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Classifying bases for 6D F-theory models

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ABSTRACT: We classify six-dimensional F-theory compactifications in terms of simple features of the divisor structure of the base surface of the elliptic fibration. This structure controls the minimal spectrum of the theory. We determine all irreducible configurations of divisors ("clusters") that are required to carry nonabelian gauge group factors based on the intersections of the divisors with one another and with the canonical class of the base. All 6D F-theory models are built from combinations of these irreducible configurations. Physically, this geometric structure characterizes the gauge algebra and matter that can remain in a 6D theory after maximal Higgsing. These results suggest that all 6D supergravity theories realized in F-theory have a maximally Higgsed phase in which the gauge algebra is built out of summands of the types $\mathfrak{su}(3), \mathfrak{so}(8), \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_8, \mathfrak{e}_7, (\mathfrak{g}_2 \oplus \mathfrak{su}(2))$, and $\mathfrak{su}(2) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(2)$, with minimal matter content charged only under the last three types of summands, corresponding to the non-Higgsable cluster types identified through F-theory geometry. Although we have identified all such geometric clusters, we have not proven that there cannot be an obstruction to Higgsing to the minimal gauge and matter configuration for any possible F-theory model. We also identify bounds on the number of tensor fields allowed in a theory with any fixed gauge algebra; we use this to bound the size of the gauge group (or algebra) in a simple class of F-theory bases.

Contents

1. Introduction

In a series of recent works, progress has been made in systematically analyzing the space of possible consistent six-dimensional $\mathcal{N} = 1$ supergravity theories [1]-[13]. A key part of this work has been the close correspondence relating the spectrum and Green–Schwarz couplings of 6D supergravity theories to geometric features of F-theory constructions.

F-theory [14, 15, 16] is a very powerful, nonperturbative approach to constructing string vacua in even-dimensional space-time. It is known that there is a finite set of possible combinations of gauge groups and matter content that can arise in F-theory constructions of 6D supergravity theories [17, 18, 5]. Nonetheless, there is no systematic global characterization of the space of such theories. 6D F-theory constructions are based on a choice of elliptically fibered Calabi–Yau manifold over a base B that is a complex surface. In general, for any given base B, there is an enormous range of possible gauge groups and matter content that can be realized in the corresponding 6D supergravity theory. The massless gauge group and matter content of the gravity theory can be changed by giving expectation values to charged matter fields; this reduces the gauge symmetry and massless matter content

through the familiar "Higgsing" process. In the F-theory picture, models with different spectra over the same F-theory base can be arranged by tuning the coefficients in a Weierstrass description of the model to arrange for certain codimension one and codimension two singularities in the elliptic fibration. Relaxing the constraints on coefficients needed for larger gauge symmetry corresponds to Higgsing in the 6D supergravity theory. Several recent papers [6, 8, 10] explore the range of gauge groups and matter content available for F-theory models that are realized over the simplest base surface, $B = \mathbb{P}^2$.

Just as 6D theories with different spectra are connected by Higgsing transitions, more exotic transitions involving tensionless strings connect the F-theory models on different base manifolds [19, 20, 16]. In this paper we carry out a systematic analysis of the kinds of F-theory models that can arise by restricting attention to the most generic theory over each base manifold B. Over any given base B, there is a unique gauge algebra and matter representation for the generic theory, characterized physically by Higgsing (i.e., giving a vacuum expectation value to) all possible matter fields. In some cases, maximal Higgsing completely removes any gauge symmetry. In other cases, maximal Higgsing removes all charged matter fields from the massless spectrum. And in yet other cases, some small amount of residual charged matter fields remain that cannot be lifted by Higgsing. We use the structure of effective divisor classes in B with negative self-intersection to determine a minimal gauge algebra and matter content that must be present for any F-theory model over each base B. By thus identifying the geometrical structures underlying all possible "non-Higgsable clusters" composed of gauge group factors and matter fields that cannot be Higgsed or factorized, we develop tools for analyzing the space of all F-theory compactifications.

6D supergravity theories are characterized by the number T of tensor multiplets in the theory. Theories with many tensor multiplets can satisfy anomaly cancellation through a generalization of the Green-Schwarz mechanism [21, 22]. The number of tensor multiplets in a 6D supergravity theory coming from an F-theory compactification is related to the topology of the base B through

$$T = h^{1,1}(B) - 1 \,. \tag{1.1}$$

As the number of tensor multiplets increases, the topological complexity of the F-theory base B grows, and the generic (completely Higgsed) models become more complicated. We show here that for any given gauge algebra $\mathfrak{g}$ and matter content that can be realized in a maximally Higgsed theory, there is a bound on the value of T for any F-theory construction with this spectrum. Combining this result with knowledge of the set of possible non-Higgsable clusters ("NHC's") suggests a path to systematic classification of all 6D F-theory models. While the finiteness of the set of F-theory models indicates that there is a maximum value possible for T compatible with a valid F-theory construction, no specific upper bound has previously been determined for this parameter. The maximum value ever found for a consistent F-theory construction or any other quantum 6D supergravity construction is $T = 193$ [23, 24]. We show here that the bounds determined on T from the gauge group limit the size of a class of related models constructed from a linear chain of divisors. In a sequel to this paper [25], we will systematically analyze and enumerate

all F-theory models over toric bases, and show that the model with $T = 193$ fits naturally into this classification.

Section 2 reviews some basic aspects of F-theory and algebraic geometry that are needed for the analysis in this paper. In Section 3 we analyze the intersection properties of divisors on the base B and determine all irreducible clusters for maximally Higgsed theories and the associated gauge algebra and matter contents. In Section 4, we determine upper bounds on T for any given gauge algebra. Section 5 contains some concluding remarks.

2. Geometric and F-theory preliminaries

Pedagogical introductions to the aspects of F-theory needed for this paper can be found in [26, 27, 28]. We review here briefly a few of the most salient points.

An F-theory compactification to six dimensions is defined by an elliptically fibered Calabi–Yau threefold with section over a base B. The fibration can be described by a Weierstrass equation

$$y^2 = x^3 + fx + g \tag{2.1}$$

where f, g are local functions on a complex surface forming the base B. Globally, f, g, and the discriminant locus

$$\Delta = 4f^3 + 27g^2 \tag{2.2}$$

are sections of line bundles

$$f \in -4K, \;\; g \in -6K, \;\; \Delta \in -12K \tag{2.3}$$

where K is the canonical class of B. The canonical class K satisfies

$$K \cdot K = 9 - T \tag{2.4}$$

where the inner product is the intersection form on $H^2(B, \mathbb{Z})$ and T is the number of tensor multiplets in the 6D supergravity theory.

The gauge group in F-theory is a compact reductive group G whose component group $\pi_0(G)$ coincides with the Tate–Shafarevich group $\text{Ш}_{X/B}$ of the fibration [29], whose fundamental group $\pi_1(G)$ coincides with the Mordell–Weil group $\text{MW}(X/B)$ of the fibration [30], and whose Lie algebra $\mathfrak{g}$ has a non-abelian part that is determined by the singular fibers in codimension 1 as described below. To avoid the subtle analysis involving the Tate–Shafarevich and Mordell–Weil groups that is necessary for the full specification of the gauge group, we focus in this paper on the nonabelian gauge algebra of the F-theory model.

Along the discriminant locus Δ the elliptic fibration is singular. Codimension one singularities carry nonabelian gauge algebra summands of the 6D theory, and codimension two singularities carry matter fields. The gauge algebra along a given component C of the discriminant locus can be determined by the Kodaira classification and the Tate algorithm [31, 16, 32, 33, 9] in terms of the degrees of vanishing of f, g, Δ along the curve C (see Table 1). In some cases further information regarding monodromy is needed to determine

ord (f)	ord (g)	ord (Δ)	singularity	nonabelian symmetry algebra
≥ 0	≥ 0	0	none	none
0	0	$n \geq 2$	A_{n-1}	$\mathfrak{su}(n)$ or $\mathfrak{sp}(\lfloor n/2 \rfloor)$
≥ 1	1	2	none	none
1	≥ 2	3	A_1	$\mathfrak{su}(2)$
≥ 2	2	4	A_2	$\mathfrak{su}(3)$ or $\mathfrak{su}(2)$
≥ 2	≥ 3	6	D_4	$\mathfrak{so}(8)$ or $\mathfrak{so}(7)$ or $\mathfrak{g}_2$
2	3	$n \geq 7$	D_{n-2}	$\mathfrak{so}(2n-4)$ or $\mathfrak{so}(2n-5)$
≥ 3	4	8	$\mathfrak{e}_6$	$\mathfrak{e}_6$ or $\mathfrak{f}_4$
3	≥ 5	9	$\mathfrak{e}_7$	$\mathfrak{e}_7$
≥ 4	5	10	$\mathfrak{e}_8$	$\mathfrak{e}_8$
≥ 4	≥ 6	≥ 12	does not occur in F-theory	

Table 1: Table of singularity types for elliptic surfaces and associated nonabelian symmetry algebras.

the precise gauge algebra; monodromies can give rise to non-simply-laced algebras in some situations [32, 9]. The possible singularity types at codimension two are not completely classified. In most simple cases, a local rank one enhancement of the gauge algebra gives matter that can be simply interpreted [32, 34, 1], but in other cases the singularities can be more complicated. Recent progress in understanding codimension two singularities and associated matter content appears in [8, 35, 9, 36].

The components C of the discriminant locus carrying nonabelian gauge algebra summands in an F-theory model are irreducible effective divisors in B. The discriminant locus itself, $\Delta = -12K$, is effective but need not be irreducible. The key feature of the algebraic geometry of surfaces that will be useful to us here relates to irreducible effective divisors of B. If C is an irreducible effective divisor of B satisfying $C \cdot C < 0$, and A is an effective divisor satisfying $A \cdot C < 0$, then C is an irreducible component of A, meaning that

$$C \cdot C < 0, \quad A \cdot C < 0 \quad \Rightarrow \quad A = C + X \tag{2.5}$$

with X effective. We will use this fact repeatedly in our analysis, generally using it to show that certain divisors must be contained in $-4K, -6K$, and $-12K$ and thus carry a minimal gauge algebra. For example, consider an irreducible effective divisor C satisfying $C \cdot C = -8$. The genus of C is fixed by the relation

$$(K + C) \cdot C = 2g - 2. \tag{2.6}$$

If C is a rational curve (topologically $\mathbb{P}^1$, with $g = 0$) having $C \cdot C = -8$, then $K \cdot C = 6$. It follows that $-4K \cdot C = -24$, so that $-4K = 3C + X_4$ with X_4 effective and $X_4 \cdot C \geq 0$. Similarly, $-6K = 5C + X_6$ and $-12K = 9C + X_{12}$. Thus, f, g, and Δ have degrees of vanishing at least $3, 5, 9$ on C, so C carries an $\mathfrak{e}_7$ gauge algebra. A similar argument shows that any irreducible effective divisor C with $C \cdot C < -2$ must carry a nonabelian gauge algebra summand. This fact is mentioned in a related physics context in [37].

A particularly simple set of F-theory bases are given by complex projective space $\mathbb{P}^2$ and the Hirzebruch surfaces $\mathbb{F}_m$. $\mathbb{P}^2$ is the only F-theory base with $T = 0$, and has a cohomology ring generated by a single divisor H with $H \cdot H = 1$. The Hirzebruch surface $\mathbb{F}_m$ is a $\mathbb{P}^1$ fibration over $\mathbb{P}^1$ that contains an effective irreducible divisor D with $D \cdot D = -m$ corresponding to a section of the fibration. Each Hirzebruch surface also contains an effective irreducible divisor F corresponding to a fiber, satisfying $D \cdot F = 1, F \cdot F = 0$. Together D and F span the second cohomology of $\mathbb{F}_m$. The Hirzebruch surfaces with $m \leq 12$ are the only F-theory bases with $T = 1$. All F-theory bases for 6D theories (except the Enriques surface, which supports models with no gauge group or matter content) can be found by blowing up a sequence of points on one of the bases just described. When the degrees of vanishing of f, g, Δ exceed $4, 6, 12$ at a point, the singularity of the elliptic fibration over B is so bad that any corresponding F-theory model would have tensionless strings. To get a good F-theory model from such a fibration, one must blow up the point in the base, increasing the number of tensor multiplets by one, and giving a "(-1)-curve" (rational curve with $C \cdot C = -1$) on which the degrees of vanishing of f, g, Δ are reduced by $4, 6, 12$ from the original singular point.[1] When the degrees of vanishing of f, g, Δ exceed $4, 6, 12$ on a curve, the singularity is even worse, and the remedy is to divide the coefficients f and g in the Weierstrass equation by appropriate powers of the equation of the curve. Unfortunately, that change alters the canonical bundle of the total space, and so leaves one with an elliptic fibration whose total space is not Calabi–Yau, so that it cannot be used in F-theory at all. This is indicated in the final line of Table 1.

3. Non-Higgsable clusters

On any F-theory base, there is a moduli space of theories in which the generic model has all possible matter fields Higgsed. This does not necessarily mean that the generic model has no matter fields, only that no further Higgsing is possible in the generic configuration. Information about the maximally Higgsed model on a given base is contained in the intersection structure of effective divisors on the base. On any base, an important feature of the geometry is the set of irreducible effective curves with negative self-intersection. As noted in the previous section, any irreducible effective divisor C with $C \cdot C < -2$ must carry a nonabelian gauge algebra summand. In this section we consider all possible intersecting combinations of irreducible effective divisors with $C \cdot C < -1$ that can arise on valid F-theory bases and the minimal gauge algebras and matter content associated with these clusters. This gives a set of "non-Higgsable clusters" of gauge algebras and matter content that can appear as factors in maximally Higgsed 6D supergravity theories.

3.1 Clusters of intersecting irreducible divisors

Each specific configuration of irreducible effective curves C_i, characterized by the self- and pairwise intersection numbers of the curves $C_i \cdot C_j$, gives rise to a specific minimal gauge and

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[1]The tensionless string transitions [19, 20, 16] connect two components of F-theory moduli space: on one of these, we allow more general polynomials f and g to avoid the highly singular point; on the other component, we blow up, increasing the number of tensors in the spectrum. The transition point itself is not considered to be an F-theory model.

matter content that can be computed by determining the multiplicity with which each C_i appears in the multiples $-4K, -6K$, and $-12K$ of the canonical class. These gauge algebras and matter content appear in the generic non-Higgsable theory on the given base. Note that for any set of curves with all self-intersections equal or greater than -2, $-nK \cdot C_i \geq 0$, and none of the curves C_i need to occur as components of $-nK$, so there is no nonabelian gauge algebra required on such a set of curves. Thus, we focus attention here on clusters of intersecting curves that include at least one curve of self-intersection -3 or below. In the following analysis all curves mentioned are irreducible and effective unless otherwise stated (except the anti-canonical divisor $-K$, and residual components denoted X, Y, which are effective but not irreducible).

3.1.1 Single irreducible divisors

As discussed above, a single irreducible effective curve C with $C \cdot C < -2$ must have sufficiently high degrees of vanishing of f, g, Δ to support a contribution to the nonabelian gauge algebra. If such a curve C does not intersect other curves with negative self-intersection, then the degrees of vanishing and the associated gauge algebra are easily computed. The results of this computation are familiar from the F-theory bases $\mathbb{F}_m$, which give theories dual to heterotic compactifications having known generic gauge algebra and matter configurations [38, 16]. Each base $\mathbb{F}_m$ has a single irreducible effective divisor D with negative self-intersection $D \cdot D = -m$; the calculation of the multiplicities with which D appears in f, g, Δ follows just as in the example $C \cdot C = -8$ below Equation (2.6). The gauge algebra and matter content for isolated curves with $C \cdot C = -m, 3 \leq m \leq 12$ are tabulated in Table 2. (Note that in some cases the degree of Δ is increased by the fact that deg $\Delta \geq$ Min (3 deg f, 2 deg g).)

A simple way to determine the multiplicities of the curve C in $-nK$ is to write $-K$ as

$$-K = \gamma C + Y , \tag{3.1}$$

where $Y \cdot C = 0$ and γ is taken over the rational numbers. From $-K \cdot C = 2 - m, C \cdot C = -m$, it follows that $\gamma = (m-2)/m$. Writing $-nK$ as an integral multiple of C plus a residual part X that satisfies $X \cdot C \geq 0$

$$-nK = cC + X , \tag{3.2}$$

it follows that $c = \lceil n(m-2)/m \rceil$. Thus, the degrees of vanishing of $-4K, -6K$, and $-12K$ over an isolated curve of self-intersection $-m$ are

$$[f] = \lceil 4(m-2)/m \rceil, \quad [g] = \lceil 6(m-2)/m \rceil, \quad [\Delta] = \lceil 12(m-2)/m \rceil . \tag{3.3}$$

Note that for $m = 9, 10, 11$ the degrees of vanishing are $4, 5, 10$, corresponding to an $\mathfrak{e}_8$ singularity on C, but the residual divisor X must have nonvanishing intersection with C; this raises the degree of vanishing at the intersection point to $4, 6, 12$ so that the intersection point must be blown up; thus the only value of $m > 8$ possible in a good F-theory base is $m = 12$. Physically this corresponds to the fact that there is no fundamental matter field for $\mathfrak{e}_8$ that would arise from the intersection of an $m = 9, 10$, or 11 curve with another

component of the discriminant locus. The derivation of the gauge algebra for individual curves with self-intersection $-m$ is described in section 3.4.

Note also that effective irreducible curves of negative self-intersection appearing in F-theory bases must be rational; higher genus curves automatically carry degrees of vanishing of f, g, and Δ that are equal to or greater than $(4, 6, 12)$, as can be verified by a simple computation. Assume that there is a curve C with $C \cdot C < 0, g > 0$. Then from (2.6) it follows that $K \cdot C \geq -C \cdot C$. The effective divisor $-nK$ must then contain the component C at least n times, since if $-nK = cC + X$, we have $X \cdot C = -nK \cdot C - cC \cdot C < 0$ when $C \cdot C < 0$ unless $c \geq n$.

3.1.2 Pairs of intersecting divisors

In addition to isolated curves, there are combinations of intersecting curves that have higher degrees of vanishing for f, g, Δ than are required by the self-intersections of the individual curves. We next consider pairs of curves A, B that each have negative self-intersection and that intersect one another in at least one point

$$A \cdot A = -x < 0 \tag{3.4}$$

$$B \cdot B = -y < 0 \tag{3.5}$$

$$A \cdot B = p > 0 \tag{3.6}$$

For example, consider the case of two curves, each with self-intersection -3, that intersect at a single point ($x = y = 3, p = 1$). In this case, writing for example $-4K = aA + bB + X$, with X effective and $X \cdot A \geq 0, X \cdot B \geq 0$ so that X does not contain A or B, we have from (2.6) that $-4K \cdot A = -4K \cdot B = -4$, from which it follows that $3a - b \geq 4, 3b - a \geq 4$ and thence that $a + b \geq 4$, so that f is of degree at least four on the intersection point $A \cdot B$. Similarly g, Δ have degrees at least 6 and 12, so the intersection point carries an elliptic fibration structure that is too singular and the base must be blown up for a valid F-theory construction. There may be a model on the blown up base with a higher value of T that is consistent, but this shows that no consistent F-theory model will have a base containing two intersecting curves of self-intersection -3.

It is straightforward to confirm that making either curve have a more negative self-intersection or increasing the number of intersection points between the curves simply makes the model more singular. Taking the general case described by (3.4-3.6), and writing again

$$-nK = aA + bB + X \tag{3.7}$$

we have

$$-nK \cdot A = n(2 - x) = -ax + bp + X \cdot A, \tag{3.8}$$

$$-nK \cdot B = n(2 - y) = ap - by + X \cdot B. \tag{3.9}$$

This leads to a pair of inequalities

$$ax - pb \geq n(x - 2), \tag{3.10}$$

$$by - pa \geq n(y - 2). \tag{3.11}$$

This gives lower bounds on a, b when $p^2 < xy$

$$a \geq \frac{n}{xy - p^2}(xy + py - 2y - 2p),$$ (3.12)

$$b \geq \frac{n}{xy - p^2}(xy + px - 2x - 2p),$$ (3.13)

and no solution if $p^2 > xy$. There are three marginal solutions where $p^2 = xy$; in the cases $x = y = p = 1$ of a (-1)-curve intersecting a (-1)-curve at a single point, and $x = y = p = 2$ of two (-2)-curves intersecting at two points the solution $a = b = 0$ gives a valid configuration, and when $x = 4, y = 1, p = 2$ there is a solution with $b = 0$ and a as for an isolated (-4)-curve. In the cases where $p^2 < xy$, the degree of $-nK$ at the intersection point $A \cdot B$ is given by

$$a + b \geq \frac{n}{xy - p^2}(2xy + p(x + y) - 4p - 2x - 2y).$$ (3.14)

For a valid F-theory base we need $a + b < n$ for at least one of $n = 4, 6, 12$, which is only possible if

$$(x + p - 2)(y + p - 2) < 4.$$ (3.15)

Thus, the only possible pairs of negative self-intersection curves that can arise with a single intersection $(p = 1)$ in an F-theory base are those with self-intersections $(-x, -y) = (-3, -2), (-2, -2)$, or $(-m, -1)$ with $m \leq 12$. Double intersections $(p = 2)$ are only possible between curves with self-intersections $(-x, -y) = (-m, -1)$ with $m = 1, 2, 3$ or 4. Triple intersections are not possible. It is easy to check that in all cases containing a (-1)-curve, the (-1)-curve need not be a component of $-nK$, so that the degrees of f, g, Δ and the nonabelian gauge algebra on the other component with self-intersection $-m$ are the same as if that curve were isolated.

The only new irreducible cluster that has thus arisen containing a pair of intersecting negative self-intersection curves where a nonabelian gauge factor must arise is the intersection of a (-3)- and (-2)-curve. The degrees of f, g, Δ on the two curves in this configuration can be computed using the minimum values of a, b satisfying (3.13). Computation of the gauge algebra and matter content on this intersecting divisor cluster requires consideration of the monodromy structure; the details of this analysis are described in Section 3.4 below. The result is that the intersecting (-3)- and (-2)-curves carry a gauge algebra $\mathfrak{g}_2 \oplus \mathfrak{su}(2)$, with 16 half-hypermultiplet matter fields transforming under the $\mathbf{7} + \mathbf{1}$ of $\mathfrak{g}_2$ and the fundamental of $\mathfrak{su}(2)$. This configuration and the associated gauge algebra is depicted in Figure 1, and listed in Table 2.

3.1.3 More than two intersecting divisors

Now we consider clusters consisting of more than two intersecting irreducible effective divisors. We consider only clusters where all divisors have self-intersection -2 or less; connection of clusters by (-1)-curves is considered in Section 3.3. Since from the results of the previous section the only pairs that can intersect are $(-2, -3)$ and $(-2, -2)$, the range of possible multiple-intersection configurations is limited. We do not consider configurations

where all curves are -2, since these need not carry any gauge group as discussed above. So we consider only configurations that contain at least one (-3)-curve. Simple enumeration of the possibilities and analysis using the same approach as in the case of simple pairs shows that there are only two new valid intersecting clusters. These are found by adding to intersecting (-3)- and (-2)-curves another (-2)-curve that either intersects the (-3)-curve or the (-2)-curve. In the first case, the (-2)-curve carries another $\mathfrak{su}(2)$ summand; the gauge algebra on the (-3)-curve becomes $\mathfrak{so}(7)$ and there are two sets of half-hypermultiplets, transforming in the spinor $\mathbf{8}$ of $\mathfrak{so}(7)$, with one transforming in the $\mathbf{2}$ of each $\mathfrak{su}(2)$ and trivially under the other $\mathfrak{su}(2)$. In the second case, the additional curve carries no gauge group but the degrees of vanishing on the first (-2)-curve increase; monodromy brings the gauge algebra to $\mathfrak{sp}(1) = \mathfrak{su}(2)$, so we again have gauge algebra $\mathfrak{g}_2 \oplus \mathfrak{su}(2)$ with the same matter as in the original configuration. These two irreducible geometric units carrying gauge groups and non-Higgsable matter are depicted in Figure 1 and included in Table 2. The detailed derivation of the spectrum on each cluster is given in subsection 3.4 below. A systematic analysis shows that no other irreducible combination of intersecting curves each with negative self-intersection -2 or below can appear in an F-theory compactification. The various other possibilities that combine valid pairwise intersections, including the linear strings of intersecting curves $(-3, -2, -2, -2), (-2, -3, -2, -2)$ and $(-3, -2, -3)$, combinations where a single (-2)- or (-3)-curve is intersected by three (-2)- or (-3)-curves, as well as a set of three curves with self-intersections $-3, -2, -2$ that each intersect pairwise, and any clusters that include these configurations as subclusters, can all be shown to give elliptic fibrations that are too singular to describe an F-theory compactification without blowing up points on the base.

For each irreducible cluster of curves appearing in Table 2, the minimal (non-Higgsable) gauge group and matter spectrum can be determined either through geometry, as we do in Section 3.4 below, or by using anomalies. Each of these non-Higgsable gauge + matter configurations has been identified in explicit string theory constructions, including those with gauge groups $\mathfrak{f}_4, \mathfrak{g}_2 \oplus \mathfrak{su}(2)$ and $\mathfrak{su}(2) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(2)$ [39], where the matter content was determined using anomaly cancellation. The analysis here shows that no other non-Higgsable configurations of gauge groups and matter content can be required by F-theory geometry.

This completes the analysis of all connected configurations of curves of self-intersection -2 or less that must contain a nonabelian gauge group or matter content when the model is maximally Higgsed. We refer to any of the irreducible clusters and their algebra + matter content (including single gauge algebra summands without matter such as the $\mathfrak{f}_4$ on a rational curve of self-intersection -5) in Table 2 as "non-Higgsable clusters", or simply NHC's.

3.2 Higgsing

In the analysis above, we identified all combinations of curves that are forced by their intersection with multiples of $-K$ to carry certain gauge algebras, in certain cases with given minimal matter content. Since the geometry forces these gauge + matter configurations to appear in the theory, there is no way in which the matter fields in the last three
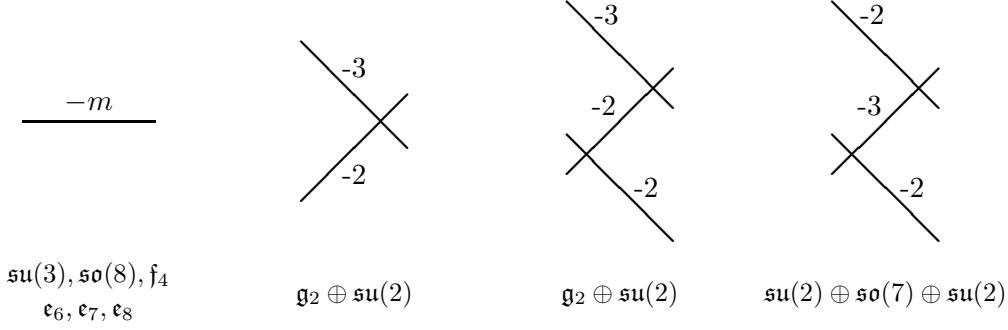
Figure 1: All possible clusters of intersecting curves with self-intersection of each curve -2 or below. For each cluster the corresponding gauge algebra is noted and the gauge algebra and matter content are listed in Table 2

Diagram	Algebra	matter	(f, g, Δ)	$\Delta T_{\max}$
-3	$\mathfrak{su}(3)$	0	$(2, 2, 4)$	$1/3$
-4	$\mathfrak{so}(8)$	0	$(2, 3, 6)$	1
-5	$\mathfrak{f}_4$	0	$(3, 4, 8)$	$16/9$
-6	$\mathfrak{e}_6$	0	$(3, 4, 8)$	$8/3$
-7	$\mathfrak{e}_7$	$\frac{1}{2}\mathbf{56}$	$(3, 5, 9)$	$57/16$
-8	$\mathfrak{e}_7$	0	$(3, 5, 9)$	$9/2$
-12	$\mathfrak{e}_8$	0	$(4, 5, 10)$	$25/3$
$-3, -2$	$\mathfrak{g}_2 \oplus \mathfrak{su}(2)$	$(\mathbf{7} + \mathbf{1}, \frac{1}{2}\mathbf{2})$	$(2, 3, 6), (1, 2, 3)$	$3/8$
$-3, -2, -2$	$\mathfrak{g}_2 \oplus \mathfrak{su}(2)$	$(\mathbf{7} + \mathbf{1}, \frac{1}{2}\mathbf{2})$	$(2, 3, 6), (2, 2, 4),$ $(1, 1, 2\ )$	$5/12$
$-2, -3, -2$	$\mathfrak{su}(2) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(2)$	$(\mathbf{1}, \mathbf{8}, \frac{1}{2}\mathbf{2})$ $+(\frac{1}{2}\mathbf{2}, \mathbf{8}, \mathbf{1})$	$(1, 2, 3), (2, 4, 6),$ $(1, 2, 3)$	$1/2$

Table 2: Irreducible geometric components (non-Higgsable clusters, or "NHC's") consisting of one or more intersecting curves associated with irreducible effective divisors each with negative self-intersection. Each cluster gives rise to a minimal gauge algebra and matter configuration.

NHC's in Table 2 can be removed by moving in the moduli space of the theory (without changing the number of tensor multiplets and moving to a different base through an extremal transition). In particular, this means that the matter in these configurations cannot be Higgsed. Indeed, analysis of the matter fields that can be Higgsed for different gauge algebras shows that Higgsing is impossible in these three situations. To use fundamentals to Higgs an $\mathfrak{su}(N)$ gauge theory, two fundamental matter fields must be simultaneously given expectation values to implement the Higgsing. A fundamental + antifundamental is needed to combine with the broken generators of the $\mathfrak{su}(N)$ in reduction to $\mathfrak{su}(N-1)$ to give the appropriate massive gauge fields. This can also be seen from the need to give a second fundamental a VEV to cancel the D-term constraints in the equations of motion. Similarly, for a gauge algebra $\mathfrak{su}(N) \oplus \mathfrak{g}$, one bifundamental field cannot be Higgsed, two are necessary. In each of the 3 NHC's described above, there is a single (half-hyper) trans-

forming under $\mathfrak{su}(2) \oplus \mathfrak{g}$ as a fundamental of $\mathfrak{su}(2)$ and an irreducible representation of $\mathfrak{g}$. Thus, it is clear that none of these configurations can be Higgsed. On the other hand, $\mathfrak{so}(7)$ can be Higgsed to $\mathfrak{g}_2$ by Higgsing a single spinor **8**, and $\mathfrak{g}_2$ can be Higgsed to $\mathfrak{su}(3)$ by a single **7**, so these configurations are not non-Higgsable without the $\mathfrak{su}(2)$ summands.

We have identified all non-Higgsable configurations that are forced by the geometry of F-theory from the intersection of the canonical class with the configurations. Though it seems plausible physically that all non-Higgsable configurations are forced in this way from geometry, this has not been rigorously proven. One could imagine some kind of conspiracy in which certain geometries force additional non-Higgsable matter configurations, for example on curves with positive self-intersection. We do not have any argument at this time that would rule out this possibility in global models, for example, if the degrees of freedom are highly constrained so that there are no charged matter field combinations available to break to the minimal gauge content indicated by the NHC configuration.

One way to show that the maximally Higgsed content of all F-theory models is captured by the NHC configuration determined from the divisor structure is to show that all other matter configurations that may arise in F-theory can be removed by Higgsing. In most cases this is fairly straightforward. For $\mathfrak{su}(N)$, a single tensor or two-index antisymmetric representation can be Higgsed, and a variety of Higgsing mechanisms are known for other common representations and gauge groups appearing in F-theory [40]. As discussed in [6, 8], however, various exotic representations may also be possible in F-theory. For example, 3-index antisymmetric representations are possible for certain $\mathfrak{su}(N)$ gauge algebras. We have not carried out a completely systematic analysis of all matter representations — indeed, it is not yet clear what matter representations can and cannot be realized in F-theory through codimension two singularities — but physically it seems likely that any Weierstrass model realizing more exotic matter will have free parameters that remove the matter by Higgsing so that exotic matter will not appear in generic F-theory constructions over any base.

3.3 Connecting clusters with (-1)-curves

In general, an F-theory base will contain a set of the non-Higgsable clusters described in Table 2. Some or all of these clusters will be connected by (-1)-curves intersecting curves from the clusters. Note that not all clusters can be connected by (-1)-curves. For example, a (-4)-curve A cannot be connected to a (-5)-curve B by a (-1)-curve C, since the degrees of vanishing of f, g, Δ on B and C would have to be 3, 4, 8, and 2, 2, 4 respectively, so that the intersection point between these curves would need to be blown up for a good F-theory base. A similar analysis can be used to identify all ways in which (-1)-curves connect NHC's.

One simple rule for when a (-1)-curve can connect a set of NHC's is that any set of curves can be connected by a (-1)-curve A when the total (including intersection multiplicities) of the degrees of f, g, Δ on the curves intersecting A is less than or equal to 4, 6, 12 respectively. To prove this consider a configuration where A intersects curves C_i with multiplicity p_i. Any multiple of $-K$ can then be written as

$$-nK = aA + \sum_i c_i C_i + X \qquad (3.16)$$

where $X \cdot A \geq 0, X \cdot C_i \geq 0$. The intersection with A is

$$-nK \cdot A = n = -a + \sum_i c_i p_i + X \cdot A. \tag{3.17}$$

The positivity of $X \cdot A$ then implies

$$a \geq \sum_i c_i p_i - n. \tag{3.18}$$

When

$$\sum_i c_i p_i \leq n \tag{3.19}$$

then this condition can be satisfied with $a = 0$. Thus, any configuration of curves satisfying the condition above for f, g, Δ (*i.e.*, satisfying (3.19) for the decomposition of $-nK$ for $n = 4, 6, 12$) can be connected by a (-1)-curve A that is not contained in any multiple of $-K$ and does not affect the gauge group or matter content of the theory.

As an example of this condition, a (-1)-curve can connect two (-4)-curves and does not carry any degree of vanishing for f, g, Δ. On the other hand, a (-1)-curve cannot connect a (-4)-curve to a (-5)-curve, as mentioned above. A (-1)-curve can only connect a -12 curve to another (-1)-curve or a (-2)-curve that does not carry a gauge group.

There are a limited number of ways in which a (-1)-curve can intersect a single NHC. Using the analysis of which pairs of curves can intersect from Section 3.1.2, we know that a (-1)-curve can intersect a curve of self-intersection -5 or less only once, and a curve of self-intersection $-4, -3$, or -2 either once or twice. Performing a case-by-case analysis shows that there are 31 distinct ways in which a (-1)-curve can intersect a single NHC in a consistent F-theory base. It is then possible to systematically analyze all possible combinations of NHC's that can be intersected by a single (-1)-curve. The details of this analysis are presented in the Appendix.

While the relation (3.19) holding for $n = 4, 6$, and 12 is a sufficient condition for an intersection configuration to be allowed, the opposite is not true. In almost all cases, when (3.19) is violated for $n = 12$ the configuration of NHC's connected by a (-1)-curve becomes singular and is not allowed. The only exceptions to this are when the (-1)-curve A intersects a (-5)-curve and the (-3)-curve of either a $(-3, -2)$ NHC or a $(-3, -2, -2)$ NHC. In these cases, $\sum_i c_i = 14$ for $n = 12$ and the vanishing degree of Δ on A is 2, but there is no 4, 6, 12 singularity at any point. There is, however, no nonabelian gauge group factor carried on A, and the algebra structure of the connected NHC's is unchanged. Thus, all configurations of NHC's connected by (-1)-curves give rise to the same gauge algebra and matter content as if they were isolated.

There are also a number of marginal cases, where a (-1)-curve A can connect NHC's in such a way that A must appear in f or g but need not contribute to the discriminant $\Delta = -12K$ (and therefore again need not carry any gauge group). For example, consider the case where A connects a (-3)-curve to a (-5)-curve. For g, Δ, (3.19) is satisfied, as can be readily confirmed from the data in Table 2. On the other hand, f has vanishing degrees of 2 and 3 on the (-3)- and (-5)-curves respectively and thus $a \geq 1$ from (3.18).

It is also not the case that any NHC combination can be connected where (3.19) is satisfied for $n = 12$. As a counterexample, consider a (-1)-curve connecting four $(-3, -2)$ type NHC's by intersecting each along the corresponding (-2)-curve $C_1, \ldots C_4$. In this case, (3.19) is satisfied for f and Δ but not for g. In fact, this is not a valid configuration for an F-theory base. To see this, write

$$-6K = aA + \sum_i c_i C_i + \sum_i d_i D_i + X \tag{3.20}$$

where D_i are the four (-3)-curves. The intersection product with A gives

$$-6K \cdot A = 6 = \sum_i c_i - a + X \cdot A, \tag{3.21}$$

so $a \geq \sum_i c_i - 6$. The intersection product with C_i gives

$$-6K \cdot C_i = 0 = d_i + a - 2c_i + X \cdot C_i, \tag{3.22}$$

so (since $d_i \geq 3$), $c_i \geq (3 + a)/2$. Combining these two equations gives $a \geq 2a$, and since $c_i \geq 2$ it must also be the case that $a > 0$. These equations cannot be simultaneously satisfied so there is no good F-theory base with this geometry.

The complete analysis of all ways in which a single (-1)-curve can connect to one or more NHC's is summarized in the Appendix. The total number of distinct ways that a (-1)-curve can connect k NHC's is:

31 configurations with a (-1)-curve intersecting 1 NHC
100 configurations with a (-1)-curve intersecting 2 NHC's
46 configurations with a (-1)-curve intersecting 3 NHC's
6 configurations with a (-1)-curve intersecting 4 NHC's

There are no ways of connecting more than 4 NHC's with a single (-1)-curve. This gives a total of 183 ways in which a (-1)-curve can connect to a combination of NHC's. In all of these configurations, the (-1)-curve carries no nonabelian gauge group, and the total gauge group is the product of factors from the NHC's.

3.4 Minimal spectra on irreducible clusters

We have determined above the neccessary orders of vanishing of f, g, Δ along the components of each irreducible cluster, but we have not yet explained how to determine the gauge algebra and charged matter content in each case. We do that in this section, using the methods of [9].

The data of the orders of vanishing of f, g and Δ along a curve C must in many cases be supplemented by some additional information in order to determine the gauge algebra. In [9], this is expressed in terms of a "monodromy cover" of C given by a polynomial equation $\mu(\psi) = 0$ of degree 2 or 3, and it must be determined whether this is an irreducible cover, and if reducible, how many components it has. When the monodromy cover has degree 2 there is a natural condition which determines this: one needs to check if the ramification divisor (which is the divisor of zeros of the discriminant of $\mu(\psi)$) is divisible by two or not.

When the monodromy cover has degree 3, however, the situation is much more subtle; we will return to this case shortly.

Closely related to the monodromy analysis is one of the two types of contributions to the charged matter content of the F-theory compactification: the "non-local matter." The non-local matter consists of genus(C) adjoint hypermultiplets, together with genus($\widetilde{C}$) copies of another representation if there is a component $\widetilde{C}$ of the monodromy cover which is not isomorphic to C. In addition to the non-local matter, there are "local" contributions to the matter representation from each zero of the residual discrimimant $\Delta_0 = (\Delta/z^n)|_C$ (although a local contribution may be trivial if the corresponding zero is also in the ramification divisor of the monodromy cover).

In the case of an isolated curve C which is an NHC, we write $-12K = mC + X$ where m is the order of vanishing of Δ, so that $\Delta_0 = X \cdot C$. It is easy to check that $X \cdot C = 0$ except in two cases. For a (-7)-curve C, we write $-12K = 9C + X$ and find that $X \cdot C = 3$. In this case, $\Delta_0 = ((f/z^3)|_C)^3$, so Δ_0 has a single zero of multiplicity three: it corresponds to a $\frac{1}{2}\mathbf{56}$ hypermultiplet of $\mathfrak{e}_7$, according to [9].

Similarly, for a (-5)-curve C, we write $-12K = 8C + X$ and find that $X \cdot C = 4$. In this case, according to [9], we have $\Delta_0 = (\mathrm{disc}(\mu(\psi)))^2$, so the ramification divisor of the monodromy cover has two zeros. If those zeros are not distinct, then the cover splits into two components, the gauge algebra is $\mathfrak{e}_6$, and the matter representation is a single $\mathbf{27}$ hypermultiplet. From the geometry side, we can vary coefficients in the equation to guarantee that the zeros of $\mathrm{disc}(\mu(\psi))$ become distinct; from the physics side, we can Higgs the $\mathbf{27}$. Thus, from either approach we see that this is not the correct description of the NHC.

The alternative is that the two zeros of the ramification divisor are distinct. Then the gauge algebra is $\mathfrak{f}_4$, we have genus($\widetilde{C}$) $= 0$ so there is no non-local matter, and the zeros of $\mathrm{disc}(\mu(\psi))$ do not contribute to the local matter either. Thus, we find an NHC as stated in Table 2: the gauge algebra is $\mathfrak{f}_4$ and there is no charged matter.

In all other cases of an NHC with a single curve, $X \cdot C = 0$ and there is no possibility of localized matter. For $C^2 = -3$, -4, or -6, we must determine the monodromy cover in order to determine the gauge algebra. However, in all of these cases, $X \cdot C = 0$ implies that the monodromy polynomial $\mu(\psi)$ has constant coefficients and so it must factor completely. This leads to the maximum possible gauge algebra in these three cases, namely $\mathfrak{su}(3)$, $\mathfrak{so}(8)$, and $\mathfrak{e}_6$. Moreover, there cannot be any non-local matter in these cases because genus(C) $= 0$.

We note in passing that this discussion illuminates our earlier explanation of why curves C with $C^2 = -9$, -10, or -11 cannot be NHC's. In these cases, we write $-12K = 10C + X$ and find that $X \cdot C = 6$, 4, or 2. But any nonzero value of $X \cdot C$ with gauge algebra $\mathfrak{e}_8$ leads to a point of intersection where the multiplicities exceed $(4, 6, 12)$ and so such curves are not allowed on an F-theory base.

We return now to a closer analysis of monodromy covers of degree 3. The only case in which a monodromy cover of degree 3 occurs is when f vanishes to order at least 2, g vanishes to order at least 3, and Δ vanishes to order exactly 6. In this case, the covering

polynomial is

$$\mu(\psi) = \psi^3 + (f/z^2)\psi + (g/z^3), \tag{3.23}$$

where $z = 0$ is a local defining equation for C. The discriminant of this polynomial is

$$\mathrm{disc}(\mu) = 4(f^3/z^6) + 27(g^2/z^6) = \Delta/z^6 = \Delta_0. \tag{3.24}$$

Let us observe some things about the three possible cases, in which the monodromy cover has various numbers of components.

If the monodromy cover has three components, then the covering polynomial factors as

$$\mu(\psi) = (\psi - a)(\psi - b)(\psi - c), \tag{3.25}$$

where $a + b + c = 0$. This is the case of gauge algebra $\mathfrak{so}(8)$. The discriminant of this polynomial is

$$\Delta_0 = (a - b)^2(a - c)^2(b - c)^2, \tag{3.26}$$

which is a perfect square. Moreover, the three factors of $\sqrt{\Delta_0}$ are associated to three different local contributions to the charged matter representation: each zero of $a - b$ is associated to an 8_v, each zero of $a - c$ is associated to an 8_+ and each zero of $b - c$ is associated to an 8_- (up to the permutation among these representations induced by triality of $\mathfrak{so}(8)$).

If the monodromy cover has two components, then the polynomial $\mu(\psi)$ factors as

$$\mu(\psi) = (\psi - a)(\psi^2 + d\psi + e) \tag{3.27}$$

(where $a - d = 0$), and the quadratic factor must define an irreducible cover, i.e.,

$$\mathrm{disc}(\psi^2 + a\psi + b) = a^2 - 4b \tag{3.28}$$

must not be a square. This is the case of gauge algebra $\mathfrak{so}(7)$. In this case, we can write

$$\Delta_0 = -(a^2 - 4b)(2a^2 - b)^2, \tag{3.29}$$

that is, $\Delta_0 = \alpha\beta^2$ with α not a square.

The zeros of β are associated to spinor representations $\mathbf{8}$ of $\mathfrak{so}(7)$. The zeros of α do not directly determine matter, but they do determine the genus of the nontrivial component $\widetilde{C}$ in the monodromy cover, and there are genus($\widetilde{C}$) copies of the vector representation $\mathbf{7}_v$ of $\mathfrak{so}(7)$.

Finally, if the polynomial $\mu(\psi)$ is irreducible, the gauge algebra is $\mathfrak{g}_2$ and there are no local contributions to the matter. However, the zeros of the discriminant will determine the genus of the cover $\widetilde{C}$, and there are genus($\widetilde{C}$) copies of the 7-dimensional representation.

Let us now use these facts to finish our analysis of the NHC's in the last three lines of Table 2. In all three cases, we have a curve C_1 of self-intersection -3 with (f, g, Δ) multiplicities of $(2, 3, 6)$ or $(2, 4, 6)$. Thus, we are in the situation with a monodromy cover of degree 3. Moreover, if we write $-12K = 6C_1 + X$, then $X \cdot C_1 = 6$ so there is room for some charged matter. If the factor of the gauge algebra corresponding to C_1 is $\mathfrak{so}(8)$, then

$X \cdot C_1$ will be twice a divisor of degree 3, and the matter representation will be $\mathbf{8}_v \oplus \mathbf{8}_+ \oplus \mathbf{8}_-$. (Note that the matter may be charged with respect to other summands of the gauge algebra as well: we will come back to that point.) If the gauge algebra is $\mathfrak{so}(7)$, then $X \cdot C_1$ takes the form $D_1 + 2D_2$ with each D_i having degree 2, and the matter consists of two copies of $\mathbf{8}$ (because in this case, genus$(\widetilde{C}) = 0$). And finally, if the gauge algebra is $\mathfrak{g}_2$ then the charged matter represention is genus$(\widetilde{C}) = 1$ copy of the 7-dimensional representation.

The NHC clusters $(-3, -2)$ and $(-2, -3, -2)$ have a second curve C_2 of self-intersection -2 with $(f, g, \Delta) = (1, 2, 3)$ which meets C_1. We can write

$$-12K = 6C_1 + 3C_2 + Y \qquad (3.30)$$

and we see the previous $X \cdot C_1$ break up into $3P + Y \cdot C_1$ where P is the intersection point of C_1 and C_2. Thus, if the gauge algebra factor for C_1 is $\mathfrak{so}(7)$ we must have $Y \cdot C_1 = P + Q_1 + Q_2$ and if it is $\mathfrak{so}(8)$ we must have $Y \cdot C_1 = P + 2Q$. (Both of these are non-generic, so we expect geometrically to get to $\mathfrak{g}_2$ by choosing f and g generically.)

On the other hand, $(6C_1 + Y) \cdot C_2 = 6P + Y \cdot C_2$ should be three times a divisor (according to [9] for type III) and as is easy to calculate, $Y \cdot C_2 = 0$, so the localized matter is associated with the divisor $D = 2P$. The spectrum consists of $2 \deg D$ fundamentals, i.e. 4 fundamentals, or 8 half-fundamental hypermultiplets. Thus, the combined matter associated to the point P adds up to

$$(8, \frac{1}{2}\mathbf{2}) \qquad (3.31)$$

as a representation of $\mathfrak{g}(C_1) \oplus \mathfrak{su}(2)$, where 8 is an appropriate 8-dimensional representation of $\mathfrak{so}(8)$, $\mathfrak{so}(7)$ or $\mathfrak{g}_2$. (In the latter case, the representation can be written $\mathbf{7} \oplus \mathbf{1}$.)

In the $(-3, -2)$ case, if the gauge algebra is larger there will be additional matter of the form $(8, \mathbf{1})$ (one such multiplet for $\mathfrak{so}(7)$ and two from $\mathfrak{so}(8)$) but these can be Higgsed, leaving the NHC with gauge algebra $\mathfrak{g}_2 \oplus \mathfrak{su}(2)$ and matter $(\mathbf{7} \oplus \mathbf{1}, \frac{1}{2}\mathbf{2})$.

In the $(-2, -3, -2)$ case, there is an additional curve C_0 leading to a second local matter contribution of the form $(8, \frac{1}{2}(\mathbf{2}))$, this time charged under the $\mathfrak{su}(2)$ associated to C_0. Thus, in this case the minimum gauge algebra which can occur is $\mathfrak{so}(7)$ and if $\mathfrak{so}(8)$ occurred there would be an extra multiplet neutral under both $\mathfrak{su}(2)$'s, which could be Higgsed. The conclusion is that the minimum gauge algebra is $\mathfrak{su}(2) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(2)$.

Finally, in the $(-3, -2, -2)$ case, the curve C_2 adjacent to C_1 is a (-2)-curve with $(f, g, \Delta) = (2, 2, 4)$, and the third curve C_3 with $(f, g, \Delta) = (1, 1, 2)$ makes no contribution to the gauge algebra. In this case, we write

$$-12K = 6C_1 + 4C_2 + 2C_3 + Z, \qquad (3.32)$$

and we find $Z \cdot C_1 = 2$ while $Z \cdot C_2 = Z \cdot C_3 = 0$. According to [9], the residual discriminant for the C_2 component satisfies $\Delta_0 = (\mathrm{disc}(\mu(\psi)))^2$ in this type IV case, so it has divisor $3P + Q$ where $P = C_1 \cap C_2$ and $Q = C_2 \cap C_3$. In particular, the ramification divisor is not even, so the gauge algebra is $\mathfrak{su}(2)$ rather than $\mathfrak{su}(3)$. Anomaly cancellation implies that the charged matter for this $\mathfrak{su}(2)$ is 4 fundamentals, so we have a very similar spectrum to the $(-3, -2)$ case. Note that in this case as well, varying coefficients in f and g or Higgsing the extra 8-dimensional representations (which are neutral under $\mathfrak{su}(2)$) should reduce that gauge algebra summand to $\mathfrak{g}_2$.

4. Bounding the number of tensors

From the analysis of the previous section, we expect that all F-theory models will have a maximally Higgsed gauge group and matter content that decomposes into factors associated with the non-Higgsable clusters (NHC's) listed in Table 2. This gives a simple way of classifying theories according to gauge algebra and matter content. In this section we show that a theory with any given combination of NHC's has a maximum number of tensors.

4.1 Theories without vector multiplets

We begin by considering F-theory constructions of theories without gauge groups. Such models arise as generic (maximally Higgsed) theories on F-theory bases with no irreducible effective divisors that are rational curves with $C \cdot C < -2$. Thus, on such a base $-K \cdot C \geq 0$ for all rational curves. The analysis of more general F-theory models follows a similar logic, though the details are more complicated; we consider general models in the following section.

We assert that for any F-theory model without vector multiplets the number of tensor multiplets is bounded above by

$$T < 10 , \quad \text{if the gauge group } G \text{ is trivial.} \tag{4.1}$$

This can be proven from F-theory as follows: decompose $-K$ into a sum of irreducible effective curves as

$$-K = \sum_i l_i C_i \tag{4.2}$$

If $K \cdot K < 0$ then for some i, $-K \cdot C_i < 0$ and there is a nonabelian gauge group factor. Thus, for theories with trivial gauge group $K \cdot K = 9 - T \geq 0$ so $T < 10$.

Another way to understand this bound is from the gravitational anomaly cancellation condition

$$H - V = 273 - 29T \tag{4.3}$$

In this relation, H is the total number of hypermultiplets, V is the number of vector multiplets, and T is the number of tensors. For theories with no vector multiplets, we have $H = 273 - 29T$, which must be positive, so $T \leq 9$. The argument based on (4.2) carries the same information as the gravitational anomaly bound, though from a rather different geometric perspective. Note that the first argument depends upon an F-theory realization, while the second argument only depends upon quantum consistency of the supergravity theory. On the other hand, the second argument assumes that there are no $U(1)$ factors in the gauge group, while the first argument is independent of the existence of $U(1)$ factors. $U(1)$ factors add significant subtleties to analysis from the supergravity point of view, as discussed in [11, 12].

There is a complete classification of the possible F-theory bases without nonabelian gauge group factors in the generic maximally Higgsed phase. Surfaces on which all rational curves are (-1)-curves are del Pezzo surfaces. The del Pezzo surface dP_k is constructed by blowing up k generic points on $\mathbb{P}^2$. Surfaces containing rational curves with self intersection

−1 and −2 are *generalized del Pezzo* surfaces [41]. Generalized del Pezzo surfaces are limits in the moduli space of ordinary del Pezzo surfaces. Generalized del Pezzo surfaces can be classified by the intersection structure of the (−2)-curves. For a generalized del Pezzo surface with $h^{1,1}(B) = T + 1 = k + 1$, this intersection structure must form a sub-root lattice of the affine root lattice $\hat{E}_k$ [42]. Generalized del Pezzo surfaces all have $T \leq 9$, in agreement with the bound (4.1).

4.2 Bounding the number of tensors for a given gauge algebra

We can now generalize the argument from Section 4.1 to an arbitrary F-theory base. Any F-theory base has a system of effective irreducible divisors characterized by a set of disjoint irreducible components (NHC's) from Table 2. These components are connected by curves of self-intersection −1 that carry no gauge group (with additional possible clusters of (−2)-curves appearing that carry no gauge group as in the generalized del Pezzo models discussed above). For each NHC component, the bound on the number of tensors increases. For example, consider an $\mathfrak{so}(8)$ summand associated with an isolated (−4)-curve C. Since −12K must include 6 copies of C, and satisfies $-12K \cdot C = -24$, we have

$$-12K = 6C + X \;\rightarrow\; 144K^2 = 36C^2 + X^2 = -144 + X^2\,, \tag{4.4}$$

where we have used $X \cdot C = 0$. Following the same reasoning as above, if the only gauge algebra summand is the single $\mathfrak{so}(8)$, then $X^2 \geq 0$, so $K^2 > -1$. We have proven that

$$\mathfrak{g} = \mathfrak{so}(8) \;\Rightarrow\; T \leq 10\,. \tag{4.5}$$

Each additional gauge algebra summand of $\mathfrak{so}(8)$ in the maximally Higgsed theory on a given base thus raises the bound on T by 1.

The same general argument can be carried out for each of the other minimal components. Consider an isolated (−m)-curve C that appears in Δ with multiplicity $c = \lceil 12(m-2)/m \rceil$. We have $C^2 = -m, -K \cdot C = 2 - m$, so writing $-12K = cC + X$ we have $(-12K) \cdot C = cC \cdot C + X \cdot C$ and $X \cdot C = 24 - 12m + cm$. We can then substitute into

$$144K^2 = c^2 C^2 + 2cX \cdot C + X^2 = X^2 - c(24m - mc - 48)\,. \tag{4.6}$$

Thus, each such component in the maximally Higgsed gauge group increases the allowed number of tensors by

$$\Delta T_{\max} = \frac{1}{144}c(24m - mc - 48) = \frac{1}{144}(24m - m\lceil 12(m-2)/m\rceil - 48)\lceil 12(m-2)/m\rceil\,. \tag{4.7}$$

These are the terms listed in the last column of Table 2.

A similar contribution to the bound on T arises for non-Higgsable clusters containing more than one curve of self intersection −2 or below. A simple calculation along the lines of the above shows that the extra contribution to the bound on T from a general NHC is the sum of the terms from each curve, plus an additional contribution from each intersection of $2c_i c_j C_i \cdot C_j$, where we expand $-12K = \sum_i c_i C_i + X$. We have tabulated the increase in the bound on the number of tensors for each non-Higgsable cluster in Table 2.

4.3 Bounds on linear chains of curves

As an example of how the bounds described in the previous section can be combined with knowledge of possible NHC configurations to limit the range of possible F-theory bases, we consider linear chains of effective irreducible divisors with negative self-intersection. This gives a simple class of configurations of curves with large gauge groups where the upper bound on the number of tensors and NHC's is clearly evident. We consider in particular linear chains of divisors C_i each of negative self-intersection, where the only nonzero intersections between distinct divisors is $C_i \cdot C_{i+1} = 1$. We will find it particularly interesting to consider linear chains that consist of a repeated pattern of divisors. Such a pattern, for example, appears in the F-theory base for the model with the largest known value of T [24].

Consider for example a periodic chain of divisors with self-intersections

$$(\ldots, -4, -1, -4, -1, -4, \ldots) \tag{4.8}$$

(See Figure 2 (a)). For the moment we simply consider an idealized infinite divisor chain; boundary conditions for finite chains will be discussed shortly. This chain of divisors has several properties:

1. Each link in the chain is allowed in an F-theory model (*i.e.*, each link is either part of an NHC or a (-1)-curve connecting NHC's in a fashion allowed by the rules in Table 3 in the Appendix)

2. The chain is (locally) *maximal*, in the sense that no self-intersection number of any link in the chain can be increased while maintaining property (1).

3. The chain can be reduced by blowing down successive (-1)-curves until all nonabelian gauge groups are removed without separating the chain.

We call a chain with the first two properties a *maximal linear divisor chain*. To verify the third property, note that when a (-1)-curve A connecting curves B, C of self-intersection $-m$ and $-n$ is blown down, the (-1)-curve is replaced by a single point of intersection of B and C, and the self-intersections of these curves become $-m + 1$ and $-n + 1$. In the specific example of the chain (4.9), blowing down each of the (-1)-curves gives a new chain of the form $(\ldots, -2, -2, -2, \ldots)$ with no curves of self-intersection below -2 and hence no gauge group.

Now consider the gauge algebra on the finite chain

$$(-1, -4, -1, \ldots, -1, -4, -1) \tag{4.9}$$

with N (-4)-curves. From Table 1 this combination of NHC's gives N nonabelian gauge algebra summands $\mathfrak{so}(8)$. From the analysis in the previous section, the number of tensors in a model with this gauge algebra is bounded by $T \leq 9 + N$. Blowing down all $(N + 1)$ of the (-1)-curves in the chain gives a chain of length N containing only (-2)-curves. For this to be an F-theory model there must be a boundary condition that enables this chain to

be blown down further to get to either $\mathbb{F}_m$ or $\mathbb{P}^2$. Each blow-down removes one (-2)-curve. So the number of blow-downs necessary to reduce the original chain to one with a single curve of negative self-intersection is of order $\mathcal{O}(2N)$. This means that the original chain had a number of tensors of $T \sim 2N$. Comparing to the bound we have

$$T \sim 2N \leq 9 + N . \tag{4.10}$$

This means that the chain of $\mathfrak{so}(8)$ NHC's cannot contain more than on the order of 9 (-4)-curve components. This can also be seen by noting that the number of -2 factors in a linear chain in a model with no nonabelian gauge groups is < 9, from the discussion of generalized del Pezzo surfaces above. A more precise bound on N for the chain (4.9) requires a specific choice of boundary condition that allows the chain to be blown down to a surface $\mathbb{F}_m$. By choosing this boundary condition properly it can be shown that the value $N = 9$ can be realized. The basic idea is to terminate the chain on both ends with the (-4)-curves, including two extra (-1)-curves connecting to the next-to-last (-4)-curves on each end. By blowing down from one end and leaving a single terminal -4 intact, this gives a configuration that can be blown down to $\mathbb{F}_4$ with 16 blow-downs, giving $T = 17 < 9 + 9 = 18$. This example and other related examples are described in more detail in the context of a complete analysis of toric bases in a paper that will appear as a sequel to this work [25].

Now let us consider other possible periodic maximal linear divisor chains. There are only a few other possibilities that satisfy conditions (1) and (2) above. These are the periodic chains

χ_6: $(\ldots, -6, -1, -3, -1, -6, -1, -3, \ldots)$

χ_8: $(\ldots, -8, -1, -2, -3, -2, -1, -8, -1, -2, -3, -2, \ldots)$

χ_{12}: $(\ldots, -12, -1, -2, -2, -3, -1, -5, -1, -3, -2, -2, -1, -12, \ldots)$

These chains can be generated by starting with a NHC containing a single $(-m)$-curve with $m = 6, 8, 12$ and connecting iteratively to the curve of the most negative possible self-intersection. The same algorithm generates the sequence (4.9) (χ_4) when begun with $m = 4$, and sequence χ_{12} above when begun with $m = 5$. For each of the chains χ_n that satisfy conditions (1) and (2), condition (3) is satisfied as well. In each case, iteratively blowing down all possible (-1)-curves leads to a sequence of connected (-2)-curves carrying no gauge group, just as in the case of the sequence (4.9). In fact, blowing down the (-1)-curves appearing in χ_8 gives the chain χ_6, and blowing down the (-1)-curves in χ_6 gives the chain χ_4. Blowing down the (-1)-curves in χ_{12} gives the chain $(-10, -1, -2, -2, -3, -2, -2, -1, -10, \ldots)$, which does not satisfy condition (1), but blowing down the (-1)-curves in this chain gives χ_8.

It is fairly straightforward to check that any other periodic chain that satisfies condition (1) but that does not satisfy the maximality condition (2) will become disconnected when blown down and will not satisfy condition (3). For example, consider replacing the (-6)-curves in χ_6 with (-5)-curves. This does not satisfy condition (2). Blowing down all (-1)-curves in this chain would give the chain $(-1, -3, -1, -3, \ldots)$; blowing down again gives
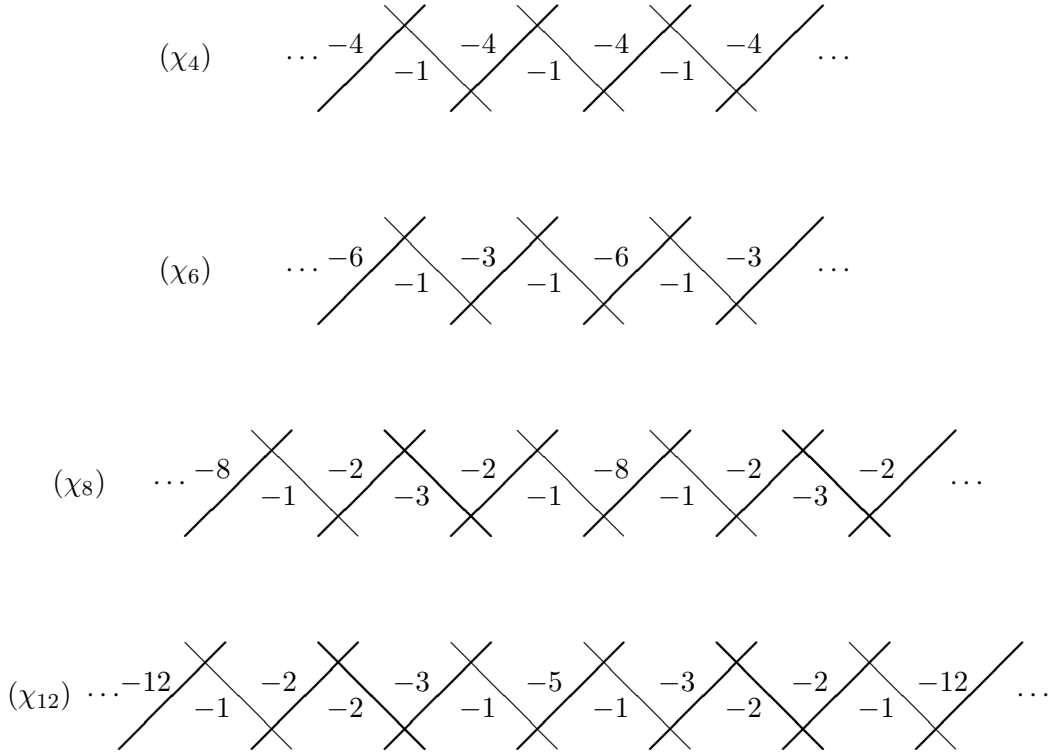
Figure 2: Periodic linear chains of divisors with simple gauge algebras. Chains shown are all those that satisfy a local maximality condition on the self-intersection numbers of the divisors. Bounds on the number of tensors given the gauge group place a limit on the size of such chains.

the chain $(-1, -1, -1, \ldots)$, which becomes disconnected when any further (-1)-curves are blown down. Thus, conditions (1) and (2) together are apparently necessary and sufficient to ensure that condition (3) holds.

Just as for the chain (4.9), we can use the bounds on the number of tensors computed above to estimate the maximum length possible for the chains χ_n. For χ_6, the gauge algebra is $N(\mathfrak{e}_6 \oplus \mathfrak{su}(3))$ or $(N \pm 1)\mathfrak{e}_6 \oplus N(\mathfrak{su}(3))$, depending upon which algebras appear on the two ends of the chain. From Table 2, the increase in the tensor bound ΔT is $8/3$ for each $\mathfrak{e}_6$ summand and $1/3$ for each $\mathfrak{su}(3)$ summand. If the number of summands is equal to N for each algebra, this gives $\Delta T = 3N$. The number of blow-downs necessary to bring each chain segment $(-6, -1, -3, -1)$ to a single (-2) is 3. So as in the example (4.9), the number of tensors needed is of order $T \sim \mathcal{O}(4N)$, and we have $T \sim 4N \leq 3N + 9$, so again the maximum number of cycles in the periodic chain will be of the order of $N \sim \mathcal{O}(9)$. A very similar analysis holds for chain χ_8, where each gauge algebra component $\mathfrak{e}_7 \oplus (\mathfrak{su}(2) \oplus \mathfrak{so}(7) \oplus \mathfrak{su}(2))$ contributes $\Delta T = 5$ and requires 5 blow-downs to bring each segment to the form (-2).

The final chain type, χ_{12}, is perhaps most interesting. This is the type of chain that appears in the F-theory realization of the model with largest known T [24]. Each segment

in the chain gives a contribution to the algebra of $\mathfrak{e}_8 \oplus \mathfrak{f}_4 \oplus 2(\mathfrak{g}_2 \oplus \mathfrak{su}(2))$, and gives an increase in ΔT of $25/3 + 16/9 + 5/6 = 10\frac{17}{18}$, while requiring 11 blow-downs to get to a single -2 from each segment in the chain. Since $\mathfrak{e}_8$ contributes the greatest part of the increase to the bound on T, to maximize the size of the chain it is desirable to use boundary conditions where an $\mathfrak{e}_8$ terminates the chain on both sides. With N copies of the basic link and this boundary condition, the gauge algebra is

$$\mathfrak{g} = (N+1)\mathfrak{e}_8 \oplus N(\mathfrak{f}_4) \oplus 2N(\mathfrak{g}_2 \oplus \mathfrak{su}(2)) \,. \tag{4.11}$$

The upper bound on the number of tensors possible for this gauge algebra is

$$T \leq 9 + N \times \frac{197}{18} + \frac{25}{3} \,, \tag{4.12}$$

while the number of blow-downs needed to reduce to a single curve of negative self-intersection is of the order of $12 \times N$ (one for each curve removed). Estimating

$$12 N_{\max} \cong \frac{52}{3} + \frac{197}{18} N_{\max} \quad \Rightarrow \quad \frac{19}{18} N_{\max} \cong \frac{52}{3} \quad \Rightarrow \quad N < N_{\max} \cong 16.4 \,. \tag{4.13}$$

In fact, this configuration can be realized for $N = 16$, with $T = 193$. As in the case of the chain (4.9), this can be done by attaching an extra pair of (-1)-curves to the next-to-last $\mathfrak{e}_8$ summands in the chain. We can check that a sequence of this type with $N = 17$ is not possible. Plugging $N = 17$ into (4.12) gives $T \leq 203.4$. But to remove all but one negative self-intersection divisors from the chain would require $17 * 12 = 204$ tensors, so we would need at least $T = 205$ even without adding additional (-1)-curves as are needed in the $N = 16$ case. Thus, the bounds we have determined here give a clear limit to the size of sequences of this type. As mentioned above, a configuration of this type with $N = 16$ was identified in [24] and represents the model with the largest known gauge group and value of T. In a sequel to this paper [25] we show that this model naturally fits into the framework of toric F-theory bases, and is indeed the maximal value of T that can be attained in that context.

5. Conclusions

We have shown that certain configurations of intersecting divisors on the base surface of a 6D F-theory model give rise to gauge algebras and matter content that cannot be removed by Higgsing scalar fields in the theory. We have tabulated all such possible "non-Higgsable clusters" (NHC's) in Table 2. Any base of a consistent 6D F-theory model contains some number of NHC's connected by (-1)-curves that do not contribute to the nonabelian gauge algebra. As for generalized del Pezzo surfaces, additional combinations of (-2)-curves that do not connect to the NHC's can appear in special limits of the moduli but do not affect the maximally Higgsed gauge algebra or matter content.

We have also determined a bound on the number of tensor fields T in a theory based on the NHC content of the theory. For models with no nonabelian gauge algebra in the maximally Higgsed phase, $T \leq 9$. Each NHC contributes a fixed positive quantity to the

upper bound on T as described in Table 2. Taken together, the classification of NHC's and the upper bound on T bound and characterize the bases possible for 6D F-theory models.

While we have identified the geometry of all divisor combinations giving NHC's, there are several things that we have not proven. First, while the gauge and matter content associated with each NHC cannot be Higgsed, we have not proven that every F-theory model has sufficient degrees of freedom to Higgs all fields down to the minimal NHC content. While we believe that this is true, it is possible in principle that there some complicated F-theory models might not have enough scalar degrees of freedom to break the gauge algebra and matter content through Higgsing to the minimal spectrum required by the NHC's. Second, our analysis has been carried out purely in the context of F-theory. The question of possible gauge groups (or algebras) and matter content in the maximally Higgsed phase can be posed more generally for the class of all quantum-consistent 6D supergravity models. There are 6D supergravity models satisfying all known quantum consistency constraints that cannot be realized in F-theory [4, 5, 6]. For example, there is an apparently-consistent 6D supergravity model containing an $\mathfrak{su}(8)$ gauge algebra and matter in the "box" (**336**) representation that violates the Kodaira constraint from F-theory. It is possible that there are "non-Higgsable" gauge algebra and matter factors that cannot appear in F-theory but that are possible in consistent 6D supergravity models. We conjecture that the NHC content identified in this paper describes all maximally Higgsed 6D supergravity theories, whether realized through F-theory or not. Further analysis of this question is left as an open problem for future work.

By combining the NHC structure identified in this paper with bounds on the number of tensors and the geometry of blow-up processes, it should be possible in principle to systematically identify all bases for 6D F-theory models. In a sequel to this paper [25] we carry out such an analysis in the case of toric bases. For each F-theory base, a wide range of different gauge groups and matter content can be realized by allowing the elliptic fibration to become more singular without requiring a blow-up of the base — this corresponds to "un-Higgsing" the theory away from the maximally Higgsed content determined from the NHC structure on the base. Recent papers describe some of the range of possibilities of models that can be realized over simple bases such as $\mathbb{P}^2$ and $\mathbb{F}_m$ [4, 6, 10, 8]. Combining analysis of bases through the methods of this paper with such analyses for each base gives tools for understanding the complete space of 6D F-theory models.

While the results of this paper are directly relevant only for six-dimensional theories, the general philosophy and ideas underlying this analysis should have an analogue in four dimensions, where the space of theories is much richer and more complex. In four dimensions there are a number of additional subtleties and issues that would need to be addressed for any kind of systematic analysis. In particular, the intersection form is a triple intersection product, and is less well understood than the intersection form on complex surfaces relevant in the 6D case. Also, in four dimensions, fluxes complicate the story by lifting moduli and modifying the set of massless degrees of freedom. Nonetheless, it may be possible in four dimensions to identify some analogue of the "non-Higgsable cluster" structures we have found in 6D that characterize many key aspects of the theories. We

leave investigation of these questions in four dimensions to future work.

A. Appendix: systematic analysis of (-1)-curves intersecting NHC's

In this Appendix we summarize the results of a systematic analysis of which combinations of NHC's can be intersected by a single (-1)-curve on a valid F-theory base. The table below gives a list of all possible ways in which a (-1)-curve can intersect a single NHC. The NHC is denoted by an ordered k-tuple of self-intersection numbers for the irreducible effective curves comprising the NHC. The number of times the (-1)-curve intersects each curve in the NHC is denoted by the number of dots over the number representing each curve. The possible intersections of a (-1)-curve with a single NHC are indexed by an integer n. For each n the possible pairs $[n, n']$ of NHC intersections by a (-1)-curve are listed (listing only values $n' \geq n$; the set of possible pairs is symmetric under interchange $n \leftrightarrow n$). Following each n' denoting a pair, a list is given of possible values $n'' \geq n'$ for which the triplet $[n, n', n'']$ of intersections is allowed. (For example, a single (-4)-curve can appear in the triplet combinations $[6, 10, 16], [6, 16, 16]$, and $[6, 16, 25]$.) Thus, the table contains all possible combinations of 1, 2, or 3 NHC's that can be intersected by a (-1)-curve. There are also 6 combinations of 4 NHC intersections possible. These are given by all configurations containing 2 intersections of type 16 (a $(-3, -2, -2)$ NHC intersected along the last -2), and another two intersections that are any combination of types 10, 16, and 25. (*i.e.*, the 6 possible configurations connecting 4 NHC's are $[10, 10, 16, 16]$, $[10, 16, 16, 16]$, $[10, 16, 16, 25]$, $[16, 16, 16, 16]$, $[16, 16, 16, 25]$, $[16, 16, 25, 25]$). This gives a total of 183 possible ways in which a single (-1)-curve can intersect a combination of 1, 2, 3, or 4 NHC's. As discussed in the main text, the gauge group and matter content of these configurations are all determined by the product of the factors associated with each separate NHC.

A few comments may be helpful regarding the analysis for anyone interested in reproducing these or similar computations. To determine the degree of vanishing of $-nK$ on any system of curves $\{C_i\}$, we must identify the minimal set of non-negative integral values c_i so that

$$-nK = \sum_i c_i C_i + X \,, \tag{A.1}$$

where $X \cdot C_j \geq 0 \; \forall j$. For some systems there is no set of values c_i that satisfy these conditions. We have used the following algorithm to determine the solution for c_i: First, we determine the minimum values $c_i^{(0)}$ needed for each curve independently (*i.e.*, ignoring intersections between distinct curves). We then iteratively increase the values c_i to compensate for intersections using the previous values at each stage. Defining $M_{ij} = C_i \cdot C_j$, we thus have

$$c_i^{(0)} = \lceil n(m-2)/m \rceil, \tag{A.2}$$

just as in the case of a single divisor discussed in Section 3. Defining $v_i = -nK \cdot C_i = n(2-m)$, we then use the difference between the desired value on the LHS and the value computed using $c_i^{(0)}$ to determine the next correction (which is not allowed to be negative)

$$c_i^{(1)} = c_i^{(0)} + \mathrm{Max}(0, \lceil \frac{v_i - M_{ij}c_j^{(0)}}{-m} \rceil) \,, \tag{A.3}$$

and so forth. When this process converges we can test to check if any intersection points have vanishing degrees 4, 6, 12 or above indicating a singularity; when the process does not converge the values continue to climb, indicating the absence of a solution. This method was used to determine all valid intersecting curve combinations described in this Appendix.

The solution to the degree equation (A.1) also can be characterized in terms of linear algebraic inequalities. Stating the conditions on the c_i's in terms of inequalities, we have the conditions

$$c_i = (M^{-1})_{ij}(v_j - z_j) \tag{A.4}$$

where $z_j \geq 0$ are non-negative integers. For any given system we can compute M^{-1}. This then defines a cone of values within $\mathbb{Q}^n$ containing the points defined by the RHS of (A.4) for different n-tuples of non-negative integers z_i; the solution c_i must lie in the intersection between this cone and the sector of non-negative integer n-tuples c_i. This characterization can be used to identify situations where there is no solution, but the analysis can be slightly subtle. For example, in the case of a $(-3, -2, -2)$ NHC intersecting a (-1)-curve along each of the 3 curves in the NHC, the linear algebra gives a rational value for $M^{-1}v$ that is positive in all components, but there is no vector of non-negative integers z_j such that the resulting c_i is non-negative itself for $-4K$, while a solution exists for $-6K$ and $-12K$. This approach also does not uniquely determine the solution for c_i. For example, consider a -1 connecting the NHC's $(-3, -2, -\dot{2})$, $(-\dot{3}), (-\dot{3})$. Working out the corresponding intersection matrix shows that (A.4) gives a solution with all $c_i = n/2$ when $z_j = 0$ for all j. This is, however, a singular configuration, and is not the minimal solution possible. The algorithm described in the preceding paragraph gives the correct solution in this case, which has vanishing degree 0 for g and Δ along the (-1)-curve and no change in the degrees of vanishing of these multiples of $-K$ on the remaining curves, as can immediately be seen from the fact that condition (3.19) is satisfied for $n = 6, 12$.

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Index (n)	NHC	pairs {triplets}
1	$(-\dot{12})$	16
2	$(-\dot{8})$	10, 16, 25
3	$(-\dot{7})$	10, 16, 25
4	$(-\dot{6})$	7, 10, 15, 16, 25
5	$(-\dot{5})$	7, 9, 10, 14, 15, 16, 25
6	$(-\dot{4})$	6, 7, 9, 10 {16}, 14, 15, 16 {16, 25}, 22, 25, 26
7	$(-\dot{3})$	7 {7, 10, 16}, 9, 10 {10, 16, 25}, 13, 14, 15, 16 {16, 25}, 22, 25 {25}, 26, 27, 29
8	$(-\ddot{3})$	
9	$(-\dot{3}, -2)$	9, 10 {10, 16}, 14, 15, 16 {16, 25}, 22, 25, 26
10	$(-3, -\dot{2})$	10 {10, 14, 15, 16, 22, 25}, 12, 13, 14 {16}, 15 {16, 25}, 16 {16, 22, 25}, 19, 20, 22 {25}, 25 {25}, 26, 27, 29
11	$(-\ddot{3}, -2)$	
12	$(-\dot{3}, -\dot{2})$	16
13	$(-3, -\ddot{2})$	15, 16 {16}, 22, 25
14	$(-\dot{3}, -2, -2)$	14, 15, 16 {16, 25}, 22, 25, 26
15	$(-3, -\dot{2}, -2)$	15, 16 {16, 25}, 22, 25 {25}, 26, 27, 29
16	$(-3, -2, -\dot{2})$	16 {16, 22, 25, 26, 27, 29}, 19, 20, 22 {25}, 25 {25}, 26, 27, 29
17	$(-\ddot{3}, -2, -2)$	
18	$(-\dot{3}, -\dot{2}, -2)$	
19	$(-\dot{3}, -2, -\dot{2})$	25
20	$(-3, -\dot{2}, -\dot{2})$	25
21	$(-3, -\ddot{2}, -2)$	
22	$(-3, -2, -\ddot{2})$	22, 25 {25}, 26, 27, 29
23	$(-\dot{3}, -2, -\ddot{2})$	
24	$(-3, -\dot{2}, -\ddot{2})$	
25	$(-\dot{2}, -3, -2)$	25 {25}, 26, 27, 29
26	$(-2, -\dot{3}, -2)$	26
27	$(-\ddot{2}, -3, -2)$	
28	$(-\dot{2}, -\dot{3}, -2)$	
29	$(-\dot{2}, -3, -\dot{2})$	
30	$(-\dot{2}, -3, -\ddot{2})$	
31	$(-\ddot{4})$	

Table 3: Table of all ways in which a (-1)-curve can intersect a single non-Higgsable cluster of irreducible effective divisors carrying a nonabelian gauge group. Each cluster (NHC) is denoted by a list of self-intersections of curves connected in a linear chain, with dots indicating intersection with the (-1)-curve. The last column of the table includes information about all pairs and triplets of NHC's that can be intersected by the (-1)-curve, as described in the main text of the Appendix.