^{-2d}:

$$\nabla_M V^M = \partial_M A^M + \Gamma_{MK}{}^M A^K \;=\; e^{2d}\partial_M(e^{-2d}A^M)\,. \tag{2.26}$$

2.3 Solving the constraints

2.3.1 The first constraint

We can derive a number of conclusions from constraint (2.15) that states the covariant constancy of the $O(D,D)$ metric η_{MN}. This constraint makes the connection antisymmetric on its last two indices. Now consider the curvature R_{MNKL} in (2.9). Using the antisymmetry condition, the last two terms are rewritten as

$$R_{MNKL} \;=\; \partial_M\Gamma_{NKL} - \partial_N\Gamma_{MKL} - \Gamma_{MLQ}\Gamma_{NK}{}^Q + \Gamma_{MKQ}\Gamma_{NL}{}^Q\,, \tag{2.27}$$

making it clear that R_{MNKL} is now also antisymmetric in the last two indices. Since it is also antisymmetric in its first two indices we have in total

$$R_{MNKL} \;=\; -R_{NMKL} \;=\; -R_{MNLK}\,. \tag{2.28}$$

Still, there is no simple relation between R_{MNKL} and R_{KLMN}. It also follows from the above and (2.13) that $\mathcal{R}$ shares those same symmetries,

$$\mathcal{R}_{MNKL} \;=\; -\mathcal{R}_{NMKL} \;=\; -\mathcal{R}_{MNLK}\,. \tag{2.29}$$

Together with (2.14) we see that $\mathcal{R}_{MNKL}$ satisfies the familiar properties of the Riemann tensor. One missing property, the algebraic Bianchi identity, will follow after the imposition of the second constraint.

2.3.2 The second constraint

Let us now see what conclusions follow from the vanishing of the generalized torsion. First, we note that the formula for the torsion in terms of the connection can be simplified. With (2.19) it follows from $\mathcal{T}_{MNK} = 0$ that

$$T_{MNK} \;=\; -\Gamma_{KMN}\,. \tag{2.30}$$

An important consequence of the first two constraints is that we have the Bianchi identity

$$\mathcal{R}_{[MNK]L} \;=\; 0\,, \tag{2.31}$$

as also noted in [19]. Given the symmetries (2.29), this is equivalent to

$$\mathcal{R}_{MNKL}+\mathcal{R}_{NKML}+\mathcal{R}_{KMNL} \;=\; 0\,. \tag{2.32}$$

In Riemannian geometry this formula follows directly from the expression for the Riemann tensor in terms of a torsion-less connection. In the present case the equation requires

$$\begin{aligned}
&R_{MNKL}+R_{NKML}+R_{KMNL}\\
&+R_{KLMN}+R_{MLNK}+R_{NLKM}\\
&+\Gamma_{QMN}\Gamma^{Q}{}_{KL}+\Gamma_{QNK}\Gamma^{Q}{}_{ML}+\Gamma_{QKM}\Gamma^{Q}{}_{NL} \;=\; 0\,.
\end{aligned} \tag{2.33}$$

This equation is readily verified using (2.7): there are twelve terms of the form $\partial\Gamma$ that combine into four groups of three terms that vanish separately, there are fifteen $\Gamma\Gamma$ terms that combine into three groups of five terms that vanish separately.

Finally, we derive a formula for the exact variation of $\mathcal{R}_{MNKL}$ upon a finite variation $\Gamma\to\Gamma+\delta\Gamma$ of the connection. Beginning with (2.9), a short calculation gives

$$\begin{aligned}
R_{MNKL}(\Gamma+\delta\Gamma) \;=\;& R_{MNKL}(\Gamma)\;+\;2\,\nabla_{[M}\delta\Gamma_{N]KL}\\
&+2\Gamma_{[MN]}{}^{P}\delta\Gamma_{PKL}+2\,\delta\Gamma_{[M|QL|}\,\delta\Gamma_{N]K}{}^{Q}\;.
\end{aligned} \tag{2.34}$$

In obtaining the above we only had to use the antisymmetry of the connection in the last two indices (constraint 1). The covariant derivatives on the above right-hand side use Γ. To obtain the analogous result for $\mathcal{R}_{MNKL}$ we use the above and (2.13). This time a short calculation gives

$$\begin{aligned}
\mathcal{R}_{MNKL}(\Gamma+\delta\Gamma) \;=\;& \mathcal{R}_{MNKL}(\Gamma)\;+\;2\,\nabla_{[M}\delta\Gamma_{N]KL}\;+\;2\,\nabla_{[K}\delta\Gamma_{L]MN}\\
&+2\,\delta\Gamma_{[M|QL|}\,\delta\Gamma_{N]K}{}^{Q}\;+\;2\,\delta\Gamma_{[K|QN|}\,\delta\Gamma_{L]M}{}^{Q}\;+\;\delta\Gamma_{QMN}\,\delta\Gamma^{Q}{}_{KL}\;.
\end{aligned} \tag{2.35}$$

In deriving this result we had to use the second constraint in the form (2.21). Note that the terms of the form $\Gamma\delta\Gamma$ in $R(\Gamma+\delta\Gamma)$ cancel out in $\mathcal{R}(\Gamma+\delta\Gamma)$.

2.3.3 The third constraint

The constraint (2.23) demands the covariant constancy of the generalized metric. To explore immediate consequences of this additional constraint consider the projectors (1.8)

$$P_{M}{}^{N} \;=\; \frac{1}{2}\big(\delta_{M}{}^{N}-\mathcal{H}_{M}{}^{N}\big)\,, \qquad \bar{P}_{M}{}^{N} \;=\; \frac{1}{2}\big(\delta_{M}{}^{N}+\mathcal{H}_{M}{}^{N}\big)\,, \tag{2.36}$$

which satisfy

$$P\,\bar{P} \;=\; 0\,, \qquad P^2 \;=\; P\,, \qquad \bar{P}^2 \;=\; \bar{P}\,. \tag{2.37}$$

Since η is covariantly constant by constraint (1) and $\mathcal{H}$ is covariantly constant by constraint (3), the projectors are also covariantly constant:

$$\nabla_K P_{M}{}^{N} \;=\; \nabla_K \bar{P}_{M}{}^{N} \;=\; 0\,. \tag{2.38}$$

We now discuss how to use this result to solve completely the constraint. For this purpose we will introduce a notation for indices that are projected. We will have two kinds of indices: barred, with a dash on top, and un-barred, or more properly, under-barred, with a dash below. The index type depends on the projector that is used to obtain it from the un-projected index. The barred index is associated with the $\bar{P}$ projector and the under-barred index is associated with the P projector. Thus, we will have

$$\begin{aligned} W_{\underline{M}} &\equiv P_M{}^N W_N\,, \\ W_{\bar{M}} &\equiv \bar{P}_M{}^N W_N\,. \end{aligned} \tag{2.39}$$

Note that this implies that

$$W_M = W_{\underline{M}} + W_{\bar{M}}\,. \tag{2.40}$$

We raise or lower projected indices with the metric η:

$$\begin{aligned} W^{\underline{M}} &\equiv \eta^{MQ} W_{\underline{Q}} = \eta^{MQ}\, P_Q{}^N W_N = P^{MN} W_N\,, \\ W^{\bar{M}} &\equiv \eta^{MQ} W_{\bar{Q}} = \eta^{MQ}\, \bar{P}_Q{}^N W_N = \bar{P}^{MN} W_N\,, \end{aligned} \tag{2.41}$$

so that one can simply use the projector with indices up or down to define a projected index. Contraction of projected indices of different types vanish. For example,

$$W^{\underline{M}} Y_{\bar{M}} = 0\,. \tag{2.42}$$

Contraction of like-wise projected indices can be done with a single projector:

$$\begin{aligned} W^{\bar{M}} Y_{\bar{M}} &= \bar{P}^{MQ} \bar{P}_M{}^R W_Q Y_R = \bar{P}^{QR} W_Q Y_R\,, \\ W^{\underline{M}} Y_{\underline{M}} &= P^{MQ} P_M{}^R W_Q Y_R = P^{QR} W_Q Y_R\,. \end{aligned} \tag{2.43}$$

A contraction of unprojected indices can be written as a sum of contractions of like-wise projected indices. Indeed,

$$W^M Y_M = (W^{\underline{M}} + W^{\bar{M}})(Y_{\underline{M}} + Y_{\bar{M}}) = W^{\underline{M}} Y_{\underline{M}} + W^{\bar{M}} Y_{\bar{M}}\,. \tag{2.44}$$

We will occasionally use tensors with mixed indices. So for example, we could have an object

$$W_{M\underline{N}K} = P_N{}^Q W_{MQK}\,. \tag{2.45}$$

There is no possible confusion: an index without a bar or under-bar is unprojected. As a final remark on the use of these indices we note that in any tensor equality with a number of free unprojected indices (appearing both on the left-hand side and the right-hand side) we can simply replace any unprojected index by like-wise projected indices on both sides of the equality. Thus, for example, $W_{MN} = Y_{MN}$ implies $W_{\underline{M}\bar{N}} = Y_{\underline{M}\bar{N}}$, as well as several other equalities.

When dealing with objects with projected indices, we will say that the object is of type (k,l) if it has k under-barred indices and l barred indices. Thus, for example, given an $O(D,D)$ tensor A_{MNP} we have

$$\text{Type } (3,0): \quad A_{\underline{M}\underline{N}\underline{P}}\,, \qquad \text{Type } (2,1): \quad A_{\bar{M}\underline{N}\underline{P}}\,,\ A_{\underline{M}\bar{N}\underline{P}}\,,\ A_{\underline{M}\underline{N}\bar{P}}\,,\ \text{etc.} \tag{2.46}$$

Let us now consider the connection Γ_{MNK}. By repeated use of (2.40) on each index we have

$$\begin{aligned}\Gamma_{MNK} &= \Gamma_{\underline{M}\underline{N}\underline{K}} + \Gamma_{\underline{M}\underline{N}\bar{K}} + \Gamma_{\underline{M}\bar{N}\underline{K}} + \Gamma_{\underline{M}\bar{N}\bar{K}} \\ &\quad + \Gamma_{\bar{M}\underline{N}\underline{K}} + \Gamma_{\bar{M}\underline{N}\bar{K}} + \Gamma_{\bar{M}\bar{N}\underline{K}} + \Gamma_{\bar{M}\bar{N}\bar{K}} \,.\end{aligned} \tag{2.47}$$

From the comments above it follows that the symmetries of Γ arising from the first two constraints carry over to the projected Γ. Thus, for example, $\Gamma_{\underline{M}\underline{N}\bar{K}} = -\Gamma_{\underline{M}\bar{K}\underline{N}}$. The cyclicity condition on the three indices also holds for any choice of index type.

Using the symmetry conditions on Γ we can rewrite (2.47) as follows:

$$\begin{aligned}\Gamma_{MNK} &= \Gamma_{\underline{M}\underline{N}\underline{K}} + \Gamma_{\underline{M}\underline{N}\bar{K}} - \Gamma_{\underline{M}\underline{K}\bar{N}} - (-\Gamma_{\bar{N}\underline{M}\bar{K}} + \Gamma_{\bar{K}\underline{M}\bar{N}}) \\ &\quad - (\Gamma_{\underline{N}\underline{K}\bar{M}} - \Gamma_{\underline{K}\underline{N}\bar{M}}) + \Gamma_{\bar{M}\underline{N}\bar{K}} - \Gamma_{\bar{M}\underline{K}\bar{N}} + \Gamma_{\bar{M}\bar{N}\bar{K}} \,.\end{aligned} \tag{2.48}$$

We then regroup the terms to find

$$\begin{aligned}\Gamma_{MNK} &= \Gamma_{\underline{M}\underline{N}\underline{K}} + \Gamma_{\bar{M}\bar{N}\bar{K}} \\ &\quad + \Gamma_{\underline{M}\underline{N}\bar{K}} - \Gamma_{\underline{M}\underline{K}\bar{N}} - \Gamma_{\underline{N}\underline{K}\bar{M}} + \Gamma_{\underline{K}\underline{N}\bar{M}} \\ &\quad + \Gamma_{\bar{M}\underline{N}\bar{K}} - \Gamma_{\bar{M}\underline{K}\bar{N}} - \Gamma_{\bar{K}\underline{M}\bar{N}} + \Gamma_{\bar{N}\underline{M}\bar{K}} \,.\end{aligned} \tag{2.49}$$

This shows that there are just four structures that need to be determined:

$$\Gamma_{\underline{M}\underline{N}\underline{K}}\,, \quad \Gamma_{\bar{M}\bar{N}\bar{K}}\,, \quad \Gamma_{\underline{M}\underline{N}\bar{K}}\,, \quad \text{and} \quad \Gamma_{\bar{M}\underline{N}\bar{K}}\,. \tag{2.50}$$

As we will now see, the covariant constancy of the projector determines the last two of these and leaves the first two undetermined. Indeed, consider the equation

$$\nabla_M P_K{}^L \;=\; \partial_M P_K{}^L - \Gamma_{MK}{}^Q P_Q{}^L + \Gamma_{MQ}{}^L P_K{}^Q \;=\; 0\,. \tag{2.51}$$

We write this as

$$\partial_M P_{KL} + P_L{}^Q \Gamma_{MQK} + P_K{}^Q \Gamma_{MQL} \;=\; 0\,. \tag{2.52}$$

Multiplying by $\bar{P}_N{}^K$ the last term drops out and we get

$$P_L{}^Q \bar{P}_N{}^K \Gamma_{MQK} \;=\; -\bar{P}_N{}^K \partial_M P_{KL} \;=\; -(\bar{P}\partial_M P)_{NL}\,, \tag{2.53}$$

or, equivalently,

$$\Gamma_{M\underline{L}\bar{N}} \;=\; -(\bar{P}\partial_M P)_{NL}\,. \tag{2.54}$$

Acting with an additional projector we obtain,

$$\begin{aligned}\Gamma_{\underline{R}\underline{L}\bar{N}} &= -P_R{}^M(\bar{P}\partial_M P)_{NL}\,, \\ \Gamma_{\bar{R}\underline{L}\bar{N}} &= -\bar{P}_R{}^M(\bar{P}\partial_M P)_{NL}\,.\end{aligned} \tag{2.55}$$

This determined the advertised components. The totally under-barred component $\Gamma_{\underline{M}\underline{N}\underline{K}}$ of Γ_{MNK} is not determined because it drops out of (2.52). Indeed note that

$$P_L{}^Q \Gamma_{\underline{M}\underline{Q}\underline{K}} + P_K{}^Q \Gamma_{\underline{M}\underline{Q}\underline{L}} \;=\; \Gamma_{\underline{M}\underline{L}\underline{K}} + \Gamma_{\underline{M}\underline{K}\underline{L}} \;=\; 0\,, \tag{2.56}$$

because of antisymmetry on the last two indices. Of course, the totally barred components $\Gamma_{\bar{M}\bar{N}\bar{K}}$ are also not determined.

2.3.4 The fourth constraint

This constraint determines the trace of the connection:

$$\Gamma_N \;\equiv\; \Gamma_{MNK}\eta^{MK} \;=\; -2\,\partial_N d\,. \tag{2.57}$$

To begin the analysis we compute the left-hand side of this relation using (2.49). We get

$$\Gamma_N \;=\; \eta^{MK}\Gamma_{\underline{M}\underline{N}\underline{K}} + \eta^{MK}\Gamma_{\bar{M}\bar{N}\bar{K}} - \eta^{MK}\Gamma_{\underline{M}\,\underline{K}\bar{N}} + \eta^{MK}\Gamma_{\bar{M}\underline{N}\bar{K}}\,, \tag{2.58}$$

where we noted that contractions of η with Γ are nonzero only if the two indices to be contracted in the projected Γ are of the same type. Moving the undetermined components to the left-hand side and recalling (2.40) we obtain

$$\eta^{MK}\Gamma_{\underline{M}\underline{N}\underline{K}} + \eta^{MK}\Gamma_{\bar{M}\bar{N}\bar{K}} \;=\; \Gamma_{\underline{N}} - \eta^{MK}\Gamma_{\bar{M}\underline{N}\bar{K}} \;+\; \Gamma_{\bar{N}} - \eta^{MK}\Gamma_{\underline{M}\bar{N}\underline{K}}\,. \tag{2.59}$$

From the above we obtain two equations for the two undetermined components, according to the type of N index:

$$\begin{aligned} \eta^{MK}\Gamma_{\underline{M}\underline{N}\underline{K}} \;&=\; \Gamma_{\underline{N}} - \eta^{MK}\Gamma_{\bar{M}\underline{N}\bar{K}} \;\equiv\; \phi_{\underline{N}}\,. \\ \eta^{MK}\Gamma_{\bar{M}\bar{N}\bar{K}} \;&=\; \Gamma_{\bar{N}} - \eta^{MK}\Gamma_{\underline{M}\bar{N}\underline{K}} \;\equiv\; \phi_{\bar{N}}\,. \end{aligned} \tag{2.60}$$

Note that $\phi_{\underline{N}}$ and $\phi_{\bar{N}}$ are projected objects. It is useful to show that they arise from a single object ϕ_N. This is what we do now. We begin with $\phi_{\underline{N}}$ and use (2.55):

$$\begin{aligned} \phi_{\underline{N}} \;&=\; P_N{}^R\Gamma_R \;+\; \eta^{MK}\bar{P}_M{}^Q(\bar{P}\partial_Q P)_{KN} \\ &=\; P_N{}^R\Gamma_R \;-\; \bar{P}^{KQ}((\partial_Q\bar{P})P)_{KN} \\ &=\; P_N{}^R\Gamma_R \;-\; \bar{P}^{KQ}\,(\partial_Q\bar{P}_{KR})P_N{}^R \\ &=\; P_N{}^R\Big(\Gamma_R \;-\; \bar{P}_{QK}\,\partial^Q\bar{P}^K{}_R\Big) \;=\; P_N{}^R\Big(\Gamma_R \;-\; (\bar{P}\,\partial^Q\bar{P})_{QR}\Big)\,. \end{aligned} \tag{2.61}$$

We note that in the final expression the reversed index combination $(\bar{P}\partial^Q\bar{P})_{RQ}$ would give zero contribution due to the $P_N{}^R$ projector. We can thus write,

$$\phi_{\underline{N}} \;=\; P_N{}^R\Big(\Gamma_R \;-\; 2(\bar{P}\,\partial^Q\bar{P})_{[QR]}\Big)\,. \tag{2.62}$$

A completely analogous calculation gives

$$\phi_{\bar{N}} \;=\; \bar{P}_N{}^R\Big(\Gamma_R \;-\; 2(P\,\partial^Q P)_{[QR]}\Big)\,. \tag{2.63}$$

We can easily verify that the terms in parenthesis in the two equations above are equal. Indeed,

$$P\partial^Q P = -(1-\bar{P})\partial^Q\bar{P} = -\partial^Q\bar{P} + \bar{P}\partial^Q\bar{P} \quad\to\quad (P\partial^Q P)_{[QR]} \;=\; (\bar{P}\partial^Q\bar{P})_{[QR]}\,. \tag{2.64}$$

We can therefore write

$$\phi_{\underline{N}} \;=\; P_N{}^R\phi_R\,,\;\; \phi_{\bar{N}} \;=\; \bar{P}_N{}^R\phi_R\,, \quad \text{with} \;\; \phi_R \;=\; -2\big(\partial_R d + (P\partial^Q P)_{[QR]}\big)\,. \tag{2.65}$$

Let us now resume the analysis of equations (2.60). A solution of these equations is of the form

$$
\begin{aligned}
\Gamma_{\underline{M}\underline{N}\underline{K}} &= \alpha P_{M[N}P_{K]}{}^{Q}\phi_Q \;=\; \alpha\, P_M{}^R P_{[N}{}^L P_{K]}{}^Q\, \eta_{RL}\phi_Q\,, \\
\Gamma_{\bar{M}\bar{N}\bar{K}} &= \alpha \bar{P}_{M[N}\bar{P}_{K]}{}^{Q}\,\phi_Q \;=\; \alpha\, \bar{P}_M{}^R \bar{P}_{[N}{}^L \bar{P}_{K]}{}^Q\, \eta_{RL}\phi_Q\,,
\end{aligned}
\tag{2.66}
$$

where α is a constant to be determined. The last right-hand side on each line was written to make it manifest that the Γ's have the correct projections. Note that this ansatz, as required, satisfies constraints (1) and (2): $\Gamma_{\underline{M}\underline{N}\underline{K}} = -\Gamma_{\underline{M}\underline{K}\underline{N}}$ and $\Gamma_{\underline{M}\underline{N}\underline{K}} + \Gamma_{\underline{N}\underline{K}\underline{M}} + \Gamma_{\underline{K}\underline{M}\underline{N}} = 0$. The coefficient α is determined by contraction. We get

$$
\eta^{MK}\Gamma_{\underline{M}\underline{N}\underline{K}} \;=\; \frac{1}{2}\alpha(1-D)\phi_{\underline{N}} = \phi_{\underline{N}} \quad\rightarrow\quad \alpha = \frac{2}{1-D}\,. \tag{2.67}
$$

Back in (2.66) and using (2.65) the full solution is therefore

$$
\begin{aligned}
\Gamma_{\underline{M}\,\underline{N}\,\underline{K}} &= -\frac{2}{(D-1)}P_{M[N}P_{K]}{}^{R}\,\phi_R + \tilde{\Gamma}_{\underline{M}\underline{N}\underline{K}}\,, \\
\Gamma_{\bar{M}\bar{N}\bar{K}} &= -\frac{2}{(D-1)}\bar{P}_{M[N}\bar{P}_{K]}{}^{R}\,\phi_R + \tilde{\Gamma}_{\bar{M}\bar{N}\bar{K}}\,,
\end{aligned}
\tag{2.68}
$$

where $\tilde{\Gamma}$ is undetermined and satisfies

$$
\begin{aligned}
\eta^{MK}\,\tilde{\Gamma}_{\underline{M}\underline{N}\underline{K}} &= 0\,, \\
\eta^{MK}\,\tilde{\Gamma}_{\bar{M}\bar{N}\bar{K}} &= 0\,.
\end{aligned}
\tag{2.69}
$$

2.3.5 The full Christoffel connection

To write a complete expression for the Christoffel connection we begin by adding the two contributions in (2.68) and use (2.65) to find

$$
\begin{aligned}
\Gamma_{\underline{M}\underline{N}\underline{K}} + \Gamma_{\bar{M}\bar{N}\bar{K}} \;=\; & \frac{4}{(D-1)}\Big(P_{M[N}P_{K]}{}^{R} + \bar{P}_{M[N}\bar{P}_{K]}{}^{R}\Big)\big(\partial_R d + (\bar{P}\partial^Q\bar{P})_{[QR]}\big) \\
& + \tilde{\Gamma}_{\underline{M}\underline{N}\underline{K}} + \tilde{\Gamma}_{\bar{M}\bar{N}\bar{K}}\,.
\end{aligned}
\tag{2.70}
$$

The full connection is then given by (2.49) which we write as

$$
\begin{aligned}
\Gamma_{MNK} \;=\; & (\Gamma_{\underline{M}\,\underline{N}\bar{K}} + \Gamma_{\bar{M}\underline{N}\bar{K}}) - (\Gamma_{\underline{M}\,\underline{K}\bar{N}} + \Gamma_{\bar{M}\underline{K}\bar{N}}) \\
& - \Gamma_{\underline{N}\,\underline{K}\bar{M}} + \Gamma_{\underline{K}\,\underline{N}\bar{M}} - \Gamma_{\bar{K}\underline{M}\bar{N}} + \Gamma_{\bar{N}\underline{M}\bar{K}} \\
& + \frac{4}{(D-1)}\Big(P_{M[N}P_{K]}{}^{R} + \bar{P}_{M[N}\bar{P}_{K]}{}^{R}\Big)\big(\partial_R d + (P\partial^Q P)_{[QR]}\big) \\
& + \tilde{\Gamma}_{\underline{M}\underline{N}\underline{K}} + \tilde{\Gamma}_{\bar{M}\bar{N}\bar{K}}\,.
\end{aligned}
\tag{2.71}
$$

The first two lines on the above right-hand side can be evaluated using equations (2.55). These equations imply, for example, that

$$
\Gamma_{\underline{M}\underline{N}\bar{K}} + \Gamma_{\bar{M}\underline{N}\bar{K}} \;=\; -(\bar{P}\partial_M P)_{KN}\,. \tag{2.72}
$$

With this one quickly verifies that the first line in the right-hand side of (2.71) simplifies down to $-2(P\partial_M P)_{[NK]}$. A computation of the second line then yields the complete result. We write it as

$$\Gamma_{MNK} \;=\; \widehat{\Gamma}_{MNK} + \Sigma_{MNK}\,, \tag{2.73}$$

where $\widehat{\Gamma}_{MNK}$ is the determined part of the connection,

$$\begin{aligned}\widehat{\Gamma}_{MNK} \;=\; & -2(P\partial_M P)_{[NK]} - 2\big(\bar{P}_{[N}{}^P\bar{P}_{K]}{}^Q - P_{[N}{}^P P_{K]}{}^Q\big)\partial_P P_{QM} \\ & + \frac{4}{D-1}\big(P_{M[N}P_{K]}{}^Q + \bar{P}_{M[N}\bar{P}_{K]}{}^Q\big)\big(\partial_Q d + (P\partial^P P)_{[PQ]}\big)\,,\end{aligned} \tag{2.74}$$

and Σ_{MNK} is the undetermined part of the connection:

$$\Sigma_{MNK} \;=\; \tilde{\Gamma}_{\underline{M}\underline{N}\underline{K}} \;+\; \tilde{\Gamma}_{\bar{M}\bar{N}\bar{K}}\,. \tag{2.75}$$

The result (2.74) is equivalent to the ansatz given in eq. (15) of [19]. The Σ_{MNK} satisfy the traceless condition in (2.69). Given the symmetry properties of the connection, the trace taken on any two indices of the $\tilde{\Gamma}$'s vanishes. This completes our calculation of the connection. We finally give the number of undetermined connection components. Since P and $\bar{P}$ are rank-D projectors, any projected $O(D,D)$ index represents D independent components. The two undetermined $\tilde{\Gamma}$ can thus be viewed as taking values in the $(2,1)$ traceless $GL(D)$ Young tableau. The total number of undetermined components is then found to be $\frac{2}{3}D(D+2)(D-2)$, which is equal to the value in Siegel's frame-like formalism, see the discussion after eq. (2.40) in [7].

We can rewrite the above $\widehat{\Gamma}$ directly in terms of $\mathcal{H}$ and d. Using the definition of the projectors a quick calculation shows that

$$(P\partial_M P)_{PQ} \;=\; \frac{1}{4}\Big(-\partial_M\mathcal{H}_{PQ} + \mathcal{H}_{PK}\partial_M\mathcal{H}^K{}_Q\Big)\,. \tag{2.76}$$

The first term on the right-hand side is symmetric in P and Q while the second term is actually antisymmetric in P and Q. We thus have

$$(P\partial_M P)_{[PQ]} \;=\; \frac{1}{4}\mathcal{H}_{PK}\partial_M\mathcal{H}^K{}_Q\,. \tag{2.77}$$

As a result, we obtain

$$(P\partial^P P)_{[PQ]} \;=\; \frac{1}{4}\mathcal{H}_{PK}\partial^P\mathcal{H}^K{}_Q \;=\; \frac{1}{4}\mathcal{H}^{PM}\partial_M\mathcal{H}_{PQ}\,. \tag{2.78}$$

We can quickly work out the other projectors:

$$\bar{P}_{[N}{}^P\bar{P}_{K]}{}^Q - P_{[N}{}^P P_{K]}{}^Q \;=\; \frac{1}{2}\Big(\delta_{[N}{}^P\mathcal{H}_{K]}{}^Q + \mathcal{H}_{[N}{}^P\delta_{K]}{}^Q\Big)\,, \tag{2.79}$$

$$\bar{P}_{M[N}\bar{P}_{K]}{}^Q + P_{M[N}P_{K]}{}^Q \;=\; \frac{1}{2}\Big(\eta_{M[N}\delta_{K]}{}^Q + \mathcal{H}_{M[N}\mathcal{H}_{K]}{}^Q\Big)\,. \tag{2.80}$$

Back in the connection (2.74) we get

$$\begin{aligned}\widehat{\Gamma}_{MNK} \;=\; & \frac{1}{2}\mathcal{H}_{KQ}\partial_M\mathcal{H}^Q{}_N + \frac{1}{2}\Big(\delta_{[N}{}^P\mathcal{H}_{K]}{}^Q + \mathcal{H}_{[N}{}^P\delta_{K]}{}^Q\Big)\partial_P\mathcal{H}_{QM} \\ & + \frac{2}{D-1}\Big(\eta_{M[N}\delta_{K]}{}^Q + \mathcal{H}_{M[N}\mathcal{H}_{K]}{}^Q\Big)\Big(\partial_Q d + \frac{1}{4}\mathcal{H}^{PM}\partial_M\mathcal{H}_{PQ}\Big)\,.\end{aligned} \tag{2.81}$$

3 Analysis of the generalized Riemann tensor

In this section we examine the components of the generalized tensor $\mathcal{R}_{MNPQ}$ using the projected barred and under-barred indices. We show that the projections in which undetermined connections drop out vanish identically. There are four non-vanishing projections, as detailed in equation (3.14). We then show how the Ricci and scalar generalized curvatures arise from $\mathcal{R}_{MNPQ}$ by taking contractions that make all undetermined connections disappear. An analysis of the invariant action allows us to show that there is a single generalized Ricci curvature and to prove differential Bianchi identities.

3.1 The components of the Riemann tensor

Before we begin the detailed discussion of the various components of the Riemann tensor, we examine a useful property that follows from the covariant constancy of the projectors. This property implies that:

$$[\nabla_M, \nabla_N] P_K{}^L V_L \;=\; P_K{}^L [\nabla_M, \nabla_N] V_L \;, \tag{3.1}$$

so that expanding the commutators according to (2.6) we get

$$-\,R_{MNK}{}^P P_P{}^L V_L - T_{MN}{}^P \nabla_P (P_K{}^L V_L) \;=\; -P_K{}^L R_{MNL}{}^P V_P - P_K{}^L T_{MN}{}^P \nabla_P V_L \;. \tag{3.2}$$

Using the covariant constancy again we see that the torsion terms cancel on both sides. Relabeling indices and dropping the V's we obtain

$$R_{MNKP} P^P{}_L \;=\; R_{MNPL} P^P{}_K \;. \tag{3.3}$$

Multiplying by $\bar{P}^K{}_Q$ we see that the above right-hand side vanishes due to $P\bar{P} = 0$. We therefore find that

$$R_{MNKP} \bar{P}^K{}_Q P^P{}_L \;=\; 0 \quad \rightarrow \quad R_{MN\bar{Q}\underline{L}} \;=\; 0\,. \tag{3.4}$$

A curvature R with mixed projections on the last two indices vanishes.

In order to find out which components of the curvature depend on undetermined connections we use the variation formula (2.35) and the split (2.73) of the connection into a determined piece $\widehat{\Gamma}$ and an undetermined piece Σ. We find

$$\begin{aligned} \mathcal{R}_{MNKL} \;=\; & \widehat{\mathcal{R}}_{MNKL} \;+\; 2\,\widehat{\nabla}_{[M}\Sigma_{N]KL} \;+\; 2\,\widehat{\nabla}_{[K}\Sigma_{L]MN} \\ & +\; 2\,\Sigma_{[M|QL|}\Sigma_{N]K}{}^Q \;+\; 2\,\Sigma_{[K|QN|}\Sigma_{L]M}{}^Q \;+\; \Sigma_{QMN}\Sigma^Q{}_{KL} \;. \end{aligned} \tag{3.5}$$

In here all hatted quantities are ones that use $\widehat{\Gamma}$.

Let us now consider possible components of the projected curvatures $\mathcal{R}$. There is one $\mathcal{R}$ with all indices under-barred and one $\mathcal{R}$ with all indices barred – a type $(4,0)$ curvature in the notation introduced in (2.46). With three under-barred indices and one barred one there is just one $\mathcal{R}$ since the barred index can always be chosen to be the last by using the pair exchange symmetry and the antisymmetry in the last two indices. The same is true for the $\mathcal{R}$ with three barred indices and one under-barred one. Finally for an $\mathcal{R}$ with two indices of each type there are two configurations: one in which the first and last two indices are of the same type, and one where they are not. In summary,

$$\mathcal{R}_{\underline{M}\underline{N}\underline{K}\underline{L}}\,,\quad \mathcal{R}_{\underline{M}\underline{N}\underline{K}\bar{L}}\,,\quad \mathcal{R}_{\underline{M}\bar{N}\underline{K}\bar{L}}\,,\quad \mathcal{R}_{\underline{M}\underline{N}\bar{K}\bar{L}}\,,\quad \mathcal{R}_{\underline{M}\bar{N}\bar{K}\bar{L}}\,,\quad \mathcal{R}_{\bar{M}\bar{N}\bar{K}\bar{L}}\,. \tag{3.6}$$

The two type $(2,2)$ curvatures are not independent. The algebraic Bianchi identity (2.31) gives

$$0 \;=\; \mathcal{R}_{\underline{M}\underline{N}\bar{K}\bar{L}}+\mathcal{R}_{\underline{N}\bar{K}\underline{M}\bar{L}}+\mathcal{R}_{\bar{K}\underline{M}\underline{N}\bar{L}} \quad\rightarrow\quad \mathcal{R}_{\underline{M}\underline{N}\bar{K}\bar{L}} \;=\; \mathcal{R}_{\underline{M}\bar{K}\underline{N}\bar{L}}-\mathcal{R}_{\underline{N}\bar{K}\underline{M}\bar{L}}\,, \tag{3.7}$$

showing that the third curvature in (3.6) determines the fourth. The third structure, using definition (2.13), is given by

$$\mathcal{R}_{\underline{M}\bar{N}\underline{K}\bar{L}} \;=\; R_{\underline{M}\bar{N}\underline{K}\bar{L}}+R_{\underline{K}\bar{L}\underline{M}\bar{N}}+\Gamma_{Q\underline{M}\bar{N}}\Gamma^{Q}{}_{\underline{K}\bar{L}}\,. \tag{3.8}$$

The first two terms vanish because of (3.4) and the last one contains pieces of the connection determined in (2.54):

$$\mathcal{R}_{\underline{M}\bar{N}\underline{K}\bar{L}} \;=\; \Gamma_{Q\underline{M}\bar{N}}\Gamma^{Q}{}_{\underline{K}\bar{L}} \;=\; (\bar{P}\partial_Q P)_{MN}\,(\bar{P}\partial^Q P)_{KL} \;=\; 0\,, \tag{3.9}$$

using the strong constraint. The vanishing of this third structure then implies the vanishing of the fourth, as remarked above:

$$\mathcal{R}_{\underline{M}\underline{N}\bar{K}\bar{L}} \;=\; 0\,. \tag{3.10}$$

With (3.5) it is now easy to see that the first two and last two in (3.6) depend on the undetermined connections. In fact, for $\mathcal{R}_{\underline{M}\underline{N}\underline{K}\underline{L}}$ we use (3.5) together with (2.75) to get

$$\begin{aligned}\mathcal{R}_{\underline{M}\underline{N}\underline{K}\underline{L}} \;=\;& \widehat{\mathcal{R}}_{\underline{M}\underline{N}\underline{K}\underline{L}} \;+\; 2\,\widehat{\nabla}_{[\underline{M}}\,\tilde{\Gamma}_{\underline{N}]\underline{K}\underline{L}} \;+\; 2\,\widehat{\nabla}_{[\underline{K}}\tilde{\Gamma}_{\underline{L}]\underline{M}\underline{N}}\\ &+\; 2\,\tilde{\Gamma}_{[\underline{M}|Q\underline{L}|}\tilde{\Gamma}_{\underline{N}]\underline{K}}{}^{Q} \;+\; 2\,\tilde{\Gamma}_{[\underline{K}|Q\underline{N}|}\tilde{\Gamma}_{\underline{L}]\underline{M}}{}^{Q} \;+\; \tilde{\Gamma}_{Q\underline{M}\underline{N}}\,\tilde{\Gamma}^{Q}{}_{\underline{K}\underline{L}}\,.\end{aligned} \tag{3.11}$$

In here, projected indices on covariant derivatives are defined as usual: $\widehat{\nabla}_{\bar{L}} \equiv \bar{P}_L{}^Q\widehat{\nabla}_Q$. We note that all Σ_{MNK} in (3.5) were replaced by $\tilde{\Gamma}_{\underline{M}\underline{N}\underline{K}}$ because the projectors discard the $\tilde{\Gamma}_{\bar{M}\bar{N}\bar{K}}$ components. Note that the summed index Q only receives contributions from the under-barred values. Analogous remarks apply for the fully barred structure $\mathcal{R}_{\bar{M}\bar{N}\bar{K}\bar{L}}$.

For the second curvature in the list, the type $(3,1)$ tensor $\mathcal{R}_{\underline{M}\underline{N}\underline{K}\bar{L}}$, all Σ^2 terms vanish because in each of them one Σ has mixed barred/under-barred projections and there are no such undetermined connections. From the $\nabla\Sigma$ type terms, one survives:

$$\mathcal{R}_{\underline{M}\underline{N}\underline{K}\bar{L}} \;=\; \widehat{\mathcal{R}}_{\underline{M}\underline{N}\underline{K}\bar{L}}-\widehat{\nabla}_{\bar{L}}\tilde{\Gamma}_{\underline{K}\underline{M}\underline{N}}\,. \tag{3.12}$$

We thus see that $\mathcal{R}_{\underline{M}\underline{N}\underline{K}\bar{L}}$ involves undetermined connections. Similarly, we find for the $(1,3)$ type structure

$$\mathcal{R}_{\underline{M}\bar{N}\bar{K}\bar{L}} \;=\; \widehat{\mathcal{R}}_{\underline{M}\bar{N}\bar{K}\bar{L}} \;+\; \widehat{\nabla}_{\underline{M}}\tilde{\Gamma}_{\bar{N}\bar{K}\bar{L}}\,. \tag{3.13}$$

Our analysis shows that the list (3.6) has become

$$
\begin{aligned}
&\mathcal{R}_{\underline{M}\underline{N}\underline{K}\underline{L}} \quad \text{contains undetermined connections,}\\
&\mathcal{R}_{\underline{M}\underline{N}\underline{K}\bar{L}} \quad \text{contains undetermined connections,}\\
&\mathcal{R}_{\underline{M}\bar{N}\underline{K}\bar{L}} \;=\; 0\,,\\
&\mathcal{R}_{\underline{M}\underline{N}\bar{K}\bar{L}} \;=\; 0\,,\\
&\mathcal{R}_{\underline{M}\bar{N}\bar{K}\bar{L}} \quad \text{contains undetermined connections,}\\
&\mathcal{R}_{\bar{M}\bar{N}\bar{K}\bar{L}} \quad \text{contains undetermined connections.}
\end{aligned}
\tag{3.14}
$$

Thus, there is no Riemann tensor in terms of the physical fields.

3.2 Generalized Ricci and scalar curvatures

Undetermined connection components can drop out from traces of curvatures. In fact, we can define a scalar curvature and a Ricci tensor. A naive candidate for the scalar curvature is $\mathcal{R}_{MN}{}^{MN}$. Expanding the contractions in projected indices we have,

$$
\mathcal{R}_{MN}{}^{MN} \;=\; \eta^{MK}\eta^{NL}\,\mathcal{R}_{MNKL} \;=\; \mathcal{R}_{\underline{M}\underline{N}}{}^{\underline{M}\underline{N}} + \mathcal{R}_{\bar{M}\bar{N}}{}^{\bar{M}\bar{N}} + 2\,\mathcal{R}_{\underline{M}\bar{N}}{}^{\underline{M}\bar{N}}\,. \tag{3.15}
$$

The last term on the right-hand side vanishes by (3.9), so that we have

$$
\mathcal{R}_{MN}{}^{MN} \;=\; \mathcal{R}_{\underline{M}\underline{N}}{}^{\underline{M}\underline{N}} + \mathcal{R}_{\bar{M}\bar{N}}{}^{\bar{M}\bar{N}} \;\;. \tag{3.16}
$$

Recall from (2.43) that contractions on projected indices are implemented by contractions against the appropriate projector, so that

$$
\mathcal{R}_{\underline{M}\underline{N}}{}^{\underline{M}\underline{N}} \;=\; P^{MK}P^{NL}\mathcal{R}_{MNKL}\,, \qquad \mathcal{R}_{\bar{M}\bar{N}}{}^{\bar{M}\bar{N}} \;=\; \bar{P}^{MK}\bar{P}^{NL}\mathcal{R}_{MNKL}\,. \tag{3.17}
$$

Back to (3.16) we compute $\mathcal{R}_{MN}{}^{MN}$ directly from the definition (2.13) and from (2.7):

$$
\begin{aligned}
\mathcal{R}_{MN}{}^{MN} \;&=\; 2R_{MN}{}^{MN} + \Gamma_{MNK}\Gamma^{MNK}\\
&=\; 4\partial_M\Gamma^M + 2\,\Gamma_M\Gamma^M + 2\,\Gamma_{MNK}\Gamma^{KMN} + \Gamma_{MNK}\Gamma^{MNK} \;\equiv\; 0\,.
\end{aligned}
\tag{3.18}
$$

The first two terms on the right-hand side vanish using $\Gamma_M \sim \partial_M d$ and the strong constraint. The rest of the terms on the right-hand side vanish too:

$$
\begin{aligned}
\mathcal{R}_{MN}{}^{MN} \;&=\; \Gamma_{MNK}\Big(\Gamma^{KMN} + \Gamma^{KMN} + \Gamma^{MNK}\Big)\\
&=\; \Gamma_{MNK}\Big(\Gamma^{KMN} - \Gamma^{NMK} + \Gamma^{MNK}\Big) \;=\; 0\,,
\end{aligned}
\tag{3.19}
$$

because of $\Gamma_{[MNK]} = 0$. The vanishing of $\mathcal{R}_{MN}{}^{MN}$ is consistent with the vanishing of the flat-index combination $\mathcal{R}_{AB}{}^{AB}$ in Siegel's formalism [4]. Equation (3.16) and the vanishing of $\mathcal{R}_{MN}{}^{MN}$ suggest that we have to contract the fully projected tensors. We thus define the scalar curvature $\mathcal{R}$ by

$$
\mathcal{R} \;\equiv\; \mathcal{R}^{\underline{M}\underline{N}}{}_{\underline{M}\underline{N}} \;=\; -\mathcal{R}^{\bar{M}\bar{N}}{}_{\bar{M}\bar{N}}\,. \tag{3.20}
$$

We now use (3.11) to show that the undetermined connections drop out of $\mathcal{R}$. Let us do one contraction first. The tracelessness of the $\tilde{\Gamma}$ (on any pair of indices) implies that

$$\begin{aligned}\eta^{NL}\mathcal{R}_{\underline{M}\,\underline{N}\,\underline{K}\,\underline{L}} &= \widehat{\mathcal{R}}_{\underline{M}\,\underline{N}\,\underline{K}}{}^{\underline{N}} - \widehat{\nabla}^{\underline{L}}(\tilde{\Gamma}_{\underline{M}\,\underline{K}\,\underline{L}} + \tilde{\Gamma}_{\underline{K}\,\underline{M}\,\underline{L}}) \\ &\quad + \tilde{\Gamma}_{\underline{M}\underline{Q}}{}^{\underline{L}}\,\tilde{\Gamma}_{\underline{L}\underline{K}}{}^{\underline{Q}} + \tilde{\Gamma}_{\underline{K}\underline{Q}}{}^{\underline{L}}\,\tilde{\Gamma}_{\underline{L}\underline{M}}{}^{\underline{Q}} + \tilde{\Gamma}_{\underline{Q}\underline{M}\underline{L}}\,\tilde{\Gamma}^{\underline{Q}}{}_{\underline{K}}{}^{\underline{L}}\,.\end{aligned} \tag{3.21}$$

A few undetermined connection coefficients dropped out but several remain. After the second contraction with η^{MK} we get only $\tilde{\Gamma}\tilde{\Gamma}$ terms that survive, but they add up to zero:

$$\begin{aligned}\mathcal{R} &= \widehat{\mathcal{R}}^{\underline{M}\underline{N}}{}_{\underline{M}\underline{N}} + \tilde{\Gamma}_{\underline{M}\underline{Q}\underline{L}}\,\tilde{\Gamma}^{\underline{L}\underline{M}\underline{Q}} + \tilde{\Gamma}_{\underline{K}\underline{Q}\underline{N}}\,\tilde{\Gamma}^{\underline{N}\underline{K}\underline{Q}} + \tilde{\Gamma}_{\underline{Q}\underline{M}\underline{N}}\,\tilde{\Gamma}^{\underline{Q}\underline{M}\underline{N}} \\ &= \widehat{\mathcal{R}}^{\underline{M}\underline{N}}{}_{\underline{M}\underline{N}} + \tilde{\Gamma}_{\underline{M}\underline{Q}\underline{L}}\,(\tilde{\Gamma}^{\underline{L}\underline{M}\underline{Q}} + \tilde{\Gamma}^{\underline{M}\underline{Q}\underline{L}} + \tilde{\Gamma}^{\underline{L}\underline{M}\underline{Q}}) \\ &= \widehat{\mathcal{R}}^{\underline{M}\underline{N}}{}_{\underline{M}\underline{N}}\,,\end{aligned} \tag{3.22}$$

using the generalized torsion constraint. The undetermined connections dropped out and there is a well-defined scalar curvature $\mathcal{R}$. It must be proportional to the scalar curvature defined in [3]. One may fix the normalization by inserting the explicit connection components, say, focusing on the dilaton-dependent terms. We then find that (3.20) equals the curvature scalar defined in eq. (4.24) in [3].

Equation (3.21) shows that we cannot get a well-defined Ricci tensor with two under-barred (or two barred) indices. The Ricci tensor is of type $(1,1)$, and we can define such an object by contraction with η of a curvature with $(1,3)$ or $(3,1)$ index structure. We define the following objects starting with the $(3,1)$ index structure:

$$\begin{aligned}\mathcal{R}_{\underline{M}\bar{N}} &\equiv \mathcal{R}_{\underline{K}\underline{M}\bar{N}}{}^{\underline{K}} = \eta^{KL}\mathcal{R}_{\underline{K}\,\underline{M}\,\bar{N}\,\underline{L}}\,, \\ \mathcal{R}_{\bar{N}\underline{M}} &\equiv \mathcal{R}_{\underline{K}\bar{N}\underline{M}}{}^{\underline{K}} = \eta^{KL}\mathcal{R}_{\underline{K}\,\bar{N}\,\underline{M}\,\underline{L}}\,.\end{aligned} \tag{3.23}$$

In fact, the Bianchi identity implies they are equal:

$$\mathcal{R}_{\bar{N}\underline{M}} = \mathcal{R}_{\underline{K}\bar{N}\underline{M}}{}^{\underline{K}} = -\mathcal{R}_{\bar{N}\underline{M}\underline{K}}{}^{\underline{K}} - \mathcal{R}_{\underline{M}\underline{K}\bar{N}}{}^{\underline{K}} = \mathcal{R}_{\underline{K}\,\underline{M}\bar{N}}{}^{\underline{K}} = \mathcal{R}_{\underline{M}\bar{N}}\,. \tag{3.24}$$

This is the "symmetry" property of the Ricci curvature. Most importantly, undetermined connections do not appear in the Ricci curvature. Indeed, starting from the definition (3.23) and using (3.12) we have

$$\mathcal{R}_{\underline{M}\bar{N}} = \eta^{KL}\,\mathcal{R}_{\underline{M}\underline{K}\,\underline{L}\,\bar{N}} = \eta^{KL}\,(\widehat{\mathcal{R}}_{\underline{M}\underline{K}\,\underline{L}\bar{N}} - \widehat{\nabla}_{\bar{N}}\,\tilde{\Gamma}_{\underline{L}\underline{M}\underline{K}}) = \eta^{KL}\,\widehat{\mathcal{R}}_{\underline{M}\underline{K}\underline{L}\bar{N}}\,. \tag{3.25}$$

We will show in the following subsection that the Ricci tensor defined by contraction of the $(1,3)$ index structure is identical to the one obtained here.

3.3 Invariant action and differential Bianchi identities

After having defined a generalized curvature scalar $\mathcal{R}$ we can define an invariant action for double field theory. It reads

$$S = \int dx d\tilde{x}\, e^{-2d}\,\mathcal{R} = \int dx d\tilde{x}\, e^{-2d}\,\mathcal{R}_{\underline{M}\underline{N}}{}^{\underline{M}\underline{N}} = \int dx d\tilde{x} e^{-2d} P^{MK} P^{NL}\mathcal{R}_{MNKL}\,, \tag{3.26}$$

where we recalled (3.17). Since the undetermined pieces of the connection drop out (see (3.22)), we have

$$S \;=\; \int dx d\tilde{x}\, e^{-2d} P^{MK} P^{NL} \widehat{\mathcal{R}}_{MNKL}\;. \tag{3.27}$$

Of course, on account of (3.20) we also have

$$S \;=\; -\int dx d\tilde{x}\, e^{-2d}\, \mathcal{R}_{\bar{M}\bar{N}}{}^{\bar{M}\bar{N}} \;=\; -\int dx d\tilde{x} e^{-2d} \bar{P}^{MK} \bar{P}^{NL} \widehat{\mathcal{R}}_{MNKL}\;. \tag{3.28}$$

It turns out that further Bianchi-type identities for the Ricci tensor and curvature scalar can be conveniently derived using the invariance properties of this action.

We start by discussing the variational principle based on (3.27). For earlier discussions of the general variation in double field theory see [2,3,7,8,19]. Variations of the generalized metric imply variations of P or $\bar{P}$. In fact we can think of P and $\bar{P}$ as the field variables to be varied since the connection is written in terms of these projectors (see (2.74)). We must then take into account that these are constrained to satisfy $P^2 = P$, $\bar{P}^2 = \bar{P}$ and $P\bar{P} = 0$. Thus if we shift $P' = P + \delta P$ we need to satisfy

$$(P')^2 \;=\; P + P\delta P + \delta P P \;\equiv\; P' \;=\; P + \delta P\;, \tag{3.29}$$

and similarly for $\bar{P}$. Thus, we have the constraint

$$\delta P \;=\; P\delta P + \delta P P\;, \tag{3.30}$$

and similarly for $\bar{P}$. Acting on both sides with P from the left and the right we quickly see that $P\delta P P = 0$. Moreover, we also see that $\bar{P}\delta P\bar{P} = 0$. Finally, when acting from the left with P and the right with $\bar{P}$, or vice versa, we get trivially satisfied identities that imply that $P\delta P\bar{P}$ and $\bar{P}\delta P P$ are unconstrained. Thus, we can write the variation in terms of two unconstrained matrices $\mathcal{M}$ and $\mathcal{N}$ as follows

$$\delta P \;=\; \bar{P}\mathcal{M}P + P\mathcal{N}\bar{P} \;=\; -\delta\bar{P}\;, \tag{3.31}$$

where the last condition follows from $P + \bar{P} = 1$. Since P and $\bar{P}$ are symmetric, δP and $\delta\bar{P}$ should be symmetric too, requiring that $\mathcal{M}^T = \mathcal{N}$. Thus, the most general variations of P and $\bar{P}$ consistent with the constraints are

$$\delta P \;=\; \bar{P}\mathcal{M}P + P\mathcal{M}^T\bar{P} \;=\; -\delta\bar{P}\;. \tag{3.32}$$

Let us now consider the general variation of the action (3.27) for variations δP and δd. Of course such variations result in variations $\delta\,\widehat{\Gamma}$ of the determined parts of the connection. The undetermined parts need not be varied since they and their variations drop out of the action. We thus get

$$\begin{aligned}\delta S \;&=\; \delta\int dx d\tilde{x}\, e^{-2d}\, P^{MK} P^{NL} \widehat{\mathcal{R}}_{MNKL}\\ &=\; \int dx d\tilde{x}\, e^{-2d}\Big(-2\delta d\,\mathcal{R} + 2\,\delta P^{MK} P^{NL} \widehat{\mathcal{R}}_{MNKL} + 4P^{MK} P^{NL}\widehat{\nabla}_{[M}\delta\widehat{\Gamma}_{N]KL}\Big)\;,\end{aligned} \tag{3.33}$$

where we employed (2.35) since this relation holds for any shift of the connection. The covariant derivative in $\widehat{\nabla}\delta\widehat{\Gamma}$ can be partially integrated: it ignores the dilaton density and gives zero acting on the P's (note that both ∇ and $\widehat{\nabla}$ have such properties). This term is therefore a total derivative, in complete analogy to standard Einstein gravity. The variation of d then implies the vanishing of the scalar curvature, $\mathcal{R} = 0$. This is a well-known result in double field theory [2, 3], but here we understand more clearly why the variations of d inside $\mathcal{R}$ add up to a total derivative.

We focus on the remaining variation which reads with (3.32)

$$\begin{aligned} \delta S \;&=\; 2\int dx d\tilde{x}\, e^{-2d}\, \big(\bar{P}^{MP}\mathcal{M}_{PQ}P^{QK} + P^{MQ}\mathcal{M}_{PQ}\bar{P}^{PK}\big)\, P^{NL}\widehat{\mathcal{R}}_{MNKL} \\ &=\; 2\int dx d\tilde{x}\, e^{-2d}\mathcal{M}_{PQ}\left(\widehat{\mathcal{R}}^{\bar{P}\underline{L}\underline{Q}}{}_{\underline{L}} + \widehat{\mathcal{R}}^{\underline{Q}\underline{L}\bar{P}}{}_{\underline{L}}\right) \\ &=\; -2\int dx d\tilde{x}\, e^{-2d}\mathcal{M}_{PQ}\left(\mathcal{R}^{\underline{L}\bar{P}\underline{Q}}{}_{\underline{L}} + \mathcal{R}^{\underline{L}\underline{Q}\bar{P}}{}_{\underline{L}}\right) \;=\; -4\int dx d\tilde{x}\, e^{-2d}\mathcal{M}^{NM}\,\mathcal{R}^{\underline{K}}{}_{\underline{M}\bar{N}\underline{K}}\;, \end{aligned} \tag{3.34}$$

where we were able to remove the hats at the point where we know all undetermined connections drop out. In the last step we used (3.24) and relabeled indices. Thus, we get the field equation

$$\mathcal{R}_{\underline{M}\bar{N}} \;\equiv\; \mathcal{R}^{\underline{K}}{}_{\underline{M}\bar{N}\underline{K}} \;=\; 0\;, \tag{3.35}$$

recovering the Ricci tensor defined above.

An alternative definition for the Ricci tensor is obtained by tracing the curvature with $(1, 3)$ index structure (one under-barred, three barred). We will show now that the resulting object, $\mathcal{R}_{\bar{K}\underline{M}\bar{N}}{}^{\bar{K}}$, does not provide a new tensor. To this end we vary the alternative form of the action indicated in (3.28):

$$\delta S \;=\; -\delta\int dx d\tilde{x}\, e^{-2d}\,\bar{P}^{MK}\,\bar{P}^{NL}\widehat{\mathcal{R}}_{MNKL}\;. \tag{3.36}$$

Using $\delta\bar{P} = -\delta P$ we arrive at

$$\begin{aligned} \delta S \;&=\; 2\int dx d\tilde{x}\, e^{-2d}\,\delta P^{MK}\,\bar{P}^{NL}\widehat{\mathcal{R}}_{MNKL} \\ &=\; -2\int dx d\tilde{x}\, e^{-2d}\,\big(\bar{P}^{MP}\mathcal{M}_{PQ}P^{QK} + P^{MQ}\mathcal{M}_{PQ}\bar{P}^{PK}\big)\,\widehat{\mathcal{R}}^{\,\bar{L}}{}_{MK\bar{L}} \\ &=\; -2\int dx d\tilde{x}\, e^{-2d}\,\mathcal{M}_{PQ}\left(\mathcal{R}^{\bar{L}\bar{P}\underline{Q}}{}_{\bar{L}} + \mathcal{R}^{\bar{L}\underline{Q}\,\bar{P}}{}_{\bar{L}}\right) \\ &=\; -4\int dx d\tilde{x}\, e^{-2d}\mathcal{M}_{PQ}\,\mathcal{R}^{\bar{L}\underline{Q}\,\bar{P}}{}_{\bar{L}} \;=\; -4\int dx d\tilde{x}\, e^{-2d}\mathcal{M}^{NM}\mathcal{R}^{\bar{K}}{}_{\underline{M}\bar{N}\bar{K}}\;. \end{aligned} \tag{3.37}$$

Here we combined the two terms in the third line using the analogue of (3.24) and removed the hats, since the objects in question are well-defined. As this variation must agree with the variation (3.34) for all $\mathcal{M}$ we conclude

$$\mathcal{R}^{\underline{K}}{}_{\underline{M}\bar{N}\underline{K}} \;=\; \mathcal{R}^{\bar{K}}{}_{\underline{M}\bar{N}\bar{K}}\;, \tag{3.38}$$

proving that there is a single generalized Ricci tensor.

Let us relate the above definition of a Ricci tensor to a similar tensor defined in [3], where we considered the variation of $\mathcal{H}$ rather than P. The variation (3.32) implies the following variation for $\mathcal{H}$

$$\delta\mathcal{H} \;=\; -2\big(\bar{P}\mathcal{M}P + P\mathcal{M}^T\bar{P}\big)\;, \tag{3.39}$$

where we used (1.8) and $\delta\eta = 0$. Up to the factor of -2 this coincides with the variation given in eq. (4.54) in [3] if we assume $\mathcal{M}$ to be symmetric. The variation considered in [3] was not the most general, because $\mathcal{M}$ need not be symmetric, but it can be proved that the resulting field equations are equivalent to the ones obtained for general $\mathcal{M}$. To see this, consider a general action S based on a Lagrangian $\mathcal{L}(P)$ that we view as a function of P (suppressing the dependence on other fields). Using (3.32), its variation with respect to P then reads

$$\begin{aligned}\delta S &= \int dx d\tilde{x}\, e^{-2d}\, \frac{\delta\mathcal{L}}{\delta P_{KL}} \big(\bar{P}_{KM}\mathcal{M}^{MN}P_{NL} + P_{KN}\mathcal{M}^{MN}\bar{P}_{ML}\big) \\ &= \int dx d\tilde{x}\, e^{-2d}\, \mathcal{M}^{MN}\left(\bar{P}_{MK}P_{NL} + \bar{P}_{ML}P_{NK}\right)\frac{\delta\mathcal{L}}{\delta P_{KL}} \\ &= 2\int dx d\tilde{x}\, e^{-2d}\, \mathcal{M}^{MN}\, \bar{P}_{MK}P_{NL}\frac{\delta\mathcal{L}}{\delta P_{KL}}\,,\end{aligned} \tag{3.40}$$

where we used in the last step the symmetry of $\delta\mathcal{L}/\delta P_{KL}$. As $\mathcal{M}_{MN}$ is unconstrained, the field equations read

$$E_{MN} \equiv \bar{P}_{MK}P_{NL}\frac{\delta\mathcal{L}}{\delta P_{KL}} = 0\,. \tag{3.41}$$

An interesting property of tensors defined like this is that they vanish if and only if their symmetric projection $E_{(MN)}$ vanishes. For suppose

$$E_{(MN)} = \frac{1}{2}\left(\bar{P}_{MK}P_{NL} + \bar{P}_{NK}P_{ML}\right)\frac{\delta\mathcal{L}}{\delta P_{KL}} = 0\,. \tag{3.42}$$

We can then contract with $\bar{P}_R{}^M$, after which the second term vanishes by $P\bar{P} = 0$, implying $E_{RN} = 0$, as we wanted to show. Thus, the field equations obtained by variation with a symmetric or general $\mathcal{M}$ are equivalent.

After this preliminary discussion it is straightforward to relate the Ricci tensor in [3] to the one discussed here. We consider the variation of the action (3.26) under (3.39) (or, equivalently, (3.31)), with $\mathcal{M}$ symmetric,

$$\delta S \equiv -2\int dx d\tilde{x} e^{-2d}\,\mathcal{M}^{MN}\mathcal{R}_{MN} = -4\int dx d\tilde{x} e^{-2d}\,\mathcal{M}^{MN}P_M{}^K\bar{P}_N{}^L\mathcal{R}_{\bar{P}KL}{}^{\bar{P}}\,. \tag{3.43}$$

The first equality can be seen as the definition of $\mathcal{R}_{MN}$, where we included a factor of -2 such that the variation and hence the tensor $\mathcal{R}_{MN}$ have the same normalization as in [3]. For the second equality we used (3.34). Since we assumed $\mathcal{M}$ to be symmetric, $\mathcal{R}_{MN}$ is symmetric, too, and from (3.43) given by

$$\mathcal{R}_{MN} = \left(P_M{}^K\bar{P}_N{}^L + P_N{}^K\bar{P}_M{}^L\right)\mathcal{R}_{\bar{P}KL}{}^{\bar{P}}\,. \tag{3.44}$$

Writing the right-hand side out in terms of projected indices and using (3.24) we get

$$\mathcal{R}_{MN} = \mathcal{R}_{\underline{M}\bar{N}} + \mathcal{R}_{\bar{M}\underline{N}}\,. \tag{3.45}$$

The generalized tensor $\mathcal{R}_{MN}$ thus obtained has no projected indices. We can think of $\mathcal{R}_{\underline{M}\bar{N}}$ and $\mathcal{R}_{\bar{M}\underline{N}}$ as the projections of $\mathcal{R}_{MN}$. The symmetric field equation $\mathcal{R}_{MN} = 0$ is equivalent to $\mathcal{R}_{\underline{M}\bar{N}} = 0$.

We close this section by deriving a differential Bianchi identity following from the ξ^M gauge invariance of (3.27). First, we need to rewrite the gauge transformations. Using (2.26) the transformation of the dilaton reads

$$\delta_\xi e^{-2d} \;=\; \partial_M\big(e^{-2d}\xi^M\big) \;=\; e^{-2d}\nabla_M\xi^M \;. \tag{3.46}$$

For the projector P we have $\delta_\xi P_{MN} = \widehat{\mathcal{L}}_\xi P_{MN}$ because $\mathcal{H}$ transforms with a generalized Lie derivative (1.4) and $\widehat{\mathcal{L}}_\xi\eta = 0$ [3]. Due to the torsion constraint (2.17), all partial derivatives in Lie derivatives can be replaced by covariant derivatives. We thus have

$$\delta_\xi P_{MN} \;=\; \xi^K\nabla_K P_{MN} + 2\nabla_{[M}\xi_{K]}\,P^K{}_N + 2\nabla_{[N}\xi_{K]}\,P^K{}_M \;. \tag{3.47}$$

Using the covariant constancy of P this becomes

$$\delta_\xi P_{MN} \;=\; 2\,\nabla_{[M}\xi_{\underline{N}]} \;+\; 2\,\nabla_{[N}\xi_{\underline{M}]} \;. \tag{3.48}$$

Writing out the antisymmetrizations and using (2.40) we have

$$\begin{aligned}\delta_\xi P_{MN} \;&=\; \nabla_M\xi_{\underline{N}} - \nabla_{\underline{N}}\xi_M + \nabla_N\xi_{\underline{M}} - \nabla_{\underline{M}}\xi_N\\ &=\; \nabla_{\underline{M}}\xi_{\underline{N}} + \nabla_{\bar{M}}\xi_{\underline{N}} - \nabla_{\underline{N}}\xi_{\underline{M}} - \nabla_{\underline{N}}\xi_{\bar{M}}\\ &\quad\; + \nabla_{\underline{N}}\xi_{\underline{M}} + \nabla_{\bar{N}}\xi_{\underline{M}} - \nabla_{\underline{M}}\xi_{\underline{N}} - \nabla_{\underline{M}}\xi_{\bar{N}}\\ &=\; \nabla_{\bar{M}}\xi_{\underline{N}} - \nabla_{\underline{N}}\xi_{\bar{M}} + \nabla_{\bar{N}}\xi_{\underline{M}} - \nabla_{\underline{M}}\xi_{\bar{N}} \;.\end{aligned} \tag{3.49}$$

We can now write separate gauge transformations with respect to $\xi_{\underline{M}}$ and $\xi_{\bar{M}}$

$$\begin{aligned}\delta_{\underline{\xi}} P_{MN} \;&=\; \nabla_{\bar{M}}\,\xi_{\underline{N}} + \nabla_{\bar{N}}\,\xi_{\underline{M}} \;,\\ \delta_{\bar{\xi}} P_{MN} \;&=\; -\big(\nabla_{\underline{M}}\xi_{\bar{N}} + \nabla_{\underline{N}}\xi_{\bar{M}}\big) \;.\end{aligned} \tag{3.50}$$

For the dilaton we have, from (3.46),

$$\begin{aligned}\delta_{\underline{\xi}}\, e^{-2d} \;&=\; e^{-2d}\,\nabla_{\underline{M}}\xi^{\underline{M}} \;,\\ \delta_{\bar{\xi}}\, e^{-2d} \;&=\; e^{-2d}\,\nabla_{\bar{M}}\xi^{\bar{M}} \;.\end{aligned} \tag{3.51}$$

Consider now the gauge variation $\delta_{\underline{\xi}}$ of the action (3.27). Recalling that the curvature itself does not need to be varied because it contributes only total derivatives, as in (3.33), we have

$$\begin{aligned}0 \;&=\; \delta_{\underline{\xi}}\int dx d\tilde{x}\, e^{-2d} P^{MK}P^{NL}\widehat{\mathcal{R}}_{MNKL}\\ &=\; \int dx d\tilde{x}\, e^{-2d}\Big(\nabla_{\underline{P}}\xi^{\underline{P}}\,\mathcal{R} + 2\big(\nabla^{\bar{M}}\xi^{\underline{K}} + \nabla^{\bar{K}}\xi^{\underline{M}}\big)P^{NL}\widehat{\mathcal{R}}_{MNKL}\Big)\\ &=\; -\int dx d\tilde{x}\, e^{-2d}\xi^{\underline{P}}\,\Big(\nabla_{\underline{P}}\mathcal{R} + 4\nabla^{\bar{M}}\mathcal{R}_{\bar{M}\underline{N}\underline{P}}{}^{\underline{N}}\Big) \;,\end{aligned} \tag{3.52}$$

where we also used the property (3.24). The last contraction is (minus) the Ricci tensor, and since (3.52) holds for arbitrary $\xi^{\underline{P}}$ we conclude

$$\nabla_{\underline{P}}\mathcal{R} \;-\; 4\,\nabla^{\bar{M}}\mathcal{R}_{\underline{P}\bar{M}} \;=\; 0 \;. \tag{3.53}$$

Using the invariance under $\delta_{\bar{\xi}}$ we get a similar looking equation with an opposite relative sign:

$$\nabla_{\bar{P}}\mathcal{R} \,+\, 4\,\nabla^{\underline{M}}\mathcal{R}_{\underline{M}\bar{P}} \;=\; 0\,. \tag{3.54}$$

These are the differential Bianchi identities of double field theory [4, 7, 8]. We have not been able to find an *uncontracted* differential Bianchi identity for the full Riemann tensor, and we suspect that such an identity does not exist. In fact, it is not hard to convince oneself that the naive Bianchi identity $\nabla_{[M}\mathcal{R}_{NK]PQ} = 0$ does not hold by writing it out in terms of connections. As a further check, it is also straightforward to see that the double contraction of this naive Bianchi identity would give rise to an invalid contracted differential Bianchi identity.

4 Riemann-squared and the generalized metric

Here we will investigate if there exist manifestly $O(D,D)$ invariant terms quartic in derivatives and written with the generalized metric that, for $b = \phi = 0$, reduce to the square of the Riemann tensor in some T-duality frame. First, we work out the square of the Riemann tensor in terms of the metric g. Then we identify one tensor structure that cannot be reproduced from a generalized metric expression, answering the above question in the negative.

4.1 Outline of the approach

Our results of the previous section indicate that natural steps do not yield a physical Riemann tensor in double field theory. They give a four-index generalized tensor $\mathcal{R}_{MNPQ}$ that is not fully determined in terms of the physical fields, but whose contractions give physical scalar and Ricci curvatures that were expected to exist. It seems unlikely that there is a way to define a physical $\mathcal{R}_{MNPQ}$ that is an $O(D,D)$ tensor, a generalized tensor, and reduces to the Riemann tensor for particular combinations of indices.

We will show in this section that the Riemann-squared scalar, familiar in α' corrections to the low-energy effective action of string theory, cannot be obtained from a T-duality covariant expression built with the generalized metric and the dilaton. More explicitly, we claim that there is no $O(D,D)$ scalar $\mathcal{I}(\mathcal{H},d)$ such that it reduces to Riemann squared when we set $\tilde{\partial} = 0$, set the antisymmetric field b_{ij} to zero, and set the dilaton d to the value that corresponds to $\phi = 0$ in the relation $e^{-2d} = \sqrt{-g}e^{-2\phi}$, namely $d = d_* \equiv -\frac{1}{2}\ln\sqrt{-g}$. In other words, the answer to the following question is negative:

$$\text{Is there an } O(D,D) \text{ scalar } \;\mathcal{I}(\mathcal{H},d)\; \text{ such that } \quad R_{ijkl}R^{ijkl} \;=\; \mathcal{I}(\mathcal{H},d)\Big|_{\tilde{\partial}=0,\, b_{ij}=0,\, d=d_*} \;\; ? \tag{4.1}$$

This happens because certain tensor structures appearing in the square of the Riemann tensor cannot be reproduced from $O(D,D)$ invariant terms. This is a strong result, for the obstruction occurs just by demanding that $\mathcal{I}$ be an $O(D,D)$ scalar. An $\mathcal{I}(\mathcal{H},d)$ useful for double field theory would also have to be a generalized scalar. This result implies that even if there was a physical $\mathcal{R}_{MNPQ}$ that is both an $O(D,D)$ and a generalized tensor, and contained components that give the Riemann tensor (after setting $\tilde{\partial} = 0$, $b_{ij} = 0$, $d = d_*$) it could not be of use in constructing Riemann squared: $O(D,D)$ contractions would lead to canceling contributions.

Using the curvature scalar $\mathcal{R}$ and the Ricci tensor $\mathcal{R}_{\underline{M}\bar{N}}$, both of which are $O(D,D)$ tensors and generalized tensors, we can extend the double field theory action by the addition of higher-derivative terms with arbitrary coefficients a and b:

$$S_{\rm DFT} \;\longrightarrow\; S_{\rm DFT} + \alpha' \int dx d\tilde{x}\, e^{-2d}\big(a\mathcal{R}^2 + b\mathcal{R}^{\underline{M}\bar{N}}\mathcal{R}_{\underline{M}\bar{N}}\big)\;. \tag{4.2}$$

Setting $\tilde{\partial} = 0$ this reduces to terms containing the square of the conventional Ricci tensor and Ricci scalar. It is known, however, that in string theory also higher powers of the full Riemann tensor enter, and therefore (4.2) is not general enough to first order in α'. A correction ΔS proportional to Riemann squared in the low energy action would take the form

$$\Delta S \;\sim\; \alpha' \int dx\sqrt{-g}e^{-2\phi}R_{ijkl}R^{ijkl} \;=\; \alpha' \int dx\, e^{-2d}R_{ijkl}R^{ijkl}\,. \tag{4.3}$$

If there had been an $\mathcal{I}(\mathcal{H},d)$ that satisfies (4.1) the term

$$\Delta S_{\rm DFT} \;\sim\; \alpha' \int dx d\tilde{x}\, e^{-2d}\; \mathcal{I}(\mathcal{H},d)\,, \tag{4.4}$$

would have provided a suitable double field theory extension (if $\mathcal{I}$ was also a generalized tensor). In the absence of $\mathcal{I}(\mathcal{H},d)$ we can entertain some other possibilities. It may be that a variant of (4.1) holds up to terms that can be dropped from an action because they are total derivatives:

$$e^{-2d}R_{ijkl}R^{ijkl} \;=\; \mathcal{I}(\mathcal{H},d)\Big|_{\tilde{\partial}=0,\,b_{ij}=0} + \partial_i(\cdots)\,. \tag{4.5}$$

We will not explore this possibility in here, but it seems unlikely to work. It seems to us more likely that α' corrections require modifying the definition of the generalized metric, as will be explained in the discussion section.

4.2 Terms quadratic in the Riemann tensor

In this section we will compute the terms appearing in the square of the Riemann tensor that are relevant for the comparison with the generalized metric formulation to be discussed in the next subsection. In our conventions, which follow the book by Dirac [31], the Riemann tensor with all indices lowered is given by

$$R_{ijkl} \;=\; \frac{1}{2}\big(\partial_j\partial_k g_{il} - \partial_i\partial_k g_{jl} - \partial_j\partial_l g_{ik} + \partial_i\partial_l g_{jk}\big) + \Gamma_{pil}\Gamma^p{}_{jk} - \Gamma_{pik}\Gamma^p{}_{jl}\;, \tag{4.6}$$

with the Christoffel symbols

$$\Gamma_{ijk} \;=\; \frac{1}{2}(\partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk})\;, \qquad \Gamma^i{}_{jk} \;\equiv\; g^{ip}\Gamma_{pjk}\,. \tag{4.7}$$

We also write (4.6) as

$$R_{ijkl} \;=\; R^0_{ijkl} + 2\Gamma_{pi[l}\Gamma^p{}_{k]j}\;, \tag{4.8}$$

where

$$R^0_{ijkl} \;\equiv\; 2\partial_{[j}\partial_{[\underline{k}}\, g_{i]\underline{l}]}\;, \tag{4.9}$$

with $[ab] \equiv \frac{1}{2}(ab - ba)$, and we have underlined indices in order to avoid ambiguities in anti-symmetrizations. R^0 shares the symmetries of the full Riemann tensor,

$$R^0_{ijkl} \;=\; -R^0_{jikl} \;=\; -R^0_{ijlk} \;=\; R^0_{klij}\;. \tag{4.10}$$

Let us now consider the square of the Riemann tensor,

$$(\mathrm{Riem})^2 \;\equiv\; R_{ijkl}R^{ijkl} \;=\; R_{ijkl}\, g^{ir} g^{js} g^{kt} g^{lu}\, R_{rstu}\;. \tag{4.11}$$

From the definition (4.6) we infer that this square contains three different structures that are schematically

$$(\partial\partial g_{**})^2\;, \qquad (\partial\partial g_{**})(\partial g_{**})^2\;, \qquad (\partial g_{**})^4\;. \tag{4.12}$$

In order to establish our result it is sufficient to work out the first two. We will see that while the first structures can be reproduced from $O(D,D)$ invariant terms, this is not so for the second structures, proving that the full square of the Riemann tensor cannot be reproduced from an $O(D,D)$ invariant expression.

We begin by computing the terms of the first type, $(\partial\partial g_{**})^2$, from (4.11). Since terms involving the square of Christoffel symbols always contain the structure $(\partial g_{**})^2$, the terms we consider here originate only from the square of R^0. With (4.9) we then have

$$\begin{aligned}
(\mathrm{Riem})^2\Big|_{(\partial\partial g_{**})^2} \;&=\; 2\partial_{[j}\partial_{[\underline{k}}g_{i]\,\underline{l}]}g^{ir}g^{js}g^{kt}g^{lu}\,R^0_{rstu} \;=\; 2\,\partial_j\partial_k g_{il}\;g^{ir}g^{js}g^{kt}g^{lu}\,R^0_{rstu}\\
&=\; \partial_j\partial_k g_{il}\;g^{ir}g^{js}g^{kt}g^{lu}\big(\partial_s\partial_t g_{ru} - \partial_r\partial_t g_{su} - \partial_s\partial_u g_{rt} + \partial_r\partial_u g_{st}\big)\\
&=\; \partial_j\partial_k g_{il}\;g^{ir}g^{js}g^{kt}g^{lu}\;\partial_s\partial_t g_{ru} \;-\; 2\,\partial_j\partial_k g_{il}\;g^{ir}g^{js}g^{kt}g^{lu}\;\partial_r\partial_t g_{su}\\
&\qquad +\partial_j\partial_k g_{il}\;g^{ir}g^{js}g^{kt}g^{lu}\;\partial_r\partial_u g_{st}\;,
\end{aligned} \tag{4.13}$$

where we combined in the third line two terms using the symmetry properties of g and of second partial derivatives. After relabeling indices, this reads

$$\begin{aligned}
(\mathrm{Riem})^2\Big|_{(\partial\partial g_{**})^2} \;&=\; g^{ij}g^{kl}g^{mn}g^{pq}\;\partial_i\partial_k g_{mp}\;\partial_j\partial_l g_{nq} - 2\,g^{ij}g^{kl}g^{mn}g^{pq}\;\partial_k\partial_m g_{ip}\;\partial_j\partial_n g_{lq}\\
&\qquad + g^{ij}g^{kl}g^{mn}g^{pq}\;\partial_i\partial_k g_{mp}\;\partial_n\partial_q g_{jl}\\
&=\; g^{ij}g^{kl}g^{mn}g^{pq}\Big(\partial_i\partial_k g_{mp}\,\partial_j\partial_l g_{nq} - 2\,\partial_k\partial_m g_{ip}\,\partial_j\partial_n g_{lq} + \partial_i\partial_k g_{mp}\,\partial_n\partial_q g_{jl}\Big)\;.
\end{aligned} \tag{4.14}$$

Let us now turn to the second structure in (4.12). It originates from cross terms of R^0 and the Γ^2 term in (4.8). Thus,

$$\begin{aligned}
(\mathrm{Riem})^2\Big|_{(\partial\partial g_{**})(\partial g_{**})^2} \;&=\; 4R^{0\,nikm}\Gamma_{rnm}\Gamma^r{}_{ik}\\
&=\; R^{0\,nikm}\,(\partial_m g_{rn} + \partial_n g_{rm} - \partial_r g_{nm})\,g^{rs}\,(\partial_i g_{ks} + \partial_k g_{is} - \partial_s g_{ik})\;.
\end{aligned} \tag{4.15}$$

Using the symmetries (4.10) we can exchange $m \leftrightarrow n$ and $i \leftrightarrow k$ simultaneously. It then follows that several terms combine, giving

$$\begin{aligned}(\text{Riem})^2\Big|_{(\partial\partial g_{**})(\partial g_{**})^2} = R^{0\,nikm} g^{rs}\Big(&2\,\partial_m g_{rn}\,\partial_i g_{ks} + 2\,\partial_m g_{rn}\,\partial_k g_{is} \\ &- 2\,\partial_m g_{rn}\,\partial_s g_{ik} - 2\,\partial_r g_{nm}\,\partial_i g_{ks} + \partial_r g_{nm}\,\partial_s g_{ik}\Big)\,.\end{aligned} \tag{4.16}$$

4.3 Obstructions on the generalized metric formulation

We attempt now to write $O(D,D)$ invariant expressions in terms of the generalized metric that reproduce the above structures (4.14) and (4.16) when setting $\tilde{\partial}^i = 0$ and $b_{ij} = 0$. In this situation the generalized metric reads

$$\mathcal{H}_{MN} = \begin{pmatrix} \mathcal{H}^{ij} & \mathcal{H}^{i}{}_{j} \\ \mathcal{H}_{i}{}^{j} & \mathcal{H}_{ij} \end{pmatrix} = \begin{pmatrix} g^{ij} & 0 \\ 0 & g_{ij} \end{pmatrix}\,. \tag{4.17}$$

Specifically, we will see that the only candidate $O(D,D)$ invariant expression that could reproduce a certain tensor structure in the square of the Riemann tensor is actually zero as a consequence of the group properties of $\mathcal{H}_{MN}$.

We start with the $(\partial\partial g_{**})^2$ terms in (4.14). It turns out that they are reproduced by a term $\mathcal{I}^{(2,2)}(\mathcal{H})$ defined by

$$\begin{aligned}\mathcal{I}^{(2,2)}(\mathcal{H}) \equiv\; &-\frac{1}{2}\,\mathcal{H}^{IJ}\mathcal{H}^{KL}\,\partial_I\partial_K\mathcal{H}^{PQ}\,\partial_J\partial_L\mathcal{H}_{PQ} \\ &+ 2\,\mathcal{H}^{IJ}\mathcal{H}^{KL}\,\partial_I\partial_K\mathcal{H}^{MN}\,\partial_J\partial_M\mathcal{H}_{LN} + \partial_I\partial_J\mathcal{H}^{KL}\;\partial_K\partial_L\mathcal{H}^{IJ}\,.\end{aligned} \tag{4.18}$$

The superscripts on $\mathcal{I}$ indicate the derivative structure of the terms: they are the product of a factor with two derivatives and another factor with two derivatives. In order to evaluate the reduction of $\mathcal{I}^{(2,2)}(\mathcal{H})$ we set $\tilde{\partial}^i = 0$ and insert (4.17). First note that any derivative must have a lower index, $\partial_K \to \partial_k$, and the index contracted with this derivative also becomes k:

$$\begin{aligned}\mathcal{I}^{(2,2)} =\; &-\frac{1}{2}\,\mathcal{H}^{ij}\mathcal{H}^{kl}\,\partial_i\partial_k\mathcal{H}^{PQ}\,\partial_j\partial_l\mathcal{H}_{PQ} \\ &+ 2\,\mathcal{H}^{ij}\mathcal{H}^{kL}\,\partial_i\partial_k\mathcal{H}^{mN}\,\partial_j\partial_m\mathcal{H}_{LN} + \partial_i\partial_j\mathcal{H}^{kl}\;\partial_k\partial_l\mathcal{H}^{ij}\,.\end{aligned} \tag{4.19}$$

Because of the diagonal form of $\mathcal{H}$ in (4.17) any mixed-index structure $\mathcal{H}^{iK}$ will only receive contributions when K is an upper lowercase index, giving $\mathcal{H}^{ik}$, with the k also appearing elsewhere as a lower index. Thus in the second term above we can simply replace $L \to l$ and $N \to n$. In the first term there are two contributions for P and Q, one with structure $\mathcal{H}^{pq}\cdots\mathcal{H}_{pq}$ and the other $\mathcal{H}_{pq}\cdots\mathcal{H}^{pq}$. Both turn out to give the same answer and we therefore have:

$$\mathcal{I}^{(2,2)}(g) = -\,g^{ij}g^{kl}\,\partial_i\partial_k g^{pq}\,\partial_j\partial_l g_{pq} + 2g^{ij}g^{kl}\,\partial_i\partial_k g^{mn}\,\partial_j\partial_m g_{ln} + \partial_i\partial_j g^{kl}\,\partial_k\partial_l g^{ij}\,. \tag{4.20}$$

We can now transform the double derivatives of upper-indexed metrics to derivatives of lower-indexed metrics using

$$\begin{aligned}\partial_i\partial_j g^{-1} &= -\,\partial_i(g^{-1}\partial_j g g^{-1}) \\ &= g^{-1}(\partial_i g)g^{-1}(\partial_j g)g^{-1} - g^{-1}(\partial_i\partial_j g)g^{-1} + g^{-1}(\partial_j g)g^{-1}(\partial_i g)g^{-1}\,.\end{aligned} \tag{4.21}$$

In components this reads

$$\partial_k\partial_l g^{ij} \;=\; -g^{ir}g^{js}\partial_k\partial_l g_{rs} + 2g^{ip}g^{jq}g^{rs}\partial_{(\underline{k}}g_{pr}\,\partial_{\underline{l})}g_{qs}\,. \qquad (4.22)$$

We use this in (4.20) and collect only the terms with two derivatives on g:

$$\begin{aligned}\mathcal{I}^{(2,2)}(g)\Big|_{(\partial\partial g_{**})^2} \;=\;& g^{ij}g^{kl}\left(g^{ps}\partial_i\partial_k g_{st}g^{tq}\right)\partial_j\partial_l g_{pq} - 2g^{ij}g^{kl}\left(g^{ms}\partial_i\partial_k g_{st}g^{tn}\right)\partial_j\partial_m g_{ln}\\ &+ \left(g^{ks}\partial_i\partial_j g_{st}g^{tl}\right)\left(g^{iu}\partial_k\partial_l g_{uv}g^{vj}\right). \end{aligned} \qquad (4.23)$$

After a straightforward relabeling of indices one can compare with (4.14) and confirm that

$$\mathcal{I}^{(2,2)}(g)\Big|_{(\partial\partial g_{**})^2} \;=\; (\text{Riemann})^2\Big|_{(\partial\partial g_{**})^2}\,. \qquad (4.24)$$

This shows that the proposed generalized metric combination (4.18) correctly reproduces the portion of $(\text{Riemann})^2$ with two derivatives on each field. But, as we will see, it does not produce all of $(\text{Riemann})^2$.

Let us now consider the $(\partial\partial g_{**})(\partial g_{**})^2$ terms. Note that (4.21) implies that $\mathcal{I}^{(2,2)}$ produces already several terms of this type. It is convenient to begin again with (4.20) to do this systematically. Converting one of the $\partial^2 g^{**}$ metrics in the last term of $\mathcal{I}^{(2,2)}$ in (4.20) into g_{**}, we find

$$\begin{aligned}\mathcal{I}^{(2,2)}(g) \;=\;& -g^{ik}\,g^{kl}\,\partial_i\partial_k g^{pq}\big(\partial_j\partial_l g_{pq} - 2\partial_j\partial_p g_{lq} + \partial_p\partial_q g_{jl}\big)\\ &+ 2g^{ij}g^{kl}g^{rs}\,\partial_i\partial_k g^{pq}\,\partial_p g_{jr}\partial_q g_{ls}\,. \end{aligned} \qquad (4.25)$$

The terms in parenthesis are proportional to R^0 and thus we conclude

$$\mathcal{I}^{(2,2)}(g) \;=\; -2\,\partial_i\partial_k g^{pq}\,g^{ij}g^{kl}R^0_{qjlp} + 2\,g^{ij}g^{kl}g^{rs}\,\partial_i\partial_k g^{pq}\partial_p g_{jr}\partial_q g_{ls}\,. \qquad (4.26)$$

The second term in here is produced by minus $\mathcal{I}^{(2,1,1)}(\mathcal{H})$, defined by

$$\mathcal{I}^{(2,1,1)}(\mathcal{H}) \;\equiv\; -2\mathcal{H}^{IJ}\mathcal{H}^{KL}\mathcal{H}^{RS}\partial_I\partial_K\mathcal{H}^{PQ}\partial_P\mathcal{H}_{JR}\,\partial_Q\mathcal{H}_{LS}\,. \qquad (4.27)$$

We can therefore write

$$\mathcal{I}^{(2,2)}(g) + \mathcal{I}^{(2,1,1)}(g) \;=\; -2\partial_i\partial_k g^{pq}\,g^{ij}g^{kl}R^0_{qjlp}\,. \qquad (4.28)$$

We have shown that the terms on the right-hand side of this equation are reproduced from the generalized metric expression $\mathcal{I}^{(2,2)} + \mathcal{I}^{(2,1,1)}$.

We next investigate how much the right-hand side of (4.28) differs from the square of the Riemann tensor. For this purpose we first convert the leftover $\partial^2 g^{**}$ in (4.28) into g_{**},

$$\mathcal{I}^{(2,2)} + \mathcal{I}^{(2,1,1)} \;=\; 2g^{pm}g^{qn}\partial_i\partial_k g_{mn}\,g^{ij}g^{kl}R^0_{qjlp} - 4g^{pm}g^{qn}g^{rs}\partial_{(\underline{i}}g_{mr}\partial_{\underline{k})}g_{ns}\,g^{ij}g^{kl}R^0_{qjlp}\,. \qquad (4.29)$$

The $\partial^2 g_{**}$ structure in the first term inherits the antisymmetries from R^0 and so this term is actually $(R^0)^2$. Thus,

$$\mathcal{I}^{(2,2)} + \mathcal{I}^{(2,1,1)} \;=\; g^{ip}g^{jq}g^{km}g^{ln}R^0_{ijkl}R^0_{pqmn} - 2R^{0\,nikm}g^{rs}\left(\partial_i g_{mr}\,\partial_k g_{ns} + \partial_k g_{mr}\,\partial_i g_{ns}\right)\,. \qquad (4.30)$$

In here, the first term gives precisely the $(\partial\partial g_{**})^2$ terms, as discussed above, while the second one gives some of the $(\partial\partial g_{**})(\partial g_{**})^2$ terms. Comparing these with the actual terms of this type appearing in the square of the Riemann tensor (4.16) finally implies

$$\begin{aligned} R^{ijkl}R_{ijkl} &= \mathcal{I}^{(2,2)}(g) + \mathcal{I}^{(2,1,1)}(g) \\ &\quad - R^{0\,nikm} g^{rs}\big(-2\partial_k g_{mr}\,\partial_i g_{ns} - 2\partial_m g_{rn}\,\partial_k g_{is} - \partial_r g_{mn}\,\partial_s g_{ik} + 4\partial_m g_{rn}\,\partial_s g_{ik}\big) \\ &\quad + \mathcal{O}((\partial g)^4)\,. \end{aligned} \tag{4.31}$$

The first line is reproduced by the generalized metric expressions (4.18) and (4.27). We will now carefully examine the terms in the second line. We will identify one structure that cannot be written in terms of the generalized metric.

We first expand

$$\begin{aligned} &-R^{0\,nikm} g^{rs}\big(-2\partial_k g_{mr}\,\partial_i g_{ns} - 2\partial_m g_{rn}\,\partial_k g_{is} - \partial_r g_{mn}\,\partial_s g_{ik} + 4\partial_m g_{rn}\,\partial_s g_{ik}\big) \\ &\quad = -\frac{1}{2} g^{np}g^{iq}g^{kl}g^{mt}g^{rs}\,(\partial_q\partial_l g_{pt} - \partial_p\partial_l g_{qt} - \partial_q\partial_t g_{pl} + \partial_p\partial_t g_{ql}) \\ &\qquad\qquad (-2\partial_k g_{mr}\,\partial_i g_{ns} - 2\partial_m g_{rn}\,\partial_k g_{is} - \partial_r g_{mn}\,\partial_s g_{ik} + 4\partial_m g_{rn}\,\partial_s g_{ik}) \\ &\quad = -\frac{1}{2} g^{np}g^{iq}g^{kl}g^{mt}g^{rs}\big(-2\partial_k g_{mr}\,\partial_i g_{ns}\,\partial_q\partial_l g_{pt} + 2\partial_k g_{mr}\,\partial_i g_{ns}\,\partial_p\partial_l g_{qt} \\ &\qquad + 2\partial_k g_{mr}\,\partial_i g_{ns}\,\partial_q\partial_t g_{pl} - 2\partial_k g_{mr}\,\partial_i g_{ns}\,\partial_p\partial_t g_{ql} \\ &\qquad - 2\partial_m g_{rn}\,\partial_k g_{is}\,\partial_q\partial_l g_{pt} + 2\partial_m g_{rn}\,\partial_k g_{is}\,\partial_p\partial_l g_{qt} \\ &\qquad + 2\partial_m g_{rn}\,\partial_k g_{is}\,\partial_q\partial_t g_{pl} - 2\partial_m g_{rn}\,\partial_k g_{is}\,\partial_p\partial_t g_{ql} \\ &\qquad - \partial_r g_{mn}\,\partial_s g_{ik}\,\partial_q\partial_l g_{pt} + \partial_r g_{mn}\,\partial_s g_{ik}\,\partial_p\partial_l g_{qt} \\ &\qquad + \partial_r g_{mn}\,\partial_s g_{ik}\,\partial_q\partial_t g_{pl} - \partial_r g_{mn}\,\partial_s g_{ik}\,\partial_p\partial_t g_{ql} \\ &\qquad + 4\partial_m g_{rn}\,\partial_s g_{ik}\,\partial_q\partial_l g_{pt} - 4\partial_m g_{rn}\,\partial_s g_{ik}\,\partial_p\partial_l g_{qt} \\ &\qquad - 4\partial_m g_{rn}\,\partial_s g_{ik}\,\partial_q\partial_t g_{pl} + 4\partial_m g_{rn}\,\partial_s g_{ik}\,\partial_p\partial_t g_{ql}\,\big)\,. \end{aligned} \tag{4.32}$$

Several terms in here can be combined,

$$\begin{aligned} = -\frac{1}{2} g^{np}g^{iq}g^{kl}g^{mt}g^{rs}\big(&-2\underline{\underline{\partial_k g_{mr}\,\partial_i g_{ns}\,\partial_q\partial_l g_{pt}}} + 8\,\partial_k g_{mr}\,\partial_i g_{ns}\,\partial_p\partial_l g_{qt} \\ &- 2\partial_k g_{mr}\,\partial_i g_{ns}\,\partial_p\partial_t g_{ql} \\ &- 2\,\underline{\partial_r g_{mn}\,\partial_s g_{ik}\,\partial_q\partial_l g_{pt}} + 2\,\partial_r g_{mn}\,\partial_s g_{ik}\,\partial_p\partial_l g_{qt} \\ &+ 4\partial_m g_{rn}\,\partial_s g_{ik}\,\partial_q\partial_l g_{pt} - 4\partial_m g_{rn}\,\partial_s g_{ik}\,\partial_p\partial_l g_{qt} \\ &- 4\partial_m g_{rn}\,\partial_s g_{ik}\,\partial_q\partial_t g_{pl} + 4\underline{\partial_m g_{rn}\,\partial_s g_{ik}\,\partial_p\partial_t g_{ql}} \\ &- 4\partial_m g_{rn}\,\partial_k g_{is}\,\partial_q\partial_l g_{pt}\,\big)\,, \end{aligned} \tag{4.33}$$

where we have grouped the terms according to the index structure of the first $\partial_* g_{**}$ factor.

In (4.33) we have underlined three terms that deserve special consideration. All other terms can be reproduced by simply replacing metrics for generalized metrics and partial derivatives by $O(D,D)$ covariant partial derivatives. This happens because the indices on derivatives (that are lower, lowercase, when $\tilde{\partial}=0$) are contracted in such a way that they force all other indices to become lowercase once we recall that the generalized metric is diagonal. To make this point more transparent, consider the second term in (4.33):

$$-\,4\,g^{np}g^{iq}g^{kl}g^{mt}g^{rs}\;\partial_k g_{mr}\,\partial_i g_{ns}\,\partial_p\partial_l g_{qt}\,. \tag{4.34}$$

Its $O(D,D)$ covariant extension is simply

$$-\,4\,\mathcal{H}^{NP}\mathcal{H}^{IQ}\mathcal{H}^{KL}\mathcal{H}^{MT}\mathcal{H}^{RS}\;\partial_K\mathcal{H}_{MR}\,\partial_I\mathcal{H}_{NS}\,\partial_P\partial_L\mathcal{H}_{QT}\,. \tag{4.35}$$

To see that this works we just follow the indices on derivatives (which must be lower, lowercase) and how they force indices to become lowercase. From ∂_K we have $K\to L$, that is, we get k,l. From the second derivative we get $I\to Q\to T\to M\to R\to S\to N\to P$, and all indices become as in (4.34). One can readily check that the same happens for all other non-underlined terms.

Let us now consider the underlined terms in (4.33). We start with the two terms with a single underline, which take the form

$$X_1(g)\;=\;g^{np}g^{iq}g^{kl}g^{mt}g^{rs}\,\partial_r g_{mn}\,\partial_s g_{ik}\,\partial_q\partial_l g_{pt}\;-\;2g^{np}g^{iq}g^{kl}g^{mt}g^{rs}\,\partial_m g_{rn}\,\partial_s g_{ik}\;\partial_p\partial_t g_{ql}\,. \tag{4.36}$$

On each of the terms, each of the ∂g factors can be transformed into derivatives of inverse metrics via the identity $\partial g^{-1}=-g^{-1}(\partial g)g^{-1}$,

$$X_1(g)\;=\;g^{rs}\,\partial_r g^{pt}\,\partial_s g^{ql}\;\partial_q\partial_l g_{pt}\;-\;2\,g^{mt}\,\partial_m g^{ps}\,\partial_s g^{ql}\;\partial_p\partial_t g_{ql}\,. \tag{4.37}$$

These two structures can also be reproduced from an expression in terms of $\mathcal{H}$, with due care to double counting and extra terms that are to be thought of as higher order. We claim that the following is the answer

$$\mathcal{I}_{X_1}(\mathcal{H})\;=\;\frac{1}{2}\mathcal{H}^{RS}\partial_R\mathcal{H}^{PT}\,\partial_S\mathcal{H}^{QL}\,\partial_Q\partial_L\mathcal{H}_{PT}\;-\;\mathcal{H}^{MT}\partial_M\mathcal{H}^{PS}\partial_S\mathcal{H}^{QL}\partial_P\partial_T\mathcal{H}_{QL}\,. \tag{4.38}$$

The first step in the reduction gives

$$\mathcal{I}_{X_1}\;=\;\frac{1}{2}\,g^{rs}\partial_r\mathcal{H}^{PT}\,\partial_s g^{ql}\,\partial_q\partial_l\mathcal{H}_{PT}\;-\;g^{mt}\partial_m g^{ps}\partial_s\mathcal{H}^{QL}\partial_p\partial_t\mathcal{H}_{QL}\,. \tag{4.39}$$

This time we are left with contractions that give rise to two terms each,

$$\begin{aligned}\mathcal{I}_{X_1}(g)\;=\;&\frac{1}{2}\,g^{rs}\,\partial_r g^{ql}\,\partial_s g^{pt}\partial_q\partial_l g_{pt}+\frac{1}{2}\,g^{rs}\,\partial_r g^{ql}\,\partial_s g_{pt}\,\partial_q\partial_l g^{pt}\\&-\;g^{mt}\partial_m g^{ps}\,\partial_s g^{ql}\partial_p\partial_t g_{ql}\;-\;g^{mt}\partial_m g^{ps}\,\partial_s g_{ql}\,\partial_p\partial_t g^{ql}\,.\end{aligned} \tag{4.40}$$

Using (4.21) one can readily see that the second term on each line equals the first, up to $(\partial g)^4$ terms. Thus, we have

$$\mathcal{I}_{X_1}(g)\;=\;X_1(g)+\mathcal{O}((\partial g)^4)\,. \tag{4.41}$$

This shows that the terms with a single underline can be reproduced using the generalized metric, up to $(\partial g)^4$ terms that must be considered once all $\partial^2 g(\partial g)^2$ terms are under control.

Let us finally consider the double-underlined term in (4.33), which turns out to be problematic. The term is

$$Z(g) \;=\; g^{np}g^{iq}g^{kl}g^{mt}g^{rs}\;\partial_k g_{mr}\,\partial_i g_{ns}\,\partial_q\partial_l g_{pt} \;=\; -g^{np}g^{iq}g^{kl}\partial_k g^{ts}\;\partial_i g_{ns}\partial_q\partial_l g_{pt}\;, \tag{4.42}$$

where we rewrote the leftmost ∂g in terms of ∂g^{-1}. The only candidate $O(D,D)$ invariant term that could reproduce this structure is proportional to

$$\mathcal{I}_Z \;=\; \mathcal{H}^{NP}\mathcal{H}^{IQ}\mathcal{H}^{KL}\partial_K\mathcal{H}^{TS}\partial_I\mathcal{H}_{NS}\,\partial_Q\partial_L\mathcal{H}_{PT}\;. \tag{4.43}$$

The claim is that $\mathcal{I}_Z$ is in fact zero up to terms $(\partial\mathcal{H})^4$ — which in turn give rise to structures involving $(\partial g)^4$ and are thus of different type. To see this we raise and lower indices using η on the one hand and the analogue of (4.21) for $\mathcal{H}$ on the other:

$$\begin{aligned}
\mathcal{I}_Z \;&=\; \underline{\mathcal{H}^{NP}}\,\mathcal{H}^{IQ}\mathcal{H}^{KL}\partial_K\underline{\mathcal{H}^{TS}}\,\partial_I\underline{\mathcal{H}_{NS}}\,\partial_Q\partial_L\underline{\mathcal{H}_{PT}}\\
&=\; \mathcal{H}_{NP}\mathcal{H}^{IQ}\mathcal{H}^{KL}\partial_K\mathcal{H}_{TS}\partial_I\mathcal{H}^{NS}\underline{\partial_Q\partial_L\mathcal{H}^{PT}}\\
&=\; -\underline{\mathcal{H}_{NP}}\,\mathcal{H}^{IQ}\mathcal{H}^{KL}\underline{\mathcal{H}^{PR}}\,\mathcal{H}^{TM}\partial_K\mathcal{H}_{TS}\,\underline{\partial_I\mathcal{H}^{NS}}\,\partial_Q\partial_L\mathcal{H}_{RM}\;+\;(\partial\mathcal{H})^4\\
&=\; -\mathcal{H}^{TM}\mathcal{H}^{KL}\mathcal{H}^{IQ}\partial_I\mathcal{H}^{RS}\partial_K\mathcal{H}_{TS}\,\partial_Q\partial_L\mathcal{H}_{RM}\;+\;(\partial\mathcal{H})^4\\
&=\; -\mathcal{H}^{NP}\mathcal{H}^{IQ}\mathcal{H}^{KL}\partial_K\mathcal{H}^{TS}\partial_I\mathcal{H}_{NS}\partial_L\partial_Q\mathcal{H}_{TP}\;+\;(\partial\mathcal{H})^4\\
&=\; -\mathcal{I}_Z + (\partial\mathcal{H})^4\;.
\end{aligned} \tag{4.44}$$

As help to the reader, the underlined factors in each term denote those factors that participate in the simplification leading to the next term. In the step before the last line we relabeled indices ($I\leftrightarrow K$, $Q\leftrightarrow L$, $R\to T\to N$, $M\to P$). Thus, up to $(\partial\mathcal{H})^4$ terms, this structure is minus itself and thus zero.

One may wonder if the dilaton d can be used to help reproduce the above problematic structure. Unfortunately, this is not the case. Rather, the role of the dilaton can be understood as follows. Whenever a tensor contains the structure $g^{kl}\partial_m g_{kl}$, the generalized metric cannot be used to reproduce it. This follows because the corresponding $O(D,D)$ invariant term is minus itself by its group properties and thus vanishes:

$$\mathcal{H}^{KL}\partial_M\mathcal{H}_{KL} \;=\; -\mathcal{H}_{KL}\partial_M\mathcal{H}^{KL} \;=\; -\mathcal{H}^{KL}\partial_M\mathcal{H}_{KL}\;\equiv\;0\;. \tag{4.45}$$

In the first step we recalled that $\mathcal{H}^{KL}$ is the inverse of $\mathcal{H}_{KL}$, and in the second step we raised and lowered indices with the constant η_{MN}. In order to reproduce the structure $g^{kl}\partial_m g_{kl}$ we can use the $O(D,D)$ invariant dilaton d. Since $e^{-2d}=\sqrt{g}e^{-2\phi}$ we have, for $\tilde\partial^i=0$,

$$\partial_M d \;\;\to\;\; \partial_m d \;=\; \partial_m\phi - \frac{1}{4}g^{kl}\partial_m g_{kl}\;. \tag{4.46}$$

This means that

$$(-4\partial_M d)\Big|_{\tilde\partial=0,\phi=0} \;\to\; g^{kl}\partial_m g_{kl}\,, \tag{4.47}$$

provides the desired $O(D,D)$ covariantization of the structure. In fact, the $O(D,D)$ invariant scalar curvature given in [3] can be systematically constructed as follows. Start with the scalar curvature of Riemannian geometry written in terms of g_{ij}. For each term that can be reproduced using the generalized metric include the corresponding $O(D,D)$ covariant term. All terms that cannot be reproduced from a generalized metric expression turn out to contain the structure $g^{kl}\partial_m g_{kl}$, which is covariantized by $(-4\partial_M d)$. It can be checked that this covariantization of the Ricci scalar gives the generalized scalar $\mathcal{R}$ constructed in [3] and discussed in this paper. On the other hand, for the problematic structure (4.42) the dilaton does not help, as it contains no contractions of the $g^{kl}\partial_m g_{kl}$ type. As a side remark we point out that since the dilaton is of no use in constructing the T-duality invariant extension of the Riemann tensor-squared, this suggests that in a field basis in which the first α' correction consists only of the square of the Riemann tensor, the dilaton itself does not receive higher-derivative corrections. Intriguingly, this is confirmed by explicit computations in string theory [32].

Let us point out that for low-dimensional toy models like $D=2$ there may exist additional manipulations to rewrite the structure (4.42) such that it can be reproduced from a generalized metric or dilaton expression. In fact, in $D=2$ the Riemann tensor is fully determined by the scalar curvature and so the square of the generalized scalar $\mathcal{R}$ must contain Riemann-square. Incidentally, note that according to our formula for the number of undetermined connections given after (2.75) all connections are determined in $D=2$. In contrast, it is clear that for general D there are no additional identities that would allow for such manipulations.

Summarizing, for general D there is no $O(D,D)$ invariant expression in terms of the generalized metric that reproduces the required structure appearing in the square of the Riemann tensor. As a result there is no $O(D,D)$ invariant term fourth-order in derivatives that reproduces the square of the full Riemann tensor.

5 Discussion: T-duality and α' corrections

In this paper we have investigated the possible existence of a double field theory Riemann tensor $\mathcal{R}_{MNPQ}$ satisfying conditions 1) – 4) and (A), as stated in the introduction. In the first part of this paper we gave a self-contained presentation of a metric-like formalism introducing connections and invariant curvatures along the lines of the frame-like approach of Siegel [4]. The main difference with the related metric-like formalism of Jeon, Lee, and Park [18] is that we keep track of undetermined pieces in the connection and their effects on curvatures. Our analysis sheds new light on the Riemann tensor. Specifically, we showed that the components that are fully determined in terms of the physical fields vanish identically as a consequence of an algebraic Bianchi identity. Thus, within this formalism, there is no Riemann tensor meeting all conditions 1) – 4). There is a Riemann tensor satisfying conditions 1) – 3). It is an $O(D,D)$ tensor, a generalized tensor, and it determines $\mathcal{R}_{MN}$ and $\mathcal{R}$. It is not, however, fully determined in terms of the physical fields. The components of $\mathcal{R}_{MNPQ}$ that are independent of undetermined connections vanish.

In the second part of this paper we investigated a related question. We asked if there is a four-derivative $O(D,D)$ invariant function of the generalized metric and the dilaton that reduces in some T-duality frame (and with $b_{ij} = \phi = 0$) to the square of the Riemann tensor. We find that the answer is negative: for general D there is no $O(D,D)$ covariantization of Riemann-square in terms of the generalized metric and the dilaton. Such covariantization, if it existed, could be used as a Lagrangian for higher-derivative terms in double field theory. This result implies that even if a double field theory Riemann tensor satisfying conditions 1) – 4) exists, it could not provide a T-duality covariantization of Riemann-squared – its square would have to be zero.

Let us now briefly discuss the significance of this result. Suppose we had succeeded in constructing an $O(D,D)$ invariant in terms of $\mathcal{H}_{MN}$ and d that reduces to the square of the Riemann tensor in some T-duality frame. Then we would be able to write a general action with four derivatives as some arbitrary linear combination of the squares of generalized Riemann, generalized Ricci, and generalized scalar curvature. *Any* of these actions would be exactly invariant under the original forms of the T-duality and generalized diffeomorphisms that leave the original two-derivative action invariant. This would be unexpected, for the field redefinitions

$$g_{ij} \;\to\; g_{ij} + \alpha' \left(a_1 \, R_{ij} + a_2 \, g_{ij} R\right) \,, \tag{5.1}$$

that respect diffeomorphism invariance, map α'-corrected actions into each other in that they alter the coefficients of Ricci-squared and R-squared terms. After such field redefinitions the T-duality transformation of g_{ij} will acquire α' corrections, in conflict with the above implication of the (hypothetical) existence of a physical generalized Riemann tensor.

Useful insights into the structure of T-duality in double field theory to order α' are suggested by the computations of Meissner [29].[2] He considered 'cosmological' models, i.e., the reduction of gravitational actions with higher-order corrections to one dimension. The resulting theory can be written in an $O(D,D)$ invariant way only if the formula for the generalized metric in terms of the g and b fields receives α' corrections. For double field theory such a possibility would imply that the theory can be written in terms of a generalized metric $\overline{\mathcal{H}}_{MN}(g,b)$ of the form

$$\overline{\mathcal{H}}_{MN}(g,b) \;=\; \mathcal{H}_{MN}(g,b) \;+\; \alpha' \, \delta\mathcal{H}_{MN}(g,b) + \mathcal{O}(\alpha'^2) \,, \tag{5.2}$$

where $\mathcal{H}(g,b)$ is the generalized metric (1.1) and $\overline{\mathcal{H}}_{MN}(g,b)$ is a symmetric $O(D,D)$ matrix to order α'. Since (1.1) is a general parameterization of a symmetric $O(D,D)$ matrix, this means that one can write

$$\overline{\mathcal{H}}_{MN}(g,b) \;=\; \mathcal{H}_{MN}(g',b') \,, \tag{5.3}$$

where (g',b') are α' corrected versions of (g,b). The results of [29] (see eqs. (4.11)–(4.12)) suggest a redefinition of the type

$$(g')^{ij} \;=\; g^{ij} + \alpha' g^{ik} g^{jl} g^{pq} g^{rs} \big(a_1 \, \partial_r g_{kp} \, \partial_s g_{lq} + a_2 \, \partial_r b_{kp} \, \partial_s b_{lq}\big) \,. \tag{5.4}$$

It would be interesting to see if the problematic structure that we identified in the square of the Riemann tensor can be removed with such a field redefinition. Once the action is written in terms of $\mathcal{H}_{MN}(g',b')$, one could view (g',b') as the new field variables with standard (uncorrected) T-duality transformations. The redefinition (5.4) does not preserve manifest

[2] Later work of Kaloper and Meissner [30] did not use the generalized metric. It evaluated α' corrections to T-duality transformations arising in backgrounds with one abelian isometry.

general covariance because it involves first derivatives of the metric rather than tensors. Thus generalized diffeomorphisms would receive α' corrections. It would be interesting to see if the field basis suggested by string field theory has to play a special role here (see [10] for the explicit map between different field variables).

While the generalized Riemann tensor discussed in this paper is not fully determined by the physical fields, we expect it to play a crucial role in the construction of general T-duality invariant α' corrections. As discussed in section 3.1 this tensor has components of type $(4,0),(3,1),(1,3)$, and $(0,4)$:

$$\mathcal{R}_{\underline{M}\underline{N}\underline{K}\underline{L}}\,, \quad \mathcal{R}_{\underline{M}\underline{N}\underline{K}\bar{L}}\,, \quad \mathcal{R}_{\underline{M}\bar{N}\bar{K}\bar{L}}\,, \quad \mathcal{R}_{\bar{M}\bar{N}\bar{K}\bar{L}}\,, \tag{5.5}$$

all of which depend on undetermined connections. We believe that a suitable linear combination of squares of these curvatures will have the property that the undetermined part can be removed by a field redefinition.

It is amusing to speculate on the meaning of our results for the geometry that underlies string theory. The absence of a physical Riemann tensor seems to follow from the requirement of duality covariance. Since the Riemann tensor is needed for the construction of the interactions in the theory, we are forced to learn how to work with a partially physical, generalized Riemann tensor. This is all we seem to have. In Riemannian geometry a spacetime is flat if and only if the Riemannian curvature vanishes. In the absence of a physical Riemann tensor in string theory there would seem to be no obvious way to characterize flat space!

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A Relation to frame formalism

Here we explain the equivalence of the 'metric-like' formalism discussed in this paper and the 'frame-like' formalism of Siegel [4], extending the discussion given in sec. 5.3 of [7]. The vielbein $e_A{}^M$, with inverse $e_M{}^A$, carries an $O(D,D)$ index M and a flat index A with respect to the local tangent space group $GL(D)\times GL(D)$. This flat index splits as $A=(a,\bar{a})$, where a refers to the left $GL(D)$ and $\bar{a}$ to the right $GL(D)$. In order to describe only physical fields the vielbein $e_A{}^M$ needs to satisfy constraints that are written in terms of the tangent space metric $\mathcal{G}$ defined by

$$\mathcal{G}_{AB} \;\equiv\; e_A{}^M\, e_B{}^N\, \eta_{MN}\,, \quad \text{with inverse} \quad \mathcal{G}^{AB} \;=\; \eta^{MN}\, e_M{}^A\, e_N{}^B\;. \tag{A.1}$$

Flat indices are raised and lowered with $\mathcal{G}$ while $O(D,D)$ indices are raised and lowered with η. Moreover, $e_M{}^A=\eta_{MN}\mathcal{G}^{AB}e_B{}^N$. We impose the constraints

$$\mathcal{G}_{a\bar{b}} \;=\; 0\,, \qquad \mathrm{sig}(\mathcal{G}_{ab}) \;=\; (+\,-\,\dots-)\,, \quad \mathrm{sig}(\mathcal{G}_{\bar{a}\bar{b}}) \;=\; (-\,+\,\dots+)\,, \tag{A.2}$$

where 'sig' denotes the signature. Note that the signatures of $\mathcal{G}_{ab}$ and $\mathcal{G}_{\bar{a}\bar{b}}$ are opposite in order to be consistent with the (D,D) signature of $\mathcal{G}_{AB}$. The assignment of signatures here complies with the conventions of [7]. By Sylvester's theorem of inertia, the constraints (A.2) are $GL(D)\times GL(D)$ invariant.

The projectors P and $\bar{P}$ and the generalized metric $\mathcal{H}$ can be defined in terms of the frame field as in [7]:

$$P_M{}^N \;\equiv\; e_{aM}\,e^{aN}\;, \qquad \bar{P}_M{}^N \;\equiv\; e_{\bar{a}M}\,e^{\bar{a}N}\;, \qquad \mathcal{H}_M{}^N \;\equiv\; \frac{1}{2}\big(\bar{P}_M{}^N - P_M{}^N\big)\;. \tag{A.3}$$

As required, these projectors satisfy $P^2 = P$, $\bar{P}^2 = \bar{P}$ and, using the first constraint of (A.2), $P\bar{P} = 0$.

Following Siegel we may now introduce spin connections ω_{ABC} for the local $GL(D)\times GL(D)$ symmetry and impose covariant constraints in order to determine (part of) them in terms of the physical fields. These spin connections then uniquely determine Christoffel connections by means of a vielbein postulate as follows. We introduce a covariant derivative D with respect to the spin and Christoffel connection and postulate that the frame field $e_A{}^M$ is covariantly constant:

$$D_M e_A{}^N \;\equiv\; \partial_M e_A{}^N + \Gamma_{MK}{}^N e_A{}^K + \omega_{MA}{}^B e_B{}^N \;=\; 0\;. \tag{A.4}$$

Here

$$\omega_{MA}{}^B \;=\; e_M{}^C \omega_{CA}{}^B\;.$$

Note that

$$D_M \delta_A{}^B \;=\; 0\,. \tag{A.5}$$

Because of the factorized gauge group, the non-vanishing spin connections are $\omega_{Ma}{}^b$ and $\omega_{M\bar{a}}{}^{\bar{b}}$. The covariant derivative D_M reduces to the covariant derivative ∇_M discussed in this paper when acting on tensors with only curved indices. Moreover, the covariant derivative

$$D_A \;\equiv\; e_A{}^M D_M \tag{A.6}$$

reduces to the flat covariant derivative ∇_A of Siegel when acting on tensors with only $GL(D)\times GL(D)$ indices. Thus, with the vielbein being covariantly constant, any statement about 'tangent space' objects can be translated into a statement about 'world' objects and viceversa, in precise analogy to conventional Riemannian geometry. For instance, by (A.4) the Christoffel connection is determined by the frame field and the spin connection according to

$$\Gamma_{MNK} \;=\; -e_M{}^A e_N{}^B e_K{}^C \omega_{ABC} - e_N{}^A \partial_M e_{AK}\;. \tag{A.7}$$

In the following we will show that the constraints of Siegel imply via (A.4) our constraints (1)–(4) on Γ and thus that the frame formalism of Siegel is equivalent to the metric-like formalism discussed in this paper.

The frame formulation imposes the following constraints on the spin connection:

(i) The tangent space metric (A.1) is covariantly constant,

$$\nabla_A \mathcal{G}_{BC} \;=\; 0\;. \tag{A.8}$$

Since $\mathcal{G}_{BC}$ has only flat indices, the above implies that

$$D_A \mathcal{G}_{BC} \;=\; 0 \quad\rightarrow\quad D_M \mathcal{G}_{BC} \;=\; 0\,. \tag{A.9}$$

Because of (A.4) and (A.5), we have that $e_M{}^A$ is also covariantly constant and thus we can write

$$D_M\big(e_N{}^B e_K{}^C \mathcal{G}_{BC}\big) \;=\; 0 \quad\rightarrow\quad D_M \eta_{NK} \;=\; 0\,, \tag{A.10}$$

by use of (A.1). Since η only has $O(D,D)$ indices, the last equation above implies $\nabla_M \eta_{NK} = 0$, which is constraint (1). Moreover, we now readily derive the covariant constancy of P, $\bar{P}$ and therefore of $\mathcal{H}$, thus implying constraint (3). For example,

$$\nabla_M P_N{}^K \;=\; D_M P_N{}^K \;=\; D_M(e_{aN} e^{aK})\;, \tag{A.11}$$

where in the last step we noted that when D_M acts on an object $R_A{}^A$ with a contracted flat index there is no contribution from the spin connection. Given the diagonal form of the spin connection components the same is true for the action of D_M on an object of the form $R_a{}^a$ or $R_{\bar{a}}{}^{\bar{a}}$. Thus we are allowed to use the full covariant derivative D_M in the last expression above. Since D_M is a derivation and the vielbeins are covariantly constant we conclude that $\nabla_M P_N{}^K = 0$.

(ii) The second constraint requires that in the C-bracket

$$\big[\xi_1,\xi_2\big]_{\rm C}^{M} \;\equiv\; \xi_1^N \partial_N \xi_2^M - \frac{1}{2}\xi_{1N}\partial^M \xi_2^N - (1\leftrightarrow 2)\;, \tag{A.12}$$

we can flatten the indices by introducing covariant derivatives as follows,

$$\big[\xi_1,\xi_2\big]_{\rm C}^{A} \;\equiv\; e_M{}^A\big[\xi_1,\xi_2\big]_{\rm C}^{M} \;=\; \xi_1^B \nabla_B \xi_2^A - \frac{1}{2}\xi_{1B}\nabla^A \xi_2^B - (1\leftrightarrow 2)\;. \tag{A.13}$$

Since the derivatives act on flat indices we can replace ∇ by D and the constraint becomes

$$\big[\xi_1,\xi_2\big]_{\rm C}^{A} \;=\; \xi_1^B D_B \xi_2^A - \frac{1}{2}\xi_{1B}D^A \xi_2^B - (1\leftrightarrow 2)\;. \tag{A.14}$$

This constraint implies the generalized torsion constraint (2) in the form (2.17). In order to see this we recall that eqs. (3.29)–(3.30) in [3] show that the generalized Lie derivative can be written in terms of the C-bracket as

$$\widehat{\mathcal{L}}_\xi V^M \;=\; \big[\xi,V\big]_{\rm C}^{M} + \frac{1}{2}\partial^M\big(V^N\xi_N\big) \;=\; \big[\xi,V\big]_{\rm C}^{A}\, e_A{}^M + \frac{1}{2}\nabla^M\big(V^N\xi_N\big)\;, \tag{A.15}$$

where we used that the partial derivative of the scalar $V^N\xi_N$ coincides with the covariant derivative. Inserting (A.14) we obtain

$$\widehat{\mathcal{L}}_\xi V^M = \Big(\xi^B D_B V^A - V^B D_B \xi^A - \frac{1}{2}\xi_B D^A V^B + \frac{1}{2}V_B D^A \xi^B\Big)\, e_A{}^M + \frac{1}{2}\nabla^M\big(V^N\xi_N\big)\,. \tag{A.16}$$

Using the covariant constancy of the vielbein and converting all indices into curved indices we can replace D's by ∇'s and obtain

$$\begin{aligned}\widehat{\mathcal{L}}_\xi V^M \;&=\; \xi^N\nabla_N V^M - V^N\nabla_N \xi^M - \frac{1}{2}\xi_N\nabla^M V^N + \frac{1}{2}V_N\nabla^M\xi^N + \frac{1}{2}\nabla^M\big(V^N\xi_N\big)\\ &=\; \xi^N\nabla_N V^M + \big(\nabla^M\xi^N - \nabla^N\xi^M\big)V_N \;\equiv\; \widehat{\mathcal{L}}_\xi^{\nabla} V^M\;.\end{aligned} \tag{A.17}$$

We recovered (2.17) and thus constraint (2), as we wanted to show.

(iii) The third constraint requires

$$\int e^{-2d}\, V\nabla_A V^A \;=\; -\int e^{-2d}\, V^A\nabla_A V\;. \tag{A.18}$$

We can replace ∇ by D:

$$\int e^{-2d}\, VD_A V^A \;=\; -\int e^{-2d}\, V^A D_A V\;. \tag{A.19}$$

On the right-hand side we can immediately pass to $O(D,D)$ indices. On the left-hand side this requires use of (A.4). We thus find

$$\int e^{-2d}\, VD_M V^M \;=\; -\int e^{-2d}\, V^M D_M V\;. \tag{A.20}$$

Replacing D by ∇, as is allowed now, we obtain (2.24), thus implying constraint (4). Alternatively, the constraint can also be verified explicitly by inserting eq. (2.37) of [7] into the trace of (A.7), from which we recover (2.25).

In total, the constraints (i)–(iii) of the frame formalism imply, via (A.4), the constraints (1)–(4) of the metric-like formalism, thereby establishing the equivalence of both formulations.

References

[1] C. Hull, B. Zwiebach, "Double Field Theory," JHEP **0909**, 099 (2009). [arXiv:0904.4664 [hep-th]], "The Gauge algebra of double field theory and Courant brackets," JHEP **0909**, 090 (2009). [arXiv:0908.1792 [hep-th]].

[2] O. Hohm, C. Hull and B. Zwiebach, "Background independent action for double field theory," JHEP **1007** (2010) 016 [arXiv:1003.5027 [hep-th]].

[3] O. Hohm, C. Hull and B. Zwiebach, "Generalized metric formulation of double field theory," JHEP **1008** (2010) 008 [arXiv:1006.4823 [hep-th]].

[4] W. Siegel, "Superspace duality in low-energy superstrings," Phys. Rev. D **48**, 2826 (1993) [arXiv:hep-th/9305073], "Two vierbein formalism for string inspired axionic gravity," Phys. Rev. D **47**, 5453 (1993) [arXiv:hep-th/9302036].

[5] A. A. Tseytlin, "Duality Symmetric Formulation Of String World Sheet Dynamics," Phys. Lett. B **242**, 163 (1990); "Duality Symmetric Closed String Theory And Interacting Chiral Scalars," Nucl. Phys. B **350**, 395 (1991).

[6] M. J. Duff, "Duality Rotations In String Theory," Nucl. Phys. B **335**, 610 (1990), M. J. Duff and J. X. Lu, "Duality Rotations In Membrane Theory," Nucl. Phys. B **347**, 394 (1990).

[7] O. Hohm, S. K. Kwak, "Frame-like Geometry of Double Field Theory," J. Phys. A **A44**, 085404 (2011). [arXiv:1011.4101 [hep-th]],

[8] S. K. Kwak, "Invariances and Equations of Motion in Double Field Theory," JHEP **1010** (2010) 047 [arXiv:1008.2746 [hep-th]].

[9] O. Hohm, "T-duality versus Gauge Symmetry," arXiv:1101.3484 [hep-th],
B. Zwiebach, "Double Field Theory, T-Duality, and Courant Brackets," [arXiv:1109.1782 [hep-th]].

[10] O. Hohm, "On factorizations in perturbative quantum gravity," JHEP **1104**, 103 (2011). [arXiv:1103.0032 [hep-th]].

[11] O. Hohm, S. K. Kwak, "Double Field Theory Formulation of Heterotic Strings," JHEP **1106**, 096 (2011). [arXiv:1103.2136 [hep-th]].

[12] O. Hohm, S. K. Kwak, B. Zwiebach, "Unification of Type II Strings and T-duality," Phys. Rev. Lett. **107**, 171603 (2011), [arXiv:1106.5452 [hep-th]], "Double Field Theory of Type II Strings," JHEP **1109**, 013 (2011), [arXiv:1107.0008 [hep-th]].

[13] O. Hohm and S. K. Kwak, "Massive Type II in Double Field Theory," JHEP **1111** (2011) 086 [arXiv:1108.4937 [hep-th]].

[14] O. Hohm and S. K. Kwak, "N=1 Supersymmetric Double Field Theory," arXiv:1111.7293 [hep-th].

[15] C. Hillmann, "Generalized E(7(7)) coset dynamics and D=11 supergravity," JHEP **0903**, 135 (2009). [arXiv:0901.1581 [hep-th]].

[16] D. S. Berman, M. J. Perry, "Generalized Geometry and M theory," JHEP **1106**, 074 (2011). [arXiv:1008.1763 [hep-th]], D. S. Berman, H. Godazgar, M. J. Perry, "SO(5,5) duality in M-theory and generalized geometry," Phys. Lett. **B700**, 65-67 (2011). [arXiv:1103.5733 [hep-th]], D. S. Berman, E. T. Musaev, M. J. Perry, "Boundary Terms in Generalized Geometry and doubled field theory," [arXiv:1110.3097 [hep-th]], D. S. Berman, H. Godazgar, M. Godazgar, M. J. Perry, "The Local symmetries of M-theory and their formulation in generalised geometry," [arXiv:1110.3930 [hep-th]], D. S. Berman, H. Godazgar, M. J. Perry, P. West, "Duality Invariant Actions and Generalised Geometry," [arXiv:1111.0459 [hep-th]].

[17] P. West, "E_{11}, generalised space-time and IIA string theory," Phys. Lett. **B696**, 403-409 (2011). [arXiv:1009.2624 [hep-th]],
A. Rocen, P. West, "E11, generalised space-time and IIA string theory: the R-R sector," [arXiv:1012.2744 [hep-th]].

[18] I. Jeon, K. Lee, J. -H. Park, "Differential geometry with a projection: Application to double field theory," JHEP **1104**, 014 (2011). [arXiv:1011.1324 [hep-th]].

[19] I. Jeon, K. Lee, J. -H. Park, "Stringy differential geometry, beyond Riemann," Phys. Rev. **D84**, 044022 (2011). [arXiv:1105.6294 [hep-th]].

[20] I. Jeon, K. Lee, J. -H. Park, "Incorporation of fermions into double field theory," JHEP **1111**, 025 (2011). [arXiv:1109.2035 [hep-th]], "Supersymmetric Double Field Theory: Stringy Reformulation of Supergravity," arXiv:1112.0069 [hep-th].

[21] M. B. Schulz, "T-folds, doubled geometry, and the SU(2) WZW model," [arXiv:1106.6291 [hep-th]].

[22] N. B. Copland, "Connecting T-duality invariant theories," Nucl. Phys. **B854**, 575-591 (2012). [arXiv:1106.1888 [hep-th]], "A Double Sigma Model for Double Field Theory," [arXiv:1111.1828 [hep-th]].

[23] D. C. Thompson, "Duality Invariance: From M-theory to Double Field Theory," JHEP **1108**, 125 (2011). [arXiv:1106.4036 [hep-th]].

[24] C. Albertsson, S. -H. Dai, P. -W. Kao, F. -L. Lin, "Double Field Theory for Double D-branes," JHEP **1109**, 025 (2011). [arXiv:1107.0876 [hep-th]].

[25] D. Andriot, M. Larfors, D. Lust, P. Patalong, "A ten-dimensional action for non-geometric fluxes," JHEP **1109**, 134 (2011). [arXiv:1106.4015 [hep-th]], G. Aldazabal, W. Baron, D. Marques, C. Nunez, "The effective action of Double Field Theory," JHEP **1111**, 052 (2011). [arXiv:1109.0290 [hep-th]], D. Geissbuhler, "Double Field Theory and N=4 Gauged Supergravity," [arXiv:1109.4280 [hep-th]].

[26] A. Coimbra, C. Strickland-Constable, D. Waldram, "Supergravity as Generalised Geometry I: Type II Theories," [arXiv:1107.1733 [hep-th]], "$E_{d(d)} \times \mathbb{R}^+$ Generalised Geometry, Connections and M theory," arXiv:1112.3989 [hep-th].

[27] M. Gualtieri, "Branes on Poisson varieties," arXiv:0710.2719 [math.DG].

[28] I. A. Batalin and K. Bering, "Odd Scalar Curvature in Field-Antifield Formalism," J. Math. Phys. **49**, 033515 (2008) [arXiv:0708.0400 [hep-th]]. I. A. Batalin and K. Bering, "A Comparative Study of Laplacians and Schrodinger-Lichnerowicz-Weitzenbock Identities in Riemannian and Antisymplectic Geometry," J. Math. Phys. **50**, 073504 (2009) [arXiv:0809.4269 [hep-th]].

[29] K. A. Meissner, "Symmetries of higher order string gravity actions," Phys. Lett. **B392**, 298-304 (1997). [hep-th/9610131].

[30] N. Kaloper and K. A. Meissner, "Duality beyond the first loop," Phys. Rev. D **56**, 7940 (1997) [hep-th/9705193].

[31] P.A.M. Dirac, "General Theory of Relativity," Princeton University Press.

[32] C. G. Callan, Jr., I. R. Klebanov and M. J. Perry, "String Theory Effective Actions," Nucl. Phys. B **278**, 78 (1986).